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Investigate the Relationship between the Super-efficiency and Fixed Input in the Presence of Infeasibility

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***Abstract*— The fixed input allocation is an important topic in the management science field. Previous data envelopment analysis (DEA) studies consider the problem based on the constant return to scale (CRS) framework (called CCR DEA model). However, the return to scale relationship between inputs and outputs includes three cases: CRS, increasing return to scale (IRS) and decreasing return to scale (DRS). Therefore, it is necessary to study the problem based on DEA under the variable return to scale (VRS) assumption (called BCC DEA model). This paper has two contributions: one is presenting an approach to solve the infeasibility problem when a new variable is added into the super-efficiency BCC DEA model, and the other is investigating the basic relationship between the BCC efficiency scores and the allocated fixed input. Both of them are significant for allocating the fixed input under the VRS DEA framework. Finally, the proposed approach is illustrated by an example of allocating the subsidy among urban public transport enterprises.**

I. INTRODUCTION

An important problem in management science field is how to allocate fixed inputs (including resources and costs), and it has only been studied under the constant return to scale (CRS) assumption (called CCR model) [1] in previous data envelopment analysis (DEA) studies (such as [2]-[13], etc.). DEA is a well-established non-parametric methodology to measure the performance of peer decision making units (DMUs) with multiple inputs and outputs. However, different return to scale (RTS) assumptions may have different impacts on the allocation and besides, the homogeneity among DMUs also needs to take into account the RTS. Therefore, it is necessary to consider the problem under the variable return to scale (VRS) assumption (called BCC model) [14]. Nevertheless, the infeasibility problem might exist if it is solved by the traditional super-efficiency [15] BCC model. Besides, the relationship between the DEA efficiency score and the allocated input is also an important problem that is needed to be investigated before allocating the fixed input. The two problems are directly related to further studying the fixed input allocation problem under the VRS DEA framework. Therefore, this study focuses on solving the two problems and illustrates the proposed approach by an example of allocating the subsidy among urban public transport enterprises.

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This paper is organized as follows. The following section proposes the approach for evaluating the super-BCC efficiency in the presence of infeasibility in the fixed input allocation problem, and investigates the relationship between the super-BCC efficiency score and the allocated fixed input. Section 3 allocates the transportation subsidy to illustrate the proposed approach. Conclusions are given in Section 4.

II. THE METHOD

A. DEA model

Suppose there is a set of DMUs, and each DMU_j ($j = 1, 2, \ldots, n$) consumes $m+1$ inputs x_{ij} ($i = 1, 2, \ldots, m+1$) to yield s outputs y_{rj} ($r = 1, 2, \ldots, s$). Then the DEA efficiency score of any given $DMU_k (k \in \{1, 2, \cdots, n\})$ under evaluation can be calculated by the following model:

$$
\begin{aligned}
\max \quad & \sum_{r=1}^{s} u_r y_{rk} + \mu_0 = E_k \\
s.t. \quad & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m+1} \upsilon_i x_{ij} + \mu_0 \le 0, \forall j \\
& \sum_{i=1}^{m+1} \upsilon_i x_{ik} = 1 \\
& \upsilon_i,\, u_r \ge 0, \forall i, r.
\end{aligned} \tag{1}
$$

where E_k is the relative efficiency of DMU_k, and u_r, υ_i are unknown weights attached to rth output and ith input respectively. Model (1) is a CCR model when $\mu_0 = 0$ [1], and it is a BCC model when μ_0 is free [14]. They are the two basic DEA models, which assume the efficient frontier is CRS and VRS respectively. In BCC model, DMU_k is called IRS (DRS or CRS) if $\mu_0^* < 0$ ($\mu_0^* > 0$ or $\mu_0^* = 0$) in any optimal solution. That is, CRS is a special case of VRS. This study is based on the VRS assumption. Model (1) illustrates that DEA calculates the efficiency score by comparing DMU_k to a virtual DMU on the efficient frontier of the production possibility set (PPS). DMU_k is called efficient if the optimal DEA efficiency $E_k^* = 1$.

