Optimum Design of Vibrating Cantilevers: A Classical Problem Revisited[1]

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Abstract. Optimum design of vibrating cantilevers is a classical problem widely used in the literature and textbooks in structural optimization. The problem, originally formulated and solved by Karihaloo and Niordson (Ref. 5), was to find the optimal beam shape that will maximize the fundamental vibration frequency of a cantilever. Upon reexamination of the problem, it has been found that the original analysis and solution procedure can be simplified and improved substantially. Specifically, the time-consuming inner loop devised for solving the Lagrange multiplier in the original work has been proved to be totally unnecessary and thus should not be considered in the problem solution. This conclusion has led to a new set of simplified equations for the construction of iteration schemes. New asymptotic expressions for the optimum design solution have been obtained and verified by numerical results. Numerical analysis has shown a significant improvement in convergence rate by the proposed new procedure. Also some obvious numerical errors in the original paper have been identified and corrected.

Key Words. Cantilever beams, flexible manipulators, optimum design, fundamental frequency, successive iterations.

1. Introduction

Since the pioneering works by Beesack (Ref. 1), Schwarz (Refs. 2–3), and especially Niordson (Ref. 4), considerable progress has been made in

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the optimum design of vibrating elastic structures. Niordson first showed in Ref. 4 that, for simply supported beams with geometrically similar cross sections, an increase of 6.6% in the lowest frequency of vibration can be achieved through optimum shape design. Later, Karihaloo and Niordson (Ref. 5) studied the optimum design of vibrating cantilever beams and found considerably larger increases in the lowest frequency. For example, the lowest frequency of the optimum cantilever with geometrically similar cross sections is 578% larger than that of the corresponding one with a uniform cross section. Similar work and extension have also been conducted by many other researchers (Refs. 6–9). Since then, the problem has become a classical one and has been used widely in the literature and textbooks in structural optimization.

Recently, Wang (Ref. 10) has investigated the problem of the optimum shape design of flexible manipulators. The objective is to increase the fundamental vibration frequency of a flexible manipulator so that a larger bandwidth can be obtained for the manipulator control system. The problem formulation is almost the same as that in Ref. 5. However, different boundary conditions have made the optimization problem for flexible manipulators much more difficult than the corresponding one for cantilever beams.

Initially, we attempted to follow the iteration schemes in Ref. 5 in order to solve the corresponding optimization problem for flexible manipulators. However, for all cases tried, the iterative schemes of Ref. 5 did not converge. It was also found that the implicit equation (so-called inner loop) for solving the Lagrange multiplier in those schemes took a significant amount of computation time. After careful reexamination of the original problem, we found that the time-consuming inner loop in the original iteration schemes was redundant and could be removed completely in the solution process. Eliminating this redundant equation from the iteration process leads to a new formulation for the iterations, and consequently to substantial simplification of the iteration equations and significant improvement in convergence rates. For example, three simplified iteration schemes are needed in this paper to solve the optimum design problem completely, whereas five different sophisticated schemes were required in Ref. 5. These results have offered useful information for solving the optimization problem for manipulators (Refs. 10 and 11).

As in Ref. 5, we assume throughout this paper the following relationship between the moment of inertia I and the area A of a cross section of the beam:

$$I = cA^p(x), \qquad p \geq 1, \tag{1}$$

where c is a constant. Three cases (viz., $p = 1, 2, 3$) are especially interesting to us, since they correspond to beams with rectangular cross sections of

given uniform height, geometrically similar cross sections, and rectangular cross sections of given uniform width, respectively. The treatment in this paper, however, is valid for all cases with $p \geq 1$.

2. Basic Equations

Consider the small harmonic transverse vibrations of a tapered cantilever beam carrying a mass Q at its tip. If both the rotary inertia and shear deformations are neglected, the differential equation of motion and the boundary conditions can be written in the following dimensionless form:

$$(\alpha^p y'')'' - \lambda \alpha y = 0, \tag{2}$$

$$y(1) = y'(1) = 0, \qquad \alpha^p y''(0) = 0, \qquad (\alpha^p y'')'(0) = \lambda q y(0). \tag{3}$$

Here, y is the amplitude of the lateral displacement in the plane of bending and the prime indicates differentiation with respect to the dimensionless coordinate $\xi = x/l$. The dimensionless area function is denoted as $\alpha = Al/V$, in which l is the length of the beam and V is the total volume of the beam. The dimensionless eigenvalue λ and mass parameter q in the boundary conditions are defined as

$$\lambda = \omega^2 \gamma l^{p+3}/cEV^{p-1}, \qquad q = Q/\gamma V, \tag{4}$$

where ω is the natural vibration frequency and γ the mass density of the beam. From the definition, α must be nonnegative and satisfy the following constraint:

$$\int_0^1 \alpha(\xi) \, d\xi = 1. \tag{5}$$

The problem of the optimum design of vibrating cantilevers is to find the optimal area function that will maximize the fundamental vibration frequency. Using the Rayleigh quotient and variational calculus, we find the equation for determining the optimum area function as

$$p\alpha^{p-1}(y'')^2 - \lambda y^2 = \lambda a^2, \tag{6}$$

where a^2 is the Lagrange multiplier introduced for the constraint (5).

Equations (2)–(3) and (6) are basic for solving the optimum design problem. Note that, in order to be consistent with the original work by Karihaloo and Niordson, all notation used in Ref. 5 has been kept in this paper. For the detailed derivation of these equations, the reader is referred to their paper (Ref. 5).

