

Distributed algorithm for dissensus of a class of networked multiagent systems using output information

Hongwen Ma^{1,2} · Derong Liu³ · Ding Wang¹ · Xiong Yang⁴ · Hongliang Li¹

© Springer-Verlag Berlin Heidelberg 2016

Abstract In this paper, a distributed algorithm is developed to solve the dissensus of a class of networked multiagent systems only using output information. By introducing a gauge transformation, the dissensus problem is transformed to the problem of demonstrating that (A, B, C) is stabilizable and detectable. If the networked multiagent systems can reach dissensus, the signed digraph is structurally balanced containing a spanning tree. Furthermore, by solving a Riccati equation, the necessary condition becomes a necessary and sufficient condition. Finally, two examples are provided to illustrate our results. There are three main contributions in this paper: (1) a distributed algorithm with output information is introduced to deal with the difficulty of obtaining

relative full-state observations; (2) the undirected communication graph is extended to the signed digraph which is more practical in physical implementations; (3) the method established in this paper is also applicable to discrete-time networked multiagent systems by using a gauge transformation, which further demonstrates the generality of our results.

Keywords Distributed control · Dissensus · Multiagent systems · Output feedback

1 Introduction

In recent years, networked multiagent systems have been a hot topic within the control community (Cheng et al. 2010; Zhang et al. 2015a; He and Ge 2016; Liu and Kroll 2015; Quteishat et al. 2011; Cheng et al. 2014; Shen et al. 2015; Ma et al. 2015b; Liang et al. 2014; Liu and Jiang 2014; Luo et al. 2015; Ma et al. 2016a, b, 2015a; Wang et al. 2016c; Xie and Wang 2014; Wang et al. 2016b). The idea of distributed or decentralized algorithms can be traced back to Bertsekas and Tsitsiklis (1989) and Tsitsiklis (1984) to adapt to the advent of networks. Then, Jadbabaie et al. (2003) introduced nearest neighbor rules into networked multiagent systems according to Vicsek's model (Vicsek et al. 1995). After that, there has been tremendous interest in consensus of networked multiagent systems (Olfati-Saber et al. 2007; Cao et al. 2013).

There are various types of consensus in networked multiagent systems. To begin with, Lin and Ren (2014) proposed a constrained consensus with communication delays where the system can be transformed into an equivalent undelayed system to tackle the complex properties of delays. Cheng et al. (2014) discussed a mean square consensus algorithm for linear networked multiagent systems with communication noises and fixed topologies. Distributed algorithms for

Communicated by V. Loia.

✉ Derong Liu
derong@ustb.edu.cn

Hongwen Ma
mahongwen2012@ia.ac.cn

Ding Wang
ding.wang@ia.ac.cn

Xiong Yang
yangxiong1983@126.com

Hongliang Li
hongliang.li@ia.ac.cn

- ¹ The State Key Laboratory of Management and Control for Complex Systems, Institute of Automation, Chinese Academy of Sciences, 100190 Beijing, China
- ² University of Chinese Academy of Sciences, 100190 Beijing, China
- ³ School of Automation and Electrical Engineering, University of Science and Technology Beijing, 100083 Beijing, China
- ⁴ School of Electrical Engineering and Automation, Tianjin University, 300072 Tianjin, China

networked multiagent systems are widely investigated in different fields such as particle swarm optimization (Scheepers and Engelbrecht 2016), fuzzy systems (Zhang et al. 2015b), mobile robots (Liu and Jiang 2013), adaptive dynamic programming (Zhang et al. 2014; Liu et al. 2014; Wang et al. 2016a). Due to the generality of the consensus problems mentioned above, the study of dissensus is a meaningful extension to the field of networked multiagent systems.

Bipartite graph (Diestel 2000) is one of the basic concepts in graph theory, and it is appropriate for representing the communication topology of dissensus. In several physical scenarios, it is suitable to assume that some of the agents are cooperative while the rest are competitive. For example, the polarization of the community can be divided into two groups holding the opposite opinions, such as two-group political systems in Fig. 1. In traditional consensus problems, a consensus condition on when to converge and converging to what values was given in Wieland et al. (2011). Furthermore, in Ma and Zhang (2010), the consensusability of the networked multiagent systems not only depends on the agents' dynamics but also on the communication topology. Dissensus has some similar aspects to consensus problems. Therefore, the discussion about under what conditions dissensus can be reached is essential.

To the best of authors' knowledge, pioneering works of dissensus can be referred to Smith (1995). Altafini (2013) introduced the negative weights to the communication topology and demonstrated that dissensus can be reached in the presence of antagonistic interactions. However, it only dealt with the simplest situation where the dynamics of each agent was related to the distributed control without any information of the system matrix A . Subsequently, Hu and Zheng (2013) extended the dissensus to the formation control (Hu et al. 2013) and directed signed networks (Hu and Zheng 2013) with the same dynamics. Additionally, Valcher and Misra (2014) discussed a more complex situation where the dynamics of networked multiagent systems were high-order with antagonistic interactions and dissensus can be reached under the stabilizability assumption with an equilibrium between two fully competing groups.

Nevertheless, in all the papers mentioned above, the communication topologies are undirected or the distributed algorithms are based on full-state-feedback measurements which are impractical in real-world implementations. There-

fore, in this paper, a distributed algorithm is developed to guarantee the dissensus of networked multiagent systems over directed networks. Moreover, the necessary condition can be extended to the corresponding discrete-time systems.

The main contributions of this paper are listed as follows.

1. Output information is introduced to the distributed algorithm to deal with the difficulty of obtaining relative full-state observations.
2. The undirected signed graph (Valcher and Misra 2014) is extended to a structurally balanced digraph which is more practical in physical implementations.
3. The conclusions obtained in this paper can be generalized to discrete-time networked multiagent systems.

The rest of the paper is organized as follows. Basic definitions of dissensus and properties of signed graph are given in Sect. 2. Distributed algorithm with output information is developed for directed networks in Sect. 3. Implementations of dissensus are conducted to demonstrate the validity of the developed method in Sect. 4. The concluding remark is given in Sect. 5.

2 Preliminaries

2.1 Signed digraph

A triplet $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$ is called a (weighted) signed graph if $\mathcal{V} = \{1, 2, \dots, N\}$ is the set of nodes, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of edges, and $\mathcal{A} = (A_{ij}) \in \mathbb{R}^{N \times N}$ is the matrix of signed weights of \mathcal{G} . Here, we denote A_{ij} as the element of the i th row and j th column of the matrix \mathcal{A} . The i th node in a signed graph \mathcal{G} represents the i th agent, and a directed path from node i to node j is denoted as an ordered pair $(i, j) \in \mathcal{E}$ which means that agent i can directly transfer its information to agent j .

