The Synchronization Control of a Uncertainty Chaotic System

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Abstract

This paper is concerned with the robust synchronization control problem of continuous time scalar chaotic signal. It is based on nonlinear state feedback linearization, optimal control and Lyapunov method. Under some certain assumptions, we have obtained a simple controller with good performances. With the controller, chaotic system is global and asymptomatic synchronous. An application for the synchronization of a four-order autonomous chaotic system is presented.

Keywords
nonlinear chaos synchronization control robust

1 Introduction

All actual systems almost have the characteristic of uncertainty. Chaos is an especial response of a nonlinear system. So it is difficult to deal with the controlling problem of chaotic nonlinear system. Obviously it is very significant to investigate uncertainty chaotic system based on these. Since Pecora and Carroll suggested a synchronous strategy [1], people have put emphasis on theories and experiments of a chaotic synchronous system. With the appearance of the OGY chaotic control method [2], people pay more attention to studying the chaotic control. At the same time, synchronization control has became the scientific edge of chaotic study.

This paper presents an output feedback controller, which makes the output of an uncertainty chaotic system evolve to the output of a target system. Eventually, we can achieve a global and asymptomatic synchronous control of a chaotic system.

2 Description of Problem

Consider two nonlinear systems:

$$\begin{cases} \dot{x} = f(x) + \Delta f(x) + [g(x) + \Delta g(x)]u \\ y = h(x) \end{cases} \qquad (1)$$

$$\begin{cases} \dot{\hat{x}} = \hat{f}(\hat{x}) \\ \hat{y} = \hat{h}(\hat{x}) \end{cases} \qquad (2)$$

where $x, \hat{x} \in R^n$ are the states of the two systems, $u \in R$ is the control input of the system (1), $y, \hat{y} \in R$ are the outputs of them, $\Delta f(x), \Delta g(x)$ are the uncertainty factors of the two systems, $f, \hat{f} : R^n \times R \to R^n$, $g : R^n \to R^n$, $h, \hat{h} \in R^n \to R$, $h, \hat{h}$ are the output functions, and $f, \hat{f}, g, h, \hat{h} \in C^\infty$.

Here, the system (1) is a controlled one. The system (2) is a target one. If we want to synchronize them, we must find a property control rule u, which makes the following:

$$\lim_{t \to \infty} \|y - \hat{y}\| = 0$$

where y is the output of the controlled system and $\hat{y}$ is the output of the target system.

In evidence, here we don't make any restriction on f, h and $\hat{f}, \hat{h}$. That is to say, the two systems can be not the same. So it is much more important whether we can obtain such a control law.

3 Design of Output Feedback Controller

Firstly, we make the following hypotheses before designing the controller.

① Uncertainty meets the need of the following match conditions:

$$\Delta f(x) = g(x)\delta_1(x)$$
$$\Delta g(x) = g(x)\delta_2(x)$$

② The normal one of the system (1):

$$\begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) \end{cases} \qquad (3)$$

It has relative degree r [3], $r \leq n$.

Proposition1 If the system (3) has relative degree $r \leq n$, we make $\phi_1(x) = h(x)$,

$\phi_2(x) = L_f h(x), \cdots, \phi_r(x) = L_f^{r-1} h(x)$.

If $r < n$, we can get $n - r$ functions, which are $\phi_{r+1}(x), \cdots, \phi_n(x)$. They make that the Jaccobi matrix of $\Phi(x) = [\phi_1, \phi_2, \cdots, \phi_n]^T$ is non-singular on this point x_0.

Evidently, provided the Jaccobi matrix of Φ is non-singular on the point x_0, we can choose $\phi_{r+1}, \cdots, \phi_n$ at will. Specially, we can choose $\phi_{r+1}, \cdots, \phi_n$ and make $L_g \phi_i(x) = 0, \forall x \in V$ (V is some neighborhood of the point of x_0), $r + 1 \le i \le n$.

Suppose that $z = \Phi(x), \Phi(x)$ expresses the counterchange which satisfies the proposition 1, and $\lambda_i(x)$ satisfies $L_g \lambda_i(x) = 0$, $1 \le i \le n - r$. So

$$\Phi(x) = [h(x), L_f h(x), \cdots, L_f^{r-1} h(x), \lambda_1(x), \cdots, \lambda_{n-r}(x)]^T \quad (4)$$

After a coordinate change, the system (1) has been transformed into the following form:

$$\begin{cases} \dot{z}_i = z_{i+1}, i = 1, \cdots, r-1 \\ \dot{z}_r = a(x) + b(x)[\delta_1(x) + (1 + \delta_2(x))u] \\ \dot{z}_{r+j} = L_f \lambda_j(x), j = 1, \cdots, n-r \\ y = z_1 \end{cases} \quad (5)$$

where $a(x) = L_f^r h(x)$ is the r degree Li derivative of $h(x)$ along f [3], $b(x) = L_g L_f^{r-1} h(x)$.