The Rayleigh quotient can be obtained by multiplying both sides of Eq. (2) by y and integrating over the interval $[0, 1]$. Integrating by parts and taking the boundary conditions (3) into account, we have

$$\lambda = \left[\int_0^1 \alpha^p y''^2 \, d\xi \right] \bigg/ \left[\int_0^1 \alpha y^2 \, d\xi + q y^2(0) \right]. \tag{7}$$

Furthermore, by substituting Eq. (6) in (7), we obtain the following implicit equation for the Lagrange multiplier:

$$a^2 = (p - 1) \int_0^1 [\lambda(y^2 + a^2)/p(y'')^2]^{1/(p-1)} y^2 \, d\xi + pq y^2(0). \tag{8}$$

Expressions (7) and (8) can be considered as the consequences of Eqs. (2)–(3) and (5)–(6). That is, as long as y and α satisfy (2)–(3) and (5)–(6), both (7) and (8) will be satisfied automatically. Therefore, in order to solve the optimization problem, one only needs to work with Eqs. (2)–(3) and (5)–(6). This observation will serve as the basis for the development of our new formulations of iteration schemes. Note that, except for case $p = 2$, the Lagrange multiplier a^2 cannot be expressed in terms of y explicitly. For $p \neq 2$ the so-called inner loop has been used in Ref. 5 to find a^2 for a given y in their iteration schemes. The numerical simulation has indicated that most of the computation time has been spent on this inner-loop operation. Hence, by removing Eq. (8) completely from the solution process, the rate of convergence of the iteration schemes could be significantly improved.

To this end, we notice that, for a given area function and eigenvalue, y and a^2 cannot be determined uniquely from Eqs. (2)–(3) and (5)–(6). To see this, let $(y, a^2, \alpha, \lambda)$ be a solution of (2)–(3) and (5)–(6). Then, $(ny, n^2 a^2, \alpha, \lambda)$ is obviously another solution for any nonzero constant n. This nonuniqueness offers us a way to remove the Lagrange multiplier from (6) completely by selecting $n = 1/a$. In other words, for the optimization problem, we only need to find the unknown function $u = y/a$, instead of y and a^2 separately. Another method for solving the nonuniqueness problem is to impose some normalization scheme on y. This method has been used widely in structural optimization (Refs. 12–13); however, it still requires finding the Lagrange multiplier.

In terms of the new function u, Eqs. (2)–(3) and (6) can be rewritten as

$$(\alpha^p u'')'' - \lambda \alpha u = 0, \tag{9}$$

$$p \alpha^{p-1} (u'')^2 - \lambda u^2 = \lambda, \tag{10}$$

$$u(1) = u'(1) = 0, \qquad \alpha^p u''(0) = 0, \qquad (\alpha^p u'')'(0) = \lambda q u(0). \tag{11}$$

It follows from these equations that

$$\int_0^1 \alpha(\xi)\,d\xi = (p-1)\int_0^1 [\lambda(u^2+1)/p(u'')^2]^{1/(p+1)}u^2\,d\xi + pqu^2(0). \qquad (12)$$

Note that this identity does not require the constraint (5) to hold.

From (10), for $p > 1$, we can find α in terms of u and λ,

$$\alpha(\xi) = \phi_u(\xi)/\beta, \qquad \beta = (p/\lambda)^{1/(p-1)}, \qquad (13a)$$

$$\lambda = p/\beta^{p-1}, \qquad \phi_u(\xi) = [(u^2(\xi)+1)/u''^2(\xi)]^{1/(p-1)}. \qquad (13b)$$

When the constraint (5) is satisfied, we have

$$\beta = \int_0^1 \phi_u(\xi)\,d\xi. \qquad (14)$$

The following integral formulas are useful in our discussion (see Section 4):

$$\int_0^\xi \int_0^x G(s)\,ds\,dx = \xi^2 \int_0^1 (1-x)G(x\xi)\,dx, \qquad (15)$$

$$\int_\xi^1 \int_x^1 G(s)\,ds\,dx = (1-\xi)^2 \int_0^1 xG[\xi+x(1-\xi)]\,dx. \qquad (16)$$

By formal integration of (9), after satisfying the boundary conditions at $\xi = 0$, substituting α from (13) into (9), and using (15), we find that

$$u''(\xi) = \frac{[u^2(\xi)+1]^{p/(p+1)}}{\{p\xi[\beta qu(0) + \xi \int_0^1 (1-x)\phi_u(x\xi)u(x\xi)\,dx]\}^{(p-1)/(p+1)}}, \qquad (17)$$

which will be used as the basic formula for the construction of new iteration schemes.

3. Analysis of Singularity at the Free End

When $p \neq 1$, the solutions of Eqs. (9)–(11) are singular at the free end $\xi = 0$; therefore, the numerical method cannot be applied to finding the solution directly. To make the numerical solution possible, we first have to determine the behavior of the solution near the free end. This can be done by assuming that the solutions can be expanded in a power series of ξ with a characteristic term ξ^k near the free end. A standard procedure was used by Karihaloo and Niordson in Ref. 5 to derive the characteristic equations for determining the singularity k. In this paper, however, a direct and much simpler method is employed to find the singularity.

