A Stochastic Model for Budget Distribution over Time in Search Advertisements

Rui Qin[1,2,3], Yong Yuan[1,2,3], Juanjuan Li[1,2,3], Yanwu Yang[4]
[1]The State Key Laboratory of Management and Control for Complex Systems,
Institute of Automation, Chinese Academy of Sciences, Beijing, China
[2]Qingdao Academy of Intelligent Industries, Qingdao, China
[3]Beijing Engineering Research Center of Intelligent Systems and Technology,
Institute of Automation, Chinese Academy of Sciences, Beijing,China
[4]School of Management, Huazhong University of Science and Technology, Wuhan, China
Email: qinrui.isec@gmail.com, yong.yuan@ia.ac.cn, lijuanjuan.isec@gmail.com

***Abstract*—In search advertisements, advertisers have to seek for an effective allocation strategy to distribute the limited budget over a series of sequential temporal slots (e.g., days). However, advertisers usually have no sufficient knowledge to determine the optimal budget for each temporal slot, because there exist much uncertainty in search advertising markets. In this paper, we present a stochastic model for budget distribution over a series of sequential temporal slots under a finite time horizon, assuming that the best budget is a random variable. We study some properties and feasible solutions for our model, taking the best budget being characterized by either normal distribution or uniform distribution, respectively. Furthermore, we also make some experiments to evaluate our model and identify strategies with the real-world data collected from practical advertising campaigns. Experimental results show that a) our strategies outperform the baseline strategy that is commonly used in practice; b) the optimal budget is more likely to be normally distributed than uniformly distributed.**
Keywords: search advertisement; budget distribution; optimal budget; stochastic programming; budget constraint

I. Introduction

Recently, search advertisements have become more and more popular due to such advantages as lower costs and quick promotion effectiveness. Major search engines (e.g., Google) have successfully found and monetized the long tails of both advertisers and publishers. For example, the long tail advertisers contribute at least half revenue for Google [1]. Most of the long-tail advertisers are small companies and individuals that have limited resources (e.g., advertising budgets). Therefore, it is necessary for these advertisers to figure out an effective way to manipulate limited budgets in order to maximize their revenues in search markets.

Most of current works on search advertising strategies simply took the budget as constraints when determining bids over keywords of interest [2], [4], [5]. We argue that these works fall into the category of bidding strategies. With consideration of the entire lifecycle of search advertising, budget decisions in search advertisements exist at three levels [6], [7], [8]: allocation across search markets, temporal distribution over a series of slots (e.g. day) and adjustment of the remaining budget (e.g., the daily budget). This work is aimed at dealing with the budget allocation problem over a series of sequential slots (e.g., days). First, there are much uncertainty in the mapping from the budget into the advertising performance [3]. Second, advertisers have to adapt daily budgets to an optimal level according to some key factors such as cost-per-click (CPC) and click-through-rate (CTR). Third, the search marketing environment is essentially dynamic and thus it is difficult to precisely predict the optimal budget. If the allocated budget is less than the optimal budget, the advertiser will lose some potential clicks (customers), and if the allocated budget is set too high, the advertiser will waste her money on clicks without valuable actions.

In this work, we formulate the budget distribution over a series of sequential temporal slots as a stochastic programming problem. Note that this work takes one day as the temporal granularity, but can also be applied to similar decision scenarios with different temporal granularity levels. First, we consider the optimal budget as a random variable, because it can to some degree reflect the environmental randomness of budget-related decisions at the campaign level. The probability distribution of the optimal budget can be extracted from promotion logs of historical campaigns. Second, we present a stochastic model for budget distribution over a series of temporal slots (e.g., days), given the total budget in a search market is given. Third, we discuss some properties and possible solutions of our model, by considering the optimal budget as a uniform random variable and a normal random variable, respectively: (a) when the optimal budget is uniformly distributed, the proposed model is convex, and an analytic solution is presented; (b) when it is normally distributed, the proposed model is also convex. However, an analytic solution can not be obtained because it contains the standard normal distribution function Φ. For computational purposes, we provide an numerical solution that could compute Φ. Furthermore, we make some experiments to evaluate our budget model and the identified strategies. Experimental results show that the strategy driven by normal distributions outperforms the other two in terms of total effective clicks, following by the uniform distribution strategy, and then the baseline strategy that is commonly used in practice. This might be explained by the fact that the optimal budget is more likely to be normally distributed than uniformly distributed.