Suppose R is the total fixed input that should be allocated to n DMUs, and R_j is the allocated input to DMU_j such that $\sum_{j=1}^{n} R_j = R$. Assume the allocated fixed input and the input x_{m+1} have similar impacts on the outputs, then we integrate them as a single input. Thus we can obtain model

(2) as follows:

$$
\begin{aligned}
\max \quad & \sum_{r=1}^{s} u_r y_{rk} + \mu_0 = E_k \\
s.t. \quad & \sum_{r=1}^{m} \upsilon_i x_{ij} + \upsilon_{m+1}(x_{m+1,j} + R_j) \\
& -\sum_{r=1}^{s} u_r y_{rj} - \mu_0 \geq 0, \forall j \\
& \sum_{i=1}^{m} \upsilon_i x_{ik} + \upsilon_{m+1}(x_{m+1,k} + R_k) = 1 \\
& \sum_{j=1}^{n} R_j = R \\
& u_r, \upsilon_i, R_j \geq 0, \forall r, i, j; \mu_0 \text{ free.}
\end{aligned} \tag{2}
$$

Model (2) is a nonlinear programming and it is hard to calculate the optimal relative efficiency score for each DMU. To solve the problem, let $d_j = \upsilon_{m+1} \times R_j, \forall j$, then model (2) can be changed as follows

$$
\begin{aligned}
\max \quad & \sum_{r=1}^{s} u_r y_{rk} + \mu_0 = E_k \\
s.t. \quad & \sum_{r=1}^{m} \upsilon_i x_{ij} + \upsilon_{m+1} x_{m+1,j} + d_j \\
& -\sum_{r=1}^{s} u_r y_{rj} - \mu_0 \geq 0, \forall j \\
& \sum_{i=1}^{m} \upsilon_i x_{ik} + \upsilon_{m+1} x_{m+1,k} + d_k = 1 \\
& \sum_{j=1}^{n} d_j = \upsilon_{m+1} \times R \\
& \mu_r, \upsilon_i, d_j \geq 0, \forall r, i, j; \mu_0 \text{ free.}
\end{aligned} \tag{3}
$$

To investigate the relationship between the BCC efficiency score and the allocated input, the following model can be obtained:

$$
\begin{aligned}
\max \quad & \sum_{r=1}^{s} u_r y_{rk} + \mu_0 = E_k(q) \\
s.t. \quad & \sum_{r=1}^{m} \upsilon_i x_{ij} + \upsilon_{m+1} x_{m+1,j} + d_j \\
& -\sum_{r=1}^{s} u_r y_{rj} - \mu_0 \geq 0, \forall j \\
& \sum_{i=1}^{m} \upsilon_i x_{ik} + \upsilon_{m+1} x_{m+1,k} + d_k = 1 \\
& \sum_{j=1}^{n} d_j = \upsilon_{m+1} \times R \\
& d_k = q \times \upsilon_{m+1}, \forall q \in [0, R] \\
& \mu_r, \upsilon_i, d_j \geq 0, \forall r, i, j; \mu_0 \text{ free.}
\end{aligned} \tag{4}
$$

In model (4), the fixed input allocated to DMU_k and other $DMUs$ is q and $R-q$ respectively. Model (4) is a traditional BCC model, but it can not further investigate the relationship between the BCC efficiency scores of efficient $DMUs$ and the allocated fixed input. To deal with it, the traditional super-efficiency BCC DEA model can be used [15], in which a DMU under evaluation is excluded from the PPS (i.e., adding $\forall j \neq k$ at the end of the first constraint of model (4)). Therefore, the efficiency scores of efficient DMUs can be larger than 1, and the efficiency scores of inefficient DMUs will be the same as the results calculated by model (4). However, the problem of infeasibility might exist in this case ([16]-[18], etc.).

B. Solve the Infeasibility with Considering the Fixed Input

To deal with the problem of infeasibility, many researches have been done ([18]-[25], etc). In this study, we employ the following model of Cook et al. [25] :

$$
\begin{aligned}
\min \quad & \tau + M \times \beta \\
s.t. \quad & \sum_{j=1, j\neq k}^{m} \lambda_j x_{ij} \leq (1+\tau) x_{ik}, \forall i \\
& \sum_{j=1, j\neq k}^{m} \lambda_j y_{ij} \geq (1-\beta) y_{rk}, \forall r \\
& \sum_{j=1, j\neq k}^{m} \lambda_j = 1 \\
& \lambda_j \geq 0, \forall j \neq k, \tau > -1; \beta \geq 0.
\end{aligned} \tag{5}
$$

It is an input-oriented super-efficiency BCC model, where M is a relatively large number defined by users. We define $M = 10^5$ in this paper. τ, β and λ are unknown variables. The optimal solution is denoted as $(\tau^*; \beta^*; \lambda^*, \forall j)$. It deals with the infeasibility problem by allowing a micro-adjustment in outputs. That is, the infeasibility problem is occurred if and only if $\beta^* > 0$ in model (5).