\mathcal{A} is called the adjacency matrix of a signed graph \mathcal{G} with real elements, and we use the notation $\mathcal{G}(\mathcal{A})$: $A_{ij} \neq 0 \Leftrightarrow (j, i) \in \mathcal{E}$ to represent the signed graph corresponding to \mathcal{A} . Note that self-loops will not be considered in this paper, i.e., $A_{ii} = 0, i = 1, 2, \dots, N$. In a directed graph (digraph), a pair of edges sharing the same nodes $(i, j), (j, i) \in \mathcal{E}$ is called a digon (Altafini 2013). We assume that $A_{ij}A_{ji} \geq 0$, which means that all digons cannot have the opposite signs. We call this property **digon sign-symmetric**. Otherwise, we call it digon sign-nonsymmetric. Given a signed digraph $\mathcal{G}(\mathcal{A})$, \mathcal{C}_r is termed as the row connectivity matrix of \mathcal{A} with diagonal elements $c_{r,ii} = \sum_{j \in \mathcal{N}_i} |A_{ij}|$, where $\mathcal{N}_i = \{j \in \mathcal{V} | (j, i) \in \mathcal{E}\}$ is the set of neighboring nodes of agent i and other elements $c_{r,ij} = 0, i \neq j$. Antagonistic networks contain competing interactions among some

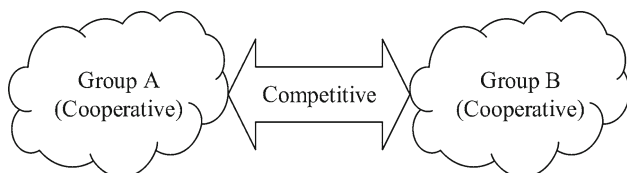


Fig. 1 Two groups with cooperative behaviors inside and competitive behaviors between each other

agents. Thus, the signed digraph $\mathcal{G}(\mathcal{A})$ is a good choice to represent the competing behaviors when $\mathcal{A}_{ij} < 0$.

2.2 Dissensus

We consider a class of networked multiagent systems consisting of N agents with continuous-time dynamics as follows:

$$\begin{aligned}\dot{x}_i(t) &= Ax_i(t) + Bu_i(t), \\ y_i(t) &= Cx_i(t), \quad i = 1, 2, \dots, N,\end{aligned}\quad (1)$$

where $x_i(t) \in \mathbb{R}^n$, $u_i(t) \in \mathbb{R}$ and $y_i(t) \in \mathbb{R}^m$, are the state, control and output of the i th agent, respectively. Here we assume that $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times 1}$ and $C \in \mathbb{R}^{m \times n}$ are constant matrices. The communication topology can be represented by a signed digraph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$ defined in Sect. 2.1. The interaction between the i th and the j th agent is cooperative if $\mathcal{A}_{ij} > 0$; otherwise, it is competitive if $\mathcal{A}_{ij} < 0$.

Following the definition of unsigned graphs in most literatures, we define the row Laplacian matrix corresponding to the adjacency matrix \mathcal{A} of the signed digraph $\mathcal{G}(\mathcal{A})$ as:

$$\mathcal{L} = \mathcal{C}_r - \mathcal{A}, \quad (2)$$

where \mathcal{C}_r is the row connectivity matrix of \mathcal{A} . Thus,

$$\mathcal{L}_{ij} = \begin{cases} \sum_{k \in \mathcal{N}_i} |\mathcal{A}_{ik}|, & \text{if } i = j; \\ -\mathcal{A}_{ij}, & \text{if } i \neq j. \end{cases} \quad (3)$$

To prove our theorems, the following two definitions are needed.

Definition 1 (Structurally balanced, cf. Altafini (2013)) A signed digraph $\mathcal{G}(\mathcal{A})$ is said to be structurally balanced if it contains \mathcal{V}_1 and \mathcal{V}_2 as a bipartition of the node set \mathcal{V} , where $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$, $\mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset$ such that $\mathcal{A}_{ij} \geq 0, \forall i, j \in \mathcal{V}_p$ ($p \in \{1, 2\}$); $\mathcal{A}_{ij} \leq 0, \forall i \in \mathcal{V}_p, j \in \mathcal{V}_q, p \neq q$ ($p, q \in \{1, 2\}$). Otherwise, it is called structurally unbalanced.

Definition 2 (Gauge transformation) A gauge transformation is a change of the orthant order in \mathbb{R}^n operated by a matrix $\mathcal{D}_1 \in \mathcal{D} = \{\mathcal{D}_1 | \mathcal{D}_1 = \text{diag}(\xi), \xi = [\xi_1, \xi_2, \dots, \xi_n]^T, \xi_i = +1 \text{ or } -1, i = 1, 2, \dots, n\}$ which contains all the gauge transformations in \mathbb{R}^n .

We assume that the signed digraph \mathcal{G} is structurally balanced and digon sign-symmetric throughout this paper. In addition, the corresponding adjacency matrix \mathcal{A} is strongly connected. According to Definition 1, this is equivalent to saying that the agents can be split into two disjoint groups, the cooperative behaviors between any two agents belonging to the same group and the competitive behaviors between any

two agents belonging to the different groups. In Valcher and Misra (2014), the multiagent system with the dynamics (1) can evolve as follows:

$$\lim_{t \rightarrow \infty} x_i(t) = \begin{cases} \varepsilon(t), & \forall i \in \mathcal{V}_1, \\ -\varepsilon(t), & \forall i \in \mathcal{V}_2. \end{cases} \quad (4)$$

Then, the multiagent system can reach **dissensus**, where $\varepsilon(t)$ is a function with some certain value related to time t .

3 Dissensus using only output information

Over the antagonistic networks, Valcher and Misra (2014) adopted the static state-feedback distributed algorithm:

$$\begin{aligned}u_i(t) &= -\tilde{K} \sum_{j \in \mathcal{N}_i} |\mathcal{A}_{ij}| [x_i(t) - \text{sgn}(\mathcal{A}_{ij})x_j(t)], \\ t &\geq 0, i = 1, 2, \dots, N,\end{aligned}\quad (5)$$

where $\tilde{K} \in \mathbb{R}^{1 \times n}$ is the feedback weighted constant vector to be designed and $\text{sgn}(\cdot)$ is the sign function. Algorithm (5) is distributed and only depends on the relative errors of the static states between the i th agent and its corresponding neighbors. In Valcher and Misra (2014), it is mentioned that with the state-feedback distributed algorithm, the N agents can reach dissensus in an undirected signed graph. However, it requires the whole information of the N agents' relative states which is usually impractical in physical implementations. Hence, it is more practical to take the available output measurements into consideration. We assume that the output information of all the agents can be measured without external noises.