The rest of this paper is organized as follows. In Section II, we present a stochastic budget distribution model over a series of temporal slots. In Section III, we discuss some properties and solutions to our budget model. In Section IV, we report some experimental results to evaluate our budget model. Section V discusses some limitations and managerial insights of our work, and Section VI concludes this paper.

II. A Stochastic Budget Distribution Model

We consider the following scenario in search advertisements: given the total budget B in a search market, an advertiser has to distribute the budget into a series of n temporal slots in order to maximize her revenue. For each temporal slot, there exists an optimal budget. If the allocated budget is less than the optimal budget, the allocated budget will be used up with effective CTR c_j. Otherwise, if the allocated budget is more than the optimal budget, then the exceeded part of the allocated budget will also be used up, but with a lower effective CTR $c_j^{'}$.

Let d_j represent the optimal budget of the jth temporal slot (e.g., day). Though the advertiser can not know the precise optimal budget since there exists much uncertainty in search advertisements, she can extract some information (e.g., the lower bound $\underline{b}$ and the upper bound $\overline{b}$) of the optimal budget from promotion logs of historical campaigns. Since the optimal budget reflects the environmental randomness of budget-related decisions at the campaign level, we can denote the optimal budget d_j as a random variable on $[\underline{b}, \overline{b}]$ with probability distribution $f(d_j)$.

Thus, in this paper, we discuss the case that the total budget B is insufficient, i.e., $n\underline{b} < B < n\overline{b}$, and the allocated budget satisfies $\underline{b} \le b_j \le \overline{b}$. Since the total budget is B, which can not be exceeded by the budget of n temporal slots, that is $\sum_{j=1}^{n} b_j \le B$.

Let $C(b_j, d_j)$ be the revenue of the jth temporal slot with budget b_j under the case that the optimal budget is d_j, then $\sum_{j=1}^{n} C(b_j, d_j)$ is the total revenue of the n temporal slots. Since d_j is a random variable, $C(b_j, d_j)$ and $\sum_{j=1}^{n} C(b_j, d_j)$ are also random variables. Thus, the purpose of the advertiser is to maximize her total expected revenue, i.e., $E[\sum_{j=1}^{n} C(b_j, d_j)]$. Since d_j, $j = 1, 2, ..., n$, are independent random variables, we have

$$E\left[\sum_{j=1}^{n} C(b_j, d_j)\right] = \sum_{j=1}^{n} E[C(b_j, d_j)].$$

Below we discuss how to compute the total expected revenue, which are characterized by total expected effective clicks.

Let p_j be clicks per unit cost of the jth temporal slot, then $p_j b_j$ represents the clicks obtained in the jth temporal slot. Let c_j be the effective CTR of the budget below the optimal budget in the jth temporal slot, and $c_j^{'}$ the effective CTR of the budget above the optimal budget in the jth temporal slot, $j = 1, 2, \ldots, n$. Since the number of potential users is limited and the optimal budget is corresponding to the potential users, if the advertiser allocates more budget than the optimal budget, the effective CTR will become smaller, i.e., $c_j > c_j^{'}$.

If $b_j < d_j$, then the effective clicks obtained by b_j is $p_j b_j c_j$, otherwise if $b_j \ge d_j$, then the effective clicks obtained by b_j can be divided into two parts, the first part d_j with effective CTR c_j, and another part $b_j - d_j$ with effective CTR $c_j^{'}$. Thus the effective clicks will be $p_j c_j d_j + p_j c_j^{'}(b_j - d_j)$. Since d_j is a random variable on $[\underline{b}, \overline{b}]$ with probability distribution $f(d_j)$, then based on the concept of expected value of random variable, for each j, we have

$$\begin{aligned} E[C(b_j, d_j)] &= \int_{\underline{b}}^{b_j} (p_j c_j d_j + p_j c_j^{'}(b_j - d_j)) f(d_j) \mathrm{d}d_j \\ &\quad + \int_{b_j}^{\overline{b}} p_j c_j b_j f(d_j) \mathrm{d}d_j. \end{aligned} \tag{1}$$

