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Finite Time Stability and Stabilization of Impulsive Dynamical Control Systems

Ruyi Yuan, Jiangqiang Yi, Guoliang Fan, Wei Zu

Abstract— **Impulsive dynamical system consists of continuous dynamics and discrete dynamics. It is possible to control the trajectories of impulsive dynamical systems to an equilibrium state in finite time by either the continuous dynamics or discrete dynamics or both of them. In this paper, we investigate the finite-time stability and stabilization of impulse dynamical control systems. We develop two sufficient conditions for finite-time stability of impulsive dynamical systems, which emphasize the effects of discrete dynamics and continuous dynamics on finite-time stability respectively. Based on these sufficient conditions, we design impulse or continuous control strategies for two class of impulsive dynamical control systems. Two numerical examples are also given to illustrate these control strategies.**

Index Terms— **Finite-time stability, Impulsive dynamical system, Finite-time stabilization.**

I. INTRODUCTION

Many evolution systems have the characteristics that at certain moments of time their states are subjected to rapid changes. The durations of these rapid changes are so short that they can be neglected compared to the whole evolution time. Hence, the state changes can be represented by state jumps (also refer as impulses) which occur at discrete times. Examples of such systems include the biological systems (e.g., sudden population of fishes changes due to fishing), electronics systems (e.g., voltage changes due to power switching), automatic control systems (e.g., agile missile with reaction jets), etc. It is difficult to mathematically model these evolution processes only by ordinary differential equations or difference equations because they exhibit both continuous-time and discrete-time behaviors. Thus impulsive differential equations, that is, differential equations involving impulse effects at discrete times, become a natural description of these processes. Impulsive differential equations can take full consideration of the effects of abrupt state jumps to the whole evolution processes. Hence it provides a convenient framework for modelling and analysing of many complex dynamical systems.

Impulses can change state rapidly, thus using impulses to control becomes natural [1]–[3]. Many impulse control methods were successfully developed under the framework of optimal control [1]. Impulse control based on impulsive differential equations is widely used because of its easy implementing, less cost and low energy consumption. In many situations, better control effects can be obtained if a continuous control is well collaborated with impulsive control. In certain circumstances, the goal can only be achieved by impulsive control. However, the dynamical behaviours of impulsive differential equations is more complex than the behaviour of dynamical equations without impulse effects. Better understanding of impulsive dynamical systems would lead to better solutions to those control problems involving impulsive control.

A Lot of researches focus on the stability for impulsive differential equations or impulsive dynamical systems and many sufficient conditions have been obtained mainly based on Lyapunov function technique. For example, by using switched Lyapunov function, Guan et al. [7] established some general criteria for exponential stability and asymptotic stability for a new class of hybrid impulsive and switching systems. In [8], Hespanha et al. introduced the concepts of input-to-state stability (ISS) and integral-ISS for impulsive systems and provided a set of Lyapunov-based sufficient conditions for establishing these ISS properties. In [9], Stamova el al. used Lyapunov function and Razumikhin technique to give sufficient conditions for the global stability of the solutions for a class of nonlinear impulsive differential equations. Based on Lyapunov function technique, lots of stability criteria in terms of two measures also be derived for impulsive differential equations [1], [4]. Based on a Lyapunov-like function technique, Hui et al. established conditions for the uniform stability and the uniform asymptotic stability of equilibria of systems with impulse effects described by systems of nonlinear, time-varying ordinary differential equations [10]. All these stability conditions are sufficient. Necessary conditions for the stability are difficult to obtained, and it can be derived only in few special situations. The skillful construction of Lyapunov functions is also still difficult for the practical impulsive dynamic systems.

There are also few researches investigating finite-time stability for impulsive dynamical systems. In system and control theory, there are two interpretations for finite-time stability, the first one is that the system is stable in finite time, i.e., for any bounded initial condition, the state norm does not exceed a certain threshold during a specified time interval [11], [12]. The second interpretation is that a system is finite-time stabile if it is stable and converges to the equilibrium point in finite time [13], [14]. In this sense, it's a stronger notation to Lyapunov stability which has infinite convergence time.

In [15], Chen et al. considered the finite-time stability of a class of impulsive dynamical systems under the first

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interpretation. By relaxing some restrictions on Lyapunov function, several sufficient conditions for finite-time stability are obtained. Based on LMI technique, Ambrosino et al. also give sufficient conditions for finite-time stability of time-driven or state-driven impulsive dynamical linear systems in the first interpretation [12], [16], [17], and in some cases, these sufficient conditions also became necessary conditions [18].

In many applications, it is desired that the dynamical system converges to a stable equilibrium state in finite time rather than asymptotically, not only because of the faster convergence, but also because of the better disturbance rejection performance. There few researches investigate finite-time stability of impulsive dynamical systems in the second interpretation. For continuous dynamical system, finite-time convergence requires non-lipschitzian dynamics [14], [19]. For impulsive dynamical systems, states can jump at each impulse moments, hence finite time stability can be achieved without the requirement of non-lipschitzian conditions on continuous dynamics . In [19], Nersesov et al. developed sufficient conditions for finite-time stability of impulsive dynamical systems using both scalar and vector Lyapunov functions. In this condition, the Lyapunov function is required to decrease between two successive impulses and not increasing when impulses occur.

