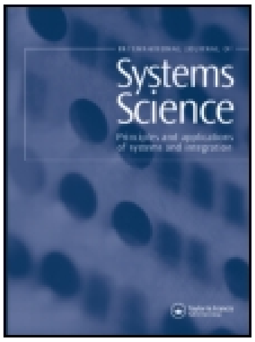

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Bipartite output consensus in networked multi-agent systems of high-order power integrators with signed digraph and input noises

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ABSTRACT

In this paper, we concentrate on investigating bipartite output consensus in networked multi-agent systems of high-order power integrators. Systems with power integrator are ubiquitous among weakly coupled, unstable and underactuated mechanical systems. In the presence of input noises, an adaptive disturbance compensator and a technique of adding power integrator are introduced to the complex nonlinear multi-agent systems to reduce the deterioration of system performance. Additionally, due to the existence of negative communication weights among agents, whether bipartite output consensus of high-order power integrators can be achieved remains unknown. Therefore, it is of great importance to study this issue. The underlying idea of designing the distributed controller is to combine the output information of each agent itself and its neighbours, the state feedback within its internal system and input adaptive noise compensator all together. When the signed digraph is structurally balanced, bipartite output consensus can be reached. Finally, numerical simulations are provided to verify the validity of the developed criteria.

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Bipartite output consensus; high-order; input noises; networked multi-agent systems; power integrator; signed digraph

1. Introduction

In the last decade, the issues about multi-agent systems have drawn a lot of attention (Cheng, Hou, Lin, Tan, & Zhang, 2011; Cheng, Wang, Hou, Tan, & Cao, 2013; Cheng, Hou, & Tan, 2014; Cheng, Wang, Hou, & Tan, 2015; Liu, Cheng, Tan, & Hou, 2015; Ma, Liu, Wang, Tan, & Li, 2015; Pan, Nian, & Guo, 2014; Ren, Beard, & Atkins, 2007; Sun, Guan, Ding, & Wang, 2013; Wang, Cheng, Ren, Hou, & Tan, 2015; Wen, Li, Duan, & Chen, 2013; Yao & Zheng, 2014; Yu & Wang, 2014; Zhang & Tian, 2014). The idea of distributed algorithms can be traced back to Tsitsiklis (1984) and Bertsekas and Tsitsiklis (1989) for dealing with the advent of networks. A novel type of phase transition in a system of self-driven particles was proposed (Vicsek, Czirók, Ben-Jacob, Cohen, & Shochet, 1995), which is the origin of the nearest neighbour rules. Then according to Vicsek's model, Jadbabaie, Lin, and Morse (2003) introduced the nearest neighbour rules into multi-agent systems. For more details, please refer to survey papers (Hespanha, Naghshtabrizi, & Xu, 2007; Olfati-Saber & Murray, 2004; Olfati-Saber, Fax, & Murray, 2007; Ren, Beard, & Atkins, 2005) and the references cited therein.

Bipartite graph (Diestel, 2000) is a basic concept in graph theory which is suitable for representing the communication topology of bipartite consensus. In several physical scenarios, it is reasonable to suppose that some of the agents are competitive, while the rest are cooperative. For instance, the polarisation of the community can be divided into two groups holding the opposite opinions, such as two competing sport teams shown in Figure 1. To the best of authors' knowledge, some pioneering works were given in Smith (1995), meanwhile Altafini (2013) was the first to propose the concept of bipartite consensus. Next, we consider a representative set of problem in the area of bipartite consensus. Altafini (2013) introduced the negative weights to the communication topology and demonstrated that bipartite consensus can be reached in the presence of antagonistic interactions. On one hand, Altafini (2013) mentioned that one of the most important requirements for the signed graph was structural balance (Cartwright & Harary, 1956). On the other hand, Altafini (2013) proposed both the linear and nonlinear Laplacian feedback distributed protocols to solve bipartite consensus. However, only the simplest situation was discussed where the dynamics of each agent were just equal to the distributed control, that is $\dot{x}_i = u_i$. Consequently, bipartite consensus was extended to formation control (Hu, Xiao, Zhou, & Yu, 2013) and directed signed networks (Hu & Zheng, 2013, 2014) with the same dynamics. In addition, Valcher and Misra (2014) discussed a more complex situation that the dynamics of multi-agent systems were high-order with antagonistic interactions, and bipartite consensus can be reached under the assumption of stabilisability with a sort of equilibrium between two fully competing groups. However, all the aspects mentioned above are associated with linear systems. In physical implementations, the power integrator system investigated by Qian and Lin (2001) is more ubiquitous. Therefore, it is of great importance to study the case where the multi-agent systems of high-order power integrators can reach bipartite output consensus.

High-order power integrator systems are both conceptually interesting because they are more complex than traditional linear systems in the aspect of analytical technique, and practically interesting because a class of weakly coupled, unstable and underactuated mechanical systems (Liu & Jiang, 2013), which are difficult to obtain stable control, are inherently nonlinear. Thus, they pose a number of challenges in terms of

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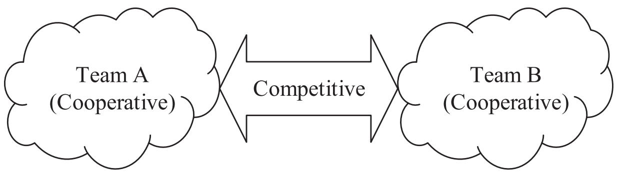

Figure 1. Two teams with cooperative behaviours inside and competitive behaviours between each other.

controllability after linearisation around the origin. In view of this, the technique of adding power integrator is a promising alternative approach to deal with the nonlinear properties of high-order power integrator systems. This technique has been widely used among the literatures. A new feedback design tool which adds a power integrator was introduced to solve the problem of global robust stabilisation when the nonlinear systems were lower triangular forms (Lin & Qian, 2000). Additionally, adding a power integrator was also introduced in Qian and Lin (2001) to deal with the global strong stabilisation with the similar form of power integrator. Moreover, in Qu (2010), network-based cooperative control of nonlinear dynamical systems was investigated and a restriction that $p_1 \geq p_2 \geq \ldots \geq p_{n-1} \geq 1$ was given, where $p_1, p_2, \ldots, p_{n-1}$ are odd integers. In Peng and Ye (2013), it shed light on cooperative output-synchronisation in multi-agent systems of high-order power integrator with input noises and undirected topology. However, we focus on a directed graph, particularly where the weights among agents are partly negative. Compared with the advances in the area of consensus (Olfati-Saber et al., 2007), less progress has been achieved in bipartite consensus and especially in bipartite output consensus. Therefore, due to the difficulty of handling the nonlinearity of power integrator and the unconventional properties of signed digraph, it is of great practical interest to investigate that under what conditions the multi-agent systems of high-order power integrators can reach bipartite output consensus.

Inspired by the above discussions, this paper aims at further investigating bipartite output consensus in networked multi-agent systems of high-order power integrators with signed digraph and input noises. By virtue of the technique of adding power integrator, we present this problem by first discussing when bipartite output consensus can be achieved in the absence of input noises, and then proceed by introducing noises to input channels, which is plausible in physical implementations. However, noises can further deteriorate performance of the entire networked systems. Thus, an adaptive noise compensator is developed to deal with noises in input channels. Finally, numerical simulations are given to validate the effectiveness of the established criteria.

The main contributions of this paper are listed as follows.

(1) Unlike conventional unsigned graph, we extend the graph to signed digraph whose communication weights are partly negative.
(2) Only the output is communicated with each other. Thus, information transferred among the multi-agent systems is not the full state vector of each agent and communication resources are highly reduced. Furthermore, performance of bipartite output consensus deteriorates in the presence of input noises which is more suitable for physical scenarios. Thus, we establish an adaptive noise compensator to minimise the negative effect of external disturbances.
(3) It is difficult to use traditional linear feedback control method to maintain the stability of multi-agent systems after linearisation around the origin. Therefore, the technique of adding power integrator is introduced to solve this problem.

The remainder of this paper is organised as follows. Basic definitions of bipartite output consensus and the properties of signed digraph are given in Section 2. By means of the technique of adding power integrator, a distributed control protocol is developed to obtain bipartite output consensus without input noises in Section 3. In view of input noises, an adaptive noise compensator is developed in Section 4 to enhance the robustness of networked multi-agent systems. In Section 5, numerical examples of bipartite output consensus are conducted to demonstrate the validity of the criteria established in Section 3 and 4. Conclusion of the whole paper is given in Section 6. Some key lemmas and propositions are provided in Appendices 1 and 2 with essential proofs.

2. Backgrounds and preliminaries

2.1. Algebraic graph theory

A triplet $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$ is called a (weighted) signed graph if $\mathcal{V} = \{1, 2, \ldots, N\}$ is the set of nodes, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of edges and $\mathcal{A} = (\mathcal{A}_{ij}) \in \mathbb{R}^{N \times N}$ is the matrix of the signed weights of $\mathcal{G}$. Here, $\mathcal{A}_{ij}$ denotes the element of the ith row and jth column of matrix $\mathcal{A}$. The ith node in signed graph $\mathcal{G}$ represents the ith agent, and a directed edge from node i to node j is denoted as an ordered pair $(i, j) \in \mathcal{E}$ which means that agent i can directly transfer its information to agent j. $\mathcal{A}$ is called the adjacency matrix of signed graph $\mathcal{G}$ with real numbers and we use the notation $\mathcal{G}(\mathcal{A}) : \mathcal{A}_{ij} \neq 0 \Leftrightarrow (j, i) \in \mathcal{E}$ to represent the signed graph corresponding to $\mathcal{A}$. Note that self-loops will not be considered in this paper, i.e. $\mathcal{A}_{ii} = 0, \forall i = 1, 2, \ldots, N$. For convenience, we introduce the following concepts. A directed cycle $\mathcal{C}$ of $\mathcal{G}(\mathcal{A})$ is a directed path with the same beginning and ending node. A cycle $\mathcal{C}$ is positive if it consists of an even number of negative edge weights: $\mathcal{A}_{w_1 w_2}\mathcal{A}_{w_2 w_3}\cdots\mathcal{A}_{w_p w_1} > 0$, where $w_1, w_2, \ldots, w_p$ belong to $\mathcal{V}$. It is negative when $\mathcal{A}_{w_1 w_2}\mathcal{A}_{w_2 w_3}\cdots\mathcal{A}_{w_p w_1} < 0$. In a directed graph (digraph), a pair of edges sharing the same nodes $(i, j), (j, i) \in \mathcal{E}$ is called a digon. We assume that $\mathcal{A}_{ij}\mathcal{A}_{ji} \geq 0$, which means that all digons cannot have the opposite signs. In this paper, we call this property digon sign-symmetric. Otherwise, we call it digon sign-nonsymmetric. Given a signed digraph $\mathcal{G}(\mathcal{A})$, $\mathcal{C}_r$ is termed as the row connectivity matrix of $\mathcal{A}$ and

$$\mathcal{C}_r = \begin{bmatrix} c_{r,11} & 0 & \cdots & 0 \\ 0 & c_{r,22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c_{r,NN} \end{bmatrix}$$

with diagonal elements $c_{r,ii} = \sum_{j \in \mathcal{N}_i} |\mathcal{A}_{ij}|$, where $\mathcal{N}_i = \{j \in \mathcal{V} | (j, i) \in \mathcal{E}\}$ is the in-degree neighbour set of node i. The column connectivity matrix $\mathcal{C}_c$ is defined likewise, where

$c_{c,ii} = \sum_{j\in\tilde{\mathcal{N}}_i} |\mathcal{A}_{ji}|$ and $\tilde{\mathcal{N}}_i = \{j \in \mathcal{V} | (i, j) \in \mathcal{E}\}$ is the out-degree neighbour set.

2.2. Bipartite consensus

The communication topology among the N agents can be represented by a signed digraph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$. The interaction between the ith agent and the jth agent is cooperative if $\mathcal{A}_{ij} > 0$; otherwise, it is antagonistic if $\mathcal{A}_{ij} < 0$. Furthermore, $\mathcal{A}_{ij} = 0$ means there is no interaction between the ith agent and the jth agent.