Thus, with the probability distribution $f(d_j)$ on $[\underline{b}, \overline{b}]$, we can formulate the following stochastic budget distribution model for a given campaign

$$\begin{aligned} \max \quad & \sum_{j=1}^{n} \Big[\int_{\underline{b}}^{b_j} (p_j c_j d_j + p_j c_j^{'}(b_j - d_j)) f(d_j) \mathrm{d}d_j \\ & \qquad + \int_{b_j}^{\overline{b}} p_j c_j b_j f(d_j) \mathrm{d}d_j \Big] \\ \text{s.t.} \quad & \sum_{j=1}^{n} b_j \le B \\ & \underline{b} \le b_j \le \overline{b}. \end{aligned} \tag{2}$$

III. Properties and Solutions

In this section, we study the properties and solutions of model (2) when the optimal budget is commonly-used random variables. We first study the properties when the optimal budget can be characterized by uniform random variable, and present an analytic solution. Then we study the case for normally distributed optimal budget, and discuss a numerical solution.

A. Uniformly Distributed Optimal Budget

When the random optimal budget for each temporal slot is uniformly distributed, we have the following theorem.

Theorem 1: If the random optimal budget for the jth temporal slot satisfies $d_j \sim U(\underline{b}, \overline{b})$, $j = 1, 2, \cdots, n$, then model (2) can be represented as

$$\begin{aligned} \max \quad & \sum_{j=1}^{n} \left[-\frac{p_j c_j - p_j c_j^{'}}{2(\overline{b} - \underline{b})} \left(b_j^2 + \underline{b}^2\right) + \frac{p_j c_j \overline{b} - p_j c_j^{'} \underline{b}}{\overline{b} - \underline{b}} b_j \right] \\ \text{s.t.} \quad & \sum_{j=1}^{n} b_j \le B \\ & \underline{b} \le b_j \le \overline{b}, \end{aligned} \tag{3}$$

and it is a convex programming.

Proof: Since $d_j \sim U(\overline{b}, \overline{b})$, $j = 1, 2, \cdots, n$, the probability distribution of d_j is

$$f(d_j) = \begin{cases} \frac{1}{\overline{b} - \underline{b}}, & \text{if } \underline{b} < x < \overline{b} \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} E[C(b_j, d_j)] &= \int_{\underline{b}}^{b_j} (p_j c_j d_j + p_j c_j^{'}(b_j - d_j)) \frac{1}{\overline{b} - \underline{b}} \mathrm{d}d_j \\ &\quad + \int_{b_j}^{\overline{b}} p_j c_j b_j \frac{1}{\overline{b} - \underline{b}} \mathrm{d}d_j \\ &= \frac{p_j c_j - p_j c_j^{'}}{2(\overline{b} - \underline{b})} \left(b_j^2 - \underline{b}^2\right) + \frac{p_j c_j^{'} b_j (b_j - \underline{b})}{\overline{b} - \underline{b}} \\ &\quad + \frac{p_j c_j b_j (\overline{b} - b_j)}{\overline{b} - \underline{b}} \\ &= -\frac{p_j c_j - p_j c_j^{'}}{2(\overline{b} - \underline{b})} \left(b_j^2 + \underline{b}_j^2\right) + \frac{p_j c_j \overline{b} - p_j c_j^{'} \underline{b}}{\overline{b} - \underline{b}} b_j. \end{aligned}$$

Thus model (2) can be written as

$$\begin{aligned}\max\quad & \sum_{j=1}^{n}\left[-\frac{p_jc_j-p_jc_j^{'}}{2(\overline{b}-\underline{b})}\left(b_j^2+\underline{b}^2\right)+\frac{p_jc_j\overline{b}-p_jc_j^{'}\underline{b}}{\overline{b}-\underline{b}}b_j\right]\\ \text{s.t.}\quad & \sum_{j=1}^{n}b_j\le B\\ & \underline{b}\le b_j\le\overline{b}.\end{aligned}$$

In the following, we prove that model (3) is a convex programming.

Because constraints of model (3) are linear, we only need to prove the objective function of programming (3) is convex.

Let

$$g(\boldsymbol{b})=\sum_{j=1}^{n}\left(-\frac{p_jc_j-p_jc_j^{'}}{2(\overline{b}-\underline{b})}\left(b_j^2+\underline{b}^2\right)+\frac{p_jc_j\overline{b}-p_jc_j^{'}\underline{b}}{\overline{b}-\underline{b}}b_j\right).$$

Differentiate $g(\boldsymbol{b})$ with b_j,

$$\begin{aligned}\frac{\partial g}{\partial b_j}&=-\frac{p_jc_j-p_jc_j^{'}}{\overline{b}-\underline{b}}b_j+\frac{p_jc_j\overline{b}-p_jc_j^{'}\underline{b}}{\overline{b}-\underline{b}}\\&=\frac{p_j}{\overline{b}-\underline{b}}(c_j(\overline{b}-b_j)+c_j^{'}(b_j-\underline{b})).\end{aligned}$$

Since $\underline{b}<b_j<\overline{b}$, then $\overline{b}-b_j>0$, $b_j-\underline{b}>0$ and $\overline{b}-\underline{b}>0$. Thus $\partial g/\partial b_j>0$. This proves that $g(\boldsymbol{b})$ is a convex function.