Go step further, (5) can be turned into the following normal form:

$$\begin{cases} \dot{z}^1 = Az^1 + B\{a(x) + b(x)[\delta_1(x) + (1 + \delta_2(x))u]\} \\ \dot{z}^2 = q(z) \\ y = [1, 0, \cdots, 0]z^1 \end{cases} \quad (6)$$

where

$$z = [z^{1T}, z^{2T}]^T, z^1 = [z_1, \cdots, z_r]^T, z^2 = [z_{r+1}, \cdots, z_n]^T,$$

$$A = \begin{bmatrix} 0 & I_{r-1} \\ 0 & 0 \end{bmatrix}_{r \times r}, B = \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix}_{r \times 1}.$$

At the same time, we make $v = a(x) + b(x)[\delta_1(x) + (1 + \delta_2(x))u]$. v is a new control input. So it appears partly linearity in (6).

$$\begin{cases} \dot{z}^1 = Az^1 + Bv \\ y = [1, 0, \cdots, 0]z^1 \end{cases}$$

③ The reference signal $\hat{y}$ is r degree derivative.

④ For all $x \in R^n$, we can get

$$\frac{1}{2} + \delta_2(x) \ge 0 \quad (7)$$

$$\left\| b(x)\delta_1(x) - a(x)\delta_2(x) - \hat{y}^{(r)} \right\| \le W$$

Meanwhile we define:

$$e = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_r \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_r \end{bmatrix} - \begin{bmatrix} \hat{y} \\ \hat{y}^{(1)} \\ \vdots \\ \hat{y}^{(r-1)} \end{bmatrix}$$

then

$$\dot{e} = \begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \vdots \\ \dot{e}_r \end{bmatrix} = \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \vdots \\ \dot{z}_r \end{bmatrix} - \begin{bmatrix} \hat{y}^{(1)} \\ \hat{y}^{(2)} \\ \vdots \\ \hat{y}^{(r)} \end{bmatrix} \quad (8)$$

So far, the synchronization control problem of (5) has been transformed into the problem that system (8) has a global and asymptotic stable balance point $e = 0$ under a proper controller u. The following is a theorem.

Theorem: For the chaotic system (1) that meets the need of hypotheses ①-④, we can design the following controller:

$$u = \frac{1}{b(x)}[-a(x) - R^{-1}B^T Pe - k_0 \operatorname{sgn}(e^T PB)] \quad (9)$$

where $k_0 \in R^+$, P, R are positive symmetrical matrices, and P, R satisfies the following Riccati equation.

$$A^T P + PA - PBR^{-1}B^T P = -Q \quad (10)$$

where Q is a positive symmetrical matrix. Therefore when $k_0 > 2W$, for all initial conditions $x(0) \in R^n$, $e(0) \in R^r$, we can get

$$\lim_{t \to \infty} e(t) = 0.$$

Identification: Choosing a Lyapunov function

$$V(e) = e^T Pe \quad (11)$$

based on (6), (8), we can get

$$\dot{e} = Ae - B\hat{y}^{(r)} + Bu' \quad (12)$$

where $u' = a(x) + b(x)[\delta_1(x) + (1 + \delta_2(x))u]$. Because the matrix P is a positive symmetrical one, $V(e)$ is a positive function for all $e \in R^n$. We derive (12) for time t along $V(e)$, and substitute (7), (9), (10) and (12) in (11). Then we obtain

$$\begin{aligned} \dot{V}(e) &= \dot{e}^T Pe + e^T P\dot{e} \\ &= (Ae - B\hat{y}^{(r)} + Bu')^T Pe + e^T P(Ae - B\hat{y}^{(r)} + Bu') \\ &= e^T(A^T P + PA)e - 2e^T PB\hat{y}^{(r)} + 2e^T PBu' \end{aligned}$$

$$= e^T(A^TP + PA)e - 2e^TPB\hat{y}^{(r)} + 2e^TPB\{a(x)$$
$$+ b(x)\delta_1(x) + [1+\delta_2(x)]$$
$$[-a(x) - R^{-1}B^TPe \qquad - k_0\,\mathrm{sgn}(e^TPB)]\}$$
$$= e^T(A^TP + PA - 2PBR^{-1}B^TP$$
$$-2\delta_2(x)PBR^{-1}B^TP)e + 2e^TPB\{[b(x)\delta_1(x)$$
$$\le e^T(A^TP + PA - PBR^{-1}B^TP)e + 2W\|e^TPB\|$$
$$-k_0e^TPB\,\mathrm{sgn}(e^TPB)$$

