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Adaptive Tracking Control of Leader-Following Linear Multi-Agent Systems With External Disturbances

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In this paper, the consensus problem for leader-following linear multi-agent systems with external disturbances is investigated. Brownian motions are used to describe exogenous disturbances. A distributed tracking controller based on Riccati inequalities with an adaptive law for adjusting coupling weights between neighboring agents is designed for leader-following multi-agent systems under fixed and switching topologies. In traditional distributed static controllers, the coupling weights depend on the communication graph. However, coupling weights associated with the feedback gain matrix in our method are updated by state errors between neighboring agents. We further present the stability analysis of leader-following multi-agent systems with stochastic disturbances under switching topology. Most traditional literature requires the graph to be connected all the time, while the communication graph is only assumed to be jointly connected in this paper. The design technique is based on Riccati inequalities and algebraic graph theory. Finally, simulations are given to show the validity of our method.

Keywords: multi-agent systems; leader-following; Riccati inequalities; adaptive control; stochastic disturbances

1. Introduction

Distributed coordination control of multi-agent networks has been a growing interest in the field of industrial control and automation. The consensus problem is the most fundamental problem in this area, which designs a distributed control protocol such that all the agents in the network asymptotically reach an agreement by interacting with their local neighbors as time goes on. Although each agent has limited resources, the interconnected system as a whole can perform complex tasks in a coordinated fashion. Therefore, comparing to conventional control systems, multi-agent systems have many advantages such as reducing cost, improving system efficiency, flexibility and reliability. The applications of consensus problems cover a wide range such as spacecraft formation flying, sensor networks and cooperative surveillance. The control of multi-agent systems is a very active area of research. The consensus problem for single-integrator agents is addressed by Olfati-Saber and Murray (2004). Distributed control has been studied for networks with and without communication delays and convergence is analyzed for directed graphs with fixed or switching topology, as well as for undirected graph. Deshpande et al. (2013) considered a network of vehicles moving in a two dimensional plane and proposed a novel distributed static output feedback methodology to maintain a desired formation. Moreover the condition for consensus is relaxed by Ren and Beard (2004), where the consensus can be achieved if the union of the directed interaction graphs across some time intervals has a spanning tree frequently enough as the system evolves. Some other relevant research topics have also been addressed, such as consensus of multiple agents under switching topologies (Olfati-Saber and Murray 2004; Ni and Cheng 2010; Wang 2013; Cheng et al. 2015), agreement over random networks (Hatano and Mesbahi 2005; Yu and Wang 2014), coordination and consensus of networked agents with

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noisy measurements (Huang and Manton 2009; Wang et al. 2015), networks with time-delays (Tian and Liu 2008; Moreau 2005; Li et al. 2014; Sakurama and Nakano 2015).

In this paper, the cooperative control task under consideration is the leader-following coordination among a group of agents, where the leader is a special agent whose motion is independent and is followed by all the other agents. The leader-following approach has been widely used in many applications such as formation control in robotic systems and unmanned aerial vehicle formation. Jadbabaie et al. (2003) considered such a leader-following consensus problem and proved that if all the agents were jointly connected with their leader, their states would converge to that of the leader as time goes on. Liu et al. (2008) regarded the leader as a particular agent acting as an external input to steer the other member agents, and controllability condition for the network with switching topology was analyzed. The problem was also considered by Ji and Egerstedt (2007), and Rahmani et al. (2009) from a graph-theoretic perspective. Distributed observer design for the leader-following control of multi-agent networks was considered by Hong et al. (2008) and Dimarogonas et al. (2009). A distributed tracking scheme with distributed estimators was developed for leader-following multi-agent systems with measurement noises and directed interconnection topology (Hu and Feng 2010).

Consensus of multi-agent systems with general linear dynamics are studied by Ni and Cheng (2010); Li et al. (2010); Seo et al. (2009); Tuna (2008, 2009); Scardovi and Sepulchre (2009); Zhang et al. (2011); Liu et al. (2015); Ji et al. (2014); Cheng et al. (2014); Matei et al. (2013); Liu et al. (2015); Zhang et al. (2014); Wang et al. (2015). In particular, different static and dynamic consensus protocols are designed by Li et al. (2010), Seo et al. (2009), Tuna (2008), requiring the knowledge of the communication graph to be known by each agent to determine the bound of the coupling weights. However, the entire communication graph is global information. In other words, these consensus protocols cannot be computed and implemented by each agent in a fully distributed fashion, for example, we just need to use the local information of its own and neighbors. In this paper, we consider the leader-following consensus problem in general linear system rather than integrators or double integrators. To handle the problem, we use Riccati inequality method to design the control gain matrix for each agent. Motivated by Li et al. (2013), we design an adaptive tracking controller for leader-following multi-agent systems. In contrast to traditional distributed static controller, the coupling weights of adaptive controller can be adjusted by state errors between neighboring agents. It is worth pointing out that we don't require the values of coupling weights beforehand, so the method is more flexible in practice.

However in real life, multi-agent systems are in open environment where exogenous disturbances widely exist and affect dynamics of the agents. External disturbances is a main source of instability and poor performance. Moreover when the interaction topologies are time-varying, consensus problem becomes much more difficult. Li et al. (2013) don't take into account any external disturbances and switching topologies. The consensus problem for multi-agent systems with exogenous disturbances are studied by Zhang and Cheng (2012); Zhang and Liu (2013). In these works, the disturbances are deterministic and assumed to be known a priori or observable. In this paper, we consider stochastic disturbances and discuss leader-following multi-agent consensus problem for general linear case with stochastic disturbances under switching topology. The stability of linear switching systems is studied by Ni and Cheng (2010), and Su and Huang (2012a) don't require the system matrix to be Hurwitz at any time instant. Moreover Su and Huang (2012b) consider the cooperative output regulation of linear multi-agent systems under switching network. In this paper, the communication graph is only assumed to be jointly connected. The stability of the closed-loop system is analyzed and simulations are given to illustrate the method to be efficient that the consensus can be reached for leader-following multi-agent systems under time-varying topologies with stochastic disturbances.

The rest of this paper is organized as follows. The problem formulation and some preliminaries about graph theory are discussed in Section 2. The leader-following consensus problem for multi-agent systems with stochastic disturbances under the fixed interconnection topology is discussed in Section 3. The results obtained in Section 3 are generalized to the case of switching topology in Section 4. The simulation examples are given in Section 5. The conclusion is presented in Section 6.

2. Problem and preliminaries

2.1. *Preliminaries*

First we introduce some notations. Let $\mathbf{R}^{n\times n}$ be the set of $n \times n$ real matrices. I_N represents the identity matrix of dimension N. For a symmetric matrix P, the matrix inequality $P > 0$ ($P \geq 0$) means that P is positive definite (positive semi-definite). $A \otimes B$ denotes the Kronecker product of matrices A and B.

In the literature, a multi-agent system is always represented as a graph. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be an undirected graph of order N, where $\mathcal{V} = \{1, 2, \dots, N\}$ is a finite set and a finite set of arcs $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of edges of the graph. An arc of $\mathcal{G}$ is denoted by (i, j), which starts from i and ends at j and represents the information flow from agent j to agent i. A path in $\mathcal{G}$ is a sequence $i_0, i_1, \dots, i_q$ of distinct vertices such that (i_{j-1}, i_j) is an arc for $j = 1, \dots, q$. If there exists a path from vertex i to vertex j, we say that vertex j is reachable from vertex i. Furthermore, if there exists a path from every vertex to vertex j, then vertex j is a globally reachable vertex of $\mathcal{G}$. The graph is undirected which means that the edges (i, j) and (j, i) in $\mathcal{E}$ are considered to be the same. Two nodes i and j are neighbors to each other if $(i, j) \in \mathcal{E}$. The set of the neighbors of node i is denoted by $\mathcal{N}_i = \{j \in \mathcal{V} : (j, i) \in \mathcal{E}, j \neq i\}$. A graph is connected if there exists a path between each pair of the nodes. A component of the graph $\mathcal{G}$ is a connected subgraph that is maximal. A nonnegative matrix $\mathscr{A} = [a_{ij}] \in \mathbf{R}^{N\times N}$ is called an adjacency matrix of graph $\mathcal{G}$ if the element a_{ij} associated with the edge (i, j) is positive, i.e., $a_{ij} = 1$ if $(i, j) \in \mathcal{E}$. Moreover, we assume that $a_{ii} = 0$ for all $i \in \mathcal{V}$. Note that the adjacency matrix $\mathscr{A}$ is a symmetric matrix for an undirected graph. A diagonal matrix $\mathscr{D} \in \mathbf{R}^{N\times N}$ is called the degree matrix whose ith diagonal element is defined as $\sum_{j=1}^{N} a_{ij}$. Then, the Laplacian matrix of the graph $\mathcal{G}$ is defined as $L = \mathscr{D} - \mathscr{A}$.