Therefore, model (3) is a convex programming. The proof is completed. ∎

Based on Theorem 1 and properties of convex programming, if b_j^* is the local optimal solution of model (3), then it is also its global optimal solution. Thus, we have the following theorem.

Theorem 2: If the random optimal daily budget for the jth temporal slot satisfies $d_j\sim U(\underline{b},\overline{b})$, $j=1,2,\cdots,n$, then the optimal solutions of model (3) without consideration of the constraint $\underline{b}\le b_j\le\overline{b}$ is

$$b_j^*=\frac{p_jc_j\overline{b}-p_jc_j^{'}\underline{b}}{p_jc_j-p_jc_j^{'}}-\frac{\overline{b}-\underline{b}}{p_jc_j-p_jc_j^{'}}\lambda,j=1,2,\cdots,n,\quad(4)$$

where

$$\lambda=(\sum_{j=1}^{n}\frac{p_jc_j\overline{b}-p_jc_j^{'}\underline{b}}{p_jc_j-p_jc_j^{'}}-B)/(\sum_{j=1}^{n}\frac{\overline{b}-\underline{b}}{p_jc_j-p_jc_j^{'}}),$$

and the corresponding optimal value is

$$\sum_{j=1}^{n}\left[-\frac{p_jc_j-p_jc_j^{'}}{2(\overline{b}-\underline{b})}\left(b_j^{*2}+\underline{b}^2\right)+\frac{p_jc_j^{(1)}\overline{b}-p_jc_j^{'}\underline{b}}{\overline{b}-\underline{b}}b_j^*\right].\quad(5)$$

Theorem 3: If for all b_j^* defined by formula (4) satisfy the constraints $\underline{b}\le b_j\le\overline{b}$, $j=1,2,\cdots,n$, then $b_1^*,b_2^*,\cdots,b_n^*$ are also the optimal solutions of model (3), and the corresponding optimal value is given by (5). Otherwise, (5) is the upper bound of the optimal value of model (3).

Proof: By removing constant items, the optimal solution of model (3) also solves the following model

$$\begin{aligned}\max\quad & \sum_{j=1}^{n}\left[-\frac{p_jc_j-p_jc_j^{'}}{2(\overline{b}-\underline{b})}b_j^2+\frac{p_jc_j\overline{b}-p_jc_j^{'}\underline{b}}{\overline{b}-\underline{b}}b_j\right]\\ \text{s.t.}\quad & \sum_{j=1}^{n}b_j\le B.\end{aligned}\quad(6)$$

Using the Lagrange method, model (6) can be transformed into the following unconstrained programming

$$\max\quad\sum_{j=1}^{n}\left[-\frac{p_jc_j-p_jc_j^{'}}{2(\overline{b}-\underline{b})}b_j^2+\frac{p_jc_j\overline{b}-p_jc_j^{'}\underline{b}}{\overline{b}-\underline{b}}b_j\right]-\lambda\left(\sum_{j=1}^{n}b_j-B\right),\quad(7)$$

where λ is the Lagrange multiplier.

We differentiate the objective function of model (7) with b_j, $j=1,2,\cdots,n$. If b_j^* $(j=1,2,\cdots,n)$ is the optimal solution, then it satisfies

$$-\frac{p_jc_j-p_jc_j^{'}}{\overline{b}-\underline{b}}b_j^*+\frac{p_jc_j\overline{b}-p_jc_j^{'}\underline{b}}{\overline{b}-\underline{b}}-\lambda=0,$$

thus

$$\lambda=-\frac{p_jc_j-p_jc_j^{'}}{\overline{b}-\underline{b}}b_j^*+\frac{p_jc_j\overline{b}-p_jc_j^{'}\overline{b}}{\overline{b}-\underline{b}},$$

and

$$b_j^*=\frac{c_j\overline{b}-c_j^{'}\underline{b}}{c_j-c_j^{'}}-\frac{\overline{b}-\underline{b}}{p_jc_j-p_jc_j^{'}}\lambda,j=1,2,\cdots,n.\quad(8)$$