3.1. Case $q \neq 0$. In this case, since u is finite at $\xi = 0$, we can see from (9)–(11) that, at $\xi = 0$,

$$\alpha^p u'' = 0, \quad (\alpha^p u'')' = \text{finite}, \quad (\alpha^p u'')'' = \text{finite}, \quad \alpha^{p-1}(u'')^2 = \text{finite}.$$

Therefore, in the neighborhood of $\xi = 0$,

$$\alpha^p u'' = a_1 \xi + a_2 \xi^2 + \cdots, \qquad \alpha^{p-1} u''^2 = b_1 + \cdots,$$

where a_1, a_2, b_1 are constants. Solving the above equations for α and u'', we get

$$\alpha(\xi) = c_1 \xi^{2/(p+1)} + \cdots, \tag{18}$$

$$u''(\xi) = d_1 \xi^{-(p-1)/(p+1)} + \cdots, \tag{19}$$

where c_1 and d_1 are two new constants. Then, the behavior of u near the free end can be found as

$$u(\xi) = d_1 \xi^{(p+3)/(p+1)} + \cdots. \tag{20}$$

Therefore, the singularity of u at $\xi = 0$ is

$$k = (p+3)/(p+1).$$

3.2. Case $q = 0$. Expand both u and α in a power series of ξ at $\xi = 0$,

$$u(\xi) = u_0 \xi^k + \cdots, \tag{21a}$$

$$\alpha(\xi) = a_0 \xi^m + \cdots. \tag{21b}$$

Substituting these expressions into (10), we get

$$2k = m(p-1) + 2(k-2), \qquad \lambda = p a_0^{p-1} k^2 (k-1)^2.$$

Similarly, from (9) we get

$$\lambda a_0 = a_0^p k(k-1)(m+k+2)(m+k+1).$$

By eliminating λ and a_0 from these equations, we find that

$$m = 4/(p-1), \tag{22}$$

$$[k(p-1) + 2(p+1)][k(p-1) + p + 3] - p(p-1)^2 k(k-1) = 0. \tag{23}$$

For $p = 2$, we find $k = -2$; and for $p = 3$, we find $k = -1$. In both cases, the results are the same as those obtained in Ref. 5 by the standard procedure.

4. Solution by Successive Iterations: New Formulation

In this section, we present a new simplified formulation for solving Eqs. (9)–(11) using successive iterations. We first discuss in detail the degenerate case in which $p = 1$ and then investigate the cases where $p > 1$, $q = 0$ and $p > 1$, $q \neq 0$, respectively.

4.1. Case $p = 1$. For $p = 1$, function α drops out of Eq. (10) and we have a degenerate case. Equations (9)–(11) now have the form

$$(\alpha u'')'' - \lambda \alpha u = 0, \tag{24}$$

$$(u'')^2 = \lambda(u^2 + 1), \tag{25}$$

$$u(1) = u'(1) = 0, \qquad \alpha u''(0) = 0, \qquad (\alpha u'')'(0) = \lambda q u(0), \tag{26}$$

and the identity (12) becomes

$$q u^2(0) = \int_0^1 \alpha(\xi) \, d\xi.$$

The above equation indicates that, for $q = 0$, the solution to the optimum design problem of a vibrating cantilever beam does not exist, since the constraint (5) cannot be satisfied by any solution of (24)–(26). Actually, as pointed out in Ref. 5, the vibration frequency of a cantilever can be increased indefinitely in this case by selecting α appropriately. For $q \neq 0$, if we choose

$$q = 1/u^2(0), \tag{27}$$

then α obtained by solving Eq. (24) with boundary conditions (26) will satisfy the constraint (5) automatically. This observation leads to an inverse approach to solve the problem in this case. In other words, starting with a given λ, we determine u by solving (25) with the first two boundary conditions in (26) and then calculate the corresponding q by (27). This will establish a relationship between the mass parameter q and the optimum eigenvalue λ, which will enable us to find λ for a given q, and hence solve the optimization problem. Note that this process does not involve at all the computation of α. Once u and q have been found for a given λ, α can be obtained by solving (24) with the last two boundary conditions in (26). It is guaranteed that the resulting α will meet the constraint (5).

For very large q, an asymptotic relationship between q and λ can be obtained. Since λ is very small in this case, we can use $\epsilon = \sqrt{\lambda}$ as a small parameter and solve (25) by the perturbation method (Ref. 14), i.e., expand u in a power series of ϵ,

$$u(\xi) = u_0(\xi) + \epsilon u_1(\xi) + \cdots.$$

Substituting the above expression into (25), we get

$$u_0 = 0, \qquad u_1 = (1/2)(1 - \xi)^2, \ldots \tag{28}$$

Hence, from (27), (24), and (26), we have

$$\lambda q \simeq 4, \qquad \alpha(\xi) \simeq 2\xi, \tag{29}$$

when q is very large.

For a cantilever beam of uniform cross section and the same length and volume as the optimum beam, $\alpha(\xi) = 1$ and the characteristic equation for the eigenvalue λ_c is (see Ref. 15)

$$1 + \cos \theta \cosh \theta + q\theta(\cos \theta \sinh \theta - \sin \theta \cosh \theta) = 0, \qquad \theta = \lambda_c^{1/4}. \tag{30}$$

From this, we find that, for very large q,

$$\lambda_c q \simeq 3. \tag{31}$$

Therefore, for a large tip mass, a relative increase of

$$\omega/\omega_c - 1 = \sqrt{\lambda/\lambda_c} - 1 = \sqrt{4/3} - 1 = 15.47\%$$

in the lowest natural frequency can be achieved with the optimum tapering of cantilever beams.

To develop a successive iteration scheme for general $q \neq 0$, we formally integrate (24) and (25) with boundary conditions (26). Application of formulas (15) and (16) leads to

$$u(\xi) = \sqrt{\lambda}(1 - \xi)^2 \int_0^1 x\sqrt{u^2[\xi + x(1 - \xi)] + 1} \, dx, \tag{32}$$

$$\alpha(\xi) = \sqrt{\lambda}\xi \, \frac{1/u(0) + \xi \int_0^1 (1 - x)\alpha(x\xi)u(x\xi) \, dx}{\sqrt{u^2(\xi) + 1}}. \tag{33}$$

Based on Eqs. (32) and (33), the iteration scheme can now be outlined as follows.