When the graph $\mathcal{G}$ is used to describe the interconnection topology of a multi-agent system consisting of one leader and N followers, the leader can be represented by vertex 0 and information is exchanged between the leader and the agents which are in the neighborhood of the leader. We define a diagonal matrix $D = \text{diag}\{d_1, d_2, \dots, d_N\} \in \mathbf{R}^{N\times N}$ to be a leader adjacency matrix, i.e., $d_i = 1$ if the ith follower is connected to the leader across the arc $(i, 0)$, otherwise $d_i = 0$. Let $\overline{\mathcal{G}}$ be the graph defined on the vertices $\{0, 1, 2, \dots, N\}$.

We define a new matrix $H = L + D \in \mathbf{R}^{N\times N}$, and the following lemma plays an important role in the sequel.

Lemma 1. *The following statements are equivalent:*

1) Vertex 0 is a globally reachable vertex for all vertices $i \in \mathcal{V}$.
2) The matrix H is symmetric and positive definite.

Remark 1. If the vertex 0 is a globally reachable vertex for all vertices $i \in \mathcal{V}$, the undirected graph $\overline{\mathcal{G}}$ is connected. Therefore H is symmetric and positive definite according to Ni and Cheng (2010).

To analyze the time-varying topologies $\mathcal{G}$ of the leader-following multi-agent system, we give the following general assumptions:

1) There exists a switching signal $\sigma\colon [t_0, \infty) \to \mathcal{P}$, which is piecewise constant. $\mathcal{P}$ is a finite set of all possible interconnection topologies of the multi-agent system and t_0 is the initial time. We denote all the possible graphs defined on the vertices $\{0, 1, 2, \dots, N\}$ by $\{\overline{\mathcal{G}}_p \colon p \in \mathcal{P}\}$, and use $\{\mathcal{G}_p \colon p \in \mathcal{P}\}$ to denote subgraphs defined on vertices $\{1, 2, \dots, N\}$.
2) The time-interval $[t_0, \infty)$ is constituted by an infinite sequence of bounded, non-overlapping, contiguous time-intervals $[t_j, t_{j+1})$ for $j = 0, 1, \dots$ with $t_0 = 0$. For each interval $[t_k, t_{k+1})$, there is a sequence of nonoverlapping subintervals

$$[t_k^0, t_k^1), [t_k^1, t_k^2), \dots, [t_k^{m_k-1}, t_k^{m_k}), t_k = t_k^0, t_{k+1} = t_k^{m_k},$$

satisfying $t_k^{j+1} - t_k^j \geq \tau$, $0 \leq j \leq m_k - 1$ for some integers $m_k \geq 1$ and a given constant $\tau \geq 0$, such that during each of such subintervals, the interconnection topology is fixed. Therefore, during each subinterval $[t_k^j, t_k^{j+1})$, the graph denoted by $\mathcal{G}_{\sigma(t)}$ is fixed and we denote it by $\mathcal{G}_k^j$.

2.2. *The leader-following problem*

In this paper, we consider the multi-agent systems consisting of an active leader and N following agents. The dynamics of each agent is represented as

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t) + \varphi_i(t, x_i(t))\upsilon(t),\ i = 1, 2, \ldots, N, \tag{1}$$

where $x_i \in \mathbf{R}^n$ is the state of the ith agent and $u_i \in \mathbf{R}^m$ is the control input of the ith agent which can only use the local information of its neighbors and itself. Let $\varphi_i(\cdot, \cdot)\colon \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^n$ be the noise intensity function vector and $\upsilon(t)$ be the scalar zero mean white Gaussian white noise. Note that the time derivative of a Brownian motion is a white noise process. Hence the dynamics of each agent can be rewritten as the following stochastic differential equation

$$\mathrm{d}x_i(t) = [Ax_i(t) + Bu_i(t)]\mathrm{d}t + \varphi_i(t, x_i(t))\mathrm{d}w(t), \tag{2}$$

where $w(t)$ is a standard one-dimensional Wiener process (Brownian motions) and $\mathbb{E}\{w(t)\} = 0$, the variance $\mathbb{E}\{[w(t)]^2\} = \mathrm{d}t$. The leader indexed as 0 is described as

$$\mathrm{d}x_0(t) = Ax_0(t)\mathrm{d}t + \varphi_0(t, x_0(t))\mathrm{d}w(t), \tag{3}$$

where x_0 represents the state of the leader, $\varphi_0(\cdot, \cdot)\colon \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^n$ is the noise intensity and $w(t)$ is a standard Wiener process. The control input of the leader is zero. That is, the leader's dynamics is independent of others. We take the system matrices for all the agents and leader to be identical because this case has practical background such as group of birds. In order to satisfy that the followers can track the leader with feedback control, the following assumptions are proposed.

Assumption 1. *The pair (A, B) is stabilizable.*

Assumption 2. *The noise intensity function vector $\varphi_i\colon \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^n$, $i = 0, 1, 2, \ldots, N$ satisfies the Lipschitz condition, i.e., there exists a constant matrix Υ such that the following inequality*

$$\|\varphi_i(t, u) - \varphi_0(t, v)\|^2 \leq (u - v)^{\mathsf{T}}\Upsilon(u - v) \tag{4}$$

holds for all $u, v \in \mathbf{R}^n$, $i = 1, 2, \ldots, N$.

In this paper, we design the distributed control for the leader-following multi-agent system under fixed or time-varying topology so that each follower can track the leader using the local information. Therefore the leader-following consensus is achieved when the closed-loop system satisfies

$$\lim_{t\to\infty} \mathbb{E}\{\|x_i(t) - x_0(t)\|^2\} = 0,$$

for any initial condition $x_i(0)$, $i = 1, 2, \ldots, N$, where $\mathbb{E}[\,\cdot\,]$ is the mathematical expectation of a given random variable.

3. Leader-following consensus under fixed topology

In this section, we focus on designing distributed control protocol for the leader-following multi-agent systems under fixed topology so that the closed-loop system can be stabilized. The following assumption is employed throughout this section.

Assumption 3. *The vertex 0 associated with the leader is a globally reachable vertex in the undirected graph $\overline{\mathcal{G}}$.*

In many literatures, based on the relative states between neighboring agents, the static distributed control protocol for leader-following linear multi-agent systems can be designed as

$$u_i = cK\left[\sum_{j=1}^{N} a_{ij}(x_i - x_j) + d_i(x_i - x_0)\right],\ i = 1, 2, \ldots, N, \tag{5}$$

where a_{ij} is the (i, j)-th entry of the adjacency matrix $\mathscr{A}$ associated with graph $\mathcal{G}$ and d_i is the i-th diagonal entry of the leader adjacency matrix D. Let $c > 0$ be the coupling weight among neighboring agents and $K \in \mathbf{R}^{m\times n}$ be a feedback gain matrix.

The coupling weight c in (5) is dependent on λ_1, where $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N$ are the eigenvalues of H. However it is not easy to compute λ_1 when the multi-agent network is of large scale. Moreover, we require to know communication topology for calculating λ_1. We attempt to design an adaptive controller without requiring explicit λ_1. The coupling weights of adaptive controller are updated according to neighboring agents. The distributed adaptive controller is presented as

$$\begin{aligned}
u_i &= K\left[\sum_{j=1}^{N} c_{ij}a_{ij}(x_i - x_j) + c_i d_i(x_i - x_0)\right],\\
\dot{c}_{ij} &= \eta_{ij}a_{ij}(x_i - x_j)^{\mathsf{T}}\Gamma(x_i - x_j),\\
\dot{c}_i &= \eta_i d_i(x_i - x_0)^{\mathsf{T}}\Gamma(x_i - x_0),\ i = 1, 2, \ldots, N.
\end{aligned} \tag{6}$$

Let c_{ij} be the time-varying coupling weight between agent i and agent j, and c_i be the time-varying coupling weight between agent i and the leader, $i = 1, 2, \ldots, N$. Let $\eta_{ij} = \eta_{ji}$, η_i be positive constants and $c_{ij}(0) = c_{ji}(0)$.

Let $\varepsilon_i = x_i - x_0$. Using (6), the dynamics of state error ε_i can be obtained as

$$\mathrm{d}\varepsilon_i(t) = A\varepsilon_i\mathrm{d}t + BK\left[\sum_{j=1}^{N} c_{ij}a_{ij}(\varepsilon_i - \varepsilon_j) + c_i d_i\varepsilon_i\right]\mathrm{d}t + \widetilde{\varphi}_i(t, \varepsilon_i)\mathrm{d}w(t),\ i = 1, 2, \ldots, N, \tag{7}$$

where $\widetilde{\varphi}_i(t, \varepsilon_i(t)) = \varphi_i(t, x_i(t)) - \varphi_0(t, x_0(t))$. Let $P > 0$ be the solutions to the following inequality

$$\begin{aligned}
&P \leq \tau I,\\
&PA + A^{\mathsf{T}}P - 2PBB^{\mathsf{T}}P + \tau\Upsilon < 0,
\end{aligned} \tag{8}$$

where $\tau > 0$ is the tuning parameter. The feedback gain matrix K in (6) is designed as

$$K = -B^{\mathsf{T}}P. \tag{9}$$

The constant gain matrix Γ in (6) can be designed as

$$\Gamma = PBB^{\mathsf{T}}P. \tag{10}$$

According to Assumption 3 and Lemma 1, we derive that the symmetric matrix H is positive definite. Since $\Gamma \geq 0$, the coupling weights c_{ij} and c_i are nondecreasing. Obviously, the leader-following consensus problem is solved by adaptive controller (6) if the state error ε converges to zero in the mean square sense.