In this paper, the distributed algorithm is designed as follows:

$$\begin{aligned}u_i(t) &= -K \sum_{j \in \mathcal{N}_i} |\mathcal{A}_{ij}| [y_i(t) - \text{sgn}(\mathcal{A}_{ij})y_j(t)], \\ t &\geq 0, i = 1, 2, \dots, N,\end{aligned}\quad (6)$$

where $K \in \mathbb{R}^{1 \times m}$ is the feedback weighted constant vector to be designed. Note that $\mathcal{A}_{ij} \neq 0 \Leftrightarrow (j, i) \in \mathcal{E}$. Thus, the equivalent form of (6) is

$$\begin{aligned}u_i(t) &= -K \sum_{j=1}^N |\mathcal{A}_{ij}| [y_i(t) - \text{sgn}(\mathcal{A}_{ij})y_j(t)], \\ t &\geq 0, i = 1, 2, \dots, N.\end{aligned}\quad (7)$$

Before proceeding, we need to specify the definition of the dissensus analogous to Ma and Zhang (2010).

Definition 3 For the networked multiagent system (1), if for any initial conditions $x_i(0)$, $i \in \mathcal{V}$, the distributed algorithm makes the following conditions hold:

$$\begin{cases} \lim_{t \rightarrow \infty} \|x_j(t) - x_i(t)\| = 0, & \forall i, j \in \mathcal{V}_1 \text{ or } \forall i, j \in \mathcal{V}_2; \\ \lim_{t \rightarrow \infty} \|x_j(t) + x_i(t)\| = 0, & \forall i \in \mathcal{V}_1 \text{ and } \forall j \in \mathcal{V}_2, \end{cases} \quad (8)$$

where \mathcal{V}_1 and \mathcal{V}_2 are distinct node sets defined in Definition 1, then we say that the networked multiagent system (1) can reach dissensus.

Remark 1 The traditional dissensus requires that the states of the agents must converge to two different certain final values. Here whether the states themselves will converge or not, the differences of the states between any pair of agents in the same group and the summations of the states between any pair of agents in the opposite groups are both required to be zero. Therefore, it is a more relaxed condition.

Now we are in the position to establish our main theorems.

Theorem 1 Consider the networked multiagent system (1) with distributed algorithm (7), where $K \in \mathbb{R}^{1 \times m}$ is a vector to be determined. If all the agents can asymptotically reach dissensus, then (A, B, C) is stabilizable and detectable, and the communication digraph $\mathcal{G}(A)$ contains a spanning tree with structural balance.

Proof Without loss of generality, we renumber the order of agents and assume that $\mathcal{V}_1 = \{1, 2, \dots, k\}$ and $\mathcal{V}_2 = \{k+1, k+2, \dots, N\}$. The agents in \mathcal{V}_1 or \mathcal{V}_2 are cooperative but antagonistic between the two groups. With the former assumptions and Lemma 1 in Altafini (2013), we can choose a gauge transformation \mathcal{D}_1 from the set \mathcal{D} defined in Definition 2, such that

$$\mathcal{D}_1 = \begin{bmatrix} I_k & \mathbf{0}_{k \times (N-k)} \\ \mathbf{0}_{(N-k) \times k} & -I_{N-k} \end{bmatrix} \quad (9)$$

satisfies that $\mathcal{D}_1 \mathcal{A} \mathcal{D}_1$ is a nonnegative matrix and $\mathbf{0}_{m \times n}$ is an $m \times n$ matrix with all elements equal to 0. Let

$$\eta = [\eta_1, \eta_2, \dots, \eta_N]^T \in \mathbb{R}^N$$

be a left eigenvector of \mathcal{L} with $\lambda_1(\mathcal{L}) = 0$, where $\lambda_1(\mathcal{L})$ denotes the zero eigenvalue of \mathcal{L} . $\eta^T \mathcal{D}_1$ is the left eigenvector of $\mathcal{D}_1 \mathcal{L} \mathcal{D}_1$, which corresponds to the zero eigenvalue, and $\mathcal{D}_1 \mathcal{L} \mathcal{D}_1$ is the Laplacian matrix related to the nonnegative matrix $\mathcal{D}_1 \mathcal{A} \mathcal{D}_1$. Thus, we assume that

$$\eta_i = \begin{cases} \frac{1}{N}, & \text{if } i \in \mathcal{V}_1; \\ -\frac{1}{N}, & \text{if } i \in \mathcal{V}_2. \end{cases}$$

Let $\mathbf{1}_n$ denote a column vector with all elements equal to 1 and \otimes denote the Kronecker product. Construct a nonsingular

matrix as follows:

$$\Phi = \begin{bmatrix} \eta_1 & \eta_2 & \cdots & \eta_k & \eta_{k+1} & \cdots & \eta_N \\ -\mathbf{1}_{k-1} & I_{k-1} & & & \mathbf{0}_{(k-1) \times (N-k)} \\ -\mathbf{1}_{N-k} & \mathbf{0}_{(N-k) \times (k-1)} & & & -I_{N-k} \end{bmatrix}, \quad (10)$$

where $k \geq 2$ and $N \geq 3$. Thus, the corresponding coordinate transformation is

$$\begin{bmatrix} \chi(t) \\ \delta_2(t) \\ \vdots \\ \delta_k(t) \\ \vdots \\ \delta_N(t) \end{bmatrix} = (\Phi \otimes I_n) x(t) \\ = \begin{bmatrix} \eta_1 I_n & \gamma_1 & \gamma_2 \\ -\mathbf{1}_{k-1} \otimes I_n & I_{n \times (k-1)} & \mathbf{0}_{n(k-1) \times n(N-k)} \\ -\mathbf{1}_{N-k} \otimes I_n & \mathbf{0}_{n(N-k) \times n(k-1)} & -I_{n \times (N-k)} \end{bmatrix} \\ \times \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_k(t) \\ \vdots \\ x_N(t) \end{bmatrix},$$

where

$$\begin{aligned} \gamma_1 &= [\eta_2 I_n \quad \eta_3 I_n \quad \cdots \quad \eta_k I_n], \\ \gamma_2 &= [\eta_{k+1} I_n \quad \eta_{k+2} I_n \quad \cdots \quad \eta_N I_n]. \end{aligned}$$

Note that $\chi(t)$ is a column vector and $\delta_i(t) = x_i(t) - x_1(t)$, $\delta_i(t) \in \mathbb{R}^n$, $\forall i \in \mathcal{V}_1$, while $\delta_i(t) = -x_i(t) - x_1(t)$, $\delta_i(t) \in \mathbb{R}^n$, $\forall i \in \mathcal{V}_2$. Thus, if all the $\delta_i(t)$, $\forall i \in \mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$, converge to zero as $t \rightarrow \infty$, then we can say that the dissensus is achieved.