In Cook et al.[25], the super-efficiency score of DMU_k is defined as $E_k^{C^*} = 1 + \tau^* + 1/(1-\beta^*)$, where $1+\tau^*$ and $1/(1-\beta^*)$ are input and output super efficiency respectively. If the infeasibility problem is not occurred ($i.e.\ \ \beta^* = 0$), model (5) could get an accurate super efficiency solution as $E_k^{C^*} = 1 + \tau^* + 1 = \theta^* + 1$, where θ^* is the traditional super BCC efficiency score. It means that the difference between $E_k^{C^*}$ and θ^* is 1 in this case. For the convenience of comparison, we provide the following definition:

Definition 1. The super-efficiency score of DMU_k calculated by model (5) is defined as $E_k^* = \tau^* + 1/(1-\beta^*)$.

Suppose $R_k = q, q \in [0, R]$ is the fixed input allocated to DMU_k, then we can change model (5) as follows:

$$
\begin{aligned}
\min \quad & \tau + M \times \beta \\
s.t. \quad & \sum_{j=1, j\neq k}^{m} \lambda_j x_{ij} \leq (1+\tau) x_{ik}, \forall i \\
& \sum_{j=1, j\neq k}^{m} \lambda_j (x_{m+1,j} + R_j) \leq (1+\tau)(x_{m+1,k} + R_k) \\
& \sum_{j=1, j\neq k}^{m} \lambda_j y_{ij} \geq (1-\beta) y_{rk}, \forall r
\end{aligned} \tag{6}
$$

$$
\begin{aligned}
&\sum_{j=1,j\neq k}^{m} \lambda_j = 1\\
&R_k = q\\
&\sum_{j=1}^{n} R_j = R\\
&\lambda_j \geq 0, \forall j \neq k, \beta \geq 0; \tau > -1.
\end{aligned}
$$

Model (6) is a nonlinear programming since the allocations R_j are variable, and it can not be directly used to calculate the super-efficiency score. If the input allocation is determined, model (6) can be treated as a linear programming, and then the optimal super-efficiency of DMU_k can be obtained by Definition 1.

C. Investigate the Relationship Between the Efficiency Score and the Allocated Input

To investigate the relationship between the DEA efficiency score and the allocated input, we get the dual model of model (5) as follows:

$$
\begin{aligned}
\max \quad & \sum_{r=1}^{s} u_r y_{rk} - \sum_{i=1}^{m+1} \upsilon_i x_{ik} + \mu_0\\
s.t. \quad & \sum_{r=1}^{m} \upsilon_i x_{ij} - \sum_{r=1}^{s} u_r y_{rj} - \mu_0 \geq 0, j \neq k\\
& \sum_{i=1}^{m+1} \upsilon_i x_{ik} = 1\\
& \sum_{r=1}^{s} u_r y_{rk} \leq M\\
& u_r, \upsilon_i \geq 0; \mu_0 \ \text{ free}
\end{aligned} \tag{7}
$$

Considering the allocated fixed input R_j, we can transform model (7) into the following model:

$$
\begin{aligned}
\max \quad & \sum_{r=1}^{s} u_r y_{rk} - \sum_{i=1}^{m} \upsilon_i x_{ik} + \upsilon_{m+1}(x_{m+1,k} + R_k) + \mu_0\\
s.t. \quad & \sum_{r=1}^{m} \upsilon_i x_{ij} + \upsilon_{m+1}(x_{m+1,j} + R_j)\\
& - \sum_{r=1}^{s} u_r y_{rj} - \mu_0 \geq 0, j \neq k\\
& \sum_{i=1}^{m} \upsilon_i x_{ik} + \upsilon_{m+1}(x_{m+1,k} + R_k) = 1\\
& \sum_{j=1}^{n} R_j = R\\
& \sum_{r=1}^{s} u_r y_{rk} \leq M\\
& \mu_r, \upsilon_i, R_j \geq 0; \mu_0 \ \text{ free}
\end{aligned} \tag{8}
$$