In this paper, we first establish two newly sufficient conditions for the finite-time stability of impulsive dynamical systems, and based on these conditions, we design finite-time stabilizing controllers for two classes of impulsive dynamical systems respectively. And finally we present two numerical examples for finite-time stabilization of impulsive dynamical systems to illustrate these controllers.

The rest of this paper is organized. In Section II, preliminaries and description of impulsive dynamical control systems are presented. In Section III, we give the sufficient conditions for finite-time stability of the impulsive dynamical systems. Then the applications of these sufficient conditions to the impulsive dynamical control systems are given in Section IV. Finally, some conclusion and further work are drawn in Section V.

II. Preliminaries And Problem Formulation

In this section, we introduce some notations and definitions. Specifically, let $\mathbb{R}$ denote the set of real numbers, $\mathbb{R}^+$ denote the set of nonnegative real numbers, $\mathbb{N}$ denote the sets of positive integers, $\mathbb{R}^m$ denote real m-dimensional Euclidean space, $\mathbb{R}^{m\times n}$ denote the set of matrices with dimension $m\times n$, I denote $m\times m$ identity matrix, $C[E,F]$ denote the set of functions $\psi : E\to F$ which are continuous in E, $\mathscr{K}$ denote the set of functions $\kappa\in C[\mathbb{R}^+,\mathbb{R}^+]$, such that $\kappa(t)$ is monotonically strictly increasing and $\kappa(0)=0$, $\mathscr{B}_\varepsilon(\alpha)$ where $\alpha\in\mathbb{R}^m$ and $\varepsilon>0$ denote the open ball centered at α with radius ε, $\|u\|$ denote the norm of $u\in\mathbb{R}^{m\times n}$, $\mathrm{diag}(k_1,\cdots,k_m)$ denote the diagonal matrix with diagonal entries k_i, A^T denote the transposed matrix of the matrix A, $\det(A)$ denote the determinant of the matrix A.

Consider the following impulsive dynamical control system

$$\begin{cases}\dot{x}(t)=f(x,t)+g(x,t)u(t) & t\neq\tau_k,k\in\mathbb{N}\\ \Delta x=F(x,t)+G(x,t)u_k & t=\tau_k,k\in\mathbb{N}\end{cases}\tag{1}$$

where $x(t)\in\mathbb{R}^m$ is the state vector, $f(x,t)$, $F(x,t)\in C[\mathbb{R}^m\times\mathbb{R}^+,\mathbb{R}^m]$, $g(x,t)$, $G(x,t)\in C[\mathbb{R}^m\times\mathbb{R}^+,\mathbb{R}^{m\times m}]$, $\Delta x(\tau_k)=x(\tau_k^+)-x(\tau_k^-)$, where $x(\tau_k^+)=\lim_{h\to 0^+}x(\tau_k+h)$ and $x(\tau_k^-)=\lim_{h\to 0^-}x(\tau_k+h)$ are the left and right limits of $x(t)$ respectively. $\Delta x(\tau_k)$ is the state jump at τ_k. $x(t)$ is assumed to be left continuous at τ_k, that is, $x(\tau_k^-)=x(\tau_k)$. $u(t)\in\mathbb{R}^m$ is the control vector of continuous dynamics, $\{\tau_k,k\in\mathbb{N}\}$, satisfying $0<\tau_{k-1}<\tau_k$ denotes the set of instants when jump occurs, $u_k=u(\tau_k),k\in\mathbb{N}$ is the impulsive control sequence.

The dynamical system (1) consists two subsystems, i.e., the continuous dynamical subsystem which has control $u(t)$ and the discrete dynamical subsystem which has impulsive control u_k. For the system (1), a function $\phi(t):[t_0,\infty]\to\mathbb{R}^m$ is said to be a solution of it if: 1) $\phi(t)$ is left continuous on $[t_0,\infty)$ for some $t_0\geq 0$; 2) $\phi(t)$ is differentiable and $\frac{\mathrm{d}\phi(t)}{\mathrm{d}(t)}=f(x,t)+g(x,t)u(t)$ everywhere on (t_0,∞) except on set $\{\tau_k,k\in\mathbb{N}\}$; 3) $\phi(\tau_k^+)=\lim\limits_{h\to 0^+}\phi(\tau_k+h)=\phi(\tau_k)+\Delta x(\tau_k),\forall t=\tau_k$. The solution $\phi(t)$ of (1) is a piecewise continuous function that has discontinuous points of the first kind at $t=\tau_k$.

The following assumptions are made for the system (1).

H1). $f(0,t)=0$, $F(0,t)=0$.

H2). $0<\delta_1\leq\inf\limits_{k\in\mathbb{N}}(\tau_{k+1}-\tau_k)\leq\sup\limits_{k\in\mathbb{N}}(\tau_{k+1}-\tau_k)<\delta_2<\infty$

The following definition gives the notion of finite-time stability for the system (1).

Definition 1([13], [14]). The zero solution $x(t)=0$ of (1) is *finite-time stable* if there exists an open neighborhood $\mathscr{N}\subseteq\mathbb{R}^n$ of the origin and a function $T:[0,\infty)\times\mathscr{N}\backslash\{0\}\to[0,\infty)$, called the setting-time function, such that the following statements hold.