According to the definition of the Laplacian matrix \mathcal{L} , 0 is an eigenvalue of \mathcal{L} . Then by simple calculation, we can make a linear transformation as follows:

$$\Phi \mathcal{L} \Phi^{-1} = \begin{bmatrix} 0 & * \\ \mathbf{0}_{(N-1) \times 1} & \tilde{\mathcal{L}} \end{bmatrix}, \quad (11)$$

where

$$\Phi^{-1} = \begin{bmatrix} 1 & -\frac{1}{N} \mathbf{1}_{k-1}^T & -\frac{1}{N} \mathbf{1}_{N-k}^T \\ \mathbf{1}_{k-1} & I_{k-1} - \frac{1}{N} F_{k-1} & -\frac{1}{N} F_{k-1, N-k} \\ -\mathbf{1}_{N-k} & \frac{1}{N} F_{N-k, k-1} & -I_{N-k} + \frac{1}{N} F_{N-k} \end{bmatrix} \quad (12)$$

and F is the corresponding matrix whose elements are all equal to 1. Furthermore, let

$$\begin{aligned}\tilde{u}(t) &= [\tilde{u}_1(t), \tilde{u}_2(t), \dots, \tilde{u}_N(t)]^\top \\ &= \Phi \times [u_1(t), u_2(t), \dots, u_N(t)]^\top.\end{aligned}$$

Then, with $y_i(t) = Cx_i(t)$, $i = 1, 2, \dots, N$, we have

$$\begin{aligned}\begin{bmatrix} \dot{\delta}_2(t) \\ \vdots \\ \dot{\delta}_N(t) \end{bmatrix} &= (I_{N-1} \otimes A) \begin{bmatrix} \delta_2(t) \\ \vdots \\ \delta_N(t) \end{bmatrix} \\ &\quad + (I_{N-1} \otimes B) \begin{bmatrix} \tilde{u}_2(t) \\ \vdots \\ \tilde{u}_N(t) \end{bmatrix},\end{aligned}\quad (13)$$

$$\begin{bmatrix} \tilde{u}_2(t) \\ \vdots \\ \tilde{u}_N(t) \end{bmatrix} = -(\tilde{\mathcal{L}} \otimes KC) \begin{bmatrix} \delta_2(t) \\ \vdots \\ \delta_N(t) \end{bmatrix}.\quad (14)$$

For convenience of analysis, we define

$$\dot{\delta}(t) \triangleq \begin{bmatrix} \dot{\delta}_2(t) \\ \vdots \\ \dot{\delta}_N(t) \end{bmatrix} = [I_{N-1} \otimes A - \tilde{\mathcal{L}} \otimes BKC] \delta(t).\quad (15)$$

If the networked multiagent system (1) can reach dissensus, then there exists a vector $K \in \mathbb{R}^{1 \times m}$ with the distributed algorithm

$$\begin{aligned}u_i(t) &= -K \sum_{j=1}^N |\mathcal{A}_{ij}| [y_i(t) - \text{sgn}(\mathcal{A}_{ij}) y_j(t)], \\ t &\geq 0, i = 1, 2, \dots, N,\end{aligned}$$

such that $\lim_{t \rightarrow \infty} \delta_i(t) = 0$, $i = 2, 3, \dots, N$. Hence, all the eigenvalues of the matrix $I_{N-1} \otimes A - \tilde{\mathcal{L}} \otimes BKC$ in (15) are in the open left half plane. Suppose that $\lambda_1 = 0$, $\lambda_2, \dots, \lambda_N$ are the eigenvalues of the Laplacian matrix \mathcal{L} . Then, the eigenvalues of $\tilde{\mathcal{L}}$ are $\lambda_2, \lambda_3, \dots, \lambda_N$. Hence, we can find an invertible matrix T such that $\tilde{\mathcal{L}}$ is similar to a Jordan canonical matrix, i.e.,

$$T^{-1} \tilde{\mathcal{L}} T = J = \text{diag}\{J_1, J_2, \dots, J_l\},$$

where J_k , $k = 1, 2, \dots, l$, are upper triangular Jordan blocks, and their diagonal elements are λ_i , $i = 2, 3, \dots, N$. Furthermore,

$$\begin{aligned}(T \otimes I_n)^{-1} [I_{N-1} \otimes A - \tilde{\mathcal{L}} \otimes BKC] \times (T \otimes I_n) \\ = I_{N-1} \otimes A - J \otimes BKC\end{aligned}$$

is also an upper triangular block matrix implying that the eigenvalues of $I_{N-1} \otimes A - \tilde{\mathcal{L}} \otimes BKC$ are given by the eigenvalues of $A - \lambda_i BKC$, $i = 2, 3, \dots, N$. Hence, the eigenvalues of $A - \lambda_i BKC$ are all in the open left half plane.

Next, we demonstrate that (A, B, C) is stabilizable and detectable. Suppose that at least one of λ_i , $i = 2, 3, \dots, N$, is real, for example, λ_2 , then (A, B, C) is stabilizable and detectable because all the eigenvalues of $A - \lambda_2 BKC$ are in the open left half plane. In addition, if λ_i , $i = 2, 3, \dots, N$, are all complex numbers, i.e., none of their imaginary parts are zeros, then since $\tilde{\mathcal{L}}$ is a real matrix, the eigenvalues will appear in conjugate pairs. Without loss of generality, we suppose that λ_2 and λ_3 are one of the corresponding conjugate pair of roots with $\lambda_2 = \sigma + j'\omega$ and $\lambda_3 = \sigma - j'\omega$ where $j'^2 = -1$. Then $\forall \lambda \in \mathbb{C}$, where \mathbb{C} is the set of complex numbers,

$$\begin{aligned}&\begin{vmatrix} \lambda I_n - (A - \sigma BKC) & -\omega BKC \\ \omega BKC & \lambda I_n - (A - \sigma BKC) \end{vmatrix} \\ &= |\lambda I_n - (A - \lambda_2 BKC)| |\lambda I_n - (A - \lambda_3 BKC)|.\end{aligned}$$

Considering the fact that all the eigenvalues of the matrices $A - \lambda_2 BKC$ and $A - \lambda_3 BKC$ are in the open left half plane, we obtain that all the eigenvalues of the matrix

$$\begin{aligned}&\begin{bmatrix} A - \sigma BKC & \omega BKC \\ -\omega BKC & A - \sigma BKC \end{bmatrix} \\ &= \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} + \begin{bmatrix} -\sigma BK & \omega BK \\ -\omega BK & -\sigma BK \end{bmatrix} \begin{bmatrix} C & 0 \\ 0 & C \end{bmatrix}\end{aligned}$$

are also in the open left half plane, which is equivalent to that $\left(\begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}, \begin{bmatrix} C & 0 \\ 0 & C \end{bmatrix}\right)$ is detectable. Thus, $\text{rank} \begin{bmatrix} sI_n - A \\ C \end{bmatrix} = n$, $\forall s \in \mathbb{C}$ and $\text{Re}(s) \geq 0$, where $\text{Re}(s)$ represents the real part of the complex number s , i.e., (A, C) is detectable.

