A NEW STOCHASTIC BUDGET DISTRIBUTION MODEL WITH RANDOM DEMANDS

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Abstract

In search auctions, when the total budget for an advertising campaign during a certain promotion period is determined, advertisers have to distribute their budgets over a series of sequential temporal slots (e.g., daily budgets). However, due to the uncertainties existed in search markets, advertisers can only obtain the value range of budget demand for each temporal slot based on promotion logs. In this paper, we present a stochastic model for budget distribution over a series of sequential temporal slots during a promotion period, considering the budget demand for each temporal slot as a random variable. We study some properties and present feasible solution algorithms for our budget model, in the case that the budget demand is characterized either by uniform random variable or normal random variable. We also conduct some experiments to evaluate our model with the empirical data. Experimental results show that the budget demand is more likely to be normal distributed than uniform distributed, and our strategy can outperform the baseline strategy commonly used in practice.

Keywords: budget distribution, budget demand, stochastic strategy, budget constraints, search advertising

1 Introduction

In search auctions, many brand managers from small companies face budget constraints due to their financial problems (Chakrabarty et al., 2007). Moreover, there are plenty of uncertainty in the mapping from the budget into the advertising performance in search auctions (Yang et al., 2013). Classical budget allocation methods usually seek maximum profits or minimum cost with known parameters. However, it is difficult for advertisers to predict necessary factors such as cost per click (CPC) and click-through rate (CTR) in search auctions.

Fortunately, an advertiser has access to some information about the budget demand for each slot (e.g., the lower bound $\underline{b}$ and the upper bound $\overline{b}$ of the daily budget) from historical reports of past promotional activities. Therefore, we can take the budget demand for each promotional slot as a random variable on $[\underline{b}, \overline{b}]$ to capture the randomness of budget decisions at the campaign level in search auctions.

Most existing works take the budget as a simple constraint for other advertising strategies (Archak et al., 2010; Feldman et al., 2007; Ozluk and Cholette, 2007). With consideration of the entire lifecycle of search advertising campaigns, budget-related decisions exist at three levels (Yang et al., 2012): allocation across search markets, temporal distribution over a series of temporal slots (e.g., days) and adjustment of the remaining budget (e.g., the daily budget). Consequently, they proposed a three-level budget optimization framework (BOF) where the budget decision is modeled as a structural problem.

In this paper, we focus on the budget distribution problem at the campaign level of the BOF framework. Due to the uncertainty existed in search markets, the budget demand of each temporal slot cannot be known in advance, and advertisers can only obtain its value range based on promotion logs. Moreover, the allocated budget of each temporal slot is in an interval constraint, due to the limitation of search auction systems (e.g., the lower bound) and the advertiser's financial constraint (e.g., the upper bound). Considering the budget demand for each temporal slot as a random variable, we utilize stochastic programming to deal with the budget distribution problem. First, we take the budget demand of each temporal slot as a random variable, because it can to some degree reflect the environmental randomness of budget-related decisions at the campaign level. The probability distribution of budget demand can be extracted from promotion logs of historical campaigns. Second, we present a stochastic model for budget distribution over a series of temporal slots (e.g., days), when the total budget in a search market is given. Third, we discuss some properties and possible solutions of our model, by taking the budget demand for each temporal slot as a uniform random variable or a normal random variable, respectively. Furthermore, we conduct experiments to evaluate our model, and the experimental results show that the strategy driven by normal distributions outperform the other two in terms of total effective clicks, followed by the uniform distribution strategy, and then the baseline strategy commonly used in practice. This can be explained by the fact that the budget demand for each temporal slot is more likely to be normal distributed than uniform distributed.

The rest of this paper is organized as follows. In Section 2, we briefly review some relevant literatures. In Section 3, we state the problem under consideration, then propose the stochastic budget distribution model over a series of temporal slots. In Section 4, we discuss some properties and solution algorithms for our model. In Section 5, we present some experimental results to evaluate our budget model. Section 6 concludes this paper.