Theorem 1. *Consider the multi-agent systems represented by (2) and (3). Let Assumptions 1–3 hold. The distributed tracking problem can be solved under the controller (6) where $P > 0$, K and Γ are the solutions to (8), (9) and (10). Moreover, coupling weights c_{ij} and c_i converge to some finite constants.*

Proof. Consider the Lyapunov function candidate

$$V_1 = \sum_{i=1}^{N} \varepsilon_i^{\mathsf{T}} P \varepsilon_i + \sum_{i=1}^{N} \sum_{j=1, j\neq i}^{N} \frac{(c_{ij}-\alpha)^2}{2\eta_{ij}} + \sum_{i=1}^{N} \frac{(c_i-\alpha)^2}{\eta_i}.$$

By (7) and Itô formula, we obtain that

$$\begin{aligned} \mathrm{d}V_1(t) = {} & \sum_{i=1}^{N} 2\varepsilon_i^{\mathsf{T}} P \mathrm{d}\varepsilon_i(t) + \sum_{i=1}^{N} \widetilde{\varphi}_i^{\mathsf{T}}(t,\varepsilon_i) P \widetilde{\varphi}_i(t,\varepsilon_i)\mathrm{d}t + \sum_{i=1}^{N} \sum_{j=1, j\neq i}^{N} \frac{(c_{ij}-\alpha)}{\eta_{ij}} \mathrm{d}c_{ij}(t) \\ & + \sum_{i=1}^{N} \frac{2(c_i-\alpha)}{\eta_i} \mathrm{d}c_i(t). \end{aligned} \tag{11}$$

Observing (6), we can obtain that $c_{ij}(t) = c_{ji}(t), \forall t \geq 0$. Using (6) and (10), we get

$$\begin{aligned} \sum_{i=1}^{N} \sum_{j=1, j\neq i}^{N} \frac{(c_{ij}-\alpha)}{\eta_{ij}} \mathrm{d}c_{ij}(t) &= \sum_{i=1}^{N} \sum_{j=1, j\neq i}^{N} (c_{ij}-\alpha) a_{ij} (\varepsilon_i - \varepsilon_j)^{\mathsf{T}} PBB^{\mathsf{T}} P (\varepsilon_i - \varepsilon_j)\mathrm{d}t \\ &= 2\sum_{i=1}^{N} \sum_{j=1, j\neq i}^{N} (c_{ij}-\alpha) a_{ij} \varepsilon_i^{\mathsf{T}} PBB^{\mathsf{T}} P (\varepsilon_i - \varepsilon_j)\mathrm{d}t. \end{aligned} \tag{12}$$

By using (7) and substituting (12) into (11), $\mathrm{d}V_1(t)$ can be rewritten as

$$\begin{aligned} \mathrm{d}V_1(t) = {} & 2\sum_{i=1}^{N} \varepsilon_i^{\mathsf{T}} PA\varepsilon_i \mathrm{d}t + \sum_{i=1}^{N} \widetilde{\varphi}_i^{\mathsf{T}}(t,\varepsilon_i) P \widetilde{\varphi}_i(t,\varepsilon_i)\mathrm{d}t + 2\sum_{i=1}^{N} \varepsilon_i^{\mathsf{T}} P \widetilde{\varphi}_i(t,\varepsilon_i)\mathrm{d}w(t) \\ & - 2\alpha \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} \varepsilon_i^{\mathsf{T}} PBB^{\mathsf{T}} P (\varepsilon_i - \varepsilon_j)\mathrm{d}t - 2\alpha \sum_{i=1}^{N} d_i \varepsilon_i^{\mathsf{T}} PBB^{\mathsf{T}} P \varepsilon_i \mathrm{d}t \end{aligned} \tag{13}$$

By using (4), we can obtain that

$$\begin{aligned} \sum_{i=1}^{N} \widetilde{\varphi}_i^{\mathsf{T}}(t,\varepsilon_i) P \widetilde{\varphi}_i(t,\varepsilon_i)\mathrm{d}t &\leq \tau \sum_{i=1}^{N} \widetilde{\varphi}_i^{\mathsf{T}}(t,\varepsilon_i) \widetilde{\varphi}_i(t,\varepsilon_i)\mathrm{d}t \\ &\leq \tau \sum_{i=1}^{N} \varepsilon_i^{\mathsf{T}} \Upsilon \varepsilon_i \mathrm{d}t, \end{aligned} \tag{14}$$

where τ is the maximum eigenvalue of the matrix P. Recalling that $\mathbb{E}\{w(t)\} = 0$, then using the Laplacian matrix L of graph $\mathcal{G}$ and substituting (14) into (13), we get that

$$
\begin{aligned}
\mathrm{d}\mathbb{E}\{V_1(t)\} \le\ & \mathbb{E}\left\{ \sum_{i=1}^{N} \varepsilon_i^{\mathsf{T}}(2PA\varepsilon_i + \tau\Upsilon)\varepsilon_i \mathrm{d}t - 2\alpha \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\varepsilon_i^{\mathsf{T}} PBB^{\mathsf{T}}P(\varepsilon_i - \varepsilon_j)\mathrm{d}t \right.\\
& \left. -2\alpha \sum_{i=1}^{N} d_i \varepsilon_i^{\mathsf{T}} PBB^{\mathsf{T}}P\varepsilon_i \mathrm{d}t \right\}\\
\le\ & \mathbb{E}\left\{ \sum_{i=1}^{N} \varepsilon_i^{\mathsf{T}}(2PA\varepsilon_i + \tau\Upsilon)\varepsilon_i \mathrm{d}t - 2\alpha \sum_{i=1}^{N}\sum_{j=1}^{N} H_{ij}\varepsilon_i^{\mathsf{T}} PBB^{\mathsf{T}}P\varepsilon_j \mathrm{d}t \right\}, \qquad (15)
\end{aligned}
$$

where H_{ij} denotes the (i,j)-th entry of the matrix $H = L + D$. By denoting $\varepsilon = (\varepsilon_1^{\mathsf{T}}, \varepsilon_2^{\mathsf{T}}, \ldots, \varepsilon_N^{\mathsf{T}})$, (15) can be rewritten in a compact form

$$
\mathrm{d}\mathbb{E}\{V_1(t)\} \le \mathbb{E}\left\{\varepsilon^{\mathsf{T}}\left[I_N \otimes (PA + A^{\mathsf{T}}P + \tau\Upsilon) - 2\alpha H \otimes PBB^{\mathsf{T}}P\right]\varepsilon \mathrm{d}t\right\}. \qquad (16)
$$

Let $H = U^{\mathsf{T}}\Lambda U$, where U is an orthogonal matrix. Denote $\delta = (U \otimes I_n)\,\varepsilon$. Then (16) becomes

$$
\begin{aligned}
\mathrm{d}\mathbb{E}\{V_1(t)\} \le\ & \mathbb{E}\left\{\delta^{\mathsf{T}}\left[I_N \otimes (A^{\mathsf{T}}P + PA + \tau\Upsilon) - 2\alpha\Lambda \otimes PBB^{\mathsf{T}}P\right]\delta \mathrm{d}t\right\}\\
\le\ & \mathbb{E}\left\{\sum_{i=1}^{N} \delta_i^{\mathsf{T}}(A^{\mathsf{T}}P + PA - 2\alpha\lambda_i PBB^{\mathsf{T}}P + \tau\Upsilon)\delta_i \mathrm{d}t\right\}, \qquad (17)
\end{aligned}
$$

where λ_i are the eigenvalues of matrix H. As long as $\alpha\lambda_i \ge 1, i = 1, 2, \ldots, N$ hold, by applying (8), (17) becomes

$$
\frac{\mathrm{d}\mathbb{E}\{V_1(t)\}}{\mathrm{d}t} < 0. \qquad (18)
$$

Thus, for any $\delta \ne 0, \mathrm{d}\mathbb{E}\{V_1(t)\} < 0$, which implies that for any $\varepsilon \ne 0, \mathrm{d}\mathbb{E}\{V_1(t)\} < 0$. Therefore, $V_1(t)$ is bounded in the mean square and so are c_{ij} and c_i. Note $\Gamma \ge 0$, so it is obvious that c_{ij} and c_i are nondecreasing. Then c_{ij} and c_i converge to some finite constants, respectively. That is, system (7) is globally stable in the mean square sense and all the agents can follow the leader. □

Remark 2. It is worth mentioning that since c_{ij} and c_i are nondecreasing, the constant α can be sufficiently large to satisfy $\alpha\lambda_i \ge 1, i = 1, 2, \ldots, N$.