1). *Finite-time convergence*. For every $x_0\in\mathscr{N}\backslash 0,t_0\in[0,\infty),\phi(t,t_0,x_0)$ is defined on $[t_0,T(t_0,x_0))$, $\phi(t,t_0,x_0)\in\mathscr{N}\backslash\{0\},\forall t\in[t_0,T(t_0,x_0))$, and $\lim_{t\to T(t_0,x_0))}=0$, and $\phi(t)=0,\forall t>T(t_0,x_0)$.

2). *Lyapunov stability*. For every $\varepsilon>0$ and $t_0\in[0,\infty)$, there exist $\delta=\delta(\varepsilon,t_0)>0$, such that $\mathscr{B}_\delta(0)\subset\mathscr{N}$ and for every $x_0\in\mathscr{B}_\delta(0)\backslash\{0\}$, $\phi(t,t_0,x_0)\in\mathscr{B}_\varepsilon(0)$ for all $t\in[t_0,T(t_0,x_0))$. If $\mathscr{N}=\mathbb{R}^m$, then the zero solution $x(t)=0$ is globally finite-time stable if it is finite-time stable.

III. Finite-time stability analysis

In this section, we investigate the sufficient conditions for finite-time stability of system (1). Let's first present some stability results for the impulsive dynamical systems.

Theorem 1([14]). Consider the nonlinear impulsive dynamical system (1). Assume there exists a continuously differentiable function $\mathscr{V}:\mathscr{N}\to\mathbb{R}^+$ satisfying $\mathscr{V}(0)=0$, $\mathscr{V}(x)>0$, $x\in\mathscr{N}\backslash\{0\}$, and

1) $\mathscr{V}'(x)f_c(x)\leq -c\mathscr{V}^\alpha(x)$, $t\neq\tau_k$, $k\in\mathbb{N}$

2) $\mathscr{V}(x+f_d(x))\leq\mathscr{V}(x)$, $t=\tau_k,k\in\mathbb{N}$

where $c>0$ and $\alpha\in(0,1)$, $f_c(x,t)=f(x,t)+g(x,t)u(t)$,

$f_d(x,t) = F(x,t) + G(x,t)u_k$. Then the zero solution $x(t) = 0$ to (1) is finite-time stable. If, in addition, $\mathscr{N} = \mathbb{R}^m$ and $\mathscr{V}(\cdot)$ are radically unbounded, then the zero solution to (1) is globally finite-time stable.

Theorem 2([10]). Consider the nonlinear impulsive dynamical system (1). Assume that there exist a $\mathscr{V} : \mathscr{N} \times \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the following conditions.

C1. There exists $\phi_1, \phi_2 \in \mathscr{K}$ such that

$$\phi_1(\|x\|) \le \mathscr{V}(x,t) \le \phi_2(\|x\|) \tag{2}$$

for all $(x,t) \in \mathbb{R}^n \times \mathbb{R}^+$.

C2. For any solution $x(t)$ of (1) which is defined on $[t_0, \infty)$, $\mathscr{V}(x(t),t)$ is left continuous on (t_0, ∞) except on the set $E = \{\tau_1, \tau_2, \cdots\}$, and that $\mathscr{V}(x(\tau_n^+), \tau_n^+)$ is nonincreasing for $n = 0, 1, \cdots$ where $\tau_0 = t_0$.

C3. Furthermore, assume that there exists an $h \in C[\mathbb{R}^+, \mathbb{R}^+]$ such that $h(0) = 0$, and

$$\mathscr{V}(x(t),t) \le h(\mathscr{V}(x(\tau_n^+)), \tau_n^+) \tag{3}$$

is true for all $t \in (\tau_n, \tau_{n+1}]$ and $n \in \mathbb{N}$.
Then the equilibrium (zero solution) $x(t) = 0$ of the system (1) is uniformly stable.

The sufficient condition for finite-time stability of the impulsive dynamical systems given in *Theorem 1* requires that a function $\mathscr{V}$ which does not increase on impulse instants and decreases with a rate prespecified by a known function between two successive impulses.

According to *Theorem 2*, the uniformly stable zero solution requires the function $\mathscr{V}$ to be bounded at $(\tau_{n-1}, \tau_n]$ by a prespecified bounded function of the right limits of $\mathscr{V}$ at τ_n and nonincreasing at impulse instants τ_n. If we strengthen the condition on $\mathscr{V}$ at impulse instants, then we can obtain newly sufficient conditions for the finite-time stability of the system (1) as follows.

Theorem 3. Assume that for the system (1), there exists a scale function $\mathscr{V} : \mathscr{N} \times \mathbb{R}^+ \to \mathbb{R}^+$, satisfying $\mathscr{V}(0,t) = 0$, $\mathscr{V}(x,t) > 0$, $x \in \mathscr{N} \backslash \{0\}$, and the conditions C1, C3 in *Theorem 2*. Furthermore, the following condition C4 is also satisfied.

C4. There exists a function $\beta : \mathbb{R}^+ \times \mathbb{N} \to \mathbb{R}^+$ that

$$D\mathscr{V}(x(\tau_n^+), \tau_n^+) \le -\beta(\mathscr{V}(x(\tau_{n-1}^+), \tau_{n-1}^+), n) \tag{4}$$

where

$$D\mathscr{V}(x(\tau_n^+), \tau_n^+) \triangleq \frac{\mathscr{V}(x(\tau_n^+), \tau_n^+) - \mathscr{V}(x(\tau_{n-1}^+), \tau_{n-1}^+)}{\tau_n - \tau_{n-1}}$$

$$\beta(y,n) = \begin{cases} \varepsilon & |y| > \varepsilon \\ \dfrac{y}{\delta_n} & |y| < \varepsilon \end{cases}$$

$\varepsilon > 0$, is a constant, $\delta_n = \tau_n - \tau_{n-1}, \tau_0 = t_0$. Then the zero solution $x(t) = 0$ is finite time stable.