2 Literature Review

Budget allocation has long been regarded as a big challenge for advertisers in traditional advertising, and it has been intensively studied by researchers (Holthausen and Assmus, 1982; Ichikawa et al., 2009; Wang and Xu, 2008; Zufryden, 1975). However, most of the existing effort in search auctions simply take the budget as constraints to develop various kinds of search advertising strategies (Borgs et al., 2007; Maille and Tuffin, 2011; Rusmevichientong and Williamson, 2006; Yao et al., 2009), e.g., to determine keyword bid price (Chaitanya and Narahari, 2010; Cholette et al., 2012). Feldman et al. (2007) explored how to spread a given budget over keywords to maximize the expected profits. Three

stochastic bidding strategies (i.e., proportional, independent, and scenario) were presented where some special cases identified are solvable in polynomial time or with improved approximate ratios (DasFupta and Muthukrishnan, 2010; Muthukrishnan et al., 2010). Empirical study of bidding behavior by Zhang & Feng (2011) showed that, in sponsored search auctions advertisers may engage in a cyclical bid adjustment process, and the bidding prices in both first price and second price search auctions may follow a cyclical pattern with price-escalating phases interrupted by price-collapsing phases. Yao & Mela (2011) developed a dynamic structural model of bidding behaviors of advertisers, to maximize the net present value of its discounted profits. Their results showed that advertiser's valuations are positively correlated with product's attributes. Due to the stochastic variability of the opponents and their bids, an advertiser's bidding conditions often vary, thus Pin & Key (2011) proposed a probabilistic model to deal with such problems, through which we can predict an advertiser's valuation given her bid and her opponents' bids.

The complex dynamic of search auctions increases the difficulty of the advertisers' decision making processes (Menache et al., 2009; Robu et al., 2009). Zhou et al. (2008) studied the non-cooperative dynamic keyword auction games with limited budgets and formulated it as an online multiple choice knapsack problem. Moreover, a greedy bidding strategy called "balanced bidding" was proposed to maximize the advertiser's revenue and minimize her competitors' advertising effectiveness. Vorobeychik & Reeves (2007) studied the equilibrium stability of dynamic bidding strategies, and their results showed that a high-payoff strategy profile is not sustainable in equilibrium, and a low-payoff profile is reasonably stable. However, when complete information about valuations and click-through-rates are available, the equilibrium with high payoffs for all players is sustainable over a range of settings. Considering the dynamics of search auctions, Gummadi et al. (2011) studied the bidding strategies for advertisers with budget constraints, and formulated it as a discounted Markov decision process. When the advertiser is involved in a large number of auctions, they proposed its explicit solutions. With consideration of the stochastic multi-slot setting, Abhishek & Hosanagar (2012) proposed two bidding policies named "myopic" policy and "forward-looking" policy, both of which are effective in multi-period bidding problems in search auctions.

As is shown by $\ddot{O}zl\ddot{u}k$ & Cholette (2007), the advertiser's budget decisions can be greatly influenced by price elasticities of click-through-rate and response functions, thus the advertiser should better invest on more keywords under a certain threshold in order to improve her profits. Babaioff et al. (2007) and Chakrabarty et al. (2007) formulated the budget optimization problem as an online (multiple-choice) knapsack problem, and designed both deterministic and randomized algorithms to achieve a provably optimal competitive ratio for advertisers. Archak et al. (2010) formulated the budget allocation problem as a Markov decision process with an optimal control model. Their main result showed that, under a reasonable assumption that online advertising has positive carryover effects on the propensity and the form of user interactions with the same advertiser in the future, there exists a simple greedy algorithm for budget allocations with the worst-case running time cubic in the number of model states (e.g., keywords). Fruchter & Dou (2005) also utilized optimal control to model the dynamical budget allocation problem between generic portal sand specialized portals, and their experimental results indicated that more budget should be spread to specialized portals in the long run. In the process of search auctions, advertising decay is an important issue faced by advertisers. In response to advertising decay, Raman (2006) studied the conditions of the optimality of three temporal scheduling patterns for budget allocation (e.g., constant spending over time, decreasing spending over time and increasing spending over time), and the results showed that these three spending patterns can emerge at optimality for the same response function dynamics under different salvage value assumptions.

3 Budget Distribution Over A Series of Promotional Slots

In this section, we first state our problems, and then present the basic budget distribution model over a series of promotional slots. After that, we discuss the the objective function with the budget demand as some special random variables. The notations used in this paper are listed in Table 1.