Remark 3. One disadvantage of the traditional static control protocol is that the coupling weight c in (5) is given a priori. We need to know the communication topology, which is not practical in many applications. Thus we present an adaptive controller with time-varying weights c_{ij} and c_i. In Theorem 1, we have proved the leader-following multi-agent consensus problem is solved and the coupling weights c_{ij} and c_i converge to some finite constants for any $i, j = 1, 2, \ldots, N$.

Remark 4. The adaptive law (6) is motivated by Li et al. (2013). However Li et al. (2013) didn't take into account any external disturbances and switching topology. In this section, we use stochastic disturbances to model external disturbances and discuss leader-following multi-agent consensus problem for general linear case with stochastic disturbances under fixed topology. Moreover, when the whole interaction topologies are time-varying, consensus problem becomes much more difficult. We will generalize the result in this section to the case of switching topology with stochastic disturbances in Section 4.

4. Leader-following consensus under switching topology

In this section, we discuss the consensus of leader-following multi-agent systems under switching topology. To ensure leader-following consensus under switching topology, there is another constraint on the dynamics of agents.

Assumption 4. *The internal dynamics matrix A has eigenvalues with negative real part.*

In the case of switching topology, the controller of agent i is designed as

$$\begin{aligned} u_i &= K\left[\sum_{j=1}^{N} c_{ij}a_{ij}(t)(x_i - x_j) + c_i d_i(t)(x_i - x_0)\right], \\ \dot{c}_{ij} &= \eta_{ij}a_{ij}(t)(x_i - x_j)^{\mathsf{T}}\Gamma(x_i - x_j), \\ \dot{c}_i &= \eta_i d_i(t)(x_i - x_0)^{\mathsf{T}}\Gamma(x_i - x_0),\ i = 1, 2, \ldots, N, \end{aligned} \tag{19}$$

where $K \in \mathbf{R}^{m\times n}$ is the feedback gain matrix and a_{ij} is the (i, j)-th entry of the adjacency matrix $\mathscr{A}$ associated with graph $\mathcal{G}$. d_i is the connection weight between agent i and the leader, and Γ is a constant gain matrix. c_{ij} is the time-varying coupling weight between agent i and agent j, and c_i is the time-varying coupling weight between agent i and the leader, $i = 1, 2, \ldots, N$. Let $\eta_{ij} = \eta_{ji}$, η_i be positive constants and $c_{ij}(0) = c_{ji}(0)$. Compared to the controller of fixed topology, it is important to note that the neighbors of each agent $a_{ij}(t)$ and $d_i(t)$ describing the neighbors of the leader vary with time. We use $L_{\sigma(t)}$ and $D_{\sigma(t)}$ to describe the time-dependency of graph topology, where σ is a switching signal defined in Section 2. To present the stability, we consider an infinite sequence of nonempty, bounded and contiguous time interval $[t_k, t_{k+1})$ defined as in Section 2. Over each time interval $[t_k, t_{k+1})$, some or all of $\mathcal{G}_k^j$ are permitted to be disconnected. We only require the graph to be jointly connected which is defined as follows.

Definition 1. The union of a collection graph is a graph whose vertex and edge sets are the unions of the vertex and edge sets of the graphs in the collection. We say that such a collection is jointly connected if the union of its members is a connected graph. The graphs are said to be jointly connected across the time interval $[t, t + T], T > 0$ if the union of the graphs $\overline{\mathcal{G}}_{\sigma(s)} \colon s \in [t, t + T]$ is jointly connected.

To ensure leader-following consensus under switching topology, we need another assumption.

Assumption 5. *The graphs are jointly connected across each interval $[t_k, t_{k+1}), k = 0, 1, \ldots$.*

Let $P > 0$ be the solution of the following inequalities

$$\begin{aligned} &PA + A^{\mathsf{T}}P - 2PBB^{\mathsf{T}}P + \tau\Upsilon < 0, \\ &PA + A^{\mathsf{T}}P + \tau\Upsilon \le 0, \\ &P \le \tau I, \end{aligned} \tag{20}$$

where τ is the tuning parameter. Then the feedback gain matrix K can be designed as

$$K = -B^{\mathsf{T}}P. \tag{21}$$

The constant gain matrix Γ can be designed as

$$\Gamma = PBB^{\mathsf{T}}P. \tag{22}$$

Before analyzing the stability of the multi-agent systems under (19) where P, K and T are the solutions of (20)-(22), we denote $L_{\sigma(t)} = L_p, D_{\sigma(t)} = D_p$ and $H_{\sigma(t)} = H_p$ where $p \in \mathcal{P}$. Then we need to label the N eigenvalues of H_p associated to graph $\overline{\mathcal{G}}_p$. The labeling process can be done using following steps.

Step 1: Find all the components of $\mathcal{G}_p$: $S_p^1, S_p^2, \ldots, S_p^{n_p}, 1 \le n_p \le N$.
Step 2: Write the Laplacian of $\mathcal{G}_p$ in the form $L_p = \operatorname{diag}(L_p^1, \ldots, L_p^{n_p})$, where $L_p^i \in \mathbf{R}^{v_p^i \times v_p^i}$ (v_p^i is the number of vertices in components S_p^i) is the Laplacian of the components $S_p^i, i = 1, \ldots, n_p$. Partition the matrix D_p accordingly as $D_p = \operatorname{diag}(D_p^1, \ldots, D_p^{n_p})$. Therefore, $H_p = \operatorname{diag}(H_1, \ldots, H_p^{n_p})$, where $H_p^i = L_p^i + D_p^i, i = 1, \ldots, n_p$.
Step 3: For each $i \in \{1, \ldots, n_p\}$, consider the matrix H_p^i associated with the component S_p^i. Suppose that the vertices $\mathcal{V}_p^i$ of components S_p^i can be written as $\mathcal{V}_p^i = \left\{i_1, i_2, \ldots, i_{v_p^i}\right\}$, where $\left\{i_1, i_2, \ldots, i_{v_p^i}\right\} \subset \{1, 2, \ldots, N\}$ and $i_1 < i_2 < \cdots < i_{v_p^i}$. Note that H_p^i is a $v_p^i \times v_p^i$ matrix, there are v_p^i eigenvalues of H_p^i, and denote them by $e_{p,1}^i \le e_{p,2}^i \le \cdots \le e_{p,v_p^i}^i$. Introduce v_p^i distinct symbols $\lambda_p^{i_1}, \lambda_p^{i_1}, \ldots, \lambda_p^{i_{v_p}}$ and label the eigenvalues of H_p^i as $\lambda_p^{i_1}(\leftrightarrow e_{p,1}^i), \lambda_p^{i_2}(\leftrightarrow e_{p,2}^i), \ldots, \lambda_p^{i_{v_p}}(\leftrightarrow e_{p,v_p^i}^i)$.
Step 4: Putting the labels for $H_p^1, H_p^2, \ldots, H_p^{n_p}$ together yields the labels for H_p.

We give an example to illustrate the rule of the labeling.

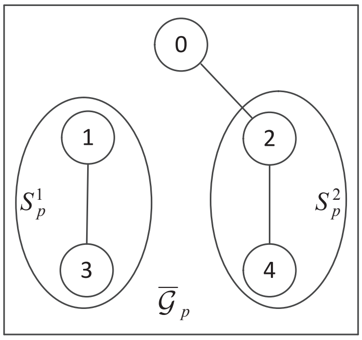


Figure 1. A graph illustrating Steps 1-4

Example. Consider the graph $\overline{\mathcal{G}}_p$ in Figure 1. There are two components S_p^1, and S_p^2 of $\mathcal{G}_p$. The matrices H_p^1 and H_p^2 associated with S_p^1, and S_p^2 are

$$H_p^1 = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad H_p^2 = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}.$$

The eigenvalues of H_p^1 are 0 and 2, then we label them as $\lambda_p^1(\leftrightarrow 0), \lambda_p^3(\leftrightarrow 2)$ using $\mathcal{V}_p^1 = \{1, 3\}$. The eigenvalues of H_p^2 are 0.382 and 2.618, then we label them as $\lambda_p^2(\leftrightarrow 0.382), \lambda_p^4(\leftrightarrow 2.618)$ using $\mathcal{V}_p^2 = \{2, 4\}$. Therefore the eigenvalues of H_p are labeled as $\lambda_p^1(\leftrightarrow 0), \lambda_p^2(\leftrightarrow 0.382), \lambda_p^3(\leftrightarrow 2), \lambda_p^4(\leftrightarrow 2.618)$.

According to the rule of labeling, for each $p \in \mathcal{P}$, there are N eigenvalues of H_p which are labeled as $\left\{\lambda_p^1, \lambda_p^2, \ldots, \lambda_p^N\right\}$. Among the N labels, we define

$$l(p) = \left\{k\colon \text{the eigenvalue associated with } \lambda_p^k \text{ is nonzero, } k = 1, 2, \ldots, N\right\}.$$

Lemma 2 (Ni and Cheng, 2010)**.** *The graphs are jointly connected across each time interval* $[t_k, t_{k+1})$ *if and only if*

$$\bigcup_{t \in [t_k, t_{k+1})} l(\sigma(t)) = \{1, 2, \ldots, N\}.$$

Next we give an example to illustrate this lemma.