Step 1.   For a given λ, select an initial $u_0(\xi)$. Update u_i until a specified accuracy is obtained,

$$u_{i+1}(\xi) = \sqrt{\lambda}(1 - \xi)^2 \int_0^1 x\sqrt{u_i^2[\xi + x(1 - \xi)] + 1} \, dx.$$

Step 2.   For u obtained in Step 1, calculate q according to Eq. (27).

Step 3.   Select an initial $\alpha_0(\xi)$ and update α_i until a specified accuracy is obtained,

$$\alpha_{i+1}(\xi) = \sqrt{\lambda}\xi \, \frac{1/u(0) + \xi \int_0^1 (1 - x)\alpha_i(x\xi)u(x\xi) \, dx}{\sqrt{u^2(\xi) + 1}}.$$

Clearly, compared with the corresponding scheme presented in Ref. 5, the new formulation in this case is simpler.

4.2. Case $p > 1$ and $q = 0$. In order to avoid the singularity of u at $\xi = 0$, we introduce two new functions,

$$f(\xi) = \xi^k u(\xi), \qquad z(\xi) = \xi^{k+2} u''(\xi), \tag{34}$$

where k is the singularity of u at $\xi = 0$ determined from (23). Both f and z are regular over the entire interval $0 \leq \xi \leq 1$. It is easy to show that

$$f(1) = f'(1) = 0, \qquad f(0) = z(0)/k(k + 1). \tag{35}$$

In terms of f and z, the function ϕ_u and parameter β can be rewritten as

$$\phi_u(\xi) = \xi^{4/(p-1)} \phi(\xi), \qquad \beta = \int_0^1 \xi^{4/(p-1)} \phi(\xi)\, d\xi, \tag{36a}$$

$$\phi(\xi) = [(f^2(\xi) + \xi^{2k})/z^2(\xi)]^{1/(p-1)}. \tag{36b}$$

From Eq. (34) and Eqs. (15), (16), and (17), we get

$$f(\xi) = \xi^k (1 - \xi)^2 \int_0^1 \frac{xz[\xi + x(1 - \xi)]}{[\xi + x(1 - \xi)]^{k+2}}\, dx, \qquad 0 < \xi \leq 1, \tag{37}$$

$$z(\xi) = \frac{[f^2(\xi) + \xi^{2k}]^{p/(p+1)}}{[p \int_0^1 (1 - x) x^{4/(p-1)-k} \phi(x\xi) f(x\xi)\, dx]^{(p-1)/(p+1)}}. \tag{38}$$

Now, the scheme for successive iterations can be specified as follows:

Step 1. Select an initial $z_0(\xi)$.

Step 2. Update $f_i(\xi)$ according to

$$f_{i+1}(\xi) = \xi^k (1 - \xi)^2 \int_0^1 \frac{xz_i[\xi + x(1 - \xi)]}{[\xi + x(1 - \xi)]^{k+2}}\, dx, \qquad 0 < \xi \leq 1,$$

$$f_{i+1}(0) = z_i(0)/k(k + 1).$$

Step 3. Update $\phi_i(\xi)$,

$$\phi_{i+1}(\xi) = [(f_{i+1}^2(\xi) + \xi^{2k})/z_i^2(\xi)]^{1/(p-1)}.$$

Step 4. Update $z_i(\xi)$,

$$z_{i+1}(\xi) = \frac{[f_{i+1}^2(\xi) + \xi^{2k}]^{p/(p+1)}}{[p \int_0^1 (1 - x) x^{4/(p-1)-k} \phi_{i+1}(x\xi) f_{i+1}(x\xi)\, dx]^{(p-1)/(p+1)}}.$$

Step 5. If a given accuracy is not obtained, go back to Step 2.

Once f and z have been obtained within the specified accuracy, one can find the parameter β. Then, the optimum eigenvalue λ, the function ϕ_u, and

the optimum α can be determined according to (36), (14), and (13), respectively. Obviously, the proposed new procedure is much simpler than the one developed in Ref. 5.

4.3. Case $p > 1$ and $q \neq 0$. In this case, we need only to take care of the singularity of u'' at $\xi = 0$. To this end, we introduce the new function

$$z(\xi) = \xi^{2-k} u''(\xi), \tag{39}$$

where $k = (p+3)/(p+1)$, according to the expression (20). The function z is regular over the entire interval $0 \leq \xi \leq 1$.

In terms of z, we can find that

$$u(\xi) = (1-\xi)^2 \int_0^1 \frac{xz[\xi + x(1-\xi)]}{[\xi + x(1-\xi)]^{2-k}} \, dx, \qquad 0 \leq \xi \leq 1. \tag{40}$$

For the purpose of numerical calculation, we need to specify $u(0)$ explicitly,

$$u(0) = \int_0^1 x^{2/(p+1)} z(x) \, dx. \tag{41}$$

Similarly, for ϕ_u and β, we have

$$\phi_u(\xi) = \xi^{2/(p+1)} \phi(\xi), \qquad \beta = \int_0^1 \xi^{2/(p+1)} \phi(\xi) \, d\xi, \tag{42a}$$

$$\phi(\xi) = [(u^2(\xi) + 1)/z^2(\xi)]^{1/(p-1)}. \tag{42b}$$

From Eq. (17), we get

$$z(\xi) = \frac{[u^2(\xi) + 1]^{p/(p+1)}}{\{p[\beta q u(0) + \xi^{(p+3)/(p+1)} \int_0^1 (1-x) x^{2/(p+1)} \phi(x\xi) u(x\xi) \, dx]\}^{(p-1)/(p+1)}}. \tag{43}$$

The iteration scheme here is similar to that of the previous case and is given below.