Following similar steps, we can verify that (A, B) is stabilizable. In summary, we have the conclusion that if the networked multiagent system (1) can reach dissensus, (A, B, C) is stabilizable and detectable.

Finally, we will focus on the latter part of the theorem, i.e., the communication digraph $\mathcal{G}(\mathcal{A})$ contains a spanning tree with structural balance. Due to the fact that linear transformation does not affect the eigenvalues of matrix, from Lemma 3.3 in Ren and Beard (2005), $\lambda_i = 0$ or $\text{Re}(\lambda_i) > 0$, $i = 2, 3, \dots, N$. By contradiction, because not all of the eigenvalues of A are in the open left half plane, then $\lambda_i \neq 0$, $i = 2, 3, \dots, N$. Otherwise, there exists $i' \in \{2, 3, \dots, N\}$ such that $\lambda_{i'} = 0$. Thus, $A = A - \lambda_i BKC$ and the eigenvalues of A are all in the open left half plane, which is a contradiction. Therefore, the signed digraph $\mathcal{G}(\mathcal{A})$ has only one zero eigenvalue, and by Lemma 3.3 in Ren and Beard (2005), it must contain a spanning tree. \square

Remark 2 Although Theorem 1 provides a necessary condition, it is useful to check whether the networked multiagent systems have the possibility of reaching dissensus. If any of the conditions is unsatisfied, i.e., (A, B, C) is not stabilizable or undetectable or the signed digraph \mathcal{G} is structurally unbalanced or \mathcal{G} contains no spanning tree, then the networked multiagent systems cannot reach dissensus over the antagonistic networks. Algorithm 1 shows the usage of the distributed algorithm for Theorem 1.

When the system matrices of (1) satisfy the following conditions in Theorem 2, the necessary condition in Theorem 1 becomes a necessary and sufficient condition. Before proceeding, Lemma 1 is introduced for demonstrating Theorem 2.

Lemma 1 (cf. Cheng and Ma (2006)) *When (A, B) is stabilizable, the following Riccati equation:*

$$A^T P + PA - PBB^T P + I_n = 0 \quad (16)$$

has a unique nonnegative definite solution P . In addition, the eigenvalues of $A - BB^T P$ are all in the open left half plane.

Algorithm 1 Distributed algorithm for Theorem 1

1. **In:** Give the initial conditions \mathcal{G} and (A, B, C) for the networked multiagent system (1) and the corresponding output information $y_i(0)$.
 2. **Out:**

$$\begin{cases} \|x_j(t) - x_i(t)\|, \forall i, j \in \mathcal{V}_1 \text{ or } \forall i, j \in \mathcal{V}_2; \\ \|x_j(t) + x_i(t)\|, \forall i \in \mathcal{V}_1 \text{ and } \forall j \in \mathcal{V}_2. \end{cases}$$
 3. **If** topology \mathcal{G} is structurally unbalanced:
 4. Terminate the algorithm.
 5. **While** dissensus is not reached:
 6. Design the distributed algorithm for every agent as

$$u_i(t) = -K \sum_{j \in \mathcal{N}_i} |\mathcal{A}_{ij}| [y_i(t) - \text{sgn}(\mathcal{A}_{ij}) y_j(t)], t \geq 0;$$
 7. Choose stabilizable and detectable (A, B, C) ;
 8. Choose K relatively large and positive;
 9. Choose $\mathcal{G}(\mathcal{A})$ which contains a spanning tree;
 10. Use input signal u_i to control agent i ;
 11. Terminate the algorithm.
-

Theorem 2 *If P is a nonnegative definite solution of equation (16) and assume that*

$$\text{rank}[C] = \text{rank} \begin{bmatrix} C \\ B^T P \end{bmatrix}, \quad (17)$$

then (A, B) is stabilizable and the signed digraph $\mathcal{G}(\mathcal{A})$ contains a spanning tree if and only if the networked multiagent system (1) can reach dissensus.

Proof Theorem 1 has demonstrated the necessary and we will focus on the proof of the sufficiency. Similar to the proof of Theorem 1, since the signed digraph $\mathcal{G}(\mathcal{A})$ contains a spanning tree, we have that $\lambda_i, i = 2, 3, \dots, N$ are all in the open right half plane, i.e., $\text{Re}(\lambda_i) > 0, i = 2, 3, \dots, N$. Let

$$\beta \triangleq \min_{2 \leq i \leq N} \{\text{Re}(\lambda_i)\}. \quad (18)$$

If condition (17) holds, then the solution of the matrix equation

$$XC = B^T P \quad (19)$$

exists. Without loss of generality, we choose one of these solutions denoted by Q . Then, let the constant gain matrix of the distributed algorithm (6) be

$$K = \max\{1, \beta^{-1}\} Q. \quad (20)$$

Thus, $A - \lambda_i BKC = A - \lambda_i \max\{1, \beta^{-1}\} BB^T P, i = 2, 3, \dots, N$. Note that for any $\sigma \geq 1$ and $\omega \in \mathbb{R}$, the eigenvalues of $A - (\sigma + j'\omega) BB^T P$ are all in the open left half plane (Ma 2009). Thus, the eigenvalues of $A - \lambda_i BKC, i = 2, 3, \dots, N$ are in the open left half plane. Thus, $\|\delta_i(t)\| = 0$, as $t \rightarrow \infty, i = 2, 3, \dots, N$, implying that the networked multiagent system (1) can reach dissensus. \square

We summarize the distributed algorithm for Theorem 2 in Algorithm 2.