Following the definition of the unsigned graph in most literature, we define the row Laplacian matrix corresponding to the adjacency matrix $\mathcal{A}$ of signed digraph $\mathcal{G}(\mathcal{A})$ as

$$\mathcal{L} = \mathcal{C}_r - \mathcal{A}, \tag{1}$$

where $\mathcal{C}_r$ is the row connectivity matrix of $\mathcal{A}$. Therefore,

$$\mathcal{L}_{ij} = \begin{cases} \sum_{k\in\mathcal{N}_i} |\mathcal{A}_{ik}|, & \text{if } i = j; \\ -\mathcal{A}_{ij}, & \text{if } i \neq j. \end{cases} \tag{2}$$

Definition 1 (Structurally balanced, cf. Altafini (2013))**:** A signed digraph $\mathcal{G}(\mathcal{A})$ is said to be structurally balanced if it contains a bipartition of the nodes $\mathcal{V}_1, \mathcal{V}_2, \mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2, \mathcal{V}_1 \cap \mathcal{V}_2 = \varnothing$ such that $\mathcal{A}_{ij} \geq 0, \forall i, j \in \mathcal{V}_p$ $(p \in \{1, 2\})$; $\mathcal{A}_{ij} \leq 0, \forall i \in \mathcal{V}_p, j \in \mathcal{V}_q, p \neq q$ $(p, q \in \{1, 2\})$. Otherwise, it is called structurally unbalanced.

In this and the subsequent sections, we assume that the signed digraph $\mathcal{G}$ is digon sign-symmetric, strongly connected and structurally balanced. In addition, all the cycles $\mathcal{C}$ in digraph $\mathcal{G}$ are positive. According to Definition 1 and the illustration in Figure 1, this is equivalent to saying that the agents can be split into two disjoint groups, where the cooperative interactions between pairs of agents exist in the same groups and the antagonistic interactions between pairs of agents exist between two different groups.

2.3. Problem formulation

Suppose that the network contains N agents and the dynamics of each agent i are given as follows:

$$\begin{aligned} \dot{x}_{i1} &= x_{i2}^{p_1} \\ \dot{x}_{i2} &= x_{i3}^{p_2} \\ &\vdots \\ \dot{x}_{i,n-1} &= x_{in}^{p_{n-1}} \\ \dot{x}_{in} &= u_i^{p_n} \\ y_i &= x_{i1} \end{aligned} \tag{3}$$

where $p_k \geq 1$, $\forall k \in \{1, 2, \dots, n\}$ are odd integers and $x_i = (x_{i1}, x_{i2}, \dots, x_{in})^\top \in \mathbb{R}^n$, $y_i \in \mathbb{R}$, $u_i \in \mathbb{R}$ are the state vector, output and control input of agent i, respectively. Before proceeding, we introduce the definition of bipartite output consensus in concert with the subsequent analyses.

Definition 2 (bipartite output consensus)**:** If for any initial condition $x_i(0)$,

$$\begin{cases} \lim_{t\to\infty} \|y_j(t) - y_i(t)\| = 0, \ \forall i, j \in \mathcal{V}_1 \text{ or } \forall i, j \in \mathcal{V}_2; \\ \lim_{t\to\infty} \|y_j(t) + y_i(t)\| = 0, \ \forall i \in \mathcal{V}_1 \text{ and } \forall j \in \mathcal{V}_2, \end{cases} \tag{4}$$

where $\mathcal{V}_1$ and $\mathcal{V}_2$ are the sets defined in Definition 1, then we say that the multi-agent system (3) can reach bipartite output consensus.

Remark 1: In this paper, we suppose that the communication capability is sufficient and the communication intensity is not related to the distance between each pair of agents. Furthermore, if there is an edge from agent i to agent j where $j \in \tilde{\mathcal{N}}_i$, then agent i can transfer its output information y_i to agent j without data loss.

3. Bipartite output consensus with directed topology

In this section, we will concentrate on designing a distributed control protocol to achieve bipartite output consensus without input noises. The main theorem is given as follows.

Theorem 1: *The dynamics of each agent in the network are given in (3) and x_{il}^*, $l = 2, 3, \dots, n$, can be seen as internal reference states. The distributed control protocols are designed as follows:*

$$\begin{aligned} &\varphi_{i1} = \sum_{j\in\mathcal{N}_i} |\mathcal{A}_{ij}|(y_i - \operatorname{sgn}(\mathcal{A}_{ij})y_j) + \sum_{j\in\tilde{\mathcal{N}}_i} |\mathcal{A}_{ji}|(y_i - \operatorname{sgn}(\mathcal{A}_{ji})y_j) \\ &x_{i2}^{*p_1} = -k_1\varphi_{i1} \qquad \varphi_{i2} = x_{i2}^{p_1} - x_{i2}^{*p_1} \\ &x_{i3}^{*p_1p_2} = -k_2\varphi_{i2} \qquad \varphi_{i3} = x_{i3}^{p_1p_2} - x_{i3}^{*p_1p_2} \\ &\qquad \vdots \qquad\qquad\qquad \vdots \\ &x_{in}^{*p_1\cdots p_{n-1}} = -k_{n-1}\varphi_{i,n-1} \qquad \varphi_{in} = x_{in}^{p_1\cdots p_{n-1}} - x_{in}^{*p_1\cdots p_{n-1}} \\ &u_i = -(k_n\varphi_{in})^{1/p_1\cdots p_n}, \end{aligned} \tag{5}$$

where $k_1, k_2, \dots, k_n$ and $p_1, p_2, \dots, p_n$ are positive constant control gains and positive odd integers, respectively. The symbol sgn *represents the sign function. That is*

$$\operatorname{sgn}(\mathcal{A}_{ij}) = \begin{cases} 1, & \text{if } \mathcal{A}_{ij} > 0; \\ 0, & \text{if } \mathcal{A}_{ij} = 0; \\ -1, & \text{if } \mathcal{A}_{ij} < 0. \end{cases}$$

Then, the multi-agent system (3) can asymptotically achieve bipartite output consensus. Furthermore, $x_{i2}, x_{i3}, \dots, x_{in}$, $\forall i \in \mathcal{V}$ are bounded and will approach zero.

Proof: Note that the basic idea of designing the distributed control protocol is borrowed from backstepping technique. φ_{i1} utilises the information of both in-degree and out-degree neighbouring nodes of agent i. Then, we obtain the internal virtual reference state $x_{i2}^{*p_1}$ which will be tracked by $x_{i2}^{p_1}$, and φ_{i1} can be seen as the feedback error to the internal virtual reference state $x_{i2}^{*p_1}$. Thus, φ_{i2} is the error between $x_{i2}^{*p_1}$ and $x_{i2}^{p_1}$, which can be fed to higher order virtual reference state $x_{i3}^{*p_1p_2}$. By using the similar designing steps, Equation (5) can easily be obtained. Furthermore, u_i is the final control signal which is used to control agent i.

Let $\hat{\mathcal{L}}_u = \mathcal{C}_u - \mathcal{A}_u$ be an undirected Laplacian matrix, where

$$\mathcal{A}_u = \frac{\mathcal{A} + \mathcal{A}^{\mathsf{T}}}{2}, \quad \mathcal{C}_u = \frac{\mathcal{C}_r + \mathcal{C}_c}{2}.$$

We first define a potential function V_1 associated with the Laplacian matrix $\hat{\mathcal{L}}_u$ as follows:

$$V_1 = \frac{1}{2}\vec{x}^{\mathsf{T}}\hat{\mathcal{L}}_u\vec{x} = \frac{1}{2}\sum_{(i,j)\in\mathcal{E}} |\mathcal{A}_{u,ij}|(x_{i1} - \mathrm{sgn}\left(\mathcal{A}_{u,ij}\right)x_{j1})^2, \quad (6)$$

where $\vec{x} = (x_{11}, x_{21}, \ldots, x_{N1})^{\mathsf{T}}$ and $\mathcal{A}_{u,ij}$ is the (i, j)th element of matrix $\mathcal{A}_u$. Then,

$$\begin{aligned}
\dot{V}_1 &= \left(\hat{\mathcal{L}}_u\vec{x}\right)^{\mathsf{T}}\dot{\vec{x}} \\
&= \frac{1}{2}\left[(\mathcal{C}_r - \mathcal{A})\vec{x} + (\mathcal{C}_c - \mathcal{A}^{\mathsf{T}})\vec{x}\right]^{\mathsf{T}}\dot{\vec{x}} \\
&= \frac{1}{2}\sum_{i=1}^{N}\left[\sum_{j\in\mathcal{N}_i}|\mathcal{A}_{ij}|(x_{i1} - \mathrm{sgn}(\mathcal{A}_{ij})x_{j1})\right. \\
&\qquad \left. + \sum_{j\in\tilde{\mathcal{N}}_i}|\mathcal{A}_{ji}|(x_{i1} - \mathrm{sgn}(\mathcal{A}_{ji})x_{j1})\right]\dot{x}_{i1} \\
&= \frac{1}{2}\sum_{i=1}^{N}\varphi_{i1}x_{i2}^{p_1} \\
&= \frac{1}{2}\sum_{i=1}^{N}\varphi_{i1}x_{i2}^{*p_1} + \frac{1}{2}\sum_{i=1}^{N}\varphi_{i1}\left(x_{i2}^{p_1} - x_{i2}^{*p_1}\right). \qquad (7)
\end{aligned}$$

Let $\varphi_{i2} = x_{i2}^{p_1} - x_{i2}^{*p_1}$. Then, by Lemma 1 in Appendix 1,

$$\begin{aligned}
\dot{V}_1 &= -\frac{k_1}{2}\sum_{i=1}^{N}\varphi_{i1}^2 + \frac{1}{2}\sum_{i=1}^{N}\varphi_{i1}\varphi_{i2} \\
&\le -\frac{k_1}{2}\sum_{i=1}^{N}\varphi_{i1}^2 + \sum_{i=1}^{N}\varphi_{i1}^2 + \frac{1}{16}\sum_{i=1}^{N}\varphi_{i2}^2 \quad \text{(by Lemma 1)} \\
&\le -b_{11}\sum_{i=1}^{N}\varphi_{i1}^2 + b_{12}\sum_{i=1}^{N}\varphi_{i2}^2, \qquad (8)
\end{aligned}$$

where b_{11} and b_{12} are two positive constants satisfying the inequality (8). In the sequel, we make use of the form of $x_{i2}^{p_1}$ to define a new scalar function

$$S_{i2} = \int_{x_{i2}^*}^{x_{i2}} \left(r^{p_1} - x_{i2}^{*p_1}\right)^{2-1/p_1}\mathrm{d}r, \quad \forall i \in \mathcal{V}. \qquad (9)$$

Referring to Proposition 1 in Appendix 2, $S_{i2} \ge 0$ and the corresponding partial derivatives of S_{i2} are

$$\begin{aligned}
\frac{\partial S_{i2}}{\partial x_{i2}} &= \varphi_{i2}^{2-1/p_1}, \\
\frac{\partial S_{i2}}{\partial x_{i1}} &= -\left(2 - \frac{1}{p_1}\right)\frac{\partial x_{i2}^{*p_1}}{\partial x_{i1}}\int_{x_{i2}^*}^{x_{i2}}\left(r^{p_1} - x_{i2}^{*p_1}\right)^{1-1/p_1}\mathrm{d}r, \\
\frac{\partial S_{i2}}{\partial x_{j1}} &= -\left(2 - \frac{1}{p_1}\right)\frac{\partial x_{i2}^{*p_1}}{\partial x_{j1}}\int_{x_{i2}^*}^{x_{i2}}\left(r^{p_1} - x_{i2}^{*p_1}\right)^{1-1/p_1}\mathrm{d}r, \quad j \in \mathcal{N}_i.
\end{aligned}$$

Similarly, define another potential function containing the information of x_{i1} and x_{i2} of all the agents as follows:

$$V_2 = V_1 + \sum_{i=1}^{N}S_{i2} = V_1 + \sum_{i=1}^{N}\int_{x_{i2}^*}^{x_{i2}}\left(r^{p_1} - x_{i2}^{*p_1}\right)^{2-1/p_1}\mathrm{d}r. \qquad (10)$$

Hence, the derivative of V_2 with respect to time t is

$$\begin{aligned}
\dot{V}_2 &= \dot{V}_1 + \sum_{i=1}^{N}\dot{S}_{i2} \\
&= \dot{V}_1 + \sum_{i=1}^{N}\frac{\partial S_{i2}}{\partial x_{i2}}\dot{x}_{i2} + \sum_{i=1}^{N}\frac{\partial S_{i2}}{\partial x_{i1}}\dot{x}_{i1} + \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\frac{\partial S_{i2}}{\partial x_{j1}}\dot{x}_{j1} \\
&= \dot{V}_1 + \sum_{i=1}^{N}\varphi_{i2}^{2-1/p_1}\left[x_{i3}^{*p_2} + \left(x_{i3}^{p_2} - x_{i3}^{*p_2}\right)\right] + \sum_{i=1}^{N}\frac{\partial S_{i2}}{\partial x_{i1}}\dot{x}_{i1} \\
&\quad + \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\frac{\partial S_{i2}}{\partial x_{j1}}\dot{x}_{j1} \\
&\le -b_{11}\sum_{i=1}^{N}\varphi_{i1}^2 + b_{12}\sum_{i=1}^{N}\varphi_{i2}^2 + \sum_{i=1}^{N}\varphi_{i2}^{2-1/p_1}x_{i3}^{*p_2} \\
&\quad + \sum_{i=1}^{N}|\varphi_{i2}|^{2-1/p_1}|x_{i3}^{p_2} - x_{i3}^{*p_2}| \\
&\quad + \sum_{i=1}^{N}\left|\frac{\partial S_{i2}}{\partial x_{i1}}\right||\dot{x}_{i1}| + \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\left|\frac{\partial S_{i2}}{\partial x_{j1}}\right||\dot{x}_{j1}|. \qquad (11)
\end{aligned}$$

Furthermore, from (5) we can derive that $x_{i3}^{*p_2} = -k_2^{1/p_1}\varphi_{i2}^{1/p_1}$ and

$$\left|x_{i3}^{p_2} - x_{i3}^{*p_2}\right| \le 2^{\frac{p_1-1}{p_1}}\left|x_{i3}^{p_1p_2} - x_{i3}^{*p_1p_2}\right|^{1/p_1} = 2^{\frac{p_1-1}{p_1}}|\varphi_{i3}|^{1/p_1}.$$

By Lemma 1, we obtain

$$\begin{aligned}
&\sum_{i=1}^{N}|\varphi_{i2}|^{2-1/p_1}\left|x_{i3}^{p_2} - x_{i3}^{*p_2}\right| \\
&\le 2^{\frac{p_1-1}{p_1}}\sum_{i=1}^{N}|\varphi_{i2}|^{2-1/p_1}|\varphi_{i3}|^{1/p_1} \le b_{22}^{'}\sum_{i=1}^{N}\varphi_{i2}^2 + b_{23}^{'}\sum_{i=1}^{N}\varphi_{i3}^2, \qquad (12)
\end{aligned}$$

where $b_{22}^{'}$ and $b_{23}^{'}$ are two positive constants.