Step 1.   Select an initial $z_0(\xi)$.

Step 2.   Update $u_i(\xi)$ according to

$$u_{i+1}(\xi) = (1-\xi)^2 \int_0^1 \frac{xz_i[\xi + x(1-\xi)]}{[\xi + x(1-\xi)]^{2-k}} \, dx, \qquad 0 \leq \xi \leq 1,$$

$$u_{i+1}(0) = \int_0^1 x^{2/(p+1)} z_i(x) \, dx.$$

Step 3.   Update $\phi_i(\xi)$,

$$\phi_{i+1}(\xi) = [(u_{i+1}^2(\xi) + 1)/z_i^2(\xi)]^{1/(p-1)}.$$

Step 4.   Update $z_i(\xi)$,

$$z_{i+1}(\xi) = \frac{[u_{i+1}^2(\xi) + 1]^{p/(p+1)}}{\{p[\beta q u_{i+1}(0) + \xi^{(p+3)/(p+1)} \int_0^1 (1-x)x^{2/(p+1)} \phi_{i+1}(x\xi)u_{i+1}(x\xi)\,dx]\}^{(p-1)/(p+1)}}.$$

Step 5.   If a given accuracy is not obtained, go back to Step 2.

Again, the new iteration scheme in this case is simpler than one used in Ref. 5.

5. Numerical Examples

To verify the correctness and efficiency of our new formulation, several numerical examples have been conducted. Some results are described in this section.

5.1. Case $p = 1$.   The iteration processes in this case are subject to the following accuracy criteria:

$$\|u_{i+1} - u_i\|/\|u_{i+1}\| < \epsilon, \qquad \|\alpha_{i+1} - \alpha_i\|/\|\alpha_{i+1}\| < \epsilon.$$

To simplify the numerical computation, we have approximated both u and α by spline functions through interpolation over their values at $N+1$ uniformly distributed discrete points in $0 \le \xi \le 1$. Throughout this section, $N = 10$ and $\epsilon = 10^{-4}$ have been used in all examples. The numerical integrations in the iterations are carried out using the recursive Simpson formula.

The iteration for u starts with u_0, found by solving (25) using the 5th order Runge–Kutta formulas. The iteration for α starts with $\alpha_0(\xi) = 1$. For various values of the nondimensional mass parameter q, Table 1 summarizes the percentage increase in the lowest frequency in comparison with that of the cantilever beam having uniform rectangular cross sections and having the same length, volume, and material as the optimum beam. The corresponding results obtained in Ref. 5 have also been included in the

Table 1.   Values of $\sqrt{\lambda/\lambda_c}$ for various values of q ($p = 1$).

	$q = 0.0375$	$q = 0.2233$	$q = 1.1027$	$q = 4 \times 10^4$
$p = 1$	2.3235	1.5025	1.2388	1.1547
$p = 1$	28.42†	2.56†	0.28†	—

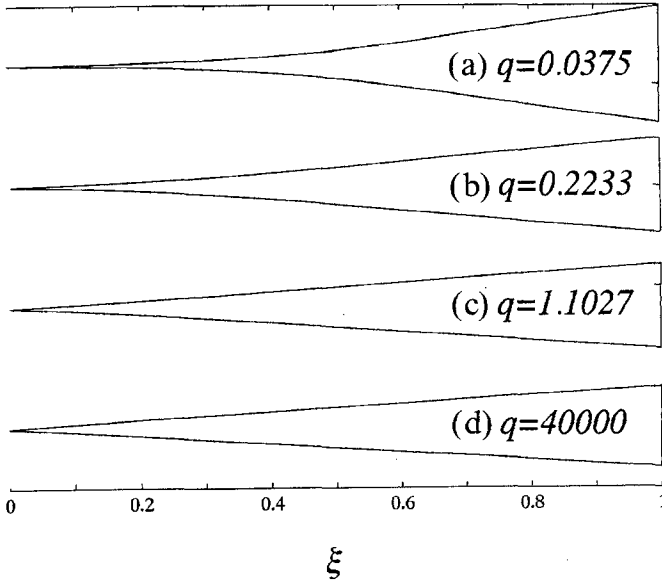†Results obtained by Karihaloo and Niordson (Ref. 5).

Fig. 1.   Optimum tapering of rectangular cross section of given height ($p = 1$).

table. As one can see, a large discrepancy exists between the two results. We believe that our results are more accurate, since they agree with the prediction from the asymptotic expressions (29) and (31); i.e., for large q,

$$\sqrt{\lambda/\lambda_c} = \sqrt{4/3} = 1.1547.$$

It is also obvious that, for $q = 1.1027$, the result $\sqrt{\lambda/\lambda_c} = 0.28$ in Ref. 5 is simply not logical, since λ should always be larger or equal to λ_c.

Figure 1 illustrates the corresponding variation in width of the rectangular cross section as a function of nondimensional coordinate ξ. Figure 2 presents the relationship between the eigenvalue λ and the mass parameter q for the optimum beam. The dashed curve in the figure is the corresponding result obtained from Eq. (30) for the cantilever beam of uniform cross section. From these curves, one can find the eigenvalue λ, and hence the lowest frequency of beams for a given mass parameter q. The percentage increase in the lowest frequency achieved through the optimum design is given in Fig. 3. Clearly, the results here have verified the asymptotic expressions (29) and (31).

