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Doubly Coprime Fractional Representations of Generalized Dynamical Systems

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Abstract—**Explicit formulas for doubly coprime fractional representations of the transfer matrix of a generalized dynamical system are given in terms of a stabilizable and detectable state-space realization of the transfer matrix. These formulas establish a possible way to use the fractional representation approach in the synthesis of generalized dynamical systems.**

I. Introduction

Both the fractional representation approach and the generalized dynamical system theory have received a great deal of attention over the past few years (see [2], [5], [6], and the references therein). It has been shown that the fractional representation approach is a powerful tool in the analysis and synthesis of linear feedback systems. A connection between state-space and doubly coprime fractional representations has been given recently in [3] by Nett, Jacobson, and Balas. However, little has been done to apply this powerful method to the corresponding problems in generalized dynamical systems. In this note we extend the result in [3] to generalized dynamical systems. This extension establishes a way to use the fractional representations for generalized dynamical systems.

II. Preliminaries

Let $\Omega \subset \boldsymbol{C}$ denote any closed superset of the closed right-half complex plane which is symmetric with respect to the real axis. Let $\boldsymbol{H}$ denote the ring of proper rational functions which are analytic in Ω. A rational matrix $X \in \boldsymbol{H}(s)^{p\times m}$ is said to be Ω-stable if it has entries in $\boldsymbol{H}$.

A pair of real matrices (E, A) is said to be Ω-Hurwitz if matrix $(sE - A)^{-1}$ exists and is Ω-stable. A triple of real matrices $(E, A, B) \in \boldsymbol{R}^{n\times n} \times \boldsymbol{R}^{n\times n} \times \boldsymbol{R}^{n\times m}$ is said to be Ω-stabilizable if there exists a $K \in \boldsymbol{R}^{m\times n}$ such that $(E, A - BK)$ is Ω-Hurwitz. Similarly, a triple of real matrices $(C, E, A) \in \boldsymbol{R}^{p\times n} \times \boldsymbol{R}^{n\times n} \times \boldsymbol{R}^{n\times n}$ is said to be Ω-detectable if there exists a matrix $F \in \boldsymbol{R}^{n\times p}$ such that $(E, A - FC)$ is Ω-Hurwitz.

III. Main Results

Consider a generalized dynamical system described by the equations

$$E\dot{x} = Ax + Bu, \tag{1}$$

$$y = Cx \tag{2}$$

where $E, A \in \boldsymbol{R}^{n\times n}$, $B \in \boldsymbol{R}^{n\times m}$, $C \in \boldsymbol{R}^{p\times n}$ are real matrices. As done by Rosenbrock [4] we henceforth make the standard assumption that $|sE - A| \not\equiv 0$. The transfer matrix of the system is $G(s) = C(sE - A)^{-1}B$. Our objective is to derive the doubly coprime factorizations of $G(s)$. The factorizations obtained are given in the following theorems.

Theorem 1: Given the system (1), (2), suppose the triples (E, A, B), (C, E, A) are Ω-stabilizable and detectable, respectively. Select matrices $K \in \boldsymbol{R}^{m\times n}$, $F \in \boldsymbol{R}^{n\times p}$ such that both $(E, A - BK)$ and $(E, A - FC)$ are Ω-Hurwitz. Define

$$N(s) = CH_c^{-1}B,\ D(s) = I - KH_c^{-1}B \tag{3}$$

$$U(s) = KH_0^{-1}F,\ V(s) = I + KH_0^{-1}B \tag{4}$$

$$\tilde{N}(s) = CH_0^{-1}B,\ \tilde{D}(s) = I - CH_0^{-1}F \tag{5}$$

$$\tilde{U}(s) = KH_c^{-1}F,\ \tilde{V}(s) = I + CH_c^{-1}F \tag{6}$$

where

$$H_c(s) = H(s) + BK,\ H_0(s) = H(s) + FC,\ H(s) = sE - A \tag{7}$$

then:

1) all eight matrices described by (3)-(6) are Ω-stable;
2) $D(s)$ and $\tilde{D}(s)$ are nonsingular;
3) $G(s) = N(s)D(s)^{-1} = \tilde{D}(s)^{-1}\tilde{N}(s)$;
4)

$$\begin{bmatrix} V(s) & U(s) \\ -\tilde{N}(s) & \tilde{D}(s) \end{bmatrix} \begin{bmatrix} D(s) & -\tilde{U}(s) \\ N(s) & \tilde{V}(s) \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