Because $\lambda(\sum_{j=1}^{n}b_j^*-B)=0$ and the total budget is limited, we have $\sum_{j=1}^{n}b_j^*-B=0$. Thus,

$$\sum_{j=1}^{n}\frac{c_j\overline{b}-c_j^{'}\underline{b}}{c_j-c_j^{'}}-\sum_{j=1}^{n}\frac{\overline{b}-\underline{b}}{p_jc_j-p_jc_j^{'}}\lambda=B.$$

From the above equation, we can obtain

$$\lambda=(\textstyle\sum_{j=1}^{n}\frac{c_j\overline{b}-c_j^{'}\underline{b}}{c_j-c_j^{'}}-B)/(\sum_{j=1}^{n}\frac{\overline{b}-\underline{b}}{p_jc_j-p_jc_j^{'}}).\quad(9)$$

We substitute the optimal solutions b_j^* into the objective function of problem (3), and the corresponding expected value is given as

$$\sum_{j=1}^{n}\left[-\frac{p_jc_j-p_jc_j^{'}}{2(\overline{b}-\underline{b})}\left(b_j^{*2}+\underline{b}^2\right)+\frac{p_jc_j\overline{b}-p_jc_j^{'}\underline{b}}{\overline{b}-\underline{b}}b_j^*\right],\quad(10)$$

where b_j^* is defined by formula (8). The proof is completed. ∎

Theorem 3 presents a method to find the optimal solutions and optimal value of model (3), which is convenient in practice. If there exists at least one b_j^* defined by formula (4) that does not satisfy the constraints $\underline{b}\le b_j\le\overline{b}$, then we can utilize traditional solution methods for nonlinear convex programming, such as the Karush-Kuhn-Tucker conditions, interior point method or external point method, to find out the approximate optimal solutions of model (3).

B. Normally Distributed Optimal Budget

In this section, we discuss the properties and solutions of model (2) when the random optimal budget of each temporal slot is normally distributed.

Theorem 4: If the random optimal budget for the jth temporal slot satisfies $d_j\sim\mathcal{N}(\mu,\sigma)$ on $[\mu-2\sigma,\mu+2\sigma]$,

$\mu > 0$, $\sigma > 0$, $j = 1, 2, \cdots, n$, then model (2) becomes

$$
\begin{aligned}
\max \quad & \sum_{j=1}^{n} \Big[-p_j(c_j - c_j^{'})(\tfrac{\sigma}{\sqrt{2\pi}}(\exp(-\tfrac{(b_j-\mu)^2}{2\sigma^2}) \\
& \quad - \exp(-2)) - (\mu - b_j)\Phi(\tfrac{b_j-\mu}{\sigma})) \\
& \quad + p_j((c_j - c_j^{'})\mu + (c_j + c_j^{'})b_j)\Phi(2) \\
& \quad - p_j(c_j - c_j^{'})\mu - p_j c_j^{'} b_j \Big] \\
\text{s.t.} \quad & \sum_{j=1}^{n} b_j \le B \\
& \mu - 2\sigma \le b_j \le \mu + 2\sigma,
\end{aligned}
\tag{11}
$$

and it is a convex programming.

Proof: Because $d_j \sim U(\mu, \sigma)$ on $[\mu - 2\sigma, \mu + 2\sigma]$, $j = 1, 2, \cdots, n$, the probability distribution of d_j is

$$
f(d_j) = \begin{cases} \frac{1}{\sqrt{2\pi}\sigma}\exp(-\frac{(d_j-\mu)^2}{2\sigma^2}), & \text{if } \mu - 2\sigma \le x \le \mu + 2\sigma \\ 0, & \text{otherwise.} \end{cases}
$$