Apparently, model (8) is also a nonlinear programming. To solve the problem, let $d_j = \upsilon_{m+1} \times R_j$, model (8) can be changed as follows:

$$
\begin{aligned}
\max \quad & \sum_{r=1}^{s} u_r y_{rk} + \mu_0 - \sum_{i=1}^{m} \upsilon_i x_{ik} - \upsilon_{m+1} x_{m+1,k} - d_k\\
s.t. \quad & \sum_{r=1}^{m} \upsilon_i x_{ij} + \upsilon_{m+1}(x_{m+1,j} + d_j) - \sum_{r=1}^{s} u_r y_{rj}\\
& - \mu_0 \geq 0, j \neq k\\
& \sum_{i=1}^{m} \upsilon_i x_{ik} + \upsilon_{m+1} x_{m+1,k} + d_k = 1\\
& \sum_{j=1}^{n} d_j = \upsilon_{m+1} \times R\\
& \sum_{r=1}^{s} \mu_r y_{rk} \leq M\\
& \mu_r, \upsilon_i, d_j \geq 0; \mu_0 \ \text{ free.}
\end{aligned} \tag{9}
$$

Model (9) can obtain the optimal solution, but it should be noteworthy that the optimal objective function value is not the optimal super-efficiency of DMU_k, which is defined by the Definition 1. To investigate the relationship, we can use the following model:

$$
\begin{aligned}
\max \quad & \sum_{r=1}^{s} u_r y_{rk} + \mu_0 - \sum_{i=1}^{m} \upsilon_i x_{ik} - \upsilon_{m+1} x_{m+1,k} - d_k\\
s.t. \quad & \sum_{r=1}^{m} \upsilon_i x_{ij} + \upsilon_{m+1}(x_{m+1,j} + d_j) - \sum_{r=1}^{s} u_r y_{rj}\\
& - \mu_0 \geq 0, j \neq k\\
& \sum_{i=1}^{m} \upsilon_i x_{ik} + \upsilon_{m+1} x_{m+1,k} + d_k = 1\\
& \sum_{j=1}^{n} d_j = \upsilon_{m+1} \times R\\
& d_k = q \times \upsilon_{m+1}, q \in [0, R]\\
& \sum_{r=1}^{s} \mu_r y_{rk} \leq M\\
& \mu_r, \upsilon_i, d_j \geq 0; \mu_0 \ \text{ free.}
\end{aligned} \tag{10}
$$

If q is determinate, model (10) would be easily solved. Denote the optimal input allocation matrix as $(R_k^* = q, R_j^*, \forall j \neq k)$ corresponding to each k. Apparently, the optimal fixed input allocation matrix here is not the final optimal fixed input allocation since it is not determinate. However, we can take it into model (6) to calculate the corresponding optimal DEA super-efficiency and investigate the relationship between them.

Algorithm 1: Calculating the super-BCC efficiency score of DMU_k based on Definition 1

Step 1. Solve model (10) corresponding to each k, and get the optimal values υ_{m+1}^* and $d_j^*, \forall j$.

Step 2. Calculate the optimal solution by the equation $R_j^* = d_j^*/\upsilon_{m+1}^*, \forall j$.

Step 3. Substitute the optimal allocation matrix $(R_k^* = q, R_j^*, \forall j \neq k)$ into model (6), and calculate the optimal values of τ^* and β^* corresponding to each k.

Step 4. Calculate the corresponding super-BCC efficiency score $E_k^*(q)$ of DMU_k based on Definition 1.

Theorem 1. In our approach, $E_k^* = E_k^*(q)$ obtained from Algorithm 1 is a monotone non-increasing function of q, $q \in [0,\, R]$.

Proof: See the Appendix 1.

III. Allocate the transportation subsidy

How to allocate the transportation subsidy is an important problem in the public transportation industry. Governments limit the ticket fares to satisfy the basic travel demands of the citizens, which lead to losses in public transportation industry. Therefore, governments should provide policies to support them. One of the policies is the economic compensation, namely subsidy. That is, the allocated subsidy can be considered as a fixed allocated resource to compensate losses from the common-weal. Since some losses of urban public transport enterprises may be caused by other factors, such as high energy waste, inefficient organization, poor management ability and so on, governments should not pay these bills. Because it is hard to obtain the real losses of urban public transport enterprises, it is significant to allocate the subsidy to urban public transport enterprises with a way that can encourage them to improve the market performance initiatively (such as decreasing operating losses, improving quality services and satisfaction degrees of passengers, etc.). To this end, we should first investigate the basic relationship between the subsidy and the BCC efficiencies.