Proof. Letting $v_n = \mathscr{V}(x(\tau_n^+), \tau_n^+)$, it follows from (4) that

$$v_n \le v_{n-1} - \beta(v_{n-1}, n)\delta_n \tag{5}$$

Thus we have

$$v_n \le v_0 - \sum_{i=1}^{n-1} \beta(v_i, i)\delta_{i+1} \tag{6}$$

Defining

$$\begin{aligned} n_1 &= \max_i \left\{ i|\ v_0 - \textstyle\sum_{j=0}^{i-1} \beta(v_j, j)\delta_{j+1} \ge \varepsilon \right\} \\ &= \max_i \left\{ i|\ v_0 - \textstyle\sum_{j=0}^{i-1} \varepsilon\delta_{j+1} \ge \varepsilon \right\} \\ &= \max_i \{ i|\ v_0 - \varepsilon\tau_i \ge \varepsilon \} \end{aligned} \tag{7}$$

then we obtain

$$v_{n_1+1} \le v_{n_1} - \beta(v_{n_1}, n_1+1)\delta_{n_1+1} = v_{n_1} - \varepsilon\delta_{n_1+1}$$

and we have $0 \le v_{n_1+1} \le \varepsilon$. It follows that

$$0 \le v_{n_1+2} \le v_{n_1+1} - \beta(v_{n_1+1})\delta_{n_1+1} = 0$$

So $v_{n_1+2} = 0$. Hence $\mathscr{V} = 0$. Then it is easy to verify that the condition C1, C2 and C3 are satisfied. Hence, according to *Theorem 2*, the zero solution $x(t) = 0$ is uniformly stable. This means that the zero solution of (1) is finite time stable, and the setting time is $T(t_0, x_0) \le \tau_{n_1+2}$.

The condition C4 for $\mathscr{V}$ on τ_n implies that the finite-time stability can be achieved by controlling the discrete dynamical subsystem. In another way, if we strengthen the condition on $\mathscr{V}$ at interval $(\tau_{n-1}, \tau_n]$, then we can obtain an another sufficient condition for the finite-time stability of the system (1) as follows.

Theorem 4. Assume that for the system (1), there exists a $\mathscr{V} : \mathscr{N} \times \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $\mathscr{V}(0,t) = 0$, $\mathscr{V}(x,t) > 0$, $x \in \mathscr{N} \backslash \{0\}$, and the condition C1 in *Theorem 2*. Furthermore the following conditions C5, C6 are also satisfied.

C5. $\mathscr{V}(x(t),t)$ is piecewise left continuous and differentiable everywhere on (t_0, ∞) except on the set $\{\tau_i, i \in \mathbb{N}\}$, and there exists a $\lambda > 0$ that

$$\dot{\mathscr{V}}(x(t),t) \le -c_{n-1}\mathscr{V}^{\alpha}(x(t),t) \quad t \in (\tau_{n-1}, \tau_n], n \in \mathbb{N} \tag{8}$$

where $\alpha \in (0,1)$, c_{n-1} is a positive constant, and satisfies the following condition.

$$c_{n-1} \ge \max\{1, a_{n-1}\} \tag{9}$$

$$a_{n-1} = \frac{(\Delta\mathscr{V}(x(\tau_{n-1}), \tau_{n-1}) + \lambda)^{1-\alpha} - \mathscr{V}^{1-\alpha}(x(\tau_{n-1}^+), \tau_{n-1}^+)}{\alpha - 1}$$

$$\Delta\mathscr{V}(x(\tau_{n-1}), \tau_{n-1}) = |\mathscr{V}(x(\tau_{n-1}^+), \tau_{n-1}^+) - \mathscr{V}(x(\tau_{n-1}), \tau_{n-1})|$$

C6. For every $n \in \mathbb{N}$, there exists an $h \in C[\mathbb{R}^+, \mathbb{R}^+]$,such that $h(0) = 0$, and

$$\mathscr{V}(x(\tau_n^+), \tau_n^+) \le \max_{t \in (\tau_{n-1}, \tau_n]} h(\mathscr{V}(x(t),t)) \tag{10}$$

Then the zero solution $x(t) = 0$ is finite time stable.

Proof. The solution for the differential inequality (8) is

$$\mathscr{V}(x(t),t) \le [c_{n-1}(\alpha-1)t + C_n]^{\frac{1}{1-\alpha}} \quad t \in (\tau_{n-1}, \tau_n], n \in \mathbb{N} \tag{11}$$

where C_n is a constant and chosen to satisfy

$$\mathscr{V}(x(\tau_{n-1}^+), \tau_{n-1}^+) = [c_{n-1}(\alpha-1)\tau_{n-1}^+ + C_n]^{\frac{1}{1-\alpha}}$$

Hence

$$C_n = \mathscr{V}(x(\tau_{n-1}^+), \tau_{n-1}^+)^{1-\alpha} - c_{n-1}(\alpha - 1)\tau_{n-1}^+$$

We obtain that

$$\begin{aligned} \mathscr{V}(x(\tau_n), \tau_n) &\leq [c_{n-1}(\alpha-1)\tau_n + C_n]^{\frac{1}{1-\alpha}} \\ &= [\mathscr{V}(x(\tau_{n-1}^+), \tau_{n-1}^+)^{1-\alpha} + c_{n-1}(\alpha-1)\delta_n]^{\frac{1}{1-\alpha}} \end{aligned} \tag{12}$$

where $\delta_n = \tau_n - \tau_{n-1}^+ = \tau_n - \tau_{n-1}$.