Notation	Definition
n	the number of temporal slots during a promotion period
B	the total budget of search advertisements for a campaign
p_j	the clicks per unit cost of the jth temporal slot, $j = 1, \dots, n$
b_j	the budget allocated in the jth temporal slot, $j = 1, \dots, n$
d_j	the random budget demand for the jth temporal slot, $j = 1,2, \dots, n$
$\underline{b}$	the lower bound of the budget demand
$\overline{b}$	the upper bound of the budget demand
c_j	the effective CTR of the jth temporal slot (below the budget demand), $j = 1, \dots, n$
c_j'	the effective CTR of the jth temporal slot (above the budget demand), $j = 1, \dots, n$
α	the lower bound of the daily budget, which is limited by search engines
β	the upper bound of the daily budget, which is given by the advertiser

Table 1. List of notations

3.1 Problem Statement

Before the statement of our problem, we first introduce the concept of "budget demand" in this paper.

Budget Demand: Since the potential search demand in each temporal slot is finite, the effective clicks cannot grow with the same speed. Generally, there exists a critical value for the allocated budget. When the allocated budget is less than the critical value, the clicks will have a higher effective CTR, while, when the allocated budget is more than the critical value, the clicks will have a lower effective CTR for the part of budget more than the critical value. In this paper, the critical value is regarded as the *budget demand*. For simplicity, we assume that the effective CTR for the part of budget less than the budget demand is c, and the part of budget more than the budget demand will also be used up, but with a lower effective CTR $c_j' < c$. This assumption is reasonable because of the fact that the allocated budget is far more than the budget demand will never happen due to the advertiser's financial conditions.

The decision problem faced by advertisers in search auctions can be described as follows:
- Given the total budget B in a search market during a certain promotional period, an advertiser has to distribute the budget to a series of n temporal slots in order to maximize her revenue.
- For each temporal slot, the budget demand cannot be precisely known in advance, however, the advertiser can obtain some useful information (e.g., the lower bound and the upper bound) of the budget demand from historical promotion logs of campaigns.
- The allocated budget in each temporal slot is restricted by both search auction system and the advertiser's financial constraints, e.g., the search auction system may set a minimum allowable budget, and the advertiser has a maximum affordable budget due to her financial constraints.

Considering the budget demand d_j of temporal slot j, we make the following assumptions: if the allocated budget b_j is lower than d_j, then the effective CTR is c_j, and if the allocated budget b_j is higher than d_j, then the effective CTR for the part of exceeded budget $b_j - d_j$ is c_j'.

3.2 The Basic Model

Suppose d_j, $j = 1,2,\cdots,n$, are mutually independent random variables on $[\underline{b}, \overline{b}]$, where $\underline{b} < \overline{b}$. Let $C(b_j, d_j)$ be the revenue obtained by allocating b_j amount of budget to the jth promotional slot where the budget demand is d_j. Then the total revenue of n slots during a given period is $\sum_{j=1}^{n} C(b_j, d_j)$. As d_j is a random variable, then both $C(b_j, d_j)$ and $\sum_{j=1}^{n} C(b_j, d_j)$ are also random variables. The objective of an advertiser is to maximize the total expected revenue during a promotional period, i.e., $E[\sum_{j=1}^{n} C(b_j, d_j)]$. As d_j, $j = 1,2,\dots,n$, are a set of independent random variables, then $E\left[\sum_{j=1}^{n} C(b_j, d_j)\right] = \sum_{j=1}^{n} E[C(b_j, d_j)]$.

Based on the above discussions, we formulate the budget distribution problem as follows,

$$\max \quad \sum_{j=1}^{n} E\left[C(b_j, d_j)\right]$$

$$\text{s.t.} \quad \sum_{j=1}^{n} b_j \le B \tag{1}$$

$$\alpha \le b_j \le \beta,$$

where α is the minimum allowable budget for a temporal slot given by search auction systems, and β is the maximum allowable budget for a temporal slot given by the advertiser due to her financial conditions.

3.3 The Objective Function

For each promotional slot j, let c_j be the effective CTR of the jth temporal slot (below the budget demand), c_j' be the effective CTR of the jth temporal slot (above the budget demand), and p_j be the clicks per unit cost of the jth temporal slot, $j = 1, 2, \dots, n$. If $b_j < d_j$, then the effective clicks obtained by b_j is $p_j b_j c_j$, otherwise if $b_j \ge d_j$, then the effective clicks obtained by b_j can be divided into two parts, the first part d_j with effective CTR c_j, and another part $b_j - d_j$ with effective CTR c_j'. Thus the effective clicks will be $p_j c_j d_j + p_j c_j'(b_j - d_j)$.