As given in Table I, there are seven DMUs with two inputs and two outputs. They are the operation data of seven urban public transport enterprises in 2009, which are collected from Jiangsu Provincial Communications Department. $X1$ and $X2$ are the number of standardized operating buses and the costs per 1000 kilometers respectively. $Y1$ and $Y2$ are the satisfaction degrees of passengers and the number of the transporting passengers per kilometer respectively. Considering the subsidy and $X2$ have similar impacts on the outputs, we combine them as a single input measure. Besides, suppose the total allocated subsidy is $R = 10000$.

Applying model (1) (BCC model) to the data set, we can obtain results as shown in the sixth and seventh column of Table I, which illustrate that only DMU D is IRS, others are DRS and no DMU is CRS.

TABLE I

Data set and relevent results

DMU		A	B	C	D	E	F	G
X_1		3453	1355	572	945	346	289	261
X_2		5642.28	5196.7	5137.98	4135.83	4171.54	3620.49	3534.53
Y_1		60.87	63.77	55.56	50.48	56.68	61.83	61.93
Y_2		2.86	2.92	2.80	2.12	2.47	1.73	1.00
μ_0^*		-0.26	-565.22	-26.52	0.56	-5.27	-67.93	-97.75
RTS		DRS	DRS	DRS	IRS	DRS	DRS	DRS
Traditional Super-BCC	$\overline{E_j^*}$	2.04	Infeasible	2.25	1.50	2.02	2.43	2.65
	$\underline{E_j^*}$	0.33	Infeasible	1.90	0.34	1.40	2.03	1.32
Algorithm 1	$\overline{E_j^*}$	2.04	2.91	2.25	1.50	2.02	2.43	2.65
	$\underline{E_j^*}$	0.33	2.54	1.90	0.34	1.40	2.03	1.32

To investigate the relationship between the subsidy and the BCC efficiencies, we set the initial value $q = 0$, which is the lower bound. Then we raise q by increasing t according to $q = \varepsilon \times t$ until $q = R$, where ε is a small positive number, and in this study we set $\varepsilon = 0.05$. The relevant results are shown in the last two columns of Table I and Fig. 1. The smaller the ε value we select, the smoother the efficiency curve will be obtained, and the selected ε value has no impact on the shape of the efficiency curve.

As shown in the last two columns of Table I, we can find that the infeasibility problem exists in the traditional super-efficiency BCC model with considering the allocated fixed subsidy, and the problem can been solved by the proposed approach (Algorithm 1). For example, DMU B can not obtain the optimal super-BCC efficiency by the traditional super-efficiency BCC model, and the problem can be solved by Algorithm 1. Furthermore, efficiencies of other DMUs calculated by Algorithm 1 are the same as the results based on the traditional super-efficiency BCC model.

As shown in Figure 1, we investigate the relationship between the BCC efficiency and the allocated subsidy under the traditional BCC model, the traditional super-efficiency BCC model and Algorithm 1, respectively. Comparing the three sub-figures, we can find that:

- If DMU_k is an inefficient DMU, then its optimal efficiencies obtained from the three approaches would be identical, such as the inefficient part of DMU A and D.
- To efficient DMUs, the traditional BCC model can not further provide the relationship between the BCC efficiencies and the allocated subsidy, such as DMU B, C, E, F and G in the first sub-figure. The traditional super-efficiency BCC model has the infeasibility problem, such as DMU B in the middle sub-figure. Algorithm 1 can deal with the infeasibility problem and can provide further relationship to efficient DMUs.
- As shown in the last sub-figure, we can find that the super-BCC efficiency scores obtained from Algorithm 1 is a monotone non-increasing function of the allocated fixed subsidy. It is not a special case as it is proved by Theorem 1. The relationship shows that increasing the allocated subsidy is disadvantageous to improve the performance of the urban public transport enterprises. Different subsidy allocation can be obtained based on the relationship and the allocation target. It provides a foundation to further extend the fixed input allocation research from the CRS assumption to the VRS assumption.
- Our proposed approach can overcome the infeasibility problem and investigate the relationship between the super-BCC efficiency score and the allocated fixed input (subsidy) effectively. Based on the basic relationship, we can further design an incentive mechanism to improve the market performance of urban public transport enterprises in future work.