Algorithm 2 Distributed algorithm for Theorem 2

1. **In:** Give the initial conditions \mathcal{G} and (A, B, C) for the networked multiagent system (1) and the corresponding output information $y_i(0)$.
2. **Out:**

$$\begin{cases} \|x_j(t) - x_i(t)\|, \forall i, j \in \mathcal{V}_1 \text{ or } \forall i, j \in \mathcal{V}_2; \\ \|x_j(t) + x_i(t)\|, \forall i \in \mathcal{V}_1 \text{ and } \forall j \in \mathcal{V}_2. \end{cases}$$
3. **If** topology \mathcal{G} is structurally unbalanced:
4. Terminate the algorithm.
5. **While** dissensus is not reached:
6. Design the distributed algorithm for every agent as

$$u_i(t) = -K \sum_{j \in \mathcal{N}_i} |\mathcal{A}_{ij}| [y_i(t) - \text{sgn}(\mathcal{A}_{ij}) y_j(t)], t \geq 0;$$

7. Choose stabilizable (A, B) such that

$$A^T P + PA - PBB^T P + I_n = 0$$

has a nonnegative definite solution P ;

8. Choose $C = B^T P$;
 9. Choose K according to (18)–(20);
 10. Choose $\mathcal{G}(\mathcal{A})$ which contains a spanning tree;
 11. Use input signal u_i to control agent i ;
 12. Terminate the algorithm.
-

Remark 3 Consider a class of discrete-time networked multiagent systems containing N agents described as follows:

$$\begin{aligned} x_i(k+1) &= Gx_i(k) + Hu_i(k), \\ y_i(k) &= Fx_i(k), \quad i = 1, 2, \dots, N, k \in \mathbb{N}^+. \end{aligned} \quad (21)$$

We can extend Theorem 1 to discrete-time networked multiagent systems by a similar proof. That is, if the agents can asymptotically reach dissensus, then (G, H, F) is stabilizable and detectable, and the communication digraph $\mathcal{G}_D(\mathcal{A})$ contains a spanning tree and is structurally balanced. However, when G is singular, stabilizable (G, H) does not imply $\text{rank}(sI_n - GH) = n, \forall s \in \mathbb{C}, |s| \geq 1$.

4 Implementations of dissensus

In this section, we investigate the implementations of dissensus to demonstrate our conclusions.

Example 1 (Dissensus with stabilizable (A, B))

We consider the networked multiagent system containing five agents with the given dynamics described as follows:

$$\begin{aligned} \dot{x}_i(t) &= \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} x_i(t) + \begin{bmatrix} 2 \\ 1 \end{bmatrix} u_i(t), \\ y_i(t) &= [17.4660 \quad -22.2645] x_i(t), \quad i = 1, 2, 3, 4, 5, \end{aligned} \quad (22)$$

where $x_i(t) \in \mathbb{R}^2$ is the state, $u_i(t) \in \mathbb{R}$ is the control and $y_i(t) \in \mathbb{R}$ is the output of the i th agent. Solving the equation (16) with function `care(·)` in MATLAB, we get

$$P = \begin{bmatrix} 42.3742 & -67.2823 \\ -67.2823 & 112.3001 \end{bmatrix},$$

which is nonnegative definite. Then, in order to satisfy the equation (17), we choose $C = B^T P = [17.4660, -22.2645]$. According to Theorem 2, we study the following two cases.

Case 1: The signed digraph $\mathcal{G}_1 = (\mathcal{V}_1, \mathcal{E}_1, \mathcal{A}_1)$ in Fig. 2 represents the topology among the five agents. In addition, $\mathcal{V}_1 = \{1, 2, 3, 4, 5\}$, $\mathcal{E}_1 = \{(1, 2), (2, 3), (3, 4), (4, 5), (5, 1)\}$, $\mathcal{A}_1 = (A_{ij})_{5 \times 5}$ where

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 \\ 4 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -0.5 & 0 & 0 \\ 0 & 0 & 0 & 1.5 & 0 \end{bmatrix}.$$

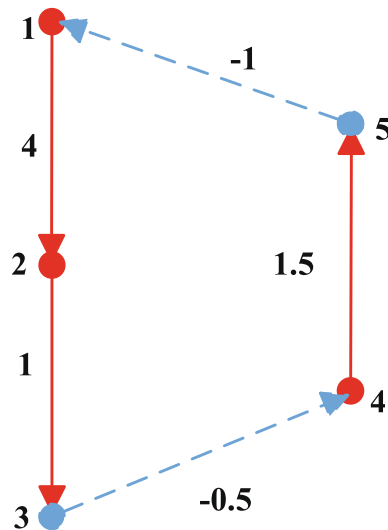


Fig. 2 The topology of signed digraph \mathcal{G}_1

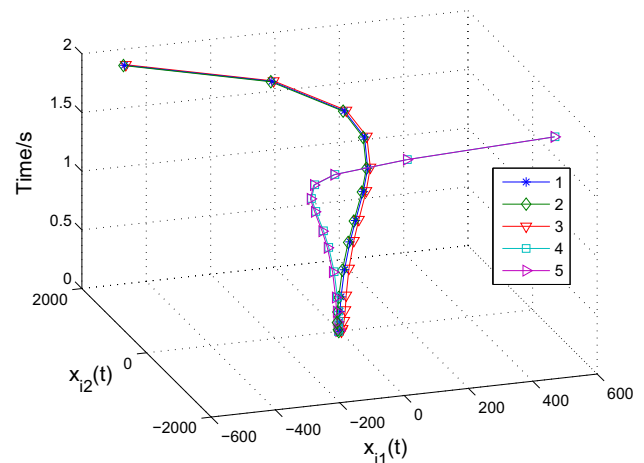


Fig. 3 Dissensus of five agents

Then,

$$\mathcal{L}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ -4 & 4 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0 & -1.5 & 1.5 \end{bmatrix}.$$

Note that the eigenvalues of \mathcal{L}_1 are $0, 0.9245 \pm 0.9194j', 3.9598, 2.1912$ which are all in the open right half plane except the zero eigenvalue. Clearly, \mathcal{G}_1 is structurally balanced containing a spanning tree.

Since we choose $C = B^T P$ in this example, according to (19) we obtain $Q = 1$. Thus, with (18) and (20) we choose $K = 1.0816$. Then, for any initial value $x_i(0), i = 1, 2, \dots, 5$, the five agents can reach dissensus and it is shown in Fig. 3, where the x-y plane contains x-axis and y-axis and the axis perpendicular to the x-y plane is the running time.