Considering the function

$$y(t) \triangleq [c_{n-1}(\alpha-1)t + \mathscr{V}(x(\tau_{n-1}^+), \tau_{n-1}^+)^{1-\alpha}]^{\frac{1}{1-\alpha}} \quad t \in (0, \delta_n]$$

Its first-order and second-order derivatives are

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = -c_{n-1}[c_{n-1}(\alpha-1)t + \mathscr{V}(x(\tau_{n-1}^+), \tau_{n-1}^+)^{1-\alpha}]^{\frac{\alpha}{1-\alpha}} < 0$$

and

$$\frac{\mathrm{d}^2y(t)}{\mathrm{d}t^2} = c_{n-1}^2\alpha[c(\alpha-1)t + \mathscr{V}(x(\tau_{n-1}^+), \tau_{n-1}^+)^{1-\alpha}]^{\frac{2\alpha-1}{1-\alpha}} > 0$$

respectively. Hence

$$\begin{aligned} y(\delta_n) &= y(\tau_{n-1}) + \int_0^{\delta_n} dy(s)ds \leq y(0) + \max_{s\in(0,\delta_n]} y'(s)\delta_n \\ &= y(0) - c_{n-1}[c_{n-1}(\alpha-1)\delta_n + \mathscr{V}(x(\tau_{n-1}^+), \tau_{n-1}^+)^{1-\alpha}]^{\frac{1}{1-\alpha}} \\ &\leq y(0) - c_{n-1}[c_{n-1}(\alpha-1)\delta_{\min} + \mathscr{V}(x(\tau_{n-1}^+), \tau_{n-1}^+)^{1-\alpha}]^{\frac{1}{1-\alpha}} \end{aligned} \tag{13}$$

where $\delta_{\min} = \min_i\{\delta_i, i \in \mathbb{N}\}$.

According to (12) and (13), we obtain

$$\begin{aligned} \mathscr{V}(x(\tau_n), \tau_n) \leq& \mathscr{V}(x(\tau_{n-1}^+), \tau_{n-1}^+) \\ &- c_{n-1}[c_{n-1}(\alpha-1)\delta_{\min} + \mathscr{V}(x(\tau_{n-1}^+), \tau_{n-1}^+)^{1-\alpha}]^{\frac{1}{1-\alpha}} \\ =& \mathscr{V}(x(\tau_{n-1}), \tau_{n-1}) + \Delta\mathscr{V}(x(\tau_{n-1}), \tau_{n-1}) \\ &- c_{n-1}[c_{n-1}(\alpha-1)\delta_{\min} + \mathscr{V}(x(\tau_{n-1}^+), \tau_{n-1}^+)^{1-\alpha}]^{\frac{1}{1-\alpha}} \end{aligned} \tag{14}$$

According to condition C6, $\mathscr{V}(x(\tau_n^+), \tau_n^+)$ is bounded by the combination of a prespecified bounded function and the maximal value of $\mathscr{V}$ in $(\tau_{n-1}, \tau_n]$. Hence, $\Delta\mathscr{V}(x(\tau_{n-1}), \tau_{n-1})$ is bounded.

According to (9),

$$\begin{aligned} &\Delta\mathscr{V}(x(\tau_{n-1}), \tau_{n-1}) \\ &\quad -c_{n-1}[c_{n-1}(\alpha-1)\delta_{\min} + \mathscr{V}(x(\tau_{n-1}^+), \tau_{n-1}^+)^a]^{\frac{1}{a}} \leq -\lambda \end{aligned} \tag{15}$$

Then

$$\mathscr{V}(x(\tau_n), \tau_n) \leq \mathscr{V}(x(\tau_{n-1}), \tau_{n-1}) - \lambda \tag{16}$$

and there exist $n_2 \in \mathbb{N}, n_2 < \infty$

$$0 \leq \mathscr{V}(x(\tau_{n_2}), \tau_{n_2}) \leq \mathscr{V}(x(\tau_0), \tau_0) - n_2\lambda \tag{17}$$

$$\mathscr{V}(x(\tau_0), \tau_0) - (n_2+1)\lambda \leq 0 \tag{18}$$

Because $\dot{\mathscr{V}} < 0$, when $\mathscr{V} > 0$, there exists $T \in (\tau_{n_2}, \tau_{n_2+1}]$

$$\mathscr{V}(V(x(T), T) = 0 \tag{19}$$

Thus the zero solution $x(t) = 0$ is finite time stable. The setting time $T \leq \tau_{n_2+1}$.

According to *Theorem 4*, the finite-time stability can also be achieved by controlling the continuous dynamical subsystem of the impulsive dynamical systems.

IV. Finite-time stabilization of Impulse Dynamical control Systems

In this section, we investigate the finite-time stabilization of two classes of impulsive dynamical control systems.