IV. Conclusions

This study presents an approach to measure the DEA super-efficiency with considering the allocated fixed input

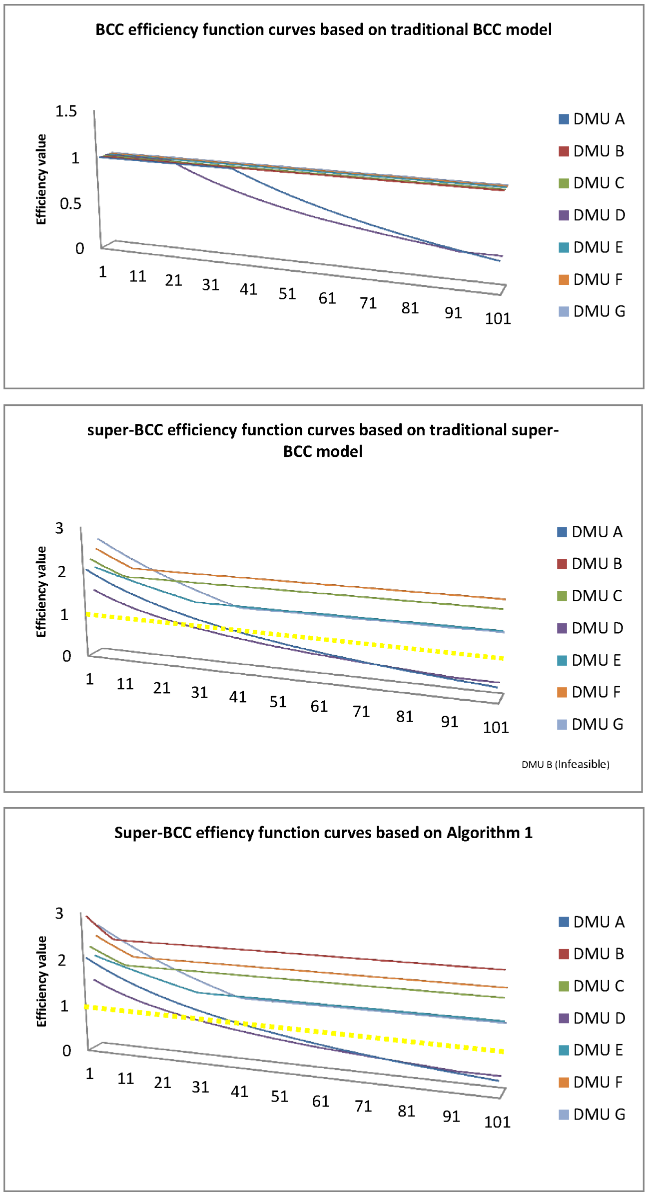

Fig. 1. The relationship between BCC efficiency values and the allocated subsidy

in the presence of infeasibility, and then investigates the relationship between the BCC DEA efficiency score and the allocated input. The results show that the infeasibility problem with considering the allocated fixed input can be solved by the proposed approach, and the super-BCC efficiency score is a monotone non-increasing function of the allocated fixed input. The relationship is significant to further extend the fixed input allocation research from the CRS framework to the VRS framework. In future research, the main work is how to present a general allocation method to obtain a unique and acceptable fixed input allocation plan.

APPENDIX

Theorem 1: In our approach, $E_k^* = E_k^*(q)$ obtained from Algorithm 1 is a monotone non-increasing function of q, $q \in [0, R]$.

Proof: For a given $R_k = q$, $q \in [0, R]$, we can obtain an optimal input allocation denoted as $(R_k = q; R_j^*, \forall j \neq k)$ from the step 1 and step 2 of Algorithm 1. Therefore, $\sum_{j=1, j\neq k}^{n} R_j^* = R - R_k = R - q$. Based on the last two steps of Algorithm 1, we can get the optimal solution of model (6) denoted as $(\tau^*, \beta^*, \lambda_j^*, \forall j \neq k)$ and the corresponding optimal super efficiency score is $E_k^*(q) = \tau^* + \frac{1}{1-\beta^*}$.