Then

$$
\begin{aligned}
& E[C(b_j, d_j)] \\
&= \tfrac{1}{\sqrt{2\pi}\sigma}\int_{\mu-2\sigma}^{b_j}(p_j c_j d_j + p_j c_j^{'}(b_j - d_j))\exp(-\tfrac{(d_j-\mu)^2}{2\sigma^2})\mathrm{d}d_j \\
&\quad + \tfrac{1}{\sqrt{2\pi}\sigma}\int_{b_j}^{\mu+2\sigma} p_j c_j b_j \exp(-\tfrac{(d_j-\mu)^2}{2\sigma^2})\mathrm{d}d_j \\
&= -\tfrac{\sigma}{\sqrt{2\pi}}p_j(c_j - c_j^{'})(\exp(-\tfrac{(b_j-\mu)^2}{2\sigma^2}) - \exp(-\tfrac{(\mu-2\sigma-\mu)^2}{2\sigma^2})) \\
&\quad + (p_j(c_j - c_j^{'})\mu + p_j c_j^{'} b_j)(\Phi(\tfrac{b_j-\mu}{\sigma}) - \Phi(\tfrac{\mu-2\sigma-\mu}{\sigma})) \\
&\quad + p_j c_j b_j(\Phi(\tfrac{\mu+2\sigma-\mu}{\sigma}) - \Phi(\tfrac{b_j-\mu}{\sigma})) \\
&= -p_j(c_j - c_j^{'})(\tfrac{\sigma}{\sqrt{2\pi}}(\exp(-\tfrac{(b_j-\mu)^2}{2\sigma^2}) - \exp(-2)) \\
&\quad - (\mu - b_j)\Phi(\tfrac{b_j-\mu}{\sigma})) + p_j((c_j - c_j^{'})\mu + (c_j + c_j^{'})b_j)\Phi(2) \\
&\quad - p_j(c_j - c_j^{'})\mu - p_j c_j^{'} b_j,
\end{aligned}
$$

where $\Phi(\cdot)$ is the standard normal probability distribution function.

Thus model (2) becomes

$$
\begin{aligned}
\max \quad & \sum_{j=1}^{n} \Big[-p_j(c_j - c_j^{'})(\tfrac{\sigma}{\sqrt{2\pi}}(\exp(-\tfrac{(b_j-\mu)^2}{2\sigma^2}) \\
& \quad - \exp(-2)) - (\mu - b_j)\Phi(\tfrac{b_j-\mu}{\sigma})) \\
& \quad + p_j((c_j - c_j^{'})\mu + (c_j + c_j^{'})b_j)\Phi(2) \\
& \quad - p_j(c_j - c_j^{'})\mu - p_j c_j^{'} b_j \Big] \\
mboxs.t. \quad & \sum_{j=1}^{n} b_j \le B \\
& \mu - 2\sigma \le b_j \le \mu + 2\sigma.
\end{aligned}
$$

In the following, we prove that model (11) is a convex programming.

Because the constraints of model (11) are linear, we only need to prove that the objective function of programming (11) is convex.

Let

$$
\begin{aligned}
g(\boldsymbol{b}) = & \sum_{j=1}^{n} \Big[-p_j(c_j - c_j^{'})(\tfrac{\sigma}{\sqrt{2\pi}}(\exp(-\tfrac{(b_j-\mu)^2}{2\sigma^2}) - \exp(-2)) \\
& - (\mu - b_j)\Phi(\tfrac{b_j-\mu}{\sigma})) \\
& + p_j((c_j - c_j^{'})\mu + (c_j + c_j^{'})b_j)\Phi(2) \\
& - p_j(c_j - c_j^{'})\mu - p_j c_j^{'} b_j \Big].
\end{aligned}
$$

Differentiate $g(\boldsymbol{b})$ with b_j,

$$
\begin{aligned}
\tfrac{\partial g}{\partial b_j} &= -p_j(c_j - c_j^{'})(-\tfrac{1}{\sqrt{2\pi}\sigma}(b_j - \mu)\exp(-\tfrac{(b_j-\mu)^2}{2\sigma^2}) \\
&\quad + \Phi(\tfrac{b_j-\mu}{\sigma}) - \tfrac{1}{\sqrt{2\pi}\sigma}(\mu - b_j)\exp(-\tfrac{(b_j-\mu)^2}{2\sigma^2})) \\
&\quad + p_j(c_j + c_j^{'})\Phi(2) - p_j c_j^{'} \\
&= -p_j(c_j - c_j^{'})\Phi(\tfrac{b_j-\mu}{\sigma}) + p_j(c_j + c_j^{'})\Phi(2) - p_j c_j^{'},
\end{aligned}
$$

and

$$
\tfrac{\partial g^2}{\partial b_j^2} = -p_j(c_j - c_j^{'})\tfrac{1}{\sqrt{2\pi}\sigma}\exp(-\tfrac{(b_j-\mu)^2}{2\sigma^2}).
$$

Because $c_j > c_j^{'}$, we have $\partial g^2/\partial b_j^2 < 0$. That is, $g(x)$ is a convex function.