Suppose d_j is a continuous random variable on $[\underline{b}, \overline{b}]$ with probability distribution $f(d_j)$, then based on the concept of expected value of continuous random variable, for each j, we have the following cases:

(1) If $\alpha \le b_j \le \underline{b}$, then

$$E[C(b_j, d_j)] = p_j c_j b_j. \tag{2}$$

(2) If $\underline{b} < b_j < \overline{b}$, then

$$E[C(b_j, d_j)] = \int_{\underline{b}}^{b_j} (p_j c_j d_j + p_j c_j'(b_j - d_j))f(d_j)\mathrm{d}d_j + \int_{b_j}^{\overline{b}} p_j\, c_j b_j f(d_j)\mathrm{d}d_j. \tag{3}$$

(3) If $\overline{b} \le b_j \le \beta$, then

$$E[C(b_j, d_j)] = \int_{\underline{b}}^{\overline{b}} (p_j c_j d_j + p_j c_j'(b_j - d_j))f(d_j)\mathrm{d}d_j. \tag{4}$$

Define $J_1 = \{j | \alpha \le b_j \le \underline{b}\}$, $J_2 = \{j | \underline{b} < b_j < \overline{b}\}$, $J_3 = \{j | \overline{b} \le b_j \le \beta\}$. Then with the probability distribution $f(d_j)$ on $[\underline{b}, \overline{b}]$, model (1) becomes

$$\max \quad \sum_{j \in J_1} p_j\, c_j b_j + \sum_{j \in J_2} \left(\int_{\underline{b}}^{b_j} (p_j c_j d_j + p_j c_j'(b_j - d_j))f(d_j)\mathrm{d}d_j \right.$$

$$\left. + \int_{b_j}^{\overline{b}} p_j\, c_j b_j f(d_j)\mathrm{d}d_j \right) + \sum_{j \in J_3} \int_{\underline{b}}^{\overline{b}} (p_j c_j d_j + p_j c_j'(b_j - d_j))f(d_j)\mathrm{d}d_j$$

$$\text{s.t.} \quad \sum_{j=1}^{n} b_j \le B \tag{5}$$

$$\alpha \le b_j \le \underline{b}, j \in J_1$$

$$\underline{b} < b_j < \overline{b}, j \in J_2$$

$$\overline{b} \le b_j \le \beta, j \in J_3.$$

4 Property Analysis and Solutions

In this section, we study some properties and present feasible solutions to model (5) with the budget demand for each promotional slot resembling either a normal or a uniform distribution.

4.1 Property Analysis

When the budget demand for each promotional slot follows uniform distribution or normal distribution, we have the following theorems.

Theorem 1. If the random budget demand for the jth temporal slot satisfies $d_j \sim U(\underline{b}, \overline{b})$, $j = 1,2,\cdots,n$, then model (5) can be represented as

$$\max \quad \sum_{j \in J_1} p_j c_j b_j + \sum_{j \in J_2} \frac{p_j}{2(\overline{b} - \underline{b})}\left[-c_j(b_j^2 + \underline{b}^2 - 2\overline{b}b_j) + c_j'(b_j - \underline{b})^2\right]$$

$$+ \sum_{j \in J_3}\left[\frac{1}{2}p_j(c_j - c_j')(\overline{b} + \underline{b}) + p_j c_j' b_j\right]$$

$$\text{s.t.} \quad \sum_{j=1}^{n} b_j \leq B \tag{6}$$

$$\alpha \leq b_j \leq \underline{b}, j \in J_1$$

$$\underline{b} < b_j < \overline{b}, j \in J_2$$

$$\overline{b} \leq b_j \leq \beta, j \in J_3.$$

and it is a convex programming.