Now, we concentrate on the latter two terms in (11). Note that with Propositions 2 and 3, the following two inequalities hold:

$$\left|\frac{\partial S_{i2}}{\partial x_{i1}}\right| |\dot{x}_{i1}| \le 4|\varphi_{i2}| \left|\frac{\partial x_{i2}^{*p_1}}{\partial x_{i1}}\dot{x}_{i1}\right| \le 4\gamma_{21}^{i}|\varphi_{i2}|(|\varphi_{i1}| + |\varphi_{i2}|),$$
$$\left|\frac{\partial S_{i2}}{\partial x_{j1}}\right| |\dot{x}_{j1}| \le 4|\varphi_{i2}| \left|\frac{\partial x_{i2}^{*p_1}}{\partial x_{j1}}\dot{x}_{j1}\right| \le 4\eta_{2j}^{i}|\varphi_{i2}|(|\varphi_{j1}| + |\varphi_{j2}|),$$
$$j \in \mathcal{N}_i,$$

where γ_{21}^{i} and η_{2j}^{i} are positive constants. By virtue of Lemma 1, we obtain

$$\sum_{i=1}^{N}\left|\frac{\partial S_{i2}}{\partial x_{i1}}\right| |\dot{x}_{i1}| + \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\left|\frac{\partial S_{i2}}{\partial x_{j1}}\right| |\dot{x}_{j1}|$$
$$\le b_{21}^{''}\sum_{i=1}^{N}\varphi_{i1}^2 + b_{22}^{''}\sum_{i=1}^{N}\varphi_{i2}^2, \tag{13}$$

where $b_{21}^{''}$ and $b_{22}^{''}$ are positive constants.

With (12) and (13), $\dot{V}_2$ can be rewritten as

$$\begin{aligned}\dot{V}_2 \le{}& -b_{11}\sum_{i=1}^{N}\varphi_{i1}^2 + b_{12}\sum_{i=1}^{N}\varphi_{i2}^2 - k_2^{1/p_1}\sum_{i=1}^{N}\varphi_{i2}^2 + b_{22}^{'}\sum_{i=1}^{N}\varphi_{i2}^2\\ &+ b_{23}^{'}\sum_{i=1}^{N}\varphi_{i3}^2 + b_{21}^{''}\sum_{i=1}^{N}\varphi_{i1}^2 + b_{22}^{''}\sum_{i=1}^{N}\varphi_{i2}^2\\ \le{}& -b_{21}\sum_{i=1}^{N}\varphi_{i1}^2 - b_{22}\sum_{i=1}^{N}\varphi_{i2}^2 + b_{23}\sum_{i=1}^{N}\varphi_{i3}^2,\end{aligned} \tag{14}$$

where k_2 and b_{11} are chosen properly such that $-b_{11} + b_{21}^{''} < 0$ and $-k_2^{1/p_1} + b_{12} + b_{22}^{'} + b_{22}^{''} < 0$, and b_{21}, b_{22}, b_{23} are positive constants.

Next, we utilise induction with similar steps above for $2 < m \le n-1$. Define

$$S_{im} = \int_{x_{im}^{*}}^{x_{im}} \left(r^{p_1\cdots p_{m-1}} - x_{im}^{*p_1\cdots p_{m-1}}\right)^{2-1/p_1\cdots p_{m-1}}\,\mathrm{d}r. \tag{15}$$

Then,

$$\begin{aligned}V_m ={}& V_{m-1} + \sum_{i=1}^{N} S_{im} = V_{m-1}\\ &+ \sum_{i=1}^{N}\int_{x_{im}^{*}}^{x_{im}} \left(r^{p_1\cdots p_{m-1}} - x_{im}^{*p_1\cdots p_{m-1}}\right)^{2-1/p_1\cdots p_{m-1}}\,\mathrm{d}r.\end{aligned} \tag{16}$$

Thus, the derivative of V_m is

$$\begin{aligned}\dot{V}_m ={}& \dot{V}_{m-1} + \sum_{i=1}^{N}\frac{\partial S_{im}}{\partial x_{im}}\dot{x}_{im} + \sum_{i=1}^{N}\sum_{l=1}^{m-1}\frac{\partial S_{im}}{\partial x_{il}}\dot{x}_{il} + \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\frac{\partial S_{im}}{\partial x_{j1}}\dot{x}_{j1}\\ ={}& \dot{V}_{m-1} + \sum_{i=1}^{N}\frac{\partial S_{im}}{\partial x_{im}}x_{i,m+1}^{*p_m} + \sum_{i=1}^{N}\frac{\partial S_{im}}{\partial x_{im}}\left(x_{i,m+1}^{p_m} - x_{i,m+1}^{*p_m}\right)\\ &+ \sum_{i=1}^{N}\sum_{l=1}^{m-1}\frac{\partial S_{im}}{\partial x_{il}}\dot{x}_{il} + \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\frac{\partial S_{im}}{\partial x_{j1}}\dot{x}_{j1}\\ \le{}& \dot{V}_{m-1} - (k_m)^{1/p_1\cdots p_{m-1}}\sum_{i=1}^{N}\varphi_{im}^2 + \sum_{i=1}^{N}\left|\varphi_{im}^{2-1/p_1\cdots p_{m-1}}\right|\\ &\times \left|x_{i,m+1}^{p_m} - x_{i,m+1}^{*p_m}\right| + \sum_{i=1}^{N}\sum_{l=1}^{m-1}\frac{\partial S_{im}}{\partial x_{il}}\dot{x}_{il} + \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\frac{\partial S_{im}}{\partial x_{j1}}\dot{x}_{j1}\\ \le{}& \dot{V}_{m-1} - (k_m)^{1/p_1\cdots p_{m-1}}\sum_{i=1}^{N}\varphi_{im}^2\\ &+ \sum_{i=1}^{N}\left(b_{im}^{'}\varphi_{im}^2 + b_{i,m+1}^{'}\varphi_{i,m+1}^2\right)\\ &+ \sum_{i=1}^{N}\sum_{l=1}^{m-1}\frac{\partial S_{im}}{\partial x_{il}}\dot{x}_{il} + \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\frac{\partial S_{im}}{\partial x_{j1}}\dot{x}_{j1}\\ \le{}& \dot{V}_{m-1} - (k_m)^{1/p_1\cdots p_{m-1}}\sum_{i=1}^{N}\varphi_{im}^2\\ &+ \sum_{i=1}^{N}\left(b_{im}^{'}\varphi_{im}^2 + b_{i,m+1}^{'}\varphi_{i,m+1}^2\right)\\ &+ \sum_{i=1}^{N}\sum_{l=1}^{m-1}4|\varphi_{im}|\left|\frac{\partial x_{im}^{*p_1\cdots p_{m-1}}}{\partial x_{il}}\dot{x}_{il}\right|\\ &+ \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}4|\varphi_{im}|\left|\frac{\partial x_{im}^{*p_1\cdots p_{m-1}}}{\partial x_{j1}}\dot{x}_{j1}\right| \text{ (by Proposition 2)}\\ \le{}& \dot{V}_{m-1} - (k_m)^{1/p_1\cdots p_{m-1}}\sum_{i=1}^{N}\varphi_{im}^2 + \sum_{i=1}^{N}\left(b_{im}^{'}\varphi_{im}^2 + b_{i,m+1}^{'}\varphi_{i,m+1}^2\right)\\ &+ \sum_{i=1}^{N}\sum_{l=1}^{m-1}4\gamma_{ml}^{i}|\varphi_{im}|(|\varphi_{i1}| + \cdots + |\varphi_{im}|)\\ &+ \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}4\eta_{mj}^{i}|\varphi_{im}|(|\varphi_{j1}| + |\varphi_{j2}|) \text{ (by Proposition 3)}\\ \le{}& -b_{m-1,1}\sum_{i=1}^{N}\varphi_{i1}^2 - b_{m-1,2}\sum_{i=1}^{N}\varphi_{i2}^2 - \cdots - b_{m-1,m-1}\\ &\times\sum_{i=1}^{N}\varphi_{i,m-1}^2 + b_{m-1,m}\sum_{i=1}^{N}\varphi_{i,m}^2\end{aligned}$$

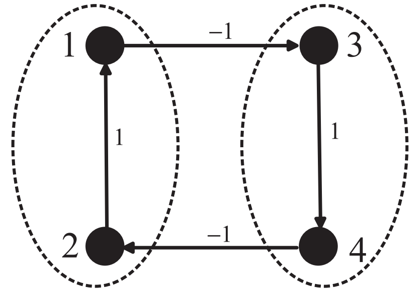

Figure 2. Communication topology of four agents.

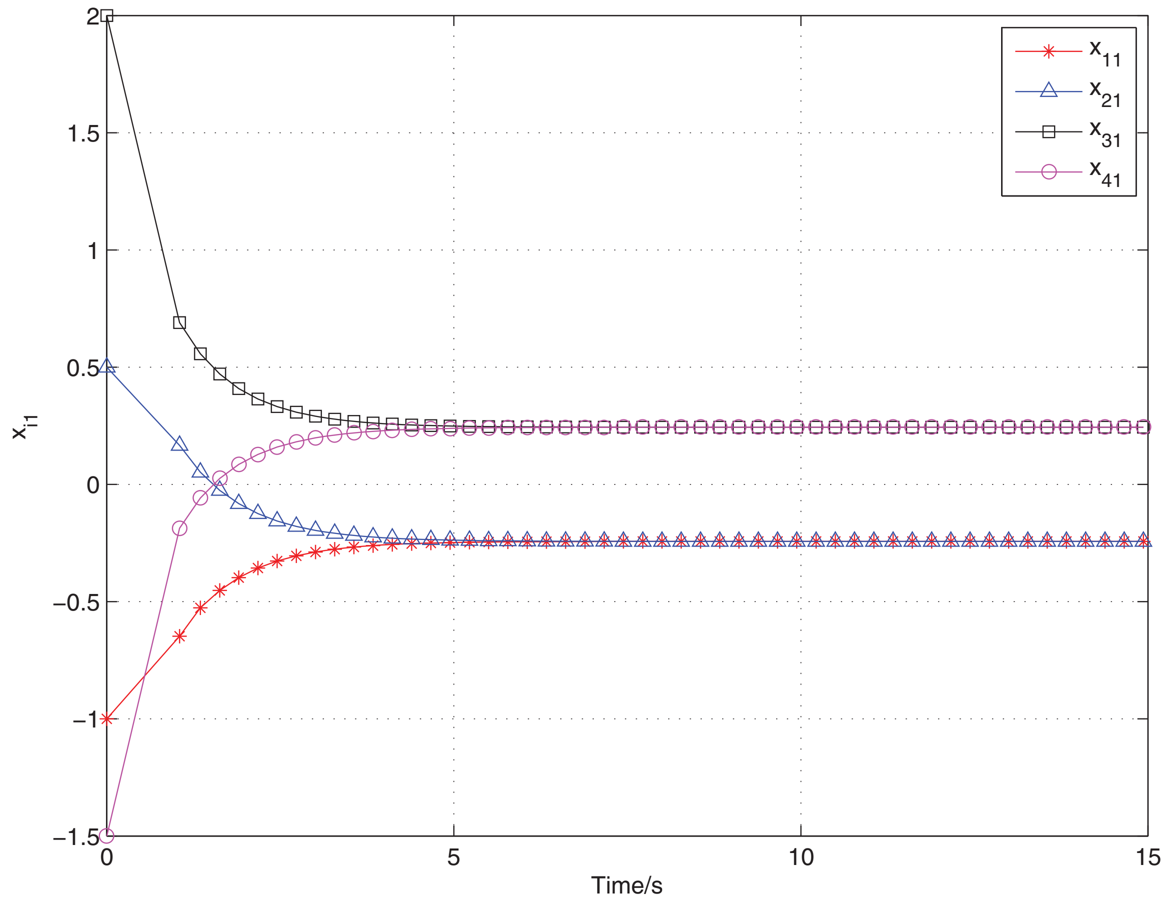

Figure 3. Output trajectories $y_i = x_{i1}$, $i = 1, 2, 3, 4$, with bipartite output consensus.