5.2. Cases $p = 2$ and $p = 3$.   The iteration processes are continued until
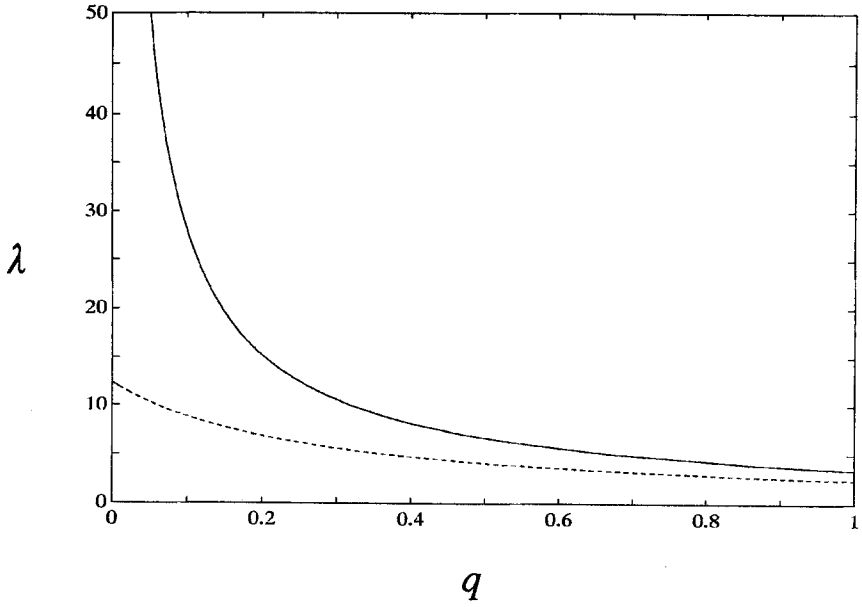
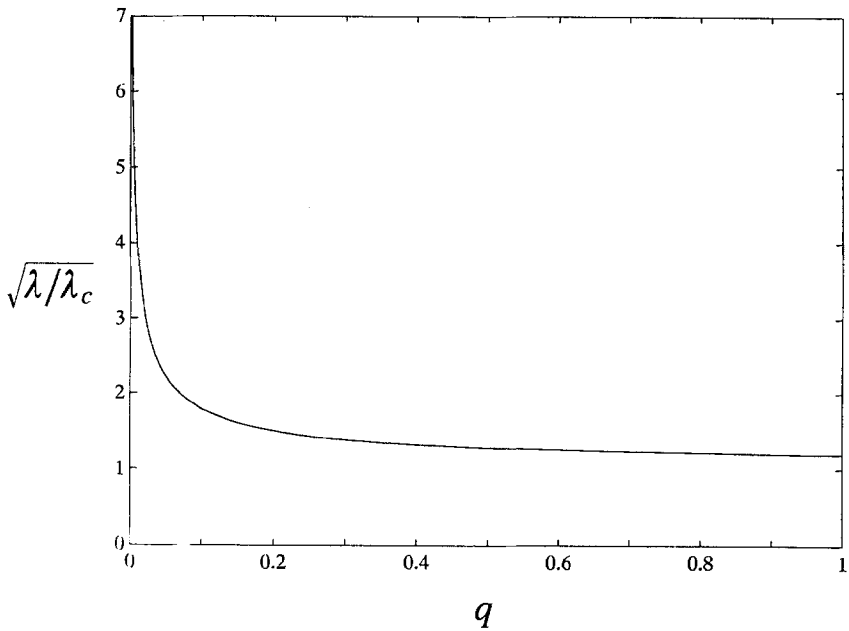Fig. 2.   Fundamental frequencies vs mass parameter q ($p = 1$).

Fig. 3.   Ratio of $\sqrt{\lambda/\lambda_c}$ vs mass parameter q ($p = 1$).

Table 2.   Values of $\sqrt{\lambda/\lambda_c}$ for various values of q ($p = 2, 3$).

$p = 2$	$q = 0$	$q = 0.0003$	$q = 0.03$	$q = 100$
$p = 2$	7.0025	4.5984	2.3450	1.2432
$p = 2$	6.78†	5.48†	3.36†	1.27†
$p = 3$	4.2937	3.3201	2.2868	1.2998
$p = 3$	4.25†	3.71†	2.30†	1.33†

†Results obtained by Karihaloo and Niordson (Ref. 5).

$$\|f_{i+1} - f_i\|/\|f_{i+1}\| \quad \text{or} \quad \|u_{i+1} - u_i\|/\|u_{i+1}\| < \epsilon,$$

$$\|z_{i+1} - z_i\|/\|z_{i+1}\| < \epsilon.$$

As in the previous case, f (or u for $q \neq 0$) and α are approximated by spline functions through interpolation over their values at $N + 1$ uniformly distributed discrete points on $0 \leq \xi \leq 1$.

The iterations always start with $z_0(\xi) = 1$ for all the cases. For various values of the nondimensional mass parameter q, Table 2 summarizes the increase in the lowest frequency in comparison with that of the corresponding cantilever beam of uniform cross section. The results of Ref. 5 have also been included.