Remarks: The above theorem is a straightforward extension of the theorem given in [3] where $E = I$, i.e., the normal dynamical system was considered. As in [3], this result is readily extended to the case $G(s) = C(sE - A)^{-1}B + W$, where $W \in H^{p\times m}$ by: 1) adding $WD(s)$ to the expression for $N(s)$; 2) adding $\tilde{D}(s)W$ to the expression for $\tilde{N}(s)$; 3) subtracting $U(s)W$ from the expression for $V(s)$; and 4) subtracting $W\tilde{U}(s)$ from the expression for $\tilde{V}(s)$.

Proof:

i) Follows immediately from the definition of an Ω-Hurwitz pair of matrices.

ii) Consider equations

$$D(s) = I - KH_c^{-1}B = I - K(H + BK)^{-1}B = (I + KH^{-1}B)^{-1}$$

and

$$|H_c| = |H + BK| = |H|\ |I + H^{-1}BK| = |H|\ |I + BKH^{-1}| = |H|\ |I + KH^{-1}B|.$$

It follows that

$$|D(s)| = \frac{|H(s)|}{|H_c(s)|}$$

which indicates that $|D(s)| \not\equiv 0$.

Similarly, one shows $|\tilde{D}(s)| = |H(s)/H_0(s)| \not\equiv 0$.

iii) Write

$$\begin{aligned} N(s)D^{-1}(s) &= CH_c^{-1}B(I - KH_c^{-1}B)^{-1} \\ &= CH_c^{-1}(I - BKH_c^{-1})^{-1}B \\ &= C((I - BKH_c^{-1})H_c)^{-1}B \\ &= C(H_c - BK)^{-1}B \\ &= CH^{-1}B = G(s) \end{aligned}$$

the matrix identity $B(I - AB)^{-1} = (I - BA)^{-1}B$ is used in the above second step.

In a similar fashion, one shows $G(s) = \tilde{D}(s)^{-1}\tilde{N}(s)$.

iv) We only need to verify three equalities, the fourth being contained

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in iii). Now,

$$\begin{aligned} U(s)N(s)+V(s)D(s) &= (KH_0^{-1}F)(CH_c^{-1}B)+(I+KH_0^{-1}B)(I-KH_c^{-1}B) \\ &= KH_0^{-1}(FCH_c^{-1}B)+I+KH_0^{-1}B-KH_c^{-1}B-KH_0^{-1}BKH_c^{-1}B \\ &= I+KH_0^{-1}(FC+H_c-H_0-BK)H_c^{-1}B \\ &= I+KH_0^{-1}(FC+BK-FC-BK)H_c^{-1}B \\ &= I. \end{aligned}$$

Also

$$\begin{aligned} V(s)\bar{U}(s) &= (I+KH_0^{-1}B)(KH_c^{-1}F) \\ &= K(I+H_0^{-1}BK)H_c^{-1}F \\ &= KH_0^{-1}(H_0+BK)H_c^{-1}F \\ &= KH_0^{-1}(H_c+FC)H_c^{-1}F \\ &= KH_0^{-1}(I+FCH_c^{-1})F \\ &= KH_0^{-1}F(I+CH_c^{-1}F) \\ &= U(s)\bar{V}(s). \qquad \square \end{aligned}$$

Finally, one shows $\bar{N}(s)\bar{U}(s) + \bar{D}(s)\bar{V}(s) = I$ by manipulations similar to those above.

The next two results were motivated by the state and state derivative feedback $u = K_1x - K_2\dot{x}$.

Theorem 2.a: Suppose there exist matrices $K_1, K_2 \in \boldsymbol{R}^{m\times n}$, $F \in \boldsymbol{R}^{n\times p}$ such that: 1) matrix $E + BK_2$ is nonsingular; 2) $(E + BK_2, A - BK_1)$ and $(E, A - FC)$ are Ω-Hurwitz; 3) $\lim_{s\to\infty} K_2(sE - A + FC)^{-1}B = 0$ and $\lim_{s\to\infty} K_2(sE - A + FC)^{-1}F = 0$. Then Theorem 1 still holds if one replaces K in the expression (3)–(7) by $K_1 + sK_2$.