$$- (k_m)^{1/p_1\cdots p_{m-1}} \sum_{i=1}^{N} \varphi_{im}^2 + \sum_{i=1}^{N} \left(b_{im}^{'} \varphi_{im}^2 + b_{i,m+1}^{'} \varphi_{i,m+1}^2 \right)$$
$$+ \sum_{i=1}^{N} \left(b_{i1}^{''} \varphi_{i1}^2 + b_{i2}^{''} \varphi_{i2}^2 + \ldots + b_{im}^{''} \varphi_{im}^2 \right). \quad (17)$$

Therefore, by appropriately choosing the parameters in (17), we can rewrite $\dot{V}_m$ as

$$\dot{V}_m \le -b_{m1} \sum_{i=1}^{N} \varphi_{i1}^2 - b_{m2} \sum_{i=1}^{N} \varphi_{i2}^2 - \cdots - b_{mm} \sum_{i=1}^{N} \varphi_{im}^2$$
$$+ b_{m,m+1} \sum_{i=1}^{N} \varphi_{i,m+1}^2. \quad (18)$$

Finally, we demonstrate $\dot{V}_n \le 0$. To that end, define

$$V_n = V_{n-1} + \sum_{i=1}^{N} S_{in}, \quad (19)$$

where

$$S_{in} = \int_{x_{in}^*}^{x_{in}} \left(r^{p_1\cdots p_{n-1}} - x_{in}^{*p_1\cdots p_{n-1}} \right)^{2-1/p_1\cdots p_{n-1}} \mathrm{d}r. \quad (20)$$

In addition, with the help of similar steps shown in calculating $\dot{V}_m$ for $2 < m \le n-1$, the derivative of V_n is

$$\dot{V}_n = \dot{V}_{n-1} + \sum_{i=1}^{N} \frac{\partial S_{in}}{\partial x_{in}} u_i^{p_n} + \sum_{i=1}^{N} \sum_{l=1}^{n-1} \frac{\partial S_{in}}{\partial x_{il}} \dot{x}_{il} + \sum_{i=1}^{N} \sum_{j\in\mathcal{N}_i} \frac{\partial S_{in}}{\partial x_{j1}} \dot{x}_{j1}$$
$$\le -b_{n-1,1} \sum_{i=1}^{N} \varphi_{i1}^2 - b_{n-1,2} \sum_{i=1}^{N} \varphi_{i2}^2 - \cdots - b_{n-1,n-1} \sum_{i=1}^{N} \varphi_{i,n-1}^2$$
$$+ b_{n-1,n} \sum_{i=1}^{N} \varphi_{in}^2 + b_{n1}^{'} \sum_{i=1}^{N} \varphi_{i1}^2 + b_{n2}^{'} \sum_{i=1}^{N} \varphi_{i2}^2 + \cdots$$
$$+ b_{nn}^{'} \sum_{i=1}^{N} \varphi_{in}^2 - k_n^{1/p_1\cdots p_{n-1}} \sum_{i=1}^{N} \varphi_{in}^2$$
$$\le -b_{n1} \sum_{i=1}^{N} \varphi_{i1}^2 - b_{n2} \sum_{i=1}^{N} \varphi_{i2}^2 - \cdots - b_{n,n-1} \sum_{i=1}^{N} \varphi_{i,n-1}^2$$
$$- b_{nn} \sum_{i=1}^{N} \varphi_{in}^2 \le 0, \quad (21)$$

where $b_{n1}, b_{n2}, \ldots, b_{nn}$ are all positive constants.

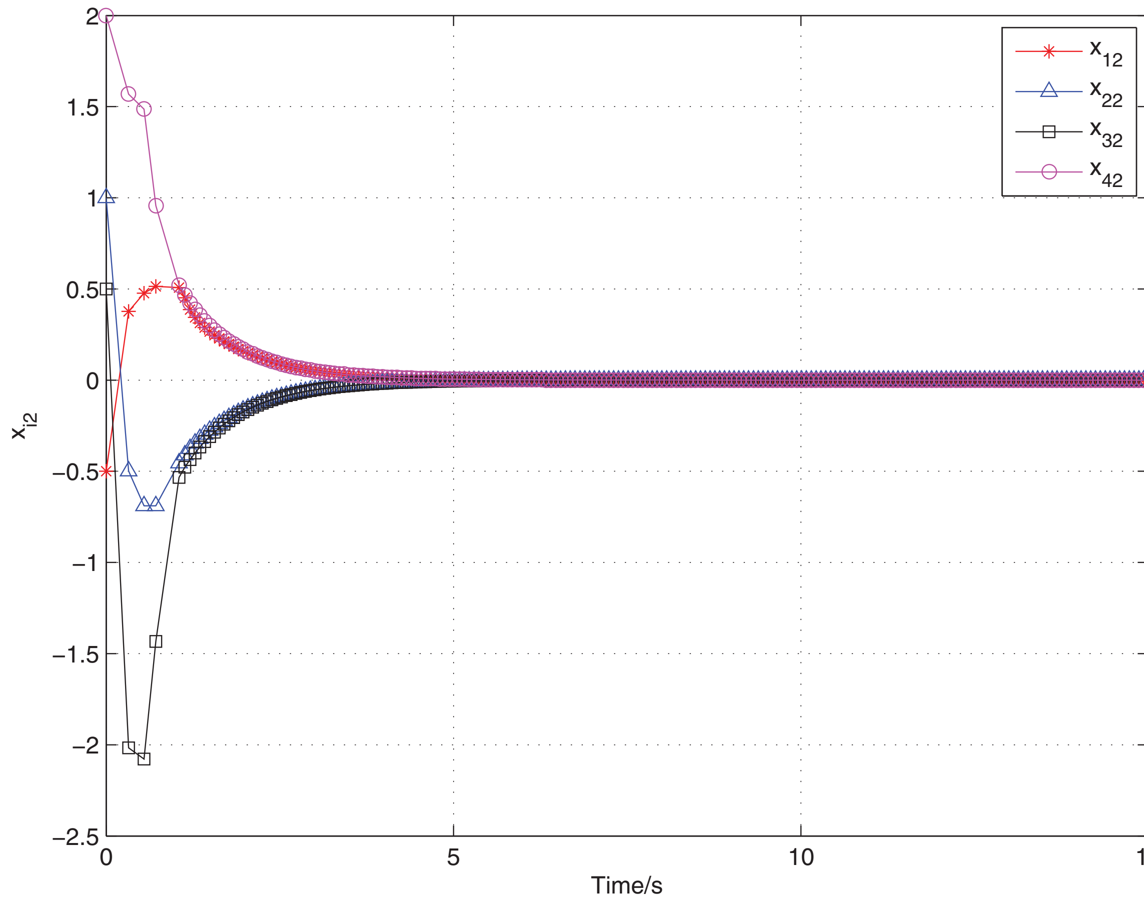

Figure 4. Trajectories of x_{i2}, $i = 1, 2, 3, 4$.

Then by integrating (21), we have

$$V_n(t) - V_n(0) \leq -b_{n1} \sum_{i=1}^{N} \int_0^t \varphi_{i1}^2(\sigma)\mathrm{d}\sigma - b_{n2} \times \sum_{i=1}^{N} \int_0^t \varphi_{i2}^2(\sigma)\mathrm{d}\sigma - \cdots - b_{nn} \times \sum_{i=1}^{N} \int_0^t \varphi_{in}^2(\sigma)\mathrm{d}\sigma \leq 0. \tag{22}$$

Therefore, $0 \leq V_n(t) \leq V_n(0)$ and $V_n(t)$ is bounded. Since $\dot{V}_n \leq 0$,

$$\lim_{t\to\infty} V_n(t) = 0. \tag{23}$$

The preceding analysis, along with $\dot{V}_n \leq 0$, yields

$$\lim_{t\to\infty} \dot{V}_n(t) = 0. \tag{24}$$

According to (21),

$$\lim_{t\to\infty} \varphi_{ik}(t) = 0, \quad \forall i \in \mathcal{V}, k = 1, 2, \ldots, n.$$

Furthermore, with regard to (5), it is clear that $x_{i2}, x_{i3}, \ldots, x_{in}$ are bounded and all approach zero when $t \to \infty$. Since $(1/2)\vec{x}^{\mathsf{T}}\hat{\mathcal{L}}_u\vec{x}, S_{i2}, S_{i3}, \ldots, S_{in}$ are all nonnegative terms and $\mathcal{G}$ is strongly connected, along with (6) and (23), we have

$$\lim_{t\to\infty} V_1(t) = 0, \tag{25}$$

and this implies that bipartite output consensus can be asymptotically achieved, which is satisfied with (4) in Definition 2. □

Remark 2: The distributed control law φ_{i1} in (5) includes two parts, i.e. the in-degree and out-degree information of agent i. Therefore, more information from neighbours can guarantee better performance of the multi-agent systems. Furthermore, although the dynamics of each agent are high-order power integrator, only output information is needed to be transferred to the neighbours around, which greatly reduces the communication overhead.

4. Bipartite output consensus with input noises

We consider the case when input channels of multi-agent system (3) are contaminated with unknown disturbances $\delta = (\delta_1, \delta_2, \ldots, \delta_N)^{\mathsf{T}} \in \mathbb{R}^N$.

Assumption 1: *There is an unknown external system*

$$\begin{aligned} \dot{\theta} &= \Gamma\theta, \\ \delta &= \Phi^{\mathsf{T}}\theta, \end{aligned} \tag{26}$$

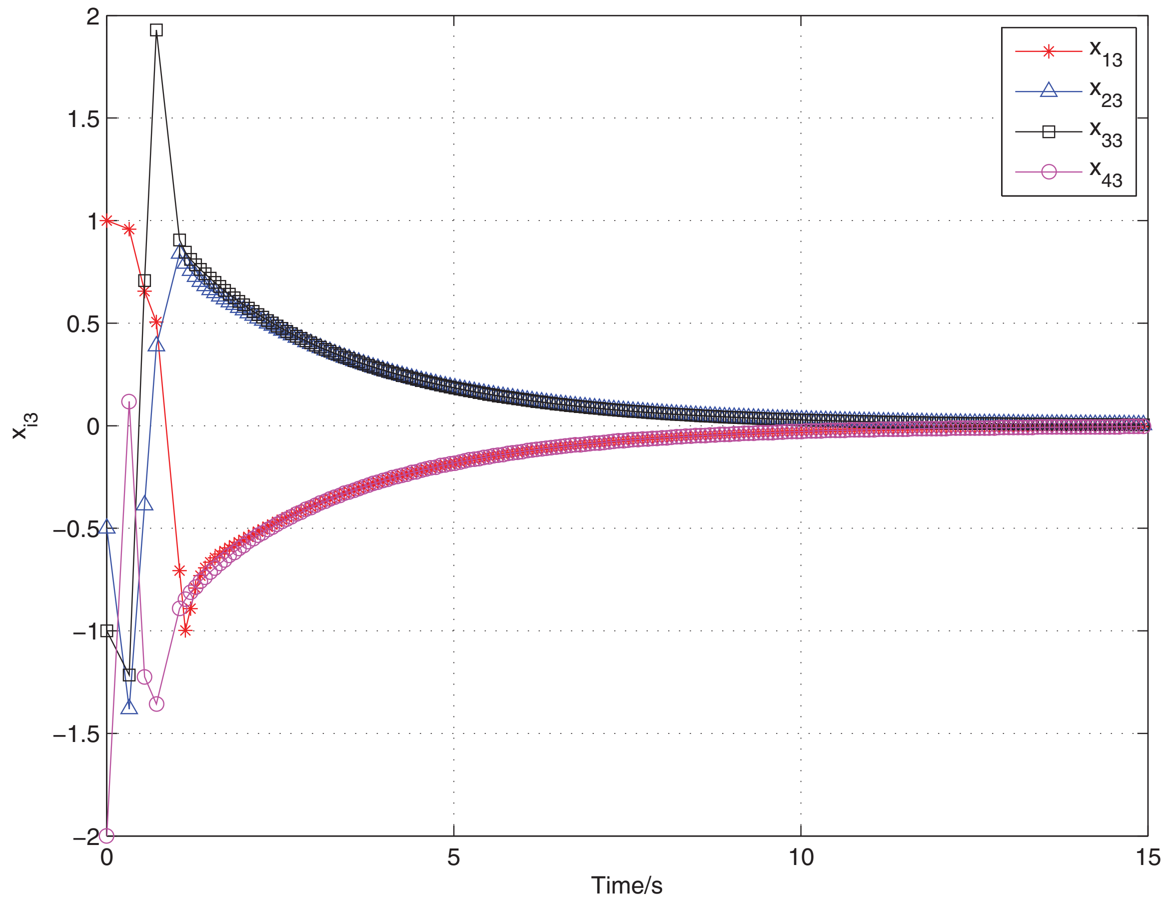

Figure 5. Trajectories of x_{i3}, $i = 1, 2, 3, 4$.

where $\theta \in \mathbb{R}^2$, $\Gamma \in \mathbb{R}^{2\times 2}$, $\Phi \in \mathbb{R}^{2\times N}$ and the eigenvalues of Γ are all on the imaginary axis. The marginal stability of the exosystem implies that δ_i is bounded by constants $\bar{\delta}_i$, i.e. $|\delta_i| \le \bar{\delta}_i$, $\forall i \in \mathcal{V}$.