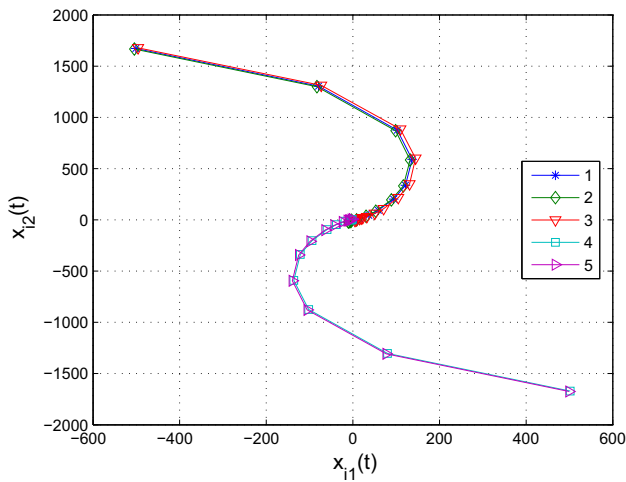


Fig. 4 Dissensus of five agents on two-dimensional plane

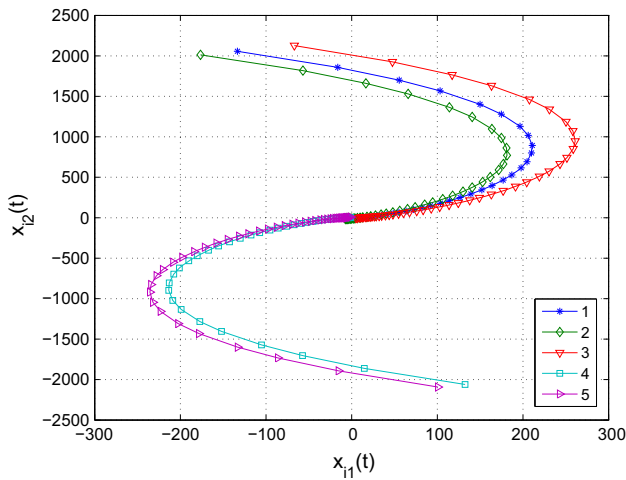


Fig. 5 Dissensus with state feedback and undirected topology

$x_{i1}(t)$ and $x_{i2}(t)$ are the two components of the state $x_i(t)$. As time goes on, agent 1, agent 2 and agent 3 reach a consensus, while agent 4 and agent 5 reach another consensus with the opposite direction. Fig. 4 shows 2-D trajectories of the five agents. It further demonstrates that the networked multiagent systems can reach dissensus if the conditions of Theorem 2 hold.

Remark 4 In order to compare the performance of the designed method in this paper with the performance of the method in Valcher and Misra (2014), we change the digraph to undirected graph in Case 1 with communication weights unchanged and use the state-feedback information to control the multiagent system (22). In Fig. 5, the running time is the same as that of Case 1 and we can see that the multiagent system has the trend to achieve the dissensus. However, in Case 1, the dissensus is obtained in 2s. Furthermore, the undirected communication topology transfers information in both directions and the state-feedback protocol provides

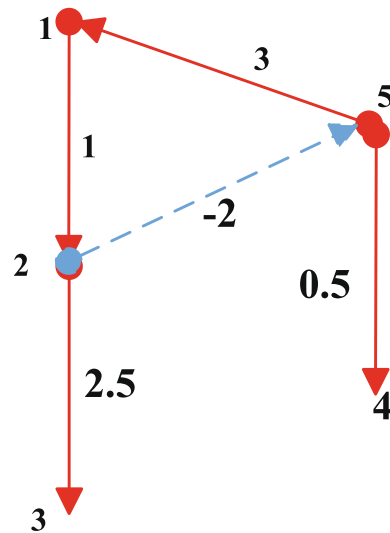


Fig. 6 The topology of signed digraph \mathcal{G}_2

more information than the protocol using only the output information in this paper. Therefore, our designed method utilizes concise output information and attains the dissensus faster than the method in Valcher and Misra (2014).

Case 2: The signed digraph $\mathcal{G}_2 = (\mathcal{V}_2, \mathcal{E}_2, \mathcal{A}_2)$ in Fig. 6 represents the topology among the five agents. In addition, $\mathcal{V}_2 = \{1, 2, 3, 4, 5\}$, $\mathcal{E}_2 = \{(1, 2), (2, 3), (2, 5), (5, 4), (5, 1)\}$, $\mathcal{A}_2 = (A_{ij})_{5 \times 5}$ where

$$A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 3 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 2.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 \\ 0 & -2 & 0 & 0 & 0 \end{bmatrix}.$$

Then,

$$\mathcal{L}_2 = \begin{bmatrix} 3 & 0 & 0 & 0 & -3 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -2.5 & 2.5 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & -0.5 \\ 0 & 2 & 0 & 0 & 2 \end{bmatrix}.$$

Note that the eigenvalues of \mathcal{L}_2 are 2.5, 0.5, 4, $1 \pm 1.4142j'$, which are all in the open right half plane. However, \mathcal{G}_2 is structurally unbalanced though containing a spanning tree.

With the similar steps in Case 1, we choose $K = 1$. Then for any initial value $x_i(0)$, $i = 1, 2, \dots, 5$, the five agents cannot reach dissensus which is shown in Fig. 7. As time goes on, the five agents reach a consensus instead of dissensus due to the **structurally unbalanced** \mathcal{G}_2 . Therefore, besides the properties of systems, communication topology is also a key factor to dissensus. The distributed algorithm in Algorithm 2 provides the guidance of how to reach dissensus.

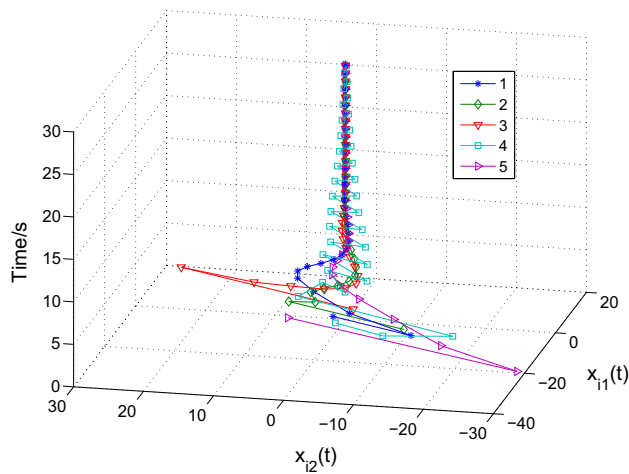


Fig. 7 Five agents without dissensus

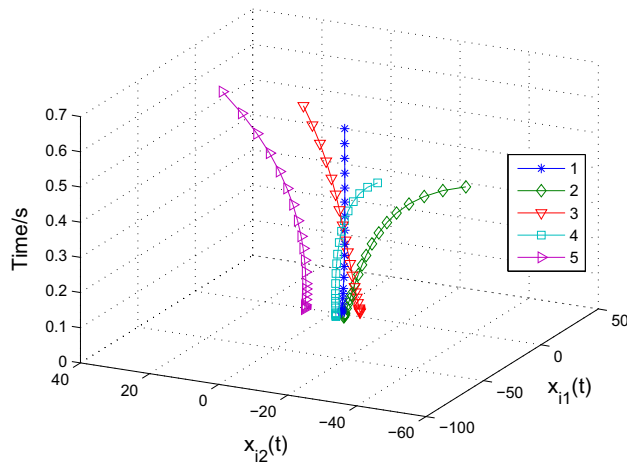


Fig. 8 Five agents with divergence

Example 2 (Five agents with unstable (A, B))

We consider the networked multiagent system containing five agents with the given dynamics described as follows:

$$\begin{aligned} \dot{x}_i(t) &= \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} x_i(t) + \begin{bmatrix} 2 \\ -1 \end{bmatrix} u_i(t), \\ y_i(t) &= [0.625 \ -1] x_i(t), \quad i = 1, 2, 3, 4, 5, \end{aligned} \quad (23)$$

where $x_i(t) \in \mathbb{R}^2$ is the state, $u_i(t) \in \mathbb{R}$ is the control and $y_i(t) \in \mathbb{R}$ is the output of the i th agent. The signed digraph is the same as in Fig. 2. Thus, we choose the same $K = 1.0816$ as in Case 1 of Example 1. However, (A, B) is unstable.