Remarks: The condition 1) in the above theorem is the necessary and sufficient condition for the generalized dynamical system (1) to be normalizable [8]. Conditions 1) and 2) are equivalent to all uncontrollable modes of (E, A, B) lying in $\boldsymbol{C} - \Omega$ [1], [8]. Condition 3) is satisfied when E is nonsingular (i.e., the normal dynamical system).

The proof of the above theorem is entirely analogous to that of Theorem 1.

The following result is dual to Theorem 2.a.

Theorem 2.b: Suppose there exist matrices $K \in \boldsymbol{R}^{m\times n}$, $F_1, F_2 \in \boldsymbol{R}^{n\times p}$ such that: 1) $(E, A - BK)$ and $(E + F_2C, A - F_1C)$ are Ω-Hurwitz; 2) matrix $E + F_2C$ is nonsingular; and 3) $\lim_{s\to\infty} C(sE - A + BK)^{-1}F_2 = 0$ and $\lim_{s\to\infty} K(sE - A + BK)^{-1}F_2 = 0$. Then Theorem 1 still holds if one replaces F in the expression (3)–(7) by $F_1 + sF_2$.

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The Explicit Structure of Inner Matrices and Its Application in H^∞-Optimization

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***Abstract*—A very simple expression for inner matrices is obtained which allows a detailed investigation of their structure. This expression is used to solve the H^∞-interpolation problem in a most simple way. A solution to this problem is obtained explicitly in terms of the interpolation parameters without solving any equation. It is found that the resulting solution encounters a reduction in its order at the critical point, and the characteristics of the resulting reduced-order solution are investigated.**

I. INTRODUCTION

One of the most important computational steps in the solution of H^∞-minimization problems in control theory is the derivation of inner matrices. These matrices are, in a sense, extensions of the single-input, single-output, all-pass networks and the calculation problem in the H^∞-minimization context is to find such an inner that satisfies given interpolation conditions [1]. These inners are also used in circuit theory [2], [3] and in the problems of LQ stabilization [4] and LQ optimal tracking [5].

Two main approaches have been suggested for solving the inners' interpolation problem. The first is the classical function-theoretic approach that is based on the Pick and Nevanlinna theory [6]. This approach has been extended to the multivariable case by [7] and [8] and recently by [9], where the interpolation condition is imposed on some given directions instead of requiring full rank matrix matching [7]. The second approach is based on the Hankel-norm approximation technique [10]–[12]. Although this state-space approach is computationally effective, it lacks the structural simplicity and the physical insight that can be gained by the first approach.

The problem with the first approach is, however, that even if one uses the directional interpolation method of [9], which resolves the overdetermination that is achieved by the matrix Pick–Nevanlinna method of [7] and applies a simpler calculation procedure, the obtained results are still rather complicated. Since they are derived by using the Schur–Nevanlinna algorithm, these results cannot be expressed in closed form and the direct effect of the elementary parameters of the interpolation problem on the structure of the resulting inner cannot be explicitly obtained.

In the present note, we introduce a state-space approach that solves the H^∞-optimization problem via the directional interpolation problem of [9]. Unlike [9], the results for the inner matrices will be found in closed-form, explicitly in terms of the interpolation directions, without using any iterative algorithm. These results will be obtained by a simple substitution of the problem parameters.

The motivation to this approach is the result that is obtained in the LQ stabilization problem, where it is required to stabilize an unstable plant without putting any weight on the system states [4]. It is found there that the return difference matrix of the optimal LQ closed-loop control scheme is an inner. This inner possesses a very simple structure whose parameters are the system elementary matrices and the corresponding Kalman gain matrix. Based on this result we introduce a general approach to the H^∞-interpolation problem which obviates the need for solving Riccati or Lyapunov type equations. By this approach we find necessary and sufficient conditions for the existence of a solution to the H^∞-interpolation problem and we derive a simple explicit expression for the resulting inner matrix.

The fundamental H^∞-optimization is formulated in Section II where the interpolation requirements on the inner matrix are derived. The structure of this matrix is investigated in Section III, where an expression

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