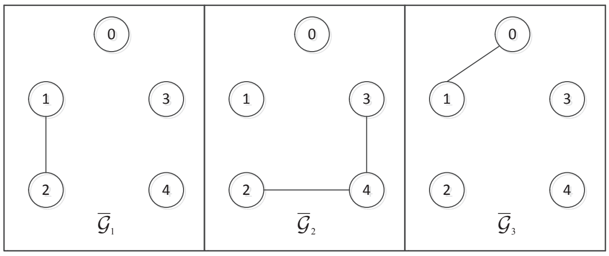


Figure 2. Three graphs illustrating Lemma 2

Example. Consider the three graphs $\overline{\mathcal{G}}_1, \overline{\mathcal{G}}_2, \overline{\mathcal{G}}_3$ in Figure 2. It can be seen that the union graph $\overline{\mathcal{G}}_1 \cup \overline{\mathcal{G}}_2 \cup \overline{\mathcal{G}}_3$ is connected. The matrices H_1, H_2, H_3 associated with graph $\overline{\mathcal{G}}_1, \overline{\mathcal{G}}_2, \overline{\mathcal{G}}_3$ can be calculated as

$$H_1 = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad H_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}, \quad H_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Carrying out the labeling procedure, we label the four eigenvalues of H_1 as $\lambda_1^1(\leftrightarrow 0), \lambda_1^2(\leftrightarrow 2), \lambda_1^3(\leftrightarrow 0), \lambda_1^4(\leftrightarrow 0)$; the eigenvalues of H_2 as $\lambda_2^1(\leftrightarrow 0), \lambda_2^2(\leftrightarrow 0), \lambda_2^3(\leftrightarrow 1), \lambda_2^4(\leftrightarrow 3)$; and the four eigenvalues of H_3 as $\lambda_3^1(\leftrightarrow 1), \lambda_3^2(\leftrightarrow 0), \lambda_3^3(\leftrightarrow 0), \lambda_3^4(\leftrightarrow 0)$. The three index sets as follows: $l(1) = \{2\}, l(2) = \{3,4\}, l(3) = \{1\}$. Thus $l(1) \cup l(2) \cup l(3) = \{1,2,3,4\}$, which illustrates the result in Lemma 2.

Theorem 2. *Consider the multi-agent systems represented by (2) and (3). Let Assumptions 1, 2 and 4 hold, and switching signal σ satisfies Assumption 5. The distributed tracking problem can be solved under the controller (19) where $P > 0$, K and Γ are the solutions to (20), (21) and (22). Moreover, each coupling weight c_{ij} and c_i converge to some finite constants.*

Proof. Let $\varepsilon_i = x_i - x_0$, $i = 1, 2, \ldots, N$. The dynamics of state errors ε_i can be obtained as

$$\mathrm{d}\varepsilon_i(t) = A\varepsilon_i \mathrm{d}t + BK\left[\sum_{j=1}^{N} c_{ij}a_{ij}(t)(\varepsilon_i - \varepsilon_j) + c_i d_i(t)\varepsilon_i\right]\mathrm{d}t + \widetilde{\varphi}_i(t,\varepsilon_i)\mathrm{d}w(t),\ i = 1, 2, \ldots, N. \quad (23)$$

where $\widetilde{\varphi}_i(t, \varepsilon_i(t)) = \varphi_i(t, x_i(t)) - \varphi_0(t, x_0(t))$. Consider Lyapunov function candidate

$$V_2 = \sum_{i=1}^{N} \varepsilon_i^{\mathsf{T}} P \varepsilon_i + \sum_{i=1}^{N}\sum_{j=1, j\neq i}^{N} \frac{(c_{ij}-\alpha)^2}{2\eta_{ij}} + \sum_{i=1}^{N} \frac{(c_i - \alpha)^2}{\eta_i}, \quad (24)$$

where $\alpha > 0$ is the positive constants to be determined.

According to Itô formula, we obtain that

$$\begin{aligned} \mathrm{d}V_2(t) = & \sum_{i=1}^{N} 2\varepsilon_i^{\mathsf{T}} P \mathrm{d}\varepsilon_i(t) + \sum_{i=1}^{N} \widetilde{\varphi}_i^{\mathsf{T}}(t,\varepsilon_i) P \widetilde{\varphi}_i(t,\varepsilon_i)\mathrm{d}t + \sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N} \frac{(c_{ij}-\alpha)}{\eta_{ij}}\mathrm{d}c_{ij}(t) \\ & + \sum_{i=1}^{N} \frac{2(c_i - \alpha)}{\eta_i}\mathrm{d}c_i(t). \end{aligned} \quad (25)$$

Observing (19), we can obtain that $c_{ij}(t) = c_{ji}(t), \forall t \geq 0$. Using (19) and (22), we get

$$\sum_{i=1}^{N} \sum_{j=1, j\neq i}^{N} \frac{(c_{ij} - \alpha)}{\eta_{ij}} \mathrm{d}c_{ij}(t) = 2 \sum_{i=1}^{N} \sum_{j=1, j\neq i}^{N} (c_{ij} - \alpha) a_{ij}(t) \varepsilon_i^\top PBB^\top P(\varepsilon_i - \varepsilon_j)\mathrm{d}t. \tag{26}$$

By using (23) and substituting (26) into (25), $\mathrm{d}V_2(t)$ can be rewritten as

$$\begin{aligned} \mathrm{d}V_2(t) = {} & 2\sum_{i=1}^{N} \varepsilon_i^\top PA\varepsilon_i \mathrm{d}t + \sum_{i=1}^{N} \widetilde{\varphi}_i^\top(t, \varepsilon_i) P \widetilde{\varphi}_i(t, \varepsilon_i)\mathrm{d}t + 2\sum_{i=1}^{N} \varepsilon_i^\top P \widetilde{\varphi}_i(t, \varepsilon_i)\mathrm{d}w(t) \\ & - 2\alpha \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}(t)\varepsilon_i^\top PBB^\top P(\varepsilon_i - \varepsilon_j)\mathrm{d}t - 2\alpha \sum_{i=1}^{N} d_i(t)\varepsilon_i^\top PBB^\top P\varepsilon_i \mathrm{d}t. \end{aligned} \tag{27}$$

By using (4), we can obtain that

$$\sum_{i=1}^{N} \widetilde{\varphi}_i^\top(t, \varepsilon_i) P \widetilde{\varphi}_i(t, \varepsilon_i)\mathrm{d}t \leq \tau \sum_{i=1}^{N} \varepsilon_i^\top \Upsilon \varepsilon_i \mathrm{d}t, \tag{28}$$

where τ is the maximum eigenvalue of the matrix P. Recalling that $\mathbb{E}\{w(t)\} = 0$, then using the Laplacian matrix L of graph $\mathcal{G}$ and substituting (28) into (27), we get that

$$\mathrm{d}\mathbb{E}\{V_2(t)\} \leq \mathbb{E}\left\{ \sum_{i=1}^{N} \varepsilon_i^\top (2PA\varepsilon_i + \tau\Upsilon)\varepsilon_i \mathrm{d}t - 2\alpha \sum_{i=1}^{N}\sum_{j=1}^{N} H_{ij}(t)\varepsilon_i^\top PBB^\top P\varepsilon_j \mathrm{d}t \right\}, \tag{29}$$

where $H_{ij}(t)$ denotes the (i,j)-th entry of the matrix $H_{\sigma(t)} = L_{\sigma(t)} + D_{\sigma(t)}$. By denoting $\varepsilon = (\varepsilon_1^\top, \varepsilon_2^\top, \ldots, \varepsilon_N^\top)$, (29) can be rewritten in a compact form

$$\mathrm{d}\mathbb{E}\{V_2(t)\} \leq \mathbb{E}\left\{ \varepsilon^\top \left[I_N \otimes (PA + A^\top P + \tau\Upsilon) - 2\alpha H \otimes PBB^\top P \right] \varepsilon \mathrm{d}t \right\}. \tag{30}$$