To obtain a concise form, we only show the different parts from (3) and (5). They are

$$\dot{x}_{in} = u_i^{p_n} + \delta_i \tag{27}$$

and

$$\begin{aligned} u_i &= -\left[(k_n\varphi_{in})^{1/p_1\cdots p_{n-1}} + \operatorname{sgn}\left(\varphi_{in}^{2-1/p_1\cdots p_{n-1}}\right)\hat{\delta}_i\right]^{1/p_n}, \\ \dot{\hat{\delta}}_i &= \kappa_i\left|\varphi_{in}^{2-1/p_1\cdots p_{n-1}}\right|, \end{aligned} \tag{28}$$

where $\hat{\delta}_i$ is the adaptive disturbance compensator and κ_i is a positive gain parameter. Based on Theorem 1 and Assumption 1, we provide the following theorem where input noises are added.

Theorem 2: *If Equations (3) and (5) are updated with (27) and (28), respectively, and other parts are kept unchanged in Theorem 1, then the multi-agent system (3) with (27) can asymptotically achieve bipartite output consensus. Moreover, $x_{i2}, x_{i3}, \ldots, x_{in}$, $\forall i \in \mathcal{V}$ are bounded and will approach zero.*

Proof: All the proof steps are similar with the steps in Section 3 except the final step from (19). For simplicity, we restrict our attention to the following different steps.

$$V_n = V_{n-1} + \sum_{i=1}^{N} S_{in} + \sum_{i=1}^{N} \frac{1}{2\kappa_i}\tilde{\delta}_i^2, \tag{29}$$

where $\tilde{\delta}_i = \bar{\delta}_i - \hat{\delta}_i$ and $\kappa_i > 0$. Then,

$$\begin{aligned} \dot{V}_n &= \dot{V}_{n-1} + \sum_{i=1}^{N} \frac{\partial S_{in}}{\partial x_{in}}\left(u_i^{p_n} + \delta_i\right) + \sum_{i=1}^{N}\sum_{l=1}^{n-1} \frac{\partial S_{in}}{\partial x_{il}}\dot{x}_{il} \\ &\quad + \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i} \frac{\partial S_{in}}{\partial x_{j1}}\dot{x}_{j1} - \sum_{i=1}^{N}\frac{1}{\kappa_i}\tilde{\delta}_i\dot{\hat{\delta}}_i \\ &\le -\tilde{b}_{n-1,1}\sum_{i=1}^{N}\varphi_{i1}^2 - \tilde{b}_{n-1,2}\sum_{i=1}^{N}\varphi_{i2}^2 - \cdots - \tilde{b}_{n-1,n-1}\sum_{i=1}^{N}\varphi_{i,n-1}^2 \\ &\quad + \tilde{b}_{n-1,n}\sum_{i=1}^{N}\varphi_{in}^2 + \tilde{b}'_{n1}\sum_{i=1}^{N}\varphi_{i1}^2 + \tilde{b}'_{n2}\sum_{i=1}^{N}\varphi_{i2}^2 + \cdots \\ &\quad + \tilde{b}'_{nn}\sum_{i=1}^{N}\varphi_{in}^2 - k_n^{1/p_1\cdots p_{n-1}}\sum_{i=1}^{N}\varphi_{in}^2 + \sum_{i=1}^{N}\varphi_{in}^{2-1/p_1\cdots p_{n-1}} \end{aligned}$$

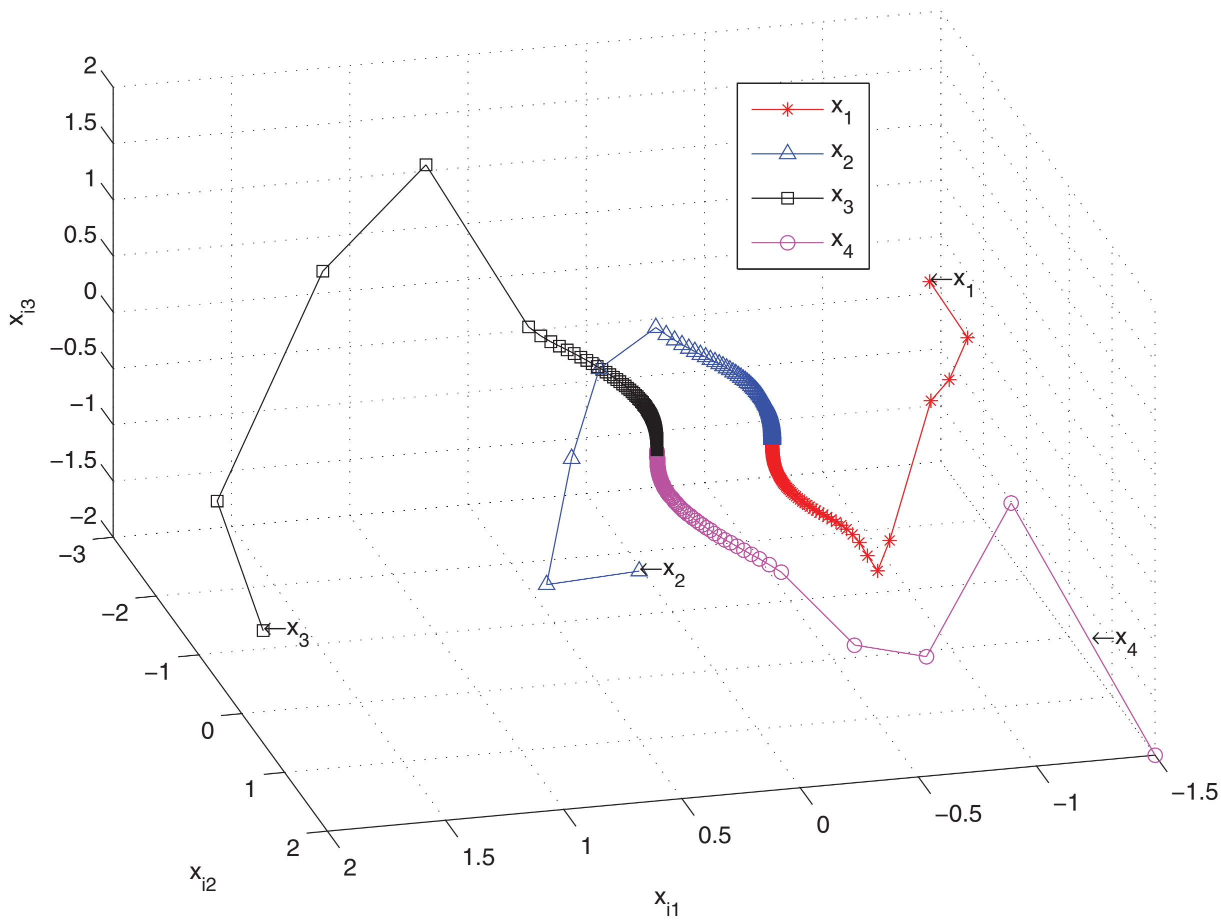

Figure 6. Trajectories of x_i, $i = 1, 2, 3, 4$ in 3D space.

$$\times\left(\delta_i - \operatorname{sgn}\left(\varphi_{in}^{2-1/p_1\cdots p_{n-1}}\right)\hat{\delta}_i\right) - \sum_{i=1}^{N}\frac{1}{\kappa_i}\tilde{\delta}_i\dot{\hat{\delta}}_i. \tag{30}$$

Note that

$$\varphi_{in}^{2-1/p_1\cdots p_{n-1}}\left(\delta_i - \operatorname{sgn}\left(\varphi_{in}^{2-1/p_1\cdots p_{n-1}}\right)\hat{\delta}_i\right)$$
$$\leq \left|\varphi_{in}^{2-1/p_1\cdots p_{n-1}}\right|\bar{\delta}_i - \left|\varphi_{in}^{2-1/p_1\cdots p_{n-1}}\right|\hat{\delta}_i = \left|\varphi_{in}^{2-1/p_1\cdots p_{n-1}}\right|\tilde{\delta}_i.$$

Therefore,

$$\dot{V}_n = \dot{V}_{n-1} + \sum_{i=1}^{N}\frac{\partial S_{in}}{\partial x_{in}}\left(u_i^{p_n} + \delta_i\right) + \sum_{i=1}^{N}\sum_{l=1}^{n-1}\frac{\partial S_{in}}{\partial x_{il}}\dot{x}_{il}$$
$$+ \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\frac{\partial S_{in}}{\partial x_{j1}}\dot{x}_{j1} - \sum_{i=1}^{N}\frac{1}{\kappa_i}\tilde{\delta}_i\dot{\hat{\delta}}_i$$
$$\leq -\tilde{b}_{n-1,1}\sum_{i=1}^{N}\varphi_{i1}^2 - \tilde{b}_{n-1,2}\sum_{i=1}^{N}\varphi_{i2}^2 - \cdots - \tilde{b}_{n-1,n-1}$$
$$\times\sum_{i=1}^{N}\varphi_{i,n-1}^2 + \tilde{b}_{n-1,n}\sum_{i=1}^{N}\varphi_{in}^2 + \tilde{b}'_{n1}\sum_{i=1}^{N}\varphi_{i1}^2$$

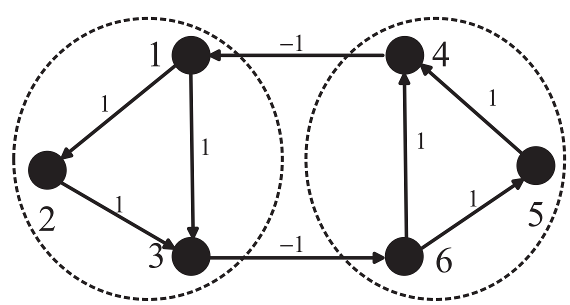

Figure 7. Communication topology of six agents.

$$+ \tilde{b}'_{n2}\sum_{i=1}^{N}\varphi_{i2}^2 + \cdots + \tilde{b}'_{nn}\sum_{i=1}^{N}\varphi_{in}^2 - k_n^{1/p_1\cdots p_{n-1}}\sum_{i=1}^{N}\varphi_{in}^2$$
$$+ \sum_{i=1}^{N}\left(\left|\varphi_{in}^{2-1/p_1\cdots p_{n-1}}\right| - \frac{1}{\kappa_i}\dot{\hat{\delta}}_i\right)\tilde{\delta}_i. \tag{31}$$

In order to remove the last item in (31), update $\hat{\delta}_i$ with

$$\dot{\hat{\delta}}_i = \kappa_i|\varphi_{in}^{2-1/p_1\cdots p_{n-1}}|. \tag{32}$$