Case a). Consider the following impulsive dynamical control system.

$$\begin{cases} \dot{x}(t) = f(x(t)) & t \neq \tau_k, k \in \mathbb{N}, x(0) = x_0 \\ \Delta x = Ex(t_k) + Fu_k & t = \tau_k, k \in \mathbb{N}, \|u_k\| \leq 1 \end{cases} \tag{20}$$

$x(t) \in \mathbb{R}^m$ is state, $f(x(t)) \in C[\mathbb{R}^m, \mathbb{R}^m]$, E, F are constant matrices, $u_k \in \mathbb{R}^m, k \in \mathbb{N}$ is the impulse control at τ_k $\tau_0 = 0$. We assume that, for all $t > t_0$, $\|x\| \leq h$, $h > 0$, we have $\|f(x(t))\| \leq \|A\|\|x\|$, where A is a constant matrix, and $\det(F) \neq 0$. Furthermore we assume that all states are observable.

According to *Theorem 3*, we have the following theorem for the system (20).

Theorem 5. The origin of the system (20) is finite-time stabilizable if there exists a matrix K satisfying

$$\|K\| \leq 1, \|I + E + \frac{FK}{\lambda}\|e^{\|A\|\delta_{\max}} \leq 1 - \frac{\varepsilon}{\lambda e^{\|A\|\delta_{\max}}} \tag{21}$$

where $0 < \varepsilon < \kappa = \min_i\{|\lambda_i|| \det(\lambda_i I + (E+I)^{-1}F) = 0\}$, $\lambda = \|x_0\|e^{\|A\|\delta_{\max}}$, $\delta_{\max} = \max_i\{\delta_i | \delta_i = \tau_i - \tau_{i-1}\}$. and u_k is chosen as

1). $u_k = K\frac{x(\tau_k)}{\lambda}$, if $\|x\| \geq \frac{\varepsilon}{e^{\|A\|\delta_{\max}}}$.

2). $u_k = -F^{-1}(I+E)x(\tau_k)$, if $\|x\| < \frac{\varepsilon}{e^{\|A\|\delta_{\max}}}$.

Proof. Let $\mathscr{V} = \|x\|$, and $\phi_1 = \phi_2 = \mathscr{V}$, $h = \lambda$. The satisfaction of C1 is straightforward. For any $t \in (\tau_k, \tau_{k+1}]$,

$$x(t) = x(\tau_k^+) + \int_{\tau_k}^t f(x(s))ds \tag{22}$$

which implies that

$$\begin{aligned} \mathscr{V}(x(t)) =& \|x(t)\| = \|x(\tau_k^+)\| + \int_{\tau_k}^t \|f(x(s))\|ds \\ &\leq \|x(\tau_k^+)\| + \int_{\tau_k}^t \|Ax(s)\|ds \\ &\leq \|x(\tau_k^+)\| + \int_{\tau_k}^t \|A\|\|x(s)\|ds \end{aligned} \tag{23}$$

According to Gronwall inequality, we have

$$\|x(t)\| \leq \|x(\tau_k^+)\| + \int_{\tau_k}^t \|A\|\|x(s)\|ds \leq \|x(\tau_k^+)\|e^{\|A\|(t-\tau_k)} \tag{24}$$

Hence

$$\begin{aligned} \|x(\tau_{k+1}^+)\| &\leq \|(I + E + \frac{FK}{\lambda})x(\tau_{k+1})\| \\ &\leq \|I + E + \frac{FK}{\lambda}\|\|e^{\|A\|(\tau_{k+1}-\tau_k)}\|x(\tau_k^+)\| \end{aligned} \tag{25}$$

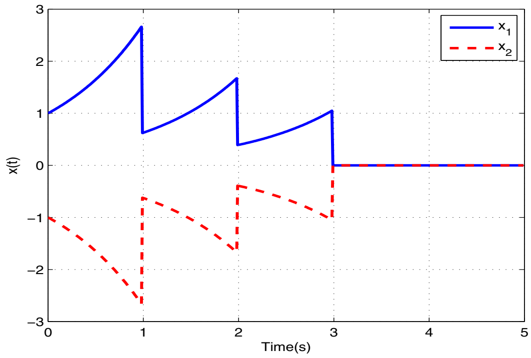

Fig. 1. State trajectory versus time for Example 1

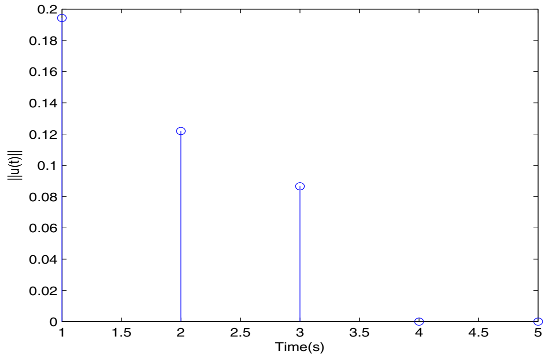

Fig. 2. Impulse control energy versus time for Example 1

Let $q = \|I+E+\frac{FK}{\lambda}\|\|e^{\|A\|\delta_{\max}}\|$ Then $0 \le q \le 1-\frac{\varepsilon}{\lambda e^{\|A\|\delta_{\max}}}$. Hence

$$\begin{aligned}\mathscr{V}(x(\tau_{k+1}^+)) &\le q\mathscr{V}(x(\tau_k^+)) \le \mathscr{V}(x(\tau_k^+)) - \frac{\varepsilon}{e^{\|A\|\delta_{\max}}} \\ &\le \mathscr{V}(x(\tau_0^+)) - \frac{k\varepsilon}{e^{\|A\|\delta_{\max}}}\end{aligned} \tag{26}$$

The condition C4 in *Theorem 3* is satisfied.