Let $H = U^\top \Lambda U$, where U is an orthogonal matrix. Denote $\delta = (U \otimes I_n)\,\varepsilon$. Then (30) becomes

$$\begin{aligned} \mathrm{d}\mathbb{E}\{V_2(t)\} & \leq \mathbb{E}\left\{ \delta^\top \left[I_N \otimes (A^\top P + PA + \tau\Upsilon) - 2\alpha\Lambda \otimes PBB^\top P \right] \delta \mathrm{d}t \right\} \\ & \leq \mathbb{E}\left\{ \sum_{i\in l(\sigma(t))} \delta_i^\top (A^\top P + PA - 2\alpha\lambda_i PBB^\top P + \tau\Upsilon)\delta_i \mathrm{d}t \right\} \\ & \leq \mathbb{E}\left\{ - \sum_{i\in l(\sigma(t))} \delta_i^\top \rho \delta_i \mathrm{d}t \right\}, \end{aligned} \tag{31}$$

where $-\rho < 0$ is the maximum eigenvalue of the matrix $PA + A^\top P - 2PBB^\top P + \tau\Upsilon$ and $\alpha\lambda_i > 1$. Because the coupling weights c_{ij} and c_i are nondecreasing, the constant α can be sufficiently large to make $\alpha\lambda_i > 1, \forall i \in l(\sigma(t))$. Therefore, V_2 has a limit in the mean square as time goes on. Next we prove that $\varepsilon(t)$ converges to 0 in the mean square. Consider the infinite sequences $V_2(t_i), i = 0, 1, \ldots$. Because $\mathbb{E}\{V_2(t)\}$ has the limit, so we can obtain that for any $\epsilon > 0$, there exists a positive number M, such that for

$\forall k \geq M$

$$\mathbb{E}\{|V_2(t_{k+1}) - V_2(t_k)|\} < \epsilon. \tag{32}$$

Therefore, considering $\varepsilon(t)$ we can get

$$\mathbb{E}\left\{\rho\left[\int_{t_k^0}^{t_k^1}\sum_{i\in l(\sigma(t_k^0))}\delta_i^{\mathsf{T}}(t)\delta_i(t)\mathrm{d}t + \cdots + \int_{t_k^{m_k-1}}^{t_k^{m_k}}\sum_{i\in l\left(\sigma(t_k^{m_k-1})\right)}\delta_i^{\mathsf{T}}(t)\delta_i(t)\mathrm{d}t\right]\right\} \leq \epsilon.$$

Thus, for $\forall k > M$, we have

$$\mathbb{E}\left\{\rho\left[\int_{t_k^0}^{t_k^0+\tau}\sum_{i\in l(\sigma(t_k^0))}\delta_i^{\mathsf{T}}(t)\delta_i(t)\mathrm{d}t + \cdots + \int_{t_k^{m_k-1}}^{t_k^{m_k-1}+\tau}\sum_{i\in l\left(\sigma(t_k^{m_k-1})\right)}\delta_i^{\mathsf{T}}(t)\delta_i(t)\mathrm{d}t\right]\right\} \leq \epsilon,$$

which implies

$$\lim_{t\to\infty}\int_t^{t+\tau}\mathbb{E}\left\{\sum_{i\in l(\sigma(t))}\delta_i(s)^{\mathsf{T}}\delta_i(s)\right\}\mathrm{d}s = 0.$$

Therefore,

$$\lim_{t\to\infty}\int_t^{t+\tau}\mathbb{E}\left\{\left[\sum_{i\in l(\sigma(t_k^0))}\delta_i^{\mathsf{T}}(s)\delta_i(s) + \cdots + \sum_{i\in l(\sigma(t_k^{m_k-1}))}\delta_i^{\mathsf{T}}(s)\delta_i(s)\right]\right\}\mathrm{d}s = 0. \tag{33}$$

According to Lemma 2 and Assumption 5, (33) can be written as

$$\lim_{t\to\infty}\int_t^{t+\tau}\mathbb{E}\left\{\left[\sum_{i=1}^{N} r_i\delta_i^{\mathsf{T}}(s)\delta_i(s)\right]\right\}\mathrm{d}s = 0,$$

where $r_1, \ldots, r_N$ are positive numbers and we can obtain that $\lim_{s\to\infty}\sum_{i=1}^{N}\mathbb{E}\{r_i\delta_i^{\mathsf{T}}(s)\delta_i(s)\} = 0$. Therefore δ_i converges to 0 in the mean square, so does ε_i, $i = 1, 2, \ldots, N$. Moreover, due to $\mathrm{d}\mathbb{E}\{V_2(t)\} \leq 0$, c_{ij} and c_i are bounded, $i = 1, 2, \ldots, N$, $j = 1, \ldots, N$. Note that c_{ij} and c_i are nondecreasing, so all of them converge to some constants. □

Remark 5. Some other relevant research on the leader-following tracking under jointly connected graphs has also been studied (Su and Huang 2012a,b). It is worth mentioning that when the stochastic disturbances are set to zero, a similar method of designing feedback gain matrix is given by Su and Huang (2012a) which only requires the second inequality of (20). Su and Huang (2012a) demonstrate the stability utilizing a generalized Barbalat's Lemma. Moreover Su and Huang (2012b) designed both dynamic state feedback and dynamic measurement output feedback controllers for leader-following under switching topology. In this paper, we consider the consensus problem for leader-following multi-agent systems under jointly connected graphs with stochastic disturbances. We propose an adaptive law for adjusting coupling weights and the stability is proved using Lyapunov's method.

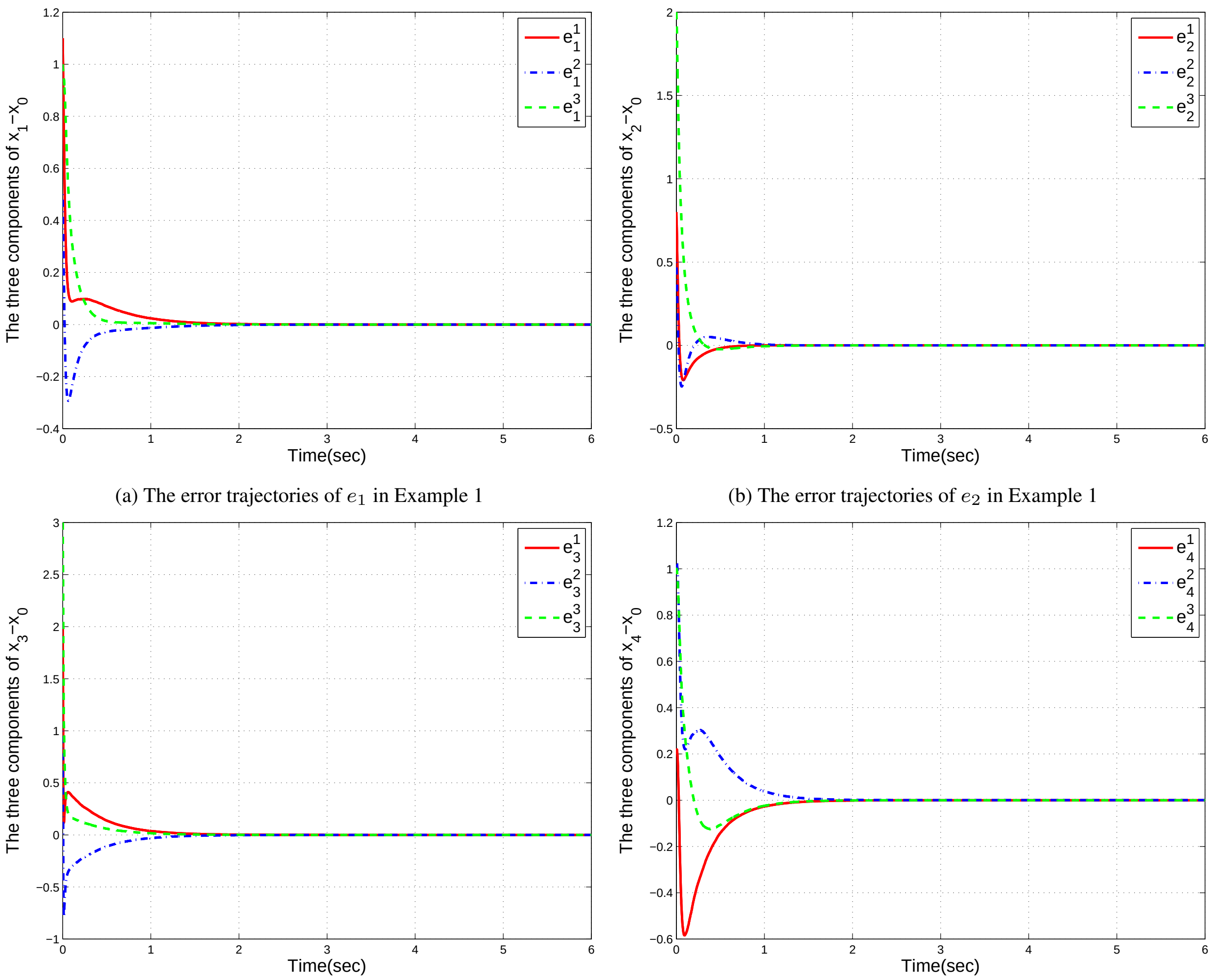
5. Simulation study

Example 1. First we consider a multi-agent system with four followers and labeled as 1, 2, 3, 4 and a leader labeled as 0. The dynamics of the ith agent is described as (2) and (3) with

$$A = \begin{bmatrix} -7 & 10 & -1 \\ -6 & 7 & 2 \\ -3 & 3 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 \\ 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