Since $e^TPB\,\mathrm{sgn}(e^TPB) = \|e^TPB\|$,

$$\dot{V}(e) \le -e^TQe - (k_0 - 2W)\|e^TPB\|.$$

So when we choose $k_0 \ge 2W$, $\dot{V}(e)$ must be a negative function. We get a conclusion that the balance point $e = 0$ of (12) is a global and asymptomatic stable one. The theorem has been proved.

Note 1: Under the condition that the system can be controlled, based on the control theory and a given positive symmetrical matrix, we can know that the Riccati matrix equation has a unique positive symmetrical matrix solution.

Note 2: The controller has some optimal control characters because the feedback is optimal under square performance index.

Note 3: The above suppositions seem to be very rigor, strong but they are pretty rational for a continuous time chaotic system and an arbitrarily optional $g(x)$. The reason is that a chaotic system having attractors is a dissipation system and its phase points always evolve in some region.

Note 4: We can change the speed of realizing synchronization by adjusting Q, k_0 under the permissive region of control quantity.

Note 5: Provided a system satisfies above suppositions, $q(z)$ in (6) doesn't affect its result of synchronization.

Note 6: The theorem only can be sure that the equilibrium point of (8) is a global and asymptotic stable one. That is to say that $y(t)$ and its r degree derivative just can respectively synchronize with $\hat{y}(t)$ and its r degree derivative eventually. We couldn't assure that $x(t)$ synchronize with $\hat{x}(t)$ entirely.

4 The Results of Computer Simulation

For checking above some conclusions, we chose a four-order autonomous chaotic circuit containing a nonlinear negative capacitor [4] as our study object. Figure 1 and figure 2 are the circuit model and the character of its nonlinear negative capacitor. The initial conditions of their states are chosen stochastically.

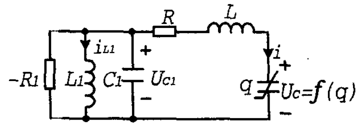

Figure 1 Circuit principle diagram

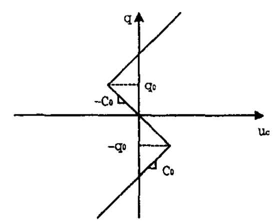

Figure 2 $q - v$ characteristic curve of the nonlinear negative capacitor

Its state equation is as follows,

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -(x_1 + |x_1 - 1| - |x_1 + 1|) - px_2 + p_0x_3 \\ \dot{x}_3 = -p_1x_2 + p_2x_3 - p_3x_4 \\ \dot{x}_4 = p_4x_3 \end{cases}$$

Because the character of the system isn't smooth, we make a commutation in order to make f, g as vector smooth functions. The changed form is:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -0.1275x_1^3 + 0.6523x_1 - px_2 + p_0x_3 \\ \dot{x}_3 = -p_1x_2 + p_2x_3 - p_3x_4 \\ \dot{x}_4 = p_4x_3 \end{cases} \quad (13)$$

From the documentary [4], we can know that the system is in a chaotic state when one of the parameters of the circuit $R = 500\Omega$. Under the condition we choose a target system and a controlled system. The controlled one is the following:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -0.1275x_1^3 + 0.6523x_1 - px_2 + p_0x_3 \\ \dot{x}_3 = -p_1x_2 + p_2x_3 - p_3x_4 + \sin(x_2) + u \\ \dot{x}_4 = p_4x_3 \\ y = x_2 \end{cases} \quad (14)$$

The target one:

$$\begin{cases} \dot{\hat{x}}_1 = \hat{x}_2 \\ \dot{\hat{x}}_2 = -0.1275\hat{x}_1^3 + 0.6523\hat{x}_1 - p\hat{x}_2 + p_0\hat{x}_3 \\ \dot{\hat{x}}_3 = -p_1\hat{x}_2 + p_2\hat{x}_3 - p_3\hat{x}_4 \\ \dot{\hat{x}}_4 = p_4\hat{x}_3 \\ \hat{y} = \hat{x}_2 \end{cases} \quad (15)$$

where u is an additional control input in (14), $\sin(x_2)$ is an uncertain factor in the controlled system. So

$$f(x) = \begin{bmatrix} x_2 \\ -0.1275x_1^3 + 0.6523x_1 - px_2 + p_0x_3 \\ -p_1x_2 + p_2x_3 - p_3x_4 \\ p_4x_3 \end{bmatrix}, g(x) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$\Delta f(x) = \sin(x_2), \Delta g(x) = 0$