Therefore, model (11) is a convex programming. The proof is completed. ■

According to Theorem 4, if b_j^* is the local optimal solution of model (11), then it is also its global optimal solution.

Since the objective function of model (11) contains standard normal distribution function Φ, it is difficult to get an analytic solution. In the following, we provide an numerical solution that can compute Φ .

From the definition of standard normal distribution function, we have

$$
\Phi(h(x)) = \int_0^{h(x)} \frac{1}{\sqrt{2\pi}}\exp(-\frac{u^2}{2})\mathrm{d}u,
$$

where $h(x)$ is a function of x. The numerical solution algorithm for evaluating $\Phi(h(x))$ is given as follows.

$N \leftarrow$ iteration times
$num \leftarrow 0$
$y \leftarrow 0$
while $num \le N$
  $q \leftarrow r \sim \mathcal{N}(0, 1)$
  if $q \le h(x)$ **then**
    $y \leftarrow y + 1$
  $num \leftarrow num + 1$
return y/N

IV. Experiments

In this section, we make some experiments with Web logs of a real ad campaign to illustrate the effectiveness of the proposed strategy, named uniform distribution strategy and strategy driven by normal distributions, which allocates the budget by assuming that the optimal budget is a uniform random variable and normal random variable, respectively. For simplicity, we represent the two strategies by StoStrategy_uniform and StoStrategy_normal, respectively.

For comparison purposes, we implement one baseline strategy, called BASE-Average, which allocates the budget to a series of temporal slots averagely. That is, it ignores the differences among these temporal slots. The reason for us to choose the BASE-Average strategy is that it is easy to implement, and thus usually adopted by advertisers.

A. Experimental Data

In this experiment, we used the data during a month (e.g., 30 days), and the data such as clicks per unit cost and effective CTR are shown in Figure 1. The total budget is $B = 3000$. From the logs, we can know the optimal budget of every day is on interval $[80, 150]$. Thus, StoStrategy_uniform and StoStrategy_normal regarding the random optimal budget for the promotion period satisfy $U(80, 150)$ and $\mathcal{N}(115, 17.5^2)$, respectively.

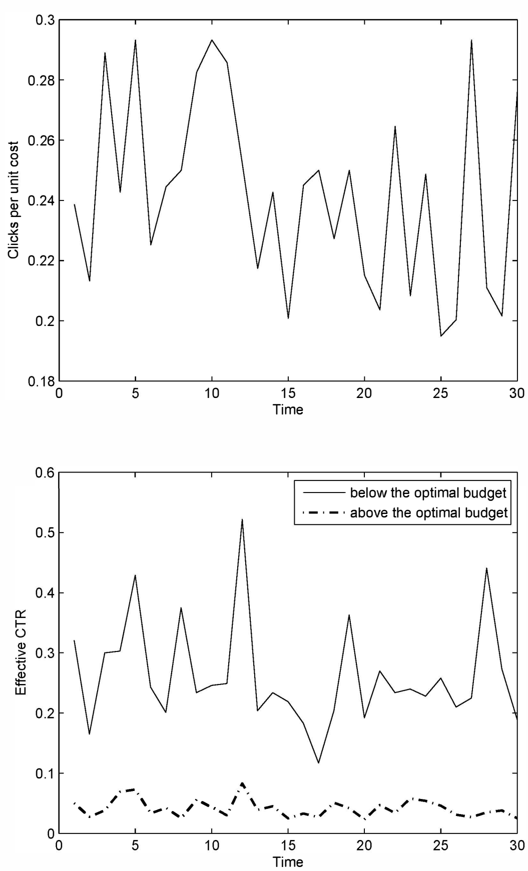

Fig. 1. Pattern of clicks per unit cost and effective CTR

B. Analysis of Experimental Results

The optimal solutions for the three strategies are shown in Figure 2, and the corresponding total cumulative effective clicks are shown in Figure 3, where "cumulative effective clicks" on the jth day represents the total effective clicks from the 1st day to the jth day, $j = 1, 2, \cdots, 30$.