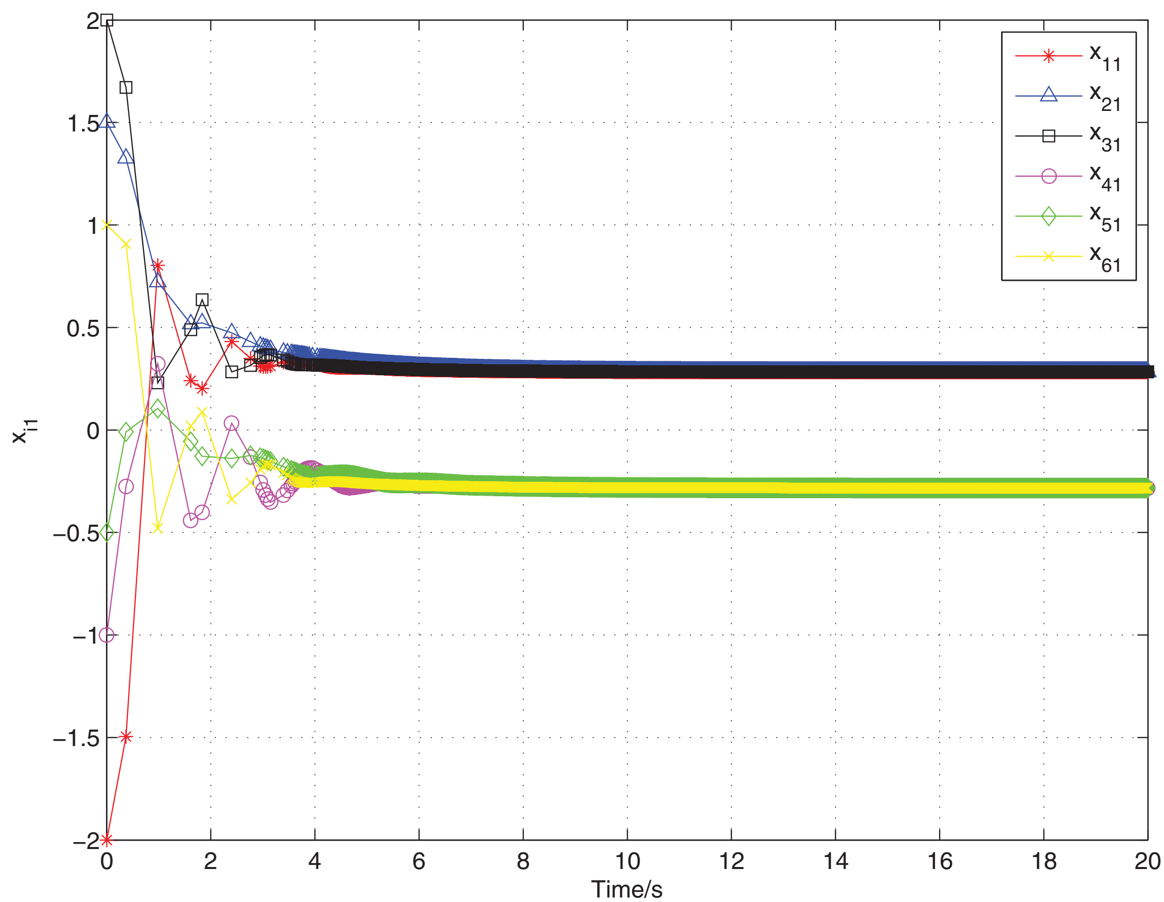

Figure 8. Output trajectories $y_i = x_{i1}$, $i = 1, 2, \ldots, 6$, with bipartite output consensus and input noises.

Then, $\dot{V}_n$ can be simplified to the following form

$$\dot{V}_n \le -\tilde{b}_{n1}\sum_{i=1}^{N}\varphi_{i1}^2 - \tilde{b}_{n2}\sum_{i=1}^{N}\varphi_{i2}^2 - \cdots - \tilde{b}_{n,n-1}\sum_{i=1}^{N}\varphi_{i,n-1}^2 - \tilde{b}_{nn}\sum_{i=1}^{N}\varphi_{in}^2 \le 0, \tag{33}$$

where $\tilde{b}_{n1}, \tilde{b}_{n2}, \ldots, \tilde{b}_{nn}$ are all positive constants. Go back to (21) and its corresponding steps (22)–(25) in the proof of Theorem 1, we can get the same conclusion that bipartite output consensus can be reached. Furthermore, $x_{i2}, x_{i3}, \ldots, x_{in}$ are bounded and all approach zero when $t \to \infty$. □

Remark 3: κ_i is aimed at adapting the amplitude of unknown disturbances δ_i. If κ_i is too large, then a small change in φ_{in} can cause a big surge in u_i, which is too sensitive. In contrary, if κ_i is too small, there will hardly be a response to the input noise δ_i. Therefore, the constant gain κ_i can affect the performance of adaptive noise compensator $\hat{\delta}_i$. Furthermore, time-varying $\kappa_i(t)$ can be considered in future works.

5. Implementations and performance analysis

We provide two examples to demonstrate the validity of the distributed control protocols established in this paper.

Example 1 (Signed digraph without input noises)**:** The dynamics of the multi-agent system are given as follows:

$$\begin{aligned} \dot{x}_{i1} &= x_{i2} \\ \dot{x}_{i2} &= x_{i3}^3 \\ \dot{x}_{i3} &= u_i^3, \quad i = 1, 2, 3, 4. \end{aligned} \tag{34}$$

There are four agents in this network and the topology is shown in Figure 2 . The graph is strongly connected, digon sign-symmetric and structurally balanced. Agents 1 and 2 are in one group, while agents 3 and 4 are in the opposite group. Let $k_1 = 0.5$, $k_2 = 10$, $k_3 = 100$ and the initial values of the four agents are

$$\begin{aligned} (x_{11}(0), x_{12}(0), x_{13}(0))^{\mathrm{T}} &= (-1, -0.5, 1)^{\mathrm{T}} \\ (x_{21}(0), x_{22}(0), x_{23}(0))^{\mathrm{T}} &= (0.5, 1, -0.5)^{\mathrm{T}} \\ (x_{31}(0), x_{32}(0), x_{33}(0))^{\mathrm{T}} &= (2, 0.5, -1)^{\mathrm{T}} \\ (x_{41}(0), x_{42}(0), x_{43}(0))^{\mathrm{T}} &= (-1.5, 2, -2)^{\mathrm{T}}. \end{aligned}$$

It is illustrated in Figure 3 that bipartite output consensus can be reached. In Figures 4 and 5 , we can see that x_{i2} and x_{i3} approach zero, which is in concert with Theorem 1. In Figure 6, we show the trajectories of x_i in three-dimensional (3D) space. Clearly, the four agents, from their initial positions, can achieve

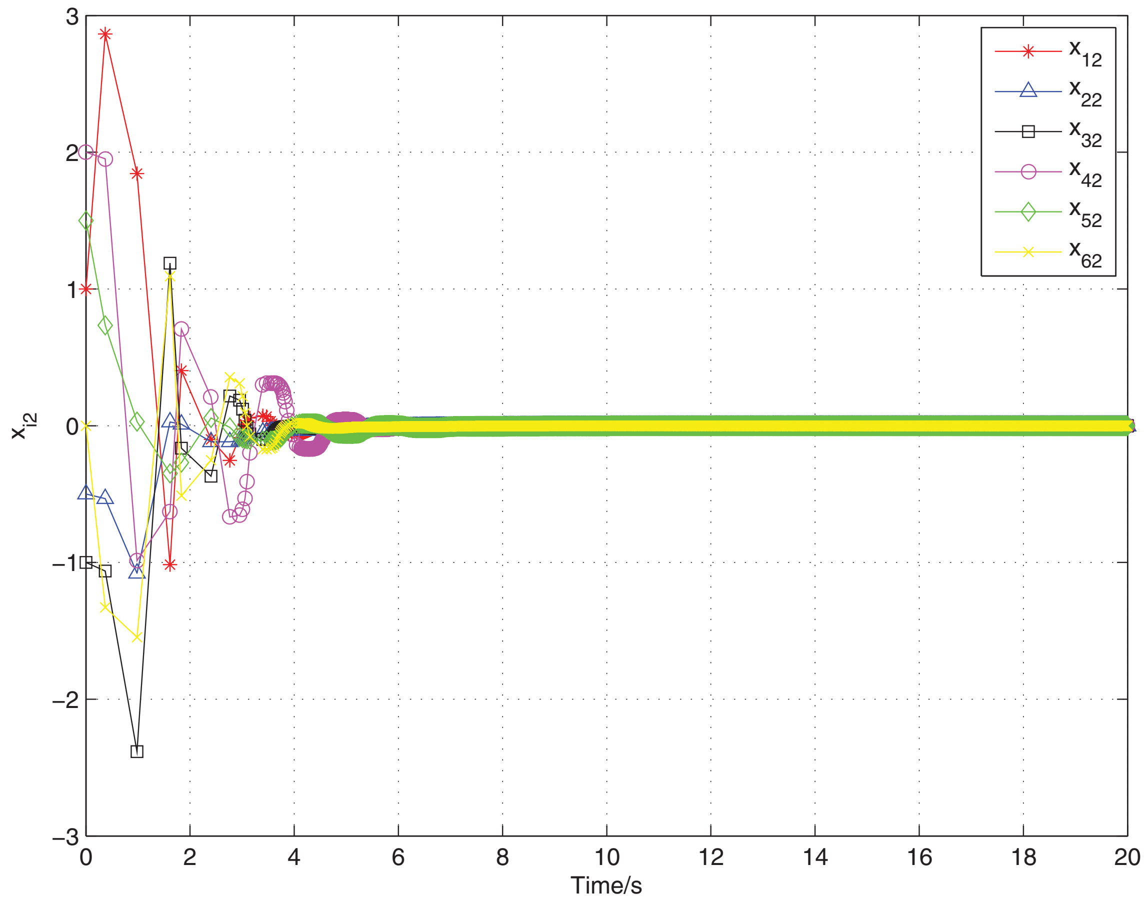

Figure 9. Trajectories of x_{i2}, $i = 1, 2, \ldots, 6$, with input noises.

bipartite output consensus by virtue of only output information, which in turn demonstrates the effectiveness of our distributed control laws (5).

Example 2 (Signed digraph with input noises)**:** In this example, the dynamics of the multi-agent system are given as follows:

$$\begin{aligned} \dot{x}_{i1} &= x_{i2} \\ \dot{x}_{i2} &= x_{i3}^3 \\ \dot{x}_{i3} &= u_i^3 + \delta_i, \ \ i = 1, 2, \ldots, 6, \end{aligned} \tag{35}$$

and

$$\Gamma = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \Phi = \begin{bmatrix} 0.1 & -0.1 & 0 & 0.3 & 0.2 & -0.2 \\ 0.2 & 0.5 & 0.05 & -0.05 & 0.1 & 0.05 \end{bmatrix}.$$

The eigenvalues of matrix Γ are $+i$ and $-i$, which are all on the imaginary axis. Thus, according to Equation (26), we can obtain the bounded oscillating signals δ_i, $i = 1, 2, \ldots, 6$, which are the input noises.

There are six agents and the topology is shown in Figure 7 . The graph is also strongly connected, digon-sign symmetric and structurally balanced. Agents 1, 2 and 3 are in one group, while agents 4, 5 and 6 are in the opposite group. Let $k_1 = 0.5$, $k_2 = 10$, $k_3 = 20$, $\kappa_i = 0.01$, $i = 1, 2, \ldots, 6$, and the initial values of the six agents are

$$\begin{aligned} (x_{11}(0), x_{12}(0), x_{13}(0))^{\mathsf{T}} &= (-2, 1, -0.5)^{\mathsf{T}} \\ (x_{21}(0), x_{22}(0), x_{23}(0))^{\mathsf{T}} &= (1.5, -0.5, 1)^{\mathsf{T}} \\ (x_{31}(0), x_{32}(0), x_{33}(0))^{\mathsf{T}} &= (2, -1, 1.5)^{\mathsf{T}} \\ (x_{41}(0), x_{42}(0), x_{43}(0))^{\mathsf{T}} &= (-1, 2, -1)^{\mathsf{T}} \\ (x_{51}(0), x_{52}(0), x_{53}(0))^{\mathsf{T}} &= (-0.5, 1.5, 0)^{\mathsf{T}} \\ (x_{61}(0), x_{62}(0), x_{63}(0))^{\mathsf{T}} &= (1, 0, 0.5)^{\mathsf{T}}. \end{aligned}$$

It is illustrated in Figure 8 that bipartite output consensus can be achieved in the presence of input noises. In Figures 9 and 10 , we can see that x_{i2} and x_{i3} are bounded and approach zero, which is in accordance with Theorem 2 and in turn verifies the validity of our distributed control protocols (28).

Remark 4: In Example 2, with several oscillations, the third dimension of each agent finally approaches zero. Thus, the developed distributed control protocol (28) can deal with the nonlinearity of high-order power integrator based merely on output information. Furthermore, due to the high nonlinearity of the third dimension in the multi-agent system, when bipartite output consensus has been achieved, x_{i3} is still varying and will vary for a period of time as shown in Figure 10.