Therefore there exists $s \in \mathbb{N}, s \le \infty$, that

$$0 \le \frac{\varepsilon}{e^{\|A\|\delta_{\max}}} \le \mathscr{V}(x(\tau_s^+)) \le \mathscr{V}(x(\tau_0^+)) - \frac{(s-1)\varepsilon}{e^{\|A\|\delta_{\max}}} \tag{27}$$

$$0 \le \mathscr{V}(x(\tau_{s+1}^+)) \le \mathscr{V}(x(\tau_0^+)) - \frac{s\varepsilon}{e^{\|A\|\delta_{\max}}} \le \frac{\varepsilon}{e^{\|A\|\delta_{\max}}} \tag{28}$$

We thus obtain

$$\mathscr{V}(x(\tau_{s+1})) \le \mathscr{V}(x(\tau_{s+1}^+))e^{\|A\|\delta_{\max}} \le \varepsilon \tag{29}$$

and for all $t \in [0, \tau_{s+1}]$, $\|x(t)\| = \mathscr{V}(x(t)) \le x_0 e^{\|A\|\delta_{\max}} = \lambda$. Hence the condition C3 is satisfied and $\|u_k\| = \|\frac{Kx(\tau_k)}{\|x(\tau_k)\|}\| \le 1$.

Define $\Omega \triangleq \{x|\ \|x\| \le \|-(E+I)^{-1}Fu_k\|, \|u_k\| \le 1\}$ and $\Lambda \triangleq \{x|\ \|x\| \le \kappa\} \subseteq \Omega$

Since $\varepsilon < \kappa$, we have $x(\tau_{s+1}) \in \Lambda$. $x(t)$ can be reset to the origin at τ_{s+1}^+ by impulsive control

$$u_{s+1} = -F^{-1}(I+E)x(\tau_{s+1}) \tag{30}$$

According to *Theorem 3*, the origin of (20) is finite-time stabilizable, and the setting time $T \le \tau_{s+1}$.

Because $x(\tau_{s+1}) \in \Lambda$ and the linear mapping $-F^{-1}(I+E)x : x \in \Lambda \to \mathbb{R}^n$ is a bijection, therefore, $\|u_{s+1}\| \le 1$.

Remark 1. Actually, the set $\Omega = \{x| \quad -x = Ex + Fu_k, \|u_k\| \le 1\}$. This implies that, whenever the state trajectory goes into this set, the state can be reset to the origin by once impulsive control.

Example 1. Consider the dynamical impulsive control system given by (20), where $x(t) = [x_1(t), x_2(t)]^T \in \mathbb{R}^2$, $f(x(t)) = [x_1(t), x_2(t)]^T$,

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, F = \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}$$

$x(0) = [1,-1]^T$, $\tau_k = k, k \in \mathbb{N}$, $\tau_0 = 0$.

Then we have $\|f(x)\| \le \|A\|\|x\|$, $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\lambda = \sqrt{2}e$, $\delta_{\max} = 1$, $\kappa = 4$. If we choose $K = \begin{bmatrix} -0.8 & 0 \\ 0 & -0.8 \end{bmatrix}$, $\varepsilon = 1$, then the conditions in (21) are satisfied. The impulsive control is chosen as

1). $u_k = \frac{Kx(\tau_k)}{\lambda}$, if $\|x\| \ge \frac{1}{e}$.

2). $u_k = -F^{-1}(I+E)x(\tau_k)$, if $\|x\| \le \frac{1}{e}$.

Then the origin is finite-time stable. The time histories of the states are given in Fig. 1., and the energy of impulsive control at time τ_k is given in Fig. 2.

Case b. Consider the following nonlinear impulse dynamical control system

$$\begin{cases} \dot{x}(t) = f(x(t)) + g(x(t))u(t) & t \ne \tau_n, n \in \mathbb{N} \\ \Delta x = Ex(\tau_k) & t = \tau_n, n \in \mathbb{N}, x(0) = x_0 \end{cases} \tag{31}$$

where $x \in \mathbb{R}^m$, $f(x(t)) \in C[\mathbb{R}^m, \mathbb{R}^m]$, $f(0) = 0$, $g(x(t)) \in C[\mathbb{R}^m, \mathbb{R}^m \times \mathbb{R}^m]$, $\det(g(x(t))) \ne 0$, E is a constant matrix, $E \ne 0$, $I + E \ne 0$.

Theorem 6. The origin of the system (31) is finite-time stabilizatable, if

$$u(t) = g^{-1}(x)(-K|x|^{2\alpha-1} - f(x)) \quad t \ne \tau_n, n \in \mathbb{N} \tag{32}$$

where $|x|^{2\alpha-1} \triangleq (\mathrm{sign}(x_1)|x_1|^{2\alpha-1}, \cdots, \mathrm{sign}(x_m)|x_m|^{2\alpha-1})^T \in \mathbb{R}^m$, $K = \mathrm{diag}(k, \cdots, k)$, $0 < \alpha < 1$. $k \ge \frac{\gamma}{1-\alpha}$, $\gamma = \|I+E\|^{2(1-\alpha)}\|x_0\|^{2(1-\alpha)}$.