$y = h(x) = x_2$

The parameters in the above formula and (13)、(14) are as follows:

$p_0 = 0.53691, p_1 = 0.3725, p_2 = 0.035355,$

$p_3 = 0.11779, p_4 = 0.84893, p = 0.1414$

So

$\delta_1(x) = \sin(x_2)$

$\delta_2(x) = 0$

$L_g h(x) = 0$

$L_f h(x) = -0.1275x_1^3 + 0.6523x_1 - 0.1414x_2 + 0.53691x_3$

$L_g L_f h(x) = 0.53691 \neq 0$

Obviously, the relative degree $r = 2 < n = 4$, then

$a(x) = L_f^2 h(x) = -0.3825x_1^2x_2 + 0.0180285x_1^3$
$- 0.0922352x_1 + 0.472295x_2 - 0.0569366x_3 - 0.0632426x_4$

$b(x) = L_g L_f h(x) = 0.53691$

At the same time

$\frac{1}{2} + \delta_2(x) = \frac{1}{2} > 0$

$\|b(x)\delta_1(x) - a(x)\delta_2(x) - \hat{x}_2^2\| \le \|b(x) - \hat{x}_2^2\| \le W$

Further more, we choose $k_0 \ge 2W$,

$R = I, Q = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, then $P = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. Hence the control law u is

$$u = \frac{1}{0.53691}\{0.3825x_1^2x_2 + 0.2369715x_1^3 - 1.2123648x_1 - 1.189495x_2 - 1.0168834x_3 + 0.0632426x_4 - 0.255\hat{x}_1^3 + 1.3046\hat{x}_1 + 0.7172\hat{x}_2 + 1.07382\hat{x}_3 - k_0\,\mathrm{sgn}[-0.255(x_1^3 - \hat{x}_1^3) + 1.3046(x_1 - \hat{x}_1) + 0.7172(x_2 - \hat{x}_2) + 1.07382(x_3 - \hat{x}_3)]\}$$

Figure 3、Figure 4 are the diagrams of synchronization stimulation. Their initial states are arbitrary. Here, we choose the same initial states in order to compare with the synchronization results of the target system and the controlled system. The initial states of them are as follows: $\hat{x}(0) = (-1., -15.67, 0.8, 1.012)^T$, $x(0) = (0.002, -0.5, 0., -1.2)^T$.

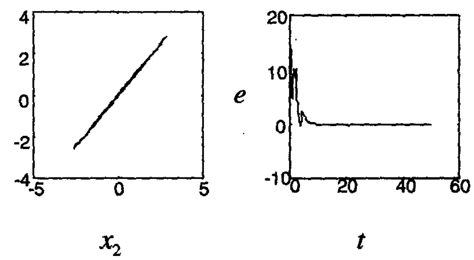

Figure 3 The Synchronization Simulation Figure when $k_0 = 60$ ($e = x_2 - \hat{x}_2$)

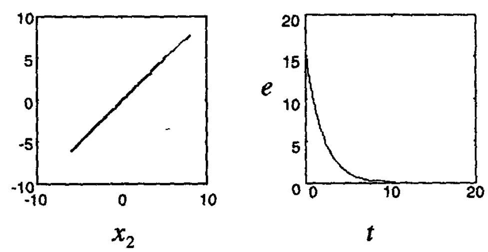

Figure 4 The Synchronization Simulation Figure when $k_0 = 200$ ($e = x_2 - \hat{x}_2$)

Figure 3、Figure 4 separately represent the synchronization simulation results $k_0 = 60$ and $k_0 = 200$. Of course we can obtain the conclusion that the controller we designed is successful. It realizes the synchronization control between two systems. At the same time, we compare Figure 1 with Figure 2, we can get the conclusion that the bigger k_0 is, the quicker synchronization speed we have under the same conditions.

5 Conclusion

This paper is concerned with the design problem of a robust chaotic synchronization controller. It is based on the feedback linearization, optimal control theory and

Lyapunov method. First we analyze the controller theory. Then we make a computer stimulation. We draw the conclusion that a given system which so long as can meet the need of the supposition always can be realized robust synchronization control. It is very important for chaotic synchronization to use in the security communication.

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