Theorem 2. If the random budget demand for the jth temporal slot satisfies $d_j \sim \mathcal{N}(\mu, \sigma)$ on $[\mu - 3\sigma, \mu + 3\sigma]$, $\mu > 0$, $\sigma > 0$, $j = 1,2,\cdots,n$, then model (5) becomes

$$\max \quad \sum_{j \in J_1} p_j c_j b_j + \sum_{j \in J_2}\left[-p_j(c_j - c_j')(\frac{\sigma}{\sqrt{2\pi}}(\exp(-\frac{(b_j - \mu)^2}{2\sigma^2}) - \exp(-3))\right.$$

$$\left.-(\mu - b_j)\Phi\left(\frac{b_j - \mu}{\sigma}\right)) + (p_j(c_j - c_j')\mu + p_j(c_j + c_j')b_j)\Phi(3)\right.$$

$$\left.-p_j(c_j - c_j')\mu - p_j c_j' b_j\right] + \sum_{j \in J_3} p_j((c_j - c_j')\mu + c_j' b_j)(2\Phi(3) - 1)$$

$$\text{s.t.} \quad \sum_{j=1}^{n} b_j \leq B \tag{7}$$

$$\alpha \leq b_j \leq \mu - 3\sigma, j \in J_1$$

$$\mu - 3\sigma < b_j < \mu + 3\sigma, j \in J_2$$

$$\mu + 3\sigma \leq b_j \leq \beta, j \in J_3,$$

and it is a convex programming.

The proofs of Theorem 1 and 2 are omitted due to page limits.

Theorem 1 and 2 present the models when the budget demand is uniform random variable and normal random variable, respectively. In both cases, the model is a convex programming. Thus, according to the properties of convex programming, its local optimal solution is also the global optimal solution. In the following section, we design the solution algorithms.

4.2　　Solution Algorithms

According to Theorem 1 and 2, if b_j^* is a local optimal solution, then it is also the global optimal solution. Since the objective function of model (6) and model (7) is a complex function of b_j, they can not be solved directly, and thus we resort to stochastic simulation.

Because the objective function of model (7) contains a standard normal distribution function Φ, we provide an algorithm to compute it.

From the definition of standard normal distribution function, we have

$$\Phi(h(x)) = \int_0^{h(x)} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right) du,$$

where $h(x)$ is a function of x. In the following we provide a numerical solution algorithm to compute Φ, as given in Table 1.

Input: the total iteration times N, $num = 0$, $y = 0$, $h(x)$,
　　　the distribution $\mathcal{N}(0,1)$
Output: f
1. Generate a number for q according to $\mathcal{N}(0,1)$
2. Compare q and $h(x)$, and if $q \leq h(x)$, then $y = y + 1$
3. $num = num + 1$. If $num \leq N$, repeat step 1
4. $f = y/N$
5. Return f

Table 2. Algorithm for evaluating $\Phi(h(x))$

5　Experimental Validation

In this section, we conduct experiments to evaluate the effectiveness of budget strategies derived from our model: the StoStrategy_uniform and the StoStrategy_normal, with the data during the period from Sep. 1st, 2009 to Sep. 30th, 2009. For comparison purpose, we implement a baseline strategy, called BASE-Average, representing a strategy to allocate the budget to a series of temporal slots averagely. That is, it neglects the differences among these temporal slots.
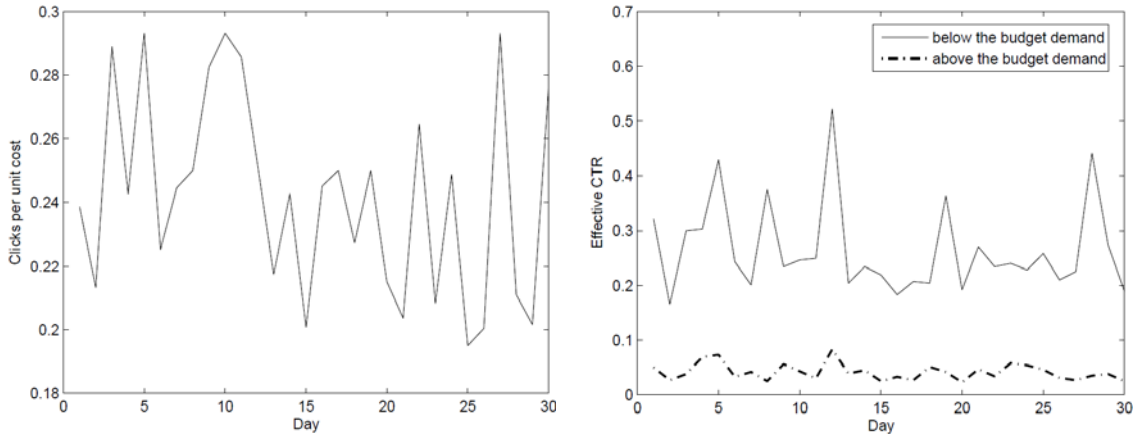