For large q, using the perturbation method, we can find that

$$\lambda_q \simeq [(p + 3)/(p + 1)]^{p+1}, \quad \alpha(\xi) \simeq [(p + 3)/(p + 1)]\xi^{2/(p+1)}. \quad (44)$$

Equation (29) is a special case of Eq. (44). From Eqs. (44) and (31), for large tip mass one can find by simple calculation that increases of 24.23% and 29.90% in the lowest natural frequency can be achieved by the optimum tapering of the cantilever beams for $p = 2$ and $p = 3$, respectively. From Table 2, the results for $q = 100$ by the new iteration scheme are quite close to these two values, whereas a relatively large discrepancy can be found for the results given in Ref. 5.

Figures 4 and 5 present the corresponding variation of linear dimension of the cross section as a function of nondimensional coordinate ξ. Figure 6 describes the changes of the eigenvalue λ versus the mass parameter q for the optimum beam ($p = 2$ only). The dashed curve in the figure is the corresponding result obtained from (30) for the cantilever beam of uniform cross section. The percentage increase in the lowest frequency achieved through optimum design is illustrated in Fig. 7 ($p = 2$ only).

Numerical computations have shown that the new successive iteration formulations converge much faster than those used in Ref. 5. For example,
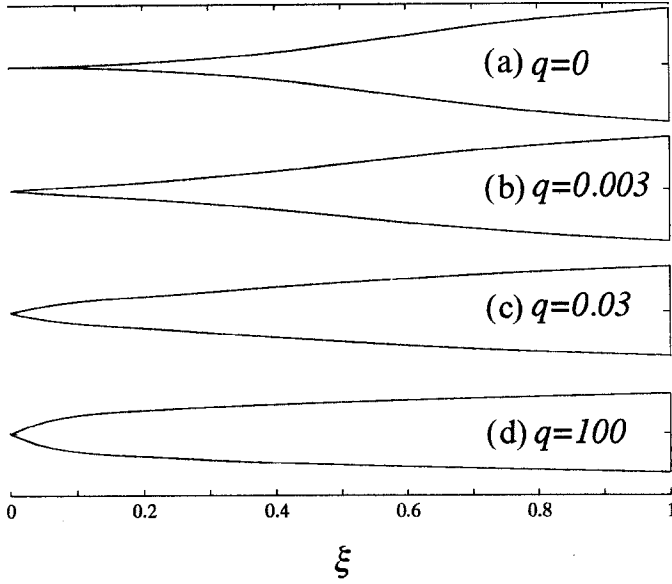
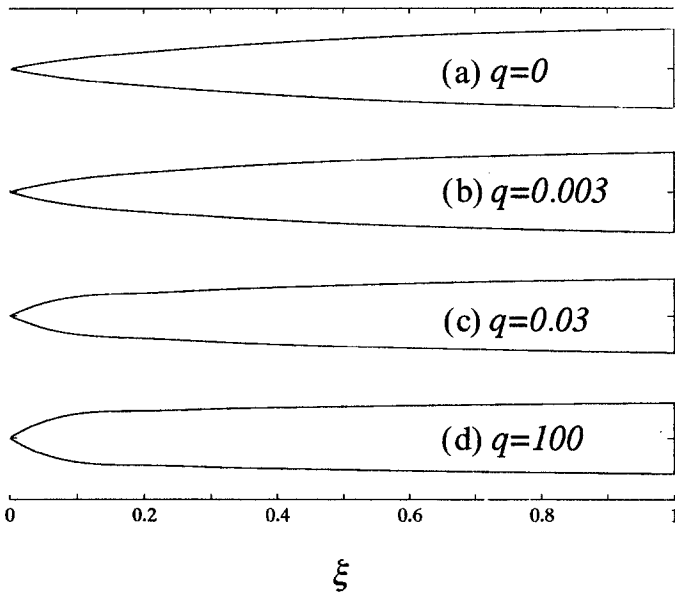Fig. 4. Optimum tapering of geometrically similar cross section ($p = 2$).

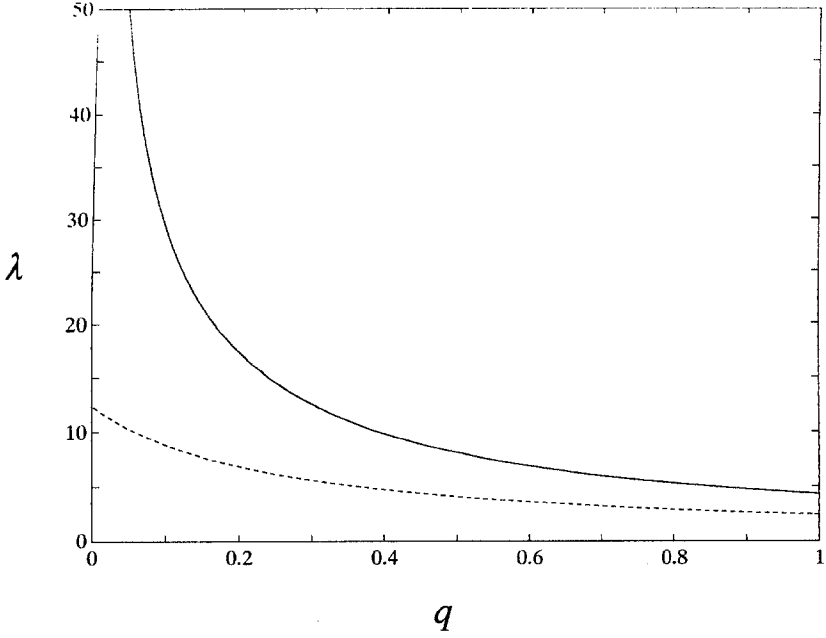Fig. 5. Optimum tapering of rectangular cross section of given width ($p = 3$).