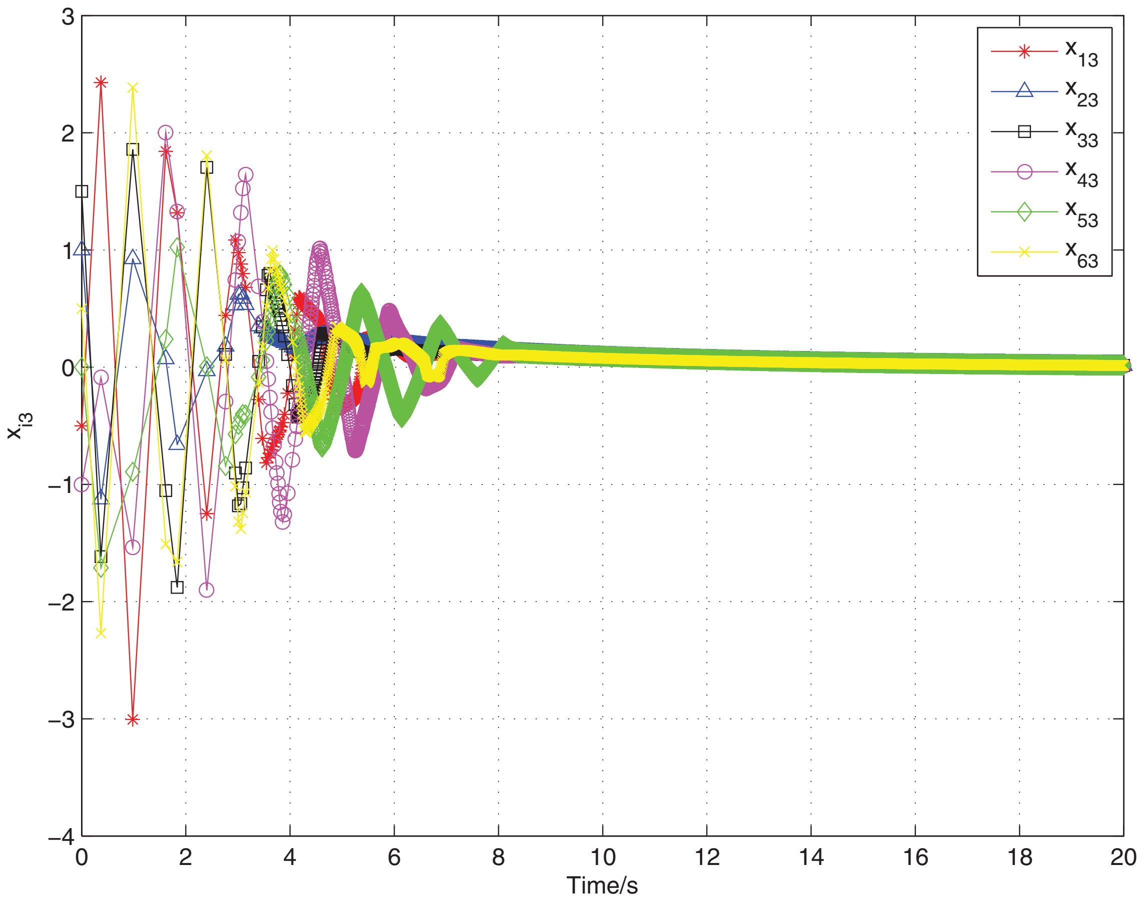

Figure 10. Trajectories of x_{i3}, $i = 1, 2, \dots, 6$, with input noises.

6. Conclusions

In this paper, we study bipartite output consensus in networked multi-agent systems of high-order power integrators with input noises and signed digraph. An adaptive disturbance compensator and the technique of adding power integrator are introduced to deal with the input noises and the nonlinearity of the multi-agent systems, respectively. In addition, the distributed controllers are divided into three parts: the output information of each agent and its neighbours, the state feedback within its internal system and the input adaptive noise compensator. By using our designed distributed control protocol, bipartite output consensus can be achieved, which is one of the ramifications in consensus problems. Note that in physical implementations, the communication intensity cannot maintain constant and the capacity of signal channels is limited. Thus, our future work will concentrate on switching topologies, time delays and packet dropouts over the networked multi-agent systems of high-order power integrators.

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Appendix 1. Frequently used important lemmas

We give three important lemmas frequently used throughout this paper.

Lemma 1(cf. Qian & Lin, 2001): *Assume x, y, m, n, α, β are all positive real numbers. Then, the following inequality holds*

$$\alpha x^m y^n \le \beta x^{m+n} + \frac{n}{m+n}\left(\frac{m+n}{n}\right)^{-\frac{m}{n}} \alpha^{\frac{m+n}{n}} \beta^{-\frac{m}{n}} y^{m+n}. \tag{A1}$$

If m, n are odd integers, x, y can be real numbers.

Lemma 2 (cf. Qian & Lin, 2001): *With $x \in \mathbb{R}$, $y \in \mathbb{R}$ and $p \ge 1$ an integer, the following two inequalities hold*

$$\left|x+y\right|^p \le 2^{p-1}\left|x^p + y^p\right|, \tag{A2}$$

$$\left(|x| + |y|\right)^{\frac{1}{p}} \le |x|^{\frac{1}{p}} + |y|^{\frac{1}{p}} \le 2^{\frac{p-1}{p}}\left(|x| + |y|\right)^{\frac{1}{p}}. \tag{A3}$$

If $p \ge 1$ is an odd integer, then

$$\left|x-y\right|^p \le 2^{p-1}\left|x^p - y^p\right|. \tag{A4}$$

Lemma 3 (cf. Qian & Lin, 2001): *Suppose that $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$ are both nonnegative real numbers and $p \ge 1$, $q \ge 1$ are integers, then*

$$\alpha^{p-1}\beta^q \le \alpha^p + \beta^{pq}. \tag{A5}$$

Appendix 2. Useful propositions

The following propositions are the foundation of the proofs for the main results in this paper.

Proposition 1: *Denote $S_{im} \ge 0$ ($i = 1, 2, \ldots, N$, $m = 2, 3, \ldots, n$) a scalar function, where*

$$S_{im} = \int_{x_{im}^*}^{x_{im}} \left(r^{p_1\cdots p_{m-1}} - x_{im}^{*p_1\cdots p_{m-1}}\right)^{2-1/p_1\cdots p_{m-1}} \mathrm{d}r. \tag{B1}$$

Then, the partial derivatives of S_{im} are

$$\begin{aligned}
\frac{\partial S_{im}}{\partial x_{im}} &= \varphi_{im}^{2-1/p_1\cdots p_{m-1}},\\
\frac{\partial S_{im}}{\partial x_{il}} &= -\left(2 - \frac{1}{p_1\cdots p_{m-1}}\right)\frac{\partial x_{im}^{*p_1\cdots p_{m-1}}}{\partial x_{il}}\\
&\quad\times \int_{x_{im}^*}^{x_{im}} \left(r^{p_1\cdots p_{m-1}} - x_{im}^{*p_1\cdots p_{m-1}}\right)^{1-1/p_1\cdots p_{m-1}} \mathrm{d}r,\\
\frac{\partial S_{im}}{\partial x_{j1}} &= -\left(2 - \frac{1}{p_1\cdots p_{m-1}}\right)\frac{\partial x_{im}^{*p_1\cdots p_{m-1}}}{\partial x_{j1}}\\
&\quad\times \int_{x_{im}^*}^{x_{im}} \left(r^{p_1\cdots p_{m-1}} - x_{im}^{*p_1\cdots p_{m-1}}\right)^{1-1/p_1\cdots p_{m-1}} \mathrm{d}r,
\end{aligned}$$

where $l = 1, 2, \ldots, m-1$, $j \in \mathcal{N}_i$ and $p_1, p_2, \ldots, p_{m-1}$ are odd integers.

Proof: According to the definition of S_{im}, when $x_{im} \ge x_{im}^*$, we have $r \ge x_{im}^*$. By virtue of Lemma 2,

$$\begin{aligned}
S_{im} &\ge \int_{x_{im}^*}^{x_{im}} \left[2^{1-p_1\cdots p_{m-1}}\left(r - x_{im}^*\right)^{p_1\cdots p_{m-1}}\right]^{2-1/p_1\cdots p_{m-1}} \mathrm{d}r\\
&= c_1\left(x_{im} - x_{im}^*\right)^{2p_1\cdots p_{m-1}} \ge 0,
\end{aligned}$$

where $c_1 > 0$. Likewise, when $x_{im} < x_{im}^*$, we can infer that

$$\begin{aligned}
S_{im} &= \int_{x_{im}^*}^{x_{im}} \left(x_{im}^{*p_1\cdots p_{m-1}} - r^{p_1\cdots p_{m-1}}\right)^{2-1/p_1\cdots p_{m-1}} \mathrm{d}(-r)\\
&\ge \int_{x_{im}^*}^{x_{im}} \left[2^{1-p_1\cdots p_{m-1}}\left(x_{im}^* - r\right)^{p_1\cdots p_{m-1}}\right]^{2-1/p_1\cdots p_{m-1}} \mathrm{d}(-r)\\
&= c_2\left(x_{im}^* - x_{im}\right)^{2p_1\cdots p_{m-1}} > 0,
\end{aligned}$$

where $c_2 > 0$. Thus, $S_{im} \ge 0$.

In the sequel, we will obtain the partial derivatives of S_{im}, $\forall\, i = 1, 2, \ldots, N$, $m = 2, 3, \ldots, n$. It is straightforward to derive that

$$\frac{\partial S_{im}}{\partial x_{im}} = \left(x_{im}^{p_1\cdots p_{m-1}} - x_{im}^{*p_1\cdots p_{m-1}}\right)^{2-1/p_1\cdots p_{m-1}} = \varphi_{im}^{2-1/p_1\cdots p_{m-1}}. \tag{B2}$$

Denote

$$\begin{aligned}
\bar{x}_{\mathcal{N}_{i1}} &= (x_{j_11}, x_{j_21}, \ldots, x_{j_{|\mathcal{N}_{i1}|}1}),\ k = 1, 2, \ldots, |\mathcal{N}_{i1}|,\\
&\qquad j_k \in \mathcal{N}_{i1},\\
\Delta_{i,m-1}^{(l)} &= \left(x_{i1}, \ldots, x_{i,l-1}, x_{il} + \Delta, x_{i,l+1}, \ldots, x_{i,m-1}, \bar{x}_{\mathcal{N}_{i1}}\right),\\
&\qquad \forall 1 \le l \le m-1\\
\tilde{x}_{i,m-1} &= \left(x_{i1}, \ldots, x_{i,l-1}, x_{il}, x_{i,l+1}, \ldots, x_{i,m-1}, \bar{x}_{\mathcal{N}_{i1}}\right),\\
&\qquad \forall 1 \le l \le m-1\\
p_M &= p_1\cdots p_{m-1}.
\end{aligned}$$

Then,

$$\begin{aligned}
\frac{\partial S_{im}}{\partial x_{il}} &= \lim_{\Delta\to 0}\frac{S_{im}\left(\Delta^{(l)}_{i,m-1},x_{im}\right)-S_{im}\left(\tilde{x}_{i,m-1},x_{im}\right)}{\Delta}\\
&= \lim_{\Delta\to 0}\frac{\int_{x^*_{im}\left(\Delta^{(l)}_{i,m-1}\right)}^{x_{im}}(r^{p_M}-x^{*p_M}_{im}(\Delta^{(l)}_{i,m-1}))^{2-1/p_M}\mathrm{d}r-\int_{x^*_{im}(\tilde{x}_{i,m-1})}^{x_{im}}(r^{p_M}-x^{*p_M}_{im}(\tilde{x}_{i,m-1}))^{2-1/p_M}\mathrm{d}r}{\Delta}\\
&= \lim_{\Delta\to 0}\frac{\int_{x^*_{im}\left(\Delta^{(l)}_{i,m-1}\right)}^{x_{im}}(r^{p_M}-x^{*p_M}_{im}(\Delta^{(l)}_{i,m-1}))^{2-1/p_M}\mathrm{d}r-\int_{x^*_{im}(\tilde{x}_{i,m-1})}^{x_{im}}(r^{p_M}-x^{*p_M}_{im}(\Delta^{(l)}_{i,m-1}))^{2-1/p_M}\mathrm{d}r}{\Delta}\\
&\quad+\lim_{\Delta\to 0}\frac{\int_{x^*_{im}(\tilde{x}_{i,m-1})}^{x_{im}}(r^{p_M}-x^{*p_M}_{im}(\Delta^{(l)}_{i,m-1}))^{2-1/p_M}\mathrm{d}r-\int_{x^*_{im}(\tilde{x}_{i,m-1})}^{x_{im}}(r^{p_M}-x^{*p_M}_{im}(\tilde{x}_{i,m-1}))^{2-1/p_M}\mathrm{d}r}{\Delta}\\
&= \lim_{\Delta\to 0}\frac{\int_{x^*_{im}(\Delta^{(l)}_{i,m-1})}^{x^*_{im}(\tilde{x}_{i,m-1})}(r^{p_M}-x^{*p_M}_{im}(\Delta^{(l)}_{i,m-1}))^{2-1/p_M}\mathrm{d}r}{\Delta}-\left(2-\frac{1}{p_M}\right)\frac{\partial x^{*p_M}_{im}}{\partial x_{il}}\int_{x^*_{im}}^{x_{im}}(r^{p_M}-x^{*p_M}_{im})^{1-1/p_M}\mathrm{d}r. \qquad \text{(B3)}
\end{aligned}$$