From Figure 2 and Figure 3, we can obtain the following results:

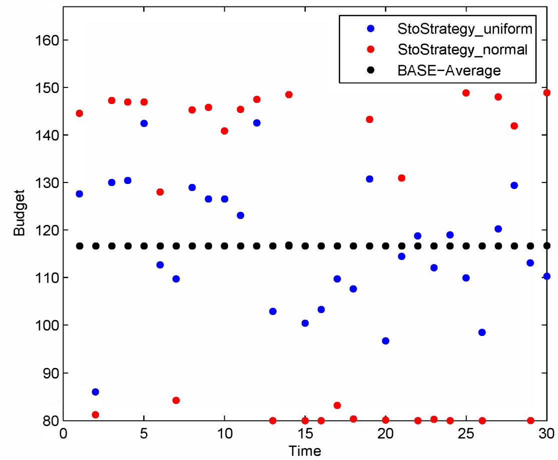

Fig. 2. Comparisons of the daily budget for the three strategies

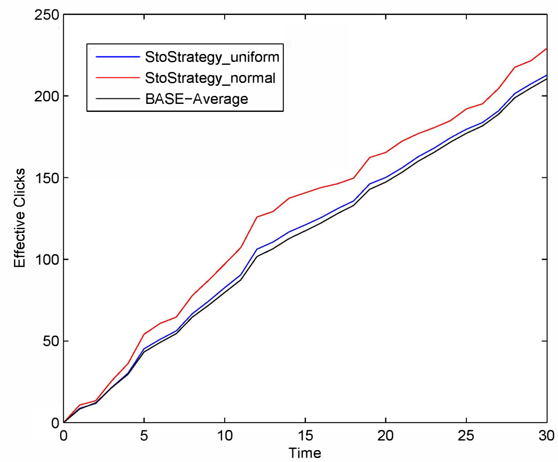

Fig. 3. Comparisons of the total cumulative effective clicks for the three strategies

(1) StoStrategy_normal and StoStrategy_uniform can obtain 229.17 and 212.80 effective clicks, respectively, and BASE-Average can obtain 210.49 effective clicks.

(2) StoStrategy_normal strategy outperforms StoStrategy_uniform strategy (about 7.69%), in terms of the total effective clicks. The reason is that the StoStrategy_normal strategy can allocate more budget in the more profitable days that have high effective clicks per unit cost (both below and above the optimal budget), and allocate less budget in the days with low effective clicks per unit cost (both below and above the optimal budget), which made the budget more profitable.

(3) Both our StoStrategy_normal strategy and StoStrategy_uniform strategy outperform BASE-Average about 8.86% and 1.10%, respectively, in terms of the total effective clicks. The reason is that both of our strategies take the differences of parameters of these days into consideration, while BASE-Average ignores the differences among these days in budget decisions.

V. Discussion and Managerial Insights

This paper reports our preliminary research on stochastic budget distribution over a series of temporal slots, by considering the randomness of the optimal budget as random variable. Two distributions (e.g., uniform and normal) are discussed in this work. Several limitations remain in our research. On the one hand, other forms of random variables might be more appropriate to describe the optimal budget. On the other hand, empirical research is necessary to find out some characteristics of random variables in budget distribution over a series of temporal slots in search advertisements.

This paper provides critical managerial insights for dealing with the budget distribution problem over a series of temporal slots in search advertisements. On the one hand, when the advertiser get some information about the upper and lower bound of the optimal budget, she can try to distribute the budget following a certain probability distribution, rather than allocating the budget averagely. On the other hand, the optimal budget is more likely to be normally distributed than uniformly distributed, thus, if the advertiser takes the optimal budget as normal distributed random variable, she can get more revenues.

VI. Conclusions and Future Work

In this paper, we formulate the budget distribution over a series of temporal slots in search advertisements as a stochastic programming problem. By considering the optimal budget for each temporal slot as a random variable to characterize uncertainty in search advertisements, we present a stochastic budget distribution model over a series of temporal slots. We also study some properties and solutions of the proposed model, considering the optimal budget as uniform random variables and normal random variables, respectively. With Web logs of real ad campaigns in search markets, we conduct some experiments to validate our model, and experimental results show that both of our strategies outperform the baseline strategy commonly used in practice and the optimal budget is more likely to be normally distributed than uniformly distributed.

In the future work, we are planning to (a) study more complex cases with the randomness from two or more factors; (b) conduct empirical research to observe the real-world distribution for the budget random variables; (c) study the optimization of solution algorithms in order to improve space and time efficiencies.

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