Figure 1. Clicks per unit cost and effective CTR

The lower bound of the daily budget limited by search engines is $\alpha = 50$, and the upper bound of the daily budget given by the advertiser is $\beta = 150$. The total budget during this period (e.g., 30 days) is $B = 3000$, and the value range of the budget demand is [80,120]. Figure 1 depicts clicks per unit cost and the effective CTR. Then for the StoStrategy_uniform strategy and the StoStrategy_normal strategy, the random budget demand for each day during this period satisfies $U(80,120)$ and $\mathcal{N}(100,20/3)$, respectively.

Optimal solutions of the daily budget distribution and the associated expected revenue (e.g., cumulative effective clicks) by these three strategies are illustrated in Figure 1. By cumulative effective clicks on the jth day we refer to the number of total effective clicks accumulated from the 1st day to the jth day, $j = 1,2,\cdots,30$.
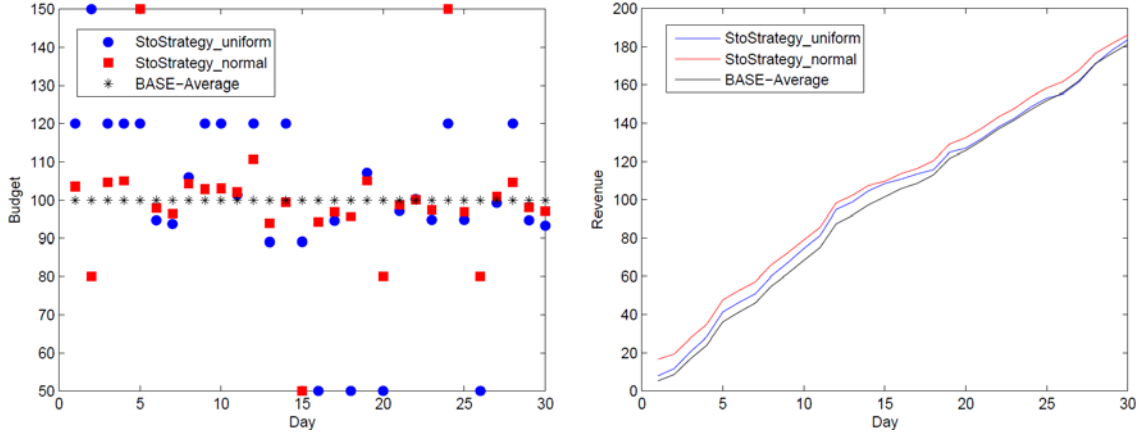

Figure 2. Comparisons of the daily budget and total effective clicks for the three strategies

From Figure 2, we can obtain the following results:

(1) The StoStrategy_normal strategy and the StoStrategy_uniform strategy obtain 187.43 and 183.51 effective clicks, respectively. The BASE-Average obtains 180.17 effective clicks. Both the StoStrategy_normal strategy and the StoStrategy_uniform strategy outperform the BASE-Average strategy about 4.03% and 1.85% in terms of cumulative effective clicks, respectively.

(2) The StoStrategy_normal strategy outperforms the StoStrategy_uniform strategy (about 2.14%), in terms of cumulative effective clicks. It implies that the budget demand is more likely to be normal distributed than uniform distributed.

(3) Most of the daily budget for StoStrategy_normal strategy and the StoStrategy_uniform strategy fall in $[80,120]$.

6 Conclusions

In this paper, we focus on the budget distribution problem for a campaign over a series of sequential temporal slots during a certain promotion period, and present a stochastic budget distribution model by assuming the budget demand for each temporal slot as a random variable within a value range. The lower bound and upper bound of the allocated budget are also considered in our model. We also study the equivalent models when the randomness is characterized by uniform random variable and normal random variable, respectively, and discuss the properties and solution algorithms. Furthermore, we conduct experiments to validate our model with the real-world data. Experimental results show that the budget demand is more likely to be normal distributed than uniform distributed, and our strategy outperforms the baseline strategy commonly used in practice in terms of effective clicks.

In the future work, we are planning to (a) investigate more complex cases with consideration of the randomness from two or more factors; (b) compare of the performance of the three strategies at different temporal granularities; and (c) determine the effect of the value range of budget demand on the revenue.

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