Proof. Define $\mathscr{V} = x^T x = \|x\|^2$, $\phi_1 = \phi_2 = \mathscr{V}$, $h = \gamma$. The satisfaction of C1 is straightforward. Furthermore, we have

$$\begin{aligned}\dot{\mathscr{V}}(x(t)) &= x^T x = \sum_{i=1}^m x_i\dot{x}_i = -k\sum_{i=1}^m |x_i|^{2\alpha} \\ &\le -k\left(\sum_{i=1}^m x_i^2\right)^\alpha = -k\mathscr{V}^\alpha \quad t \in (\tau_{n-1}, \tau_n]\end{aligned} \tag{33}$$

Define

$$a_n \triangleq \frac{\mathscr{V}^{1-\alpha}(\tau_n^+) - (\Delta\mathscr{V}(\tau_n) + \lambda_n)^{(1-\alpha)}}{1-\alpha}$$

where $\Delta\mathscr{V}(\tau_n) = |\mathscr{V}(\tau_n^+) - \mathscr{V}(\tau_n)|$, $0 < \lambda_n \le \Delta\mathscr{V}(\tau_n)$ is a positive constant. It's easy to get

$$a_n = \frac{\|I+E\|^{2(1-\alpha)}\|x(\tau_n)\|^{2(1-\alpha)} - [(\|I+E\|^2-1)\|x(\tau_n)\|^2 + \lambda_n]^{1-\alpha}}{1-\alpha}$$

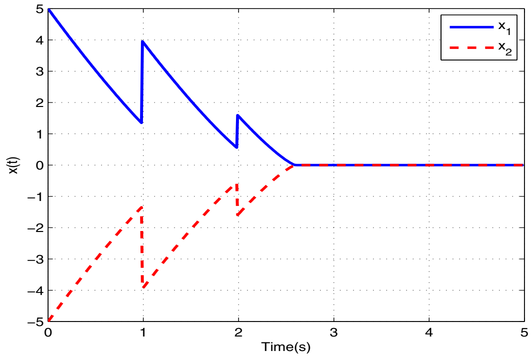

Fig. 3. State trajectory versus time for Example 2

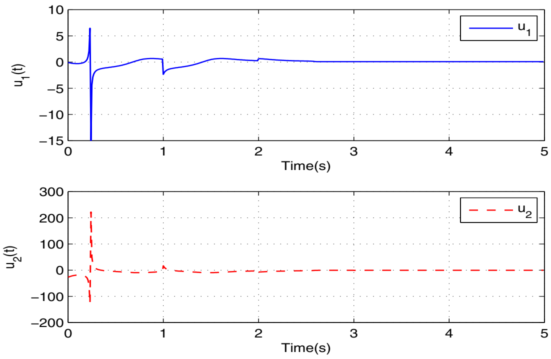

Fig. 4. Continuous control versus time for Example 2

Because $k > \frac{\gamma}{1-\alpha} = a_0$, and $\mathscr{V}(x(\tau_1)) \leq \mathscr{V}(x(\tau_0)) = \mathscr{V}(x(0))$, $k > a_0 > a_1$. Similar to the proof of *Theorem 4*, we have $\mathscr{V}(x(\tau_2)) - \mathscr{V}(x(\tau_1)) < \lambda_1$, $\mathscr{V}(\tau_2^+) \leq \|I + E\|^2\|x(\tau_2)\|^2 \leq \gamma$, and $k > a_0 > a_2$. By an induction procedure, we have $k > a_0 > a_i$, $\mathscr{V}(\tau_i)^+ \leq \gamma$, $i = 1, 2, \cdots$. The conditions C5 and C6 are satisfied. Hence, according to *Theorem 4*, the origin of the system is finite-time stabilizable.

Example 2. Consider the impulse dynamical control system (31), where $x(t) = [x_1(t), x_2(t)]^T \in \mathbb{R}^2$, $\tau_n = n, n \in \mathbb{N}$, $x(0) = [5, -5]^T$

$$f(x(t)) = \begin{bmatrix} -x_1 + x_2^2 \\ -x_1^2 \end{bmatrix}, g(x(t)) = \begin{bmatrix} 10 & \sin^2(x_2) \\ x_1^2 & 1 \end{bmatrix}, E = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

Choosing the parameters $\alpha = 0.8$, then $\gamma = \|x_0\|^{2(1-\alpha))}(\|I + E\|^{2(\alpha-1)}) \simeq 2.95$. Therefore, we choose $k = 3$, and $u(t) = -g^{-1}(x)(K|x|^{2\alpha-1} - f(x))$. Then the origin is finite-time stable. The time histories of the states are given in Fig. 3., and the control $u(t), t \neq n, n \in \mathbb{N}$ is given in Fig. 4.

V. CONCLUSIONS AND FUTURE WORKS

In this paper, we develop two sufficient conditions for the finite-time stability of impulsive dynamical systems. Based on these sufficient conditions, we also design continuous or impulsive control strategy for two class of impulsive dynamical control systems, and two numerical examples are also given to illustrate these control strategies. These conditions are obtained by adding strong constraints on function $\mathscr{V}$, so it is a bit conservative. Further research can be done on more relaxed conditions for finite-time stability and the application to the impulsive dynamical control systems.

VI. ACKNOWLEDGMENTS

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