Let $R_k' = q' = q + \Delta q, \Delta q \geq 0$ and $q' \in [0, R]$, we can obtain an optimal input allocation denoted as $(R_k' = q'; R_j', \forall j \neq k)$ from the step 1 and step 2 of Algorithm 1. Then it would be $\sum_{j=1, j\neq k}^{n} R_j' = R - R_k' = R - q' \leq \sum_{j=1, j\neq k}^{n} R_j^*$. In this case, $(\tau^*, \beta^*, \lambda_j^*, \forall j \neq k)$ must be a feasible solution of model (6) as it satisfies all constraints, such as:

$$\sum_{\substack{j=1 \\ j\neq k}}^{n} \lambda_j^*(x_{m+1,j} + R_j') \leq \sum_{\substack{j=1 \\ j\neq k}}^{n} \lambda_j^*(x_{m+1,j} + R_j^*)$$
$$\leq \quad (1+\tau^*)(x_{m+1,k} + R_k) \leq (1+\tau^*)(x_{m+1,k} + R_k'),$$

i.e., it satisfies the constraint

$$\sum_{\substack{j=1 \\ j\neq k}}^{n} \lambda_j^*(x_{m+1,j} + R_j') \leq (1+\tau^*)(x_{m+1,k} + R_k')$$

And the corresponding feasible efficiency is $E_k(q') = \tau^* + \frac{1}{1-\beta^*}$. Therefore,

$$E_k(q') = E_k^*(q). \tag{A1}$$

Based on the last two steps of Algorithm 1, we can get the optimal solution of model (6) denoted as $(\tau'^*, \beta'^*, \lambda_j'^*, \forall j \neq k)$ in this case and the corresponding optimal super efficiency score is $E_k^*(q') = \tau'^* + \frac{1}{1-\beta'^*}$.

To observe the relationship between $E_k^*(q')$ and $E_k^*(q)$, we take into consideration β'^* and β^*:

1) $\beta'^* \leq \beta^*$

i) $\tau'^* \leq \tau^*$

$$E_k^*(q') - E_k(q') = (\tau'^* - \tau^*) + \frac{(\beta'^* - \beta^*)}{[(1-\beta'^*)(1-\beta^*)]} \tag{A2}$$

Because of $\tau'^* \leq \tau^*$ and $\beta'^* \leq \beta^*$, thus,

$$E_k^*(q') \leq E_k(q'). \tag{A3}$$

From (A1) and (A3), we can get $E_k^*(q') \leq E_k^*(q)$.

ii) $\tau'^* > \tau^*$

As mentioned above, $(\tau^*, \beta^*, \lambda_j^*, \forall j \neq k)$ and $(\tau'^*, \beta'^*, \lambda_j'^*, \forall j \neq k)$ are the feasible and optimal solution of model (6) respectively when $R_k' = q' = q + \Delta q, \quad \Delta q \geq 0$. Therefore, the difference of the objective function value between them is $(\tau'^* - \tau^*) + M \cdot (\beta'^* - \beta^*) \leq 0$, i.e., $M \cdot \frac{(\beta^* - \beta'^*)}{(\tau'^* - \tau^*)} \geq 1$. Based on the definition, M is a relative large number defined by users. Let $M \geq \frac{1}{(1-\beta'^*)(1-\beta^*)}$, then we can obtain $M \cdot \frac{(\beta^* - \beta'^*)}{(\tau'^* - \tau^*)} \geq \frac{(\beta^* - \beta'^*)}{(1-\beta'^*)(1-\beta^*)(\tau'^* - \tau^*)} \geq 1$. That is, (A2)$\leq 0$. Hence, we can know from (A2) that

$E_k^*(q') \leq E_k(q')$. Based on (A1), we can get $E_k^*(q') \leq E_k^*(q)$.

2) $\beta'^* > \beta^*$

From the above, we know that $(\tau'^* - \tau^*) + M \cdot (\beta'^* - \beta^*) \leq 0$. Because of $M > 0$ and $\beta'^* > \beta^*$, we can obtain $\tau'^* \leq \tau^*$. Let $M \geq \frac{1}{(1-\beta'^*)(1-\beta^*)}$, we can obtain (A2)$\leq (\tau'^* - \tau^*) + M \cdot (\beta'^* - \beta^*) \leq 0$. Hence, we can know from (A2) that $E_k^*(q') \leq E_k(q')$. Based on (A1), we can obtain $E_k^*(q') \leq E_k^*(q)$. Q.E.D.

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