Fig. 6.   Fundamental frequencies vs mass parameter q ($p = 2$).
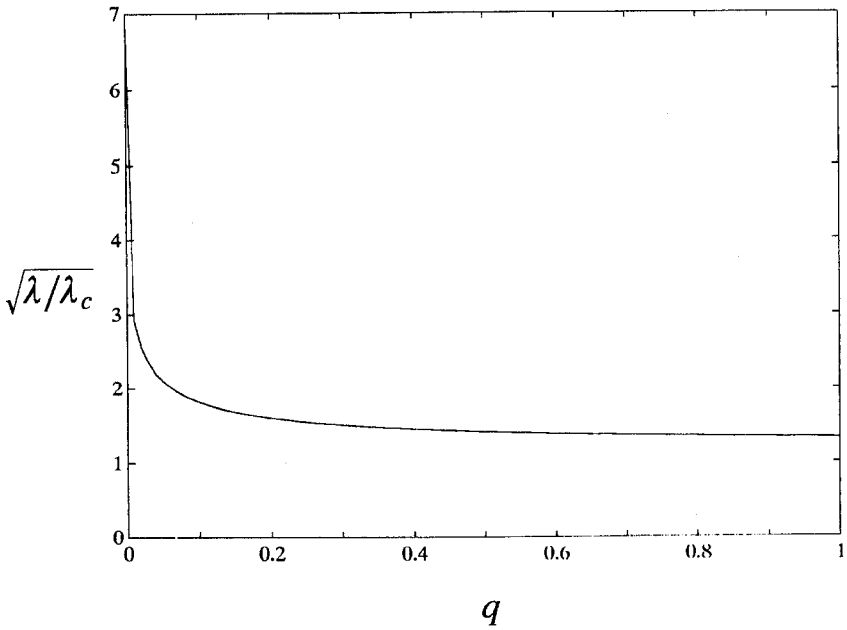

Fig. 7.   Ratio of $\sqrt{\lambda/\lambda_c}$ vs mass parameter q ($p = 2$).

in the case $p = 2$ and $q = 0$, it takes 134 iterations to achieve the specified accuracy by using the iteration scheme in Ref. 5, whereas only 33 iterations are needed with the new scheme.

6. Conclusions

By eliminating the implicit equations for the Lagrange multiplier in the iteration processes, new simplified iteration schemes have been devised for solving the problem of the optimum design of vibrating cantilever beams. The results of numerical computation have clearly indicated that a significant improvement in convergence rate has been achieved by the new schemes. The numerical results agree well with the analytical asymptotic expressions.

This investigation was begun while examining the problem of maximizing the fundamental frequency of a one-link flexible manipulator under a specified total weight constraint. The revisiting of the optimum design of vibrating cantilevers has provided us useful results for the corresponding problem for flexible manipulators.

References

1. BEESACK, P. R., *Isoperimetric Inequalities for the Nonhomogeneous Clamped Rod and Plate*, Journal of Mathematics and Mechanics, Vol. 8, pp. 471–481, 1959.
2. SCHWARZ, B., *On the Extrema of Frequencies of Nonhomogeneous Strings with Equimeasurable Density*, Journal of Mathematics and Mechanics, Vol. 10, pp. 401–409, 1961.
3. SCHAWARZ, B., *Some Results on the Frequencies of Nonhomogeneous Rods*, Journal of Mathematical Analysis and Applications, Vol. 5, pp. 169–176, 1962.
4. NIORDSON, F. I., *On the Optimal Design of a Vibrating Beam*, Quarterly of Applied Mathematics, Vol. 23, pp. 47–53, 1965.
5. KARIHALOO, B. L., and NIORDSON, F. I., *Optimum Design of Vibrating Cantilevers*, Journal of Optimization Theory and Applications, Vol. 11, pp. 638–654, 1973.
6. BRACH, R. M., *On the Extremal Fundamental Frequencies of Vibrating Beams*, International Journal of Solids and Structures, Vol. 4, pp. 667–674, 1968.
7. SHEU, C. Y., *Elastic Minimum-Weight Design for Specified Fundamental Frequency*, International Journal of Solids and Structures, Vol. 4, pp. 953–958, 1968.
8. PRAGER, W., and TAYLOR, J. E., *Problems of Optimal Structural Design*, Journal of Applied Mechanics, Vol. 35, pp. 102–106, 1968.

9. OLHOFF, N., and RASMUSSEN, S. H., *On Single and Bimodal Optimum Buckling Loads of Clamped Columns*, International Journal of Solids and Structures, Vol. 13, pp. 605–614, 1977.

10. WANG, F. Y., *On the Extremal Fundamental Frequencies of One-Link Flexible Manipulators*, International Journal of Robotics Research, Vol. 13, pp. 162–170, 1994.

11. WANG, F. Y., *Optimum Design of Flexible Manipulators: Integration of Control and Construction*, Working Paper 40-91, SIE Department, University of Arizona, Tucson, Arizona, 1991.

12. MEIROVITCH, L., *Computational Methods in Structural Dynamics*, Sijthoff and Noordhoff, Rockville, Illinois, 1980.

13. HAFTKA, R. T., GÜRDAL, Z., and KAMAT, M. P., *Elements of Structural Optimization*, 2nd Edition, Kluwer Academic Publishers, Boston, Massachusetts, 1990.

14. NAYFEH, A. H., *Perturbation Method*, John Wiley, New York, New York, 1973.

15. WANG, F. Y., and GUAN, G. G., *Influence of Rotatory Inertia, Shear Deformation, and Loading on Vibration Behaviors of Flexible Manipulators*, Journal of Sound and Vibrations, Vol. 167, pp. 171–189, 1993.