The Laplacian matrix of $\mathcal{G}$ and the adjacency matrix of leader are given by

$$L = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$



(a) The error trajectories of e_1 in Example 1

(b) The error trajectories of e_2 in Example 1

(c) The error trajectories of e_3 in Example 1

(d) The error trajectories of e_4 in Example 1

Figure 3. The error trajectories in Example 1

Note that (A, B) is stabilizable and the graph $\overline{\mathcal{G}}$ is connected. Let noise intensity function be $\varphi(t, x_i) = [1.0\cos(t)x_{i1}(t), 1.2\cos(t)x_{i2}(t), 1.4\cos(t)x_{i3}(t)]^{\mathrm{T}}$ which satisfies Assumption 2. We can get a solution to

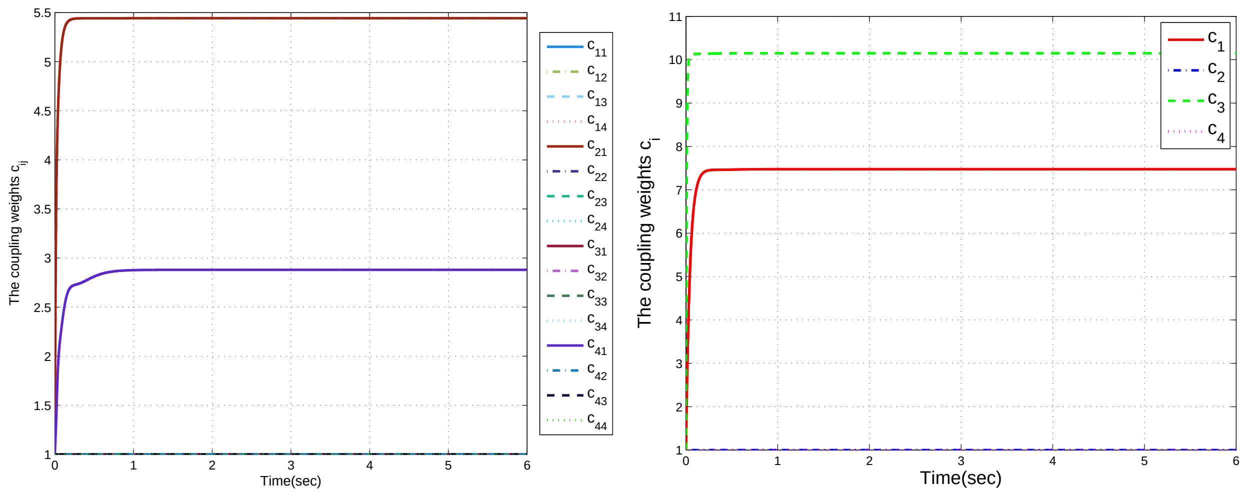
(8)–(10) as

$$P = \begin{bmatrix} 24.6826 & -12.6438 & -16.6303 \\ -12.6438 & 18.8579 & -3.1606 \\ -16.6303 & -3.1606 & 37.4419 \end{bmatrix}, \quad K = \begin{bmatrix} -7.4472 & -9.2676 & 2.1399 \\ 4.5915 & -3.0535 & -17.6510 \end{bmatrix},$$

$$\Gamma = \begin{bmatrix} 76.5437 & 54.9976 & -96.9816 \\ 54.9976 & 95.2123 & 34.0665 \\ -96.9816 & 34.0665 & 316.1381 \end{bmatrix}.$$

The initial states are set as $x_0 = (1.0, 2.0, 3.0)^{\mathsf{T}}$, $x_1 = (2.1, 2.5, 4)^{\mathsf{T}}$, $x_2 = (1.8, 2.5, 5)^{\mathsf{T}}$, $x_3 = (3.0, 3.0, 6.0)^{\mathsf{T}}$ and $x_4 = (1.2, 3, 4)^{\mathsf{T}}$. Under the control law (6), the state error trajectories of each follower $e_i = x_i - x_0, i = 1, 2, \ldots, 4$ are shown in Figure 3.

Moreover the coupling weights c_{ij} and c_i, i, $j = 1, 2, \ldots, 4$ associated with (6) converge to some constants shown in Figure 4.



(a) The coupling weights trajectories of c_{ij} in Example 1 (b) The coupling weights trajectories of c_i in Example 1

Figure 4. The coupling weights trajectories in Example 1

Example 2. Suppose there are four followers and a leader described by

$$A = \begin{bmatrix} -0.4921 & -1.5995 & -1.0676 \\ 4.9430 & 0.9422 & 0.9153 \\ -0.6021 & -1.3492 & -1.0025 \end{bmatrix}, \quad B = \begin{bmatrix} -1.1437 & 0.6380 \\ 0.4085 & -1.2665 \\ 0.4191 & 0.5462 \end{bmatrix}.$$

It is easy to check that (A, B) is stabilizable and eigenvalues of the matrix A are $-0.2316 + 2.7365i$, $-0.2316 - 2.7365i$ and -0.0893. The noise intensity function is represented as $\varphi(t, x_i) = [0.2\cos(t)x_{i1}(t), 0.2\cos(t)x_{i2}(t), 0.2\cos(t)x_{i3}(t)]^{\mathsf{T}}$ which satisfies Assumption 2. Let all the possible interaction graphs be $\{\overline{\mathcal{G}}_1, \overline{\mathcal{G}}_2, \overline{\mathcal{G}}_3, \overline{\mathcal{G}}_4, \overline{\mathcal{G}}_5, \overline{\mathcal{G}}_6\}$ shown in Figure 5. The interaction topologies are switched in order of $\overline{\mathcal{G}}_1 \to \overline{\mathcal{G}}_2 \to \overline{\mathcal{G}}_3 \to \overline{\mathcal{G}}_4 \to \overline{\mathcal{G}}_5 \to \overline{\mathcal{G}}_6 \to \overline{\mathcal{G}}_1 \to \cdots$, and each graph is kept for $1/3$ s. Because the graphs $\overline{\mathcal{G}}_1 \cup \overline{\mathcal{G}}_2 \cup \overline{\mathcal{G}}_3$ and $\overline{\mathcal{G}}_4 \cup \overline{\mathcal{G}}_5 \cup \overline{\mathcal{G}}_6$ are connected, Assumption 5 is satisfied. Using Matlab, we can

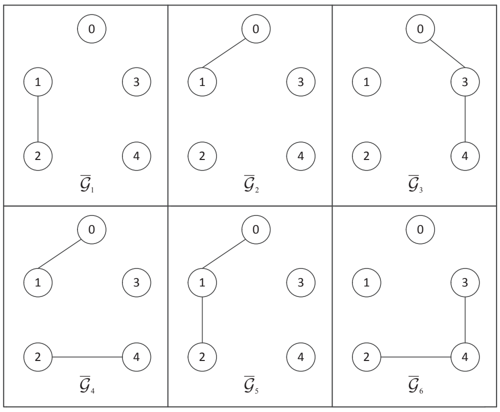


Figure 5. Possible interaction topologies in Example 2

obtain a solution to (20)–(22) as

$$P = \begin{bmatrix} 11.5516 & 0.3095 & -5.3264 \\ 0.3095 & 2.4252 & 1.7633 \\ -5.3264 & 1.7633 & 7.8493 \end{bmatrix}, \quad K = \begin{bmatrix} 15.3174 & -1.3757 & -10.1017 \\ -4.0687 & 1.9109 & 1.3442 \end{bmatrix},$$

$$\Gamma = \begin{bmatrix} 251.1781 & -28.8479 & -160.2019 \\ -28.8479 & 5.5443 & 16.4661 \\ -160.2019 & 16.4661 & 103.8520 \end{bmatrix}.$$

Under the control law (19), the state trajectories x_i, $i = 0, 1, \ldots, 4$ and states error trajectories of $e_i = x_i - x_0$, $i = 1, 2, \ldots, 4$ are shown in Figures 6 and 7. In addition the coupling weights trajectories of c_{ij}, $i = 1, 2, \ldots, 4$, $j = 1, 2, \ldots, 4$ and c_i, $i = 1, 2, \ldots, 4$ are shown in Figure 8 converging to some constants.

6. Conclusion

In this paper, we consider leader-following multi-agent consensus problem for general linear case with stochastic disturbances. In contrast to traditional distributed static controller which requires the knowledge of entire communication graph, we propose an adaptive protocol based on Riccati inequalities with an adaptive law for adjusting coupling weights between neighboring agents. We discuss multi-agent consensus problems under fixed topology and switching topology, respectively. The consensus can be reached for multi-agent systems under jointly connected graph in our method and the stability is demonstrated using Lyapunov's method. Finally, simulations are given to show our adaptive controller is efficient no matter multi-agent systems is under fixed topology or switching topology.

Future work will address directed networks with time-varying topologies and study effect of communication noise and delay acting on multi-agent systems.