Then, we consider the limit shown in (B3). Note that

$$\begin{aligned}
&\left|\frac{\int_{x^*_{im}(\Delta^{(l)}_{i,m-1})}^{x^*_{im}(\tilde{x}_{i,m-1})}(r^{p_M}-x^{*p_M}_{im}(\Delta^{(l)}_{i,m-1}))^{2-1/p_M}\mathrm{d}r}{\Delta}\right|\\
&\le\frac{\left|x^{*p_M}_{im}(\tilde{x}_{i,m-1})-x^{*p_M}_{im}(\Delta^{(l)}_{i,m-1})\right|}{|\Delta|}\\
&\times\left|x^*_{im}(\tilde{x}_{i,m-1})-x^*_{im}(\Delta^{(l)}_{i,m-1})\right|\left|x^{*p_M}_{im}(\tilde{x}_{i,m-1})-x^{*p_M}_{im}(\Delta^{(l)}_{i,m-1})\right|^{1-1/p_M},
\end{aligned}$$

and $x^{*p_M}_{im}(\Delta^{(l)}_{i,m-1})$ is $\mathcal{C}^\infty$. Therefore,

$$\lim_{\Delta\to 0}\frac{\int_{x^*_{im}(\Delta^{(l)}_{i,m-1})}^{x^*_{im}(\tilde{x}_{i,m-1})}(r^{p_M}-x^{*p_M}_{im}(\Delta^{(l)}_{i,m-1}))^{2-1/p_M}\mathrm{d}r}{\Delta}=0.$$

Thus,

$$\begin{aligned}
\frac{\partial S_{im}}{\partial x_{il}} &= -\left(2-\frac{1}{p_1\cdots p_{m-1}}\right)\frac{\partial x^{*p_1\cdots p_{m-1}}_{im}}{\partial x_{il}}\\
&\quad\times\int_{x^*_{im}}^{x_{im}}\left(r^{p_1\cdots p_{m-1}}-x^{*p_1\cdots p_{m-1}}_{im}\right)^{1-1/p_1\cdots p_{m-1}}\mathrm{d}r,\\
&\quad l=1,2,\ldots,m-1. \qquad \text{(B4)}
\end{aligned}$$

Consequently, we follow similar steps to obtain

$$\begin{aligned}
\frac{\partial S_{im}}{\partial x_{j1}} &= -\left(2-\frac{1}{p_1\cdots p_{m-1}}\right)\frac{\partial x^{*p_1\cdots p_{m-1}}_{im}}{\partial x_{j1}}\\
&\quad\times\int_{x^*_{im}}^{x_{im}}\left(r^{p_1\cdots p_{m-1}}-x^{*p_1\cdots p_{m-1}}_{im}\right)^{1-1/p_1\cdots p_{m-1}}\mathrm{d}r,\quad j\in\mathcal{N}_i. \qquad \text{(B5)}
\end{aligned}$$

□

Proposition 2: *If $S_{im}\ge 0\,(i=1,\ldots,N,\ m=2,\ldots,n)$, we can obtain*

$$\left|\frac{\partial S_{im}}{\partial x_{j1}}\right|\le 4|\varphi_{im}|\left|\frac{\partial x^{*p_1\cdots p_{m-1}}_{im}}{\partial x_{j1}}\right|,\qquad j\in\mathcal{N}_i, \qquad \text{(B6)}$$

$$\left|\frac{\partial S_{im}}{\partial x_{il}}\right|\le 4|\varphi_{im}|\left|\frac{\partial x^{*p_1\cdots p_{m-1}}_{im}}{\partial x_{il}}\right|,\qquad l=1,2,\ldots,m-1. \qquad \text{(B7)}$$

Proof: First, by using the basic properties of inequalities,

$$\begin{aligned}
\left|\frac{\partial S_{im}}{\partial x_{j1}}\right| &= \left|-\left(2-\frac{1}{p_1\cdots p_{m-1}}\right)\frac{\partial x^{*p_1\cdots p_{m-1}}_{im}}{\partial x_{j1}}\right.\\
&\quad\left.\times\int_{x^*_{im}}^{x_{im}}\left(r^{p_1\cdots p_{m-1}}-x^{*p_1\cdots p_{m-1}}_{im}\right)^{1-1/p_1\cdots p_{m-1}}\mathrm{d}r\right|\\
&\le\left(2-\frac{1}{p_1\cdots p_{m-1}}\right)\left|\frac{\partial x^{*p_1\cdots p_{m-1}}_{im}}{\partial x_{j1}}\right||\varphi_{im}|\\
&\quad\times\left|x_{im}-x^*_{im}\right||\varphi_{im}|^{-1/p_1\cdots p_{m-1}}\\
&\le\left(2-\frac{1}{p_1\cdots p_{m-1}}\right)\left|\frac{\partial x^{*p_1\cdots p_{m-1}}_{im}}{\partial x_{j1}}\right||\varphi_{im}|\left|x_{im}-x^*_{im}\right|\\
&\quad\times 2^{\frac{p_1\cdots p_{m-1}-1}{p_1\cdots p_{m-1}}}\frac{1}{\left|x_{im}-x^*_{im}\right|}\\
&\le 4|\varphi_{im}|\left|\frac{\partial x^{*p_1\cdots p_{m-1}}_{im}}{\partial x_{j1}}\right|,\qquad j\in\mathcal{N}_i. \qquad \text{(B8)}
\end{aligned}$$

Following similar steps,

$$\left|\frac{\partial S_{im}}{\partial x_{il}}\right|\le 4|\varphi_{im}|\left|\frac{\partial x^{*p_1\cdots p_{m-1}}_{im}}{\partial x_{il}}\right|,\qquad l=1,2,\ldots,m-1,$$

and this completes the proof. □

Proposition 3: *The following inequalities hold:*

$$\left|\frac{\partial x^{*p_1\cdots p_{m-1}}_{im}}{\partial x_{j1}}\dot{x}_{j1}\right|\le\eta^i_{mj}(|\varphi_{j1}|+|\varphi_{j2}|),\qquad j\in\mathcal{N}_i, \qquad \text{(B9)}$$

$$\begin{aligned}
&\left|\frac{\partial x^{*p_1\cdots p_{m-1}}_{im}}{\partial x_{il}}\dot{x}_{il}\right|\le\gamma^i_{ml}(|\varphi_{i1}|+|\varphi_{i2}|+\cdots+|\varphi_{im}|),\\
&\qquad l=1,2,\ldots,m-1, \qquad \text{(B10)}
\end{aligned}$$

where $\eta^i_{mj}\ge 0$, $\gamma^i_{ml}\ge 0$, $\forall i,j\in\mathcal{V}$, $m=2,3,\ldots,n$.

Proof: First, we demonstrate (B9) of Proposition 3. When $m = 2$,

$$\left|\frac{\partial x_{i2}^{*p_1}}{\partial x_{j1}}\dot{x}_{j1}\right| = \left|\frac{\partial x_{i2}^{*p_1}}{\partial x_{j1}}\right| |\varphi_{j2} - k_1\varphi_{j1}| = k_1(|\mathcal{A}_{ij}| + |\mathcal{A}_{ji}|)\, |\varphi_{j2} - k_1\varphi_{j1}| \le \eta_{2j}^i(|\varphi_{j1}| + |\varphi_{j2}|).$$

Consequently, when $2 < m \le n$,

$$\begin{aligned}\left|\frac{\partial x_{im}^{*p_1\cdots p_{m-1}}}{\partial x_{j1}}\dot{x}_{j1}\right| &= \left|\frac{\partial(-k_{m-1}\varphi_{i,m-1})}{\partial x_{j1}}\right| |\varphi_{j2} - k_1\varphi_{j1}| \\ &= k_1k_2\ldots k_{m-1}\,\big||\mathcal{A}_{ij}| + |\mathcal{A}_{ji}|\big|\, |\varphi_{j2} - k_1\varphi_{j1}| \\ &\le \eta_{mj}^i(|\varphi_{j1}| + |\varphi_{j2}|).\end{aligned}$$

Therefore, we can infer that

$$\left|\frac{\partial x_{im}^{*p_1\cdots p_{m-1}}}{\partial x_{j1}}\dot{x}_{j1}\right| \le \eta_{mj}^i(|\varphi_{j1}| + |\varphi_{j2}|), \quad j \in \mathcal{N}_i.$$

In what follows, we restrict our attention to demonstrating (B10) with induction.

Step 1 (Initial result)**:** When $m = 2$,

$$\left|\frac{\partial x_{i2}^{*p_1}}{\partial x_{i1}}\dot{x}_{i1}\right| = |-k_1(\mathcal{C}_{r,ii} + \mathcal{C}_{c,ii})|\, |x_{i2}^{p_1}| = k_1(\mathcal{C}_{r,ii} + \mathcal{C}_{c,ii})\, |\varphi_{i2} - k_1\varphi_{i1}| \le \gamma_{ml}^i(|\varphi_{i1}| + |\varphi_{i2}|),$$

which satisfies (B10).

Step 2 (Inductive assumption)**:** $\forall m = 3, 4, \ldots, n-1$, assume

$$\left|\frac{\partial x_{i,m-1}^{*p_1\cdots p_{m-2}}}{\partial x_{il}}\dot{x}_{il}\right| \le \gamma_{m-1,l}^i(|\varphi_{i1}| + |\varphi_{i2}| + \cdots + |\varphi_{i,m-1}|), \quad l = 1, 2, \ldots, m-1.$$

Step 3 (Validation)**:** For $m = n$, we consider two cases, i.e. $l = 1, 2, \ldots, m-2$ and $l = m-1$, respectively. First, when $l = 1, 2, \ldots, m-2$,

$$\begin{aligned}\left|\frac{\partial x_{im}^{*p_1\cdots p_{m-1}}}{\partial x_{il}}\dot{x}_{il}\right| &= \left|\frac{\partial(-k_{m-1}\varphi_{i,m-1})}{\partial x_{il}}\dot{x}_{il}\right| \\ &= k_{m-1}\left|\frac{\partial(x_{i,m-1}^{p_1\cdots p_{m-2}} - x_{i,m-1}^{*p_1\cdots p_{m-2}})}{\partial x_{il}}\dot{x}_{il}\right| \\ &\le k_{m-1}\gamma_{m-1,l}^i(|\varphi_{i1}| + |\varphi_{i2}| + \cdots + |\varphi_{i,m-1}|).\end{aligned}$$

Subsequently, when $l = m-1$ and with the aid of Lemma 3,

$$\begin{aligned}&\left|\frac{\partial x_{im}^{*p_1\cdots p_{m-1}}}{\partial x_{i,m-1}}\dot{x}_{i,m-1}\right| \\ &= k_{m-1}\left|\frac{\partial(x_{i,m-1}^{p_1\cdots p_{m-2}} - x_{i,m-1}^{*p_1\cdots p_{m-2}})}{\partial x_{i,m-1}}\dot{x}_{i,m-1}\right| \\ &= k_{m-1}p_1\cdots p_{m-2}\left|x_{i,m-1}^{p_1\cdots p_{m-2}-1}\dot{x}_{i,m-1}\right| \\ &= k_{m-1}p_1\cdots p_{m-2}\left|x_{i,m-1}^{p_1\cdots p_{m-2}-1}x_{im}^{p_{m-1}}\right| \\ &\le k_{m-1}p_1\cdots p_{m-2}\left(\left|x_{im}^{p_1\cdots p_{m-1}}\right| + \left|x_{i,m-1}^{p_1\cdots p_{m-2}}\right|\right) \text{ (by Lemma 3)} \\ &\le \gamma_{m,m-1}^i(|\varphi_{i1}| + |\varphi_{i2}| + \cdots + |\varphi_{im}|).\end{aligned}$$

In regard to the above discussions, for $m = 2, 3, \ldots, n$, the following inequalities

$$\left|\frac{\partial x_{im}^{*p_1\cdots p_{m-1}}}{\partial x_{il}}\dot{x}_{il}\right| \le \gamma_{ml}^i(|\varphi_{i1}| + |\varphi_{i2}| + \cdots + |\varphi_{im}|), \quad l = 1, 2, \ldots, m-1$$

hold, and this completes the proof. □