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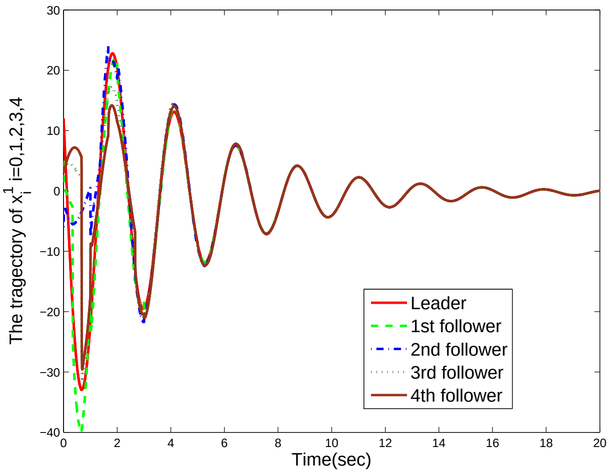
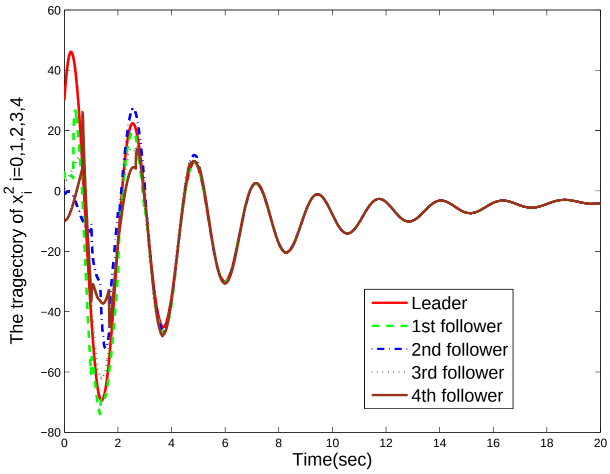
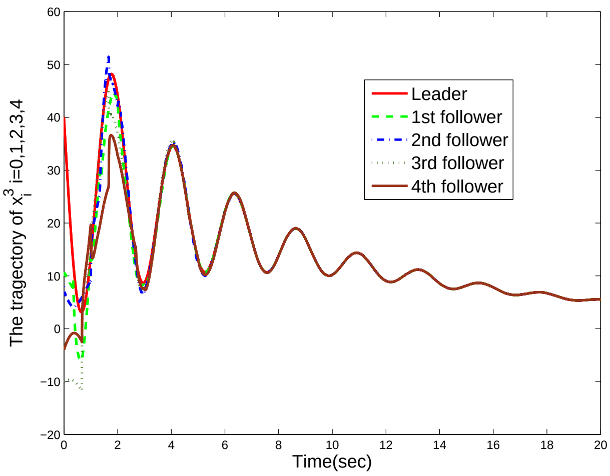


(a) The state trajectories of x_i^1 in Example 2



(b) The state trajectories of x_i^2 in Example 2



(c) The state trajectories of x_i^3 in Example 2

Figure 6. The state trajectories in Example 2

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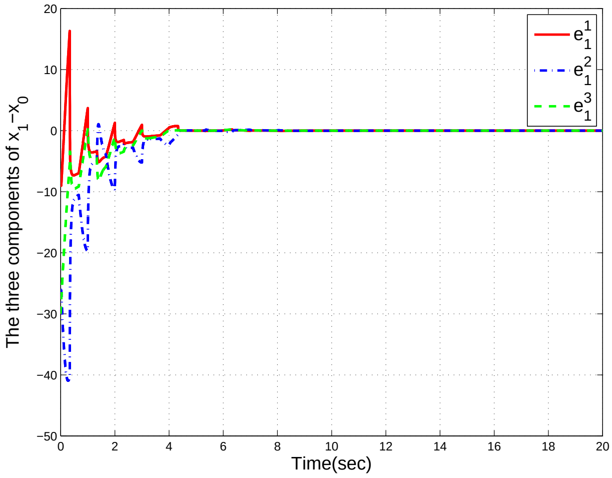
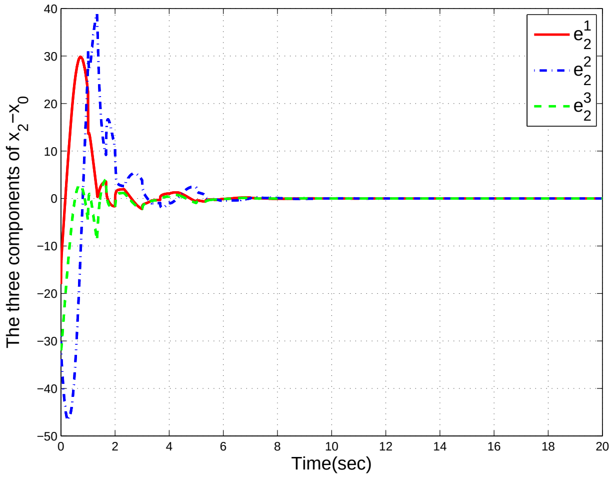
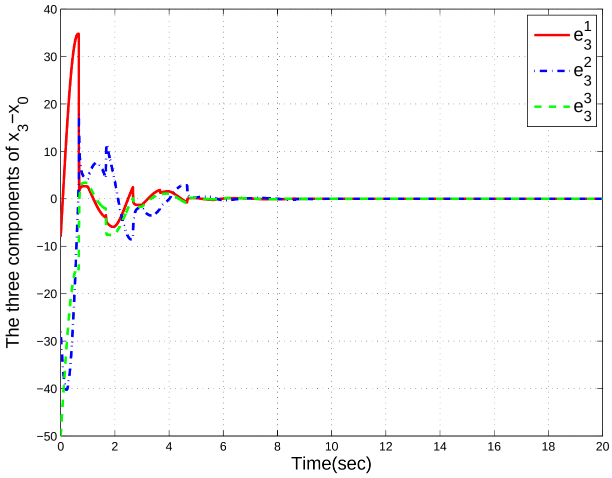


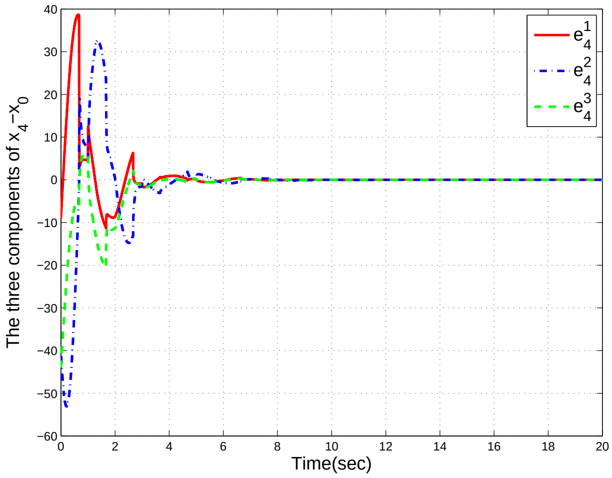
(a) The error trajectories of e_1 in Example 2



(b) The error trajectories of e_2 in Example 2

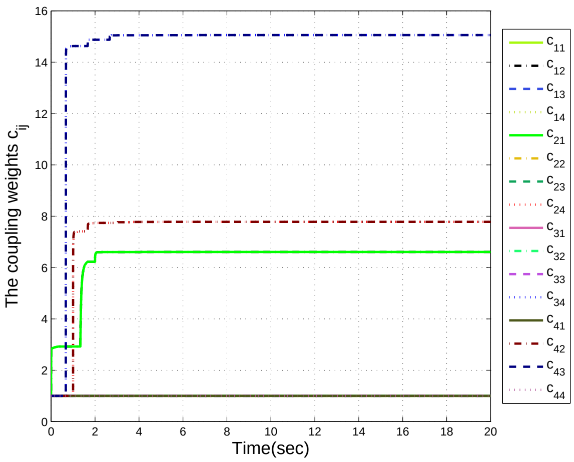


(c) The error trajectories of e_3 in Example 2

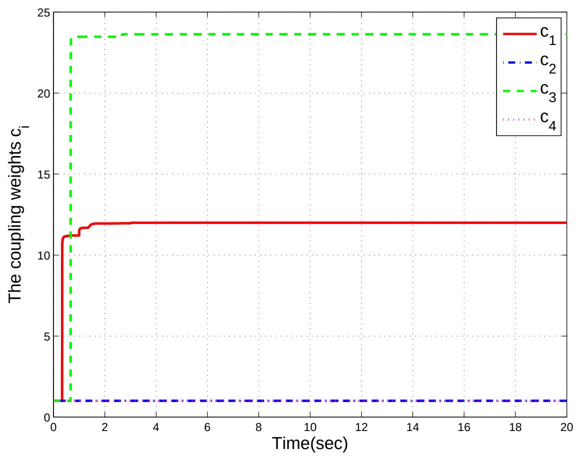


(d) The error trajectories of e_4 in Example 2

Figure 7. The error trajectories in Example 2



(a) The coupling weights trajectories of c_{ij} in Example 2



(b) The coupling weights trajectories of c_i in Example 2

Figure 8. The coupling weights trajectories in Example 2

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