Then for any initial value $x_i(0)$, $i = 1, 2, \dots, 5$, the five agents can neither reach dissensus nor consensus which is illustrated in Fig. 8. As time goes on, the five agents diverge to different directions due to the **unstable** (A, B) . Therefore, besides the communication topology, the properties of the systems are also important to the dissensus. This

demonstrates the effectiveness of the distributed algorithm in Algorithm 2.

Remark 5 It should be mentioned that structural balance is an important property of the signed digraph to guarantee dissensus. In Case 2, we can see that \mathcal{G}_2 loses the property of structural balance. Thus, consensus is achieved instead of dissensus. Compared to Altafini (2013), due to the difficulty of measuring the relative full-states in physical systems, we utilize the output information to obtain dissensus. Furthermore, using output information can reduce the communication overheads. Please refer to Algorithms 1 and 2 for more details of the distributed algorithms using output information.

5 Conclusions

A distributed algorithm is developed to solve dissensus of a class of networked multiagent systems under directed competitive networks. When the networked multiagent systems can reach dissensus, the signed digraph should be structurally balanced and contains a spanning tree. In addition, (A, B, C) is also stabilizable and detectable. Moreover, if there exists a nonnegative definite matrix P satisfying the Riccati equation, the necessary condition becomes a necessary and sufficient condition which will be a great help to design the distributed algorithm. In future work, we will focus on the situation where the dynamics is nonlinear or the communication topology is switching.

Acknowledgments This work was supported in part by the National Natural Science Foundation of China under Grants 61233001, 61273140, 61304086, 61533017, 61503379 and U1501251, in part by China Scholarship Council under the State Scholarship Fund, in part by Beijing Natural Science Foundation under Grant 4162065 and in part by the Early Career Development Award of SKLMCCS.

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

References

- Altafini C (2013) Consensus problems on networks with antagonistic interactions. *IEEE Trans Autom Control* 58(4):935–946
- Bertsekas DP, Tsitsiklis JN (1989) *Parallel and distributed computation: numerical methods*. Prentice-Hall, Englewood Cliffs
- Cao Y, Yu W, Ren W, Chen G (2013) An overview of recent progress in the study of distributed multi-agent coordination. *IEEE Trans Ind Inf* 9(1):427–438
- Cheng L, Hou ZG, Tan M, Lin Y, Zhang W (2010) Neural-network-based adaptive leader-following control for multiagent systems with uncertainties. *IEEE Trans Neural Netw* 21(8):1351–1358
- Cheng L, Hou ZG, Tan M (2014) A mean square consensus protocol for linear multi-agent systems with communication noises and fixed topologies. *IEEE Trans Autom Control* 59(1):261–267

- Cheng Z, Ma S (2006) Linear system theory. Science Press, Beijing
- Diestel R (2000) Graph theory. Springer, New York
- He W, Ge SS (2016) Cooperative control of a nonuniform gantry crane with constrained tension. *Automatica* 66(4):146–154
- Hu JP, Zheng WX (2013) Bipartite consensus for multi-agent systems on directed signed networks. In: Proceedings of the IEEE conference on decision and control. Florence, Italy, pp 3451–3456
- Hu JP, Xiao ZH, Zhou YL, Yu JY (2013) Formation control over antagonistic networks. In: Proceedings of the Chinese control conference. Xi'an, China, pp 6879–6884
- Jadbabaie A, Lin J, Morse AS (2003) Coordination of groups of mobile autonomous agents using nearest neighbor rules. *IEEE Trans Autom Control* 48(6):988–1001
- Liang H, Zhang H, Wang Z, Wang J (2014) Consensus robust output regulation of discrete-time linear multi-agent systems. *IEEE/CAA J Autom Sinica* 1(2):204–209
- Lin P, Ren W (2014) Constrained consensus in unbalanced networks with communication delays. *IEEE Trans Autom Control* 59(3):775–781
- Liu C, Kroll A (2015) Memetic algorithms for optimal task allocation in multi-robot systems for inspection problems with cooperative tasks. *Soft Comput* 19(3):567–584
- Liu D, Wang D, Li H (2014) Decentralized stabilization for a class of continuous-time nonlinear interconnected systems using online learning optimal control approach. *IEEE Trans Neural Netw Learn Syst* 25(2):418–428
- Liu T, Jiang ZP (2013) Distributed formation control of nonholonomic mobile robots without global position measurements. *Automatica* 49(2):592–600
- Liu T, Jiang ZP (2014) Distributed control of nonlinear uncertain systems: a cyclic-small-gain approach. *IEEE/CAA J Autom Sinica* 1(1):46–53
- Luo X, Feng L, Yan J, Guan X (2015) Dynamic coverage with wireless sensor and actor networks in underwater environment. *IEEE/CAA J Autom Sinica* 2(3):274–281
- Ma C (2009) System analysis and control synthesis of linear multi-agent systems. PhD thesis, Academy of Mathematics and System Science, Chinese Academy of Sciences, Beijing, China
- Ma CQ, Zhang JF (2010) Necessary and sufficient conditions for consensusability of linear multi-agent systems. *IEEE Trans Autom Control* 55(5):1263–1268
- Ma H, Liu D, Wang D (2015a) Distributed control for nonlinear time-delayed multi-agent systems with connectivity preservation using neural networks. In: Proceedings of the 22nd international conference on neural information processing. Istanbul, Turkey, pp 34–42
- Ma H, Liu D, Wang D, Tan F, Li C (2015b) Centralized and decentralized event-triggered control for group consensus with fixed topology in continuous time. *Neurocomputing* 161:267–276
- Ma H, Liu D, Wang D, Luo B (2016a) Bipartite output consensus in networked multi-agent systems of high-order power integrators with signed digraph and input noises. *Int J Syst Sci* 47(13):3116–3131
- Ma H, Wang Z, Wang D, Liu D, Yan P, Wei Q (2016b) Neural-network-based distributed adaptive robust control for a class of nonlinear multiagent systems with time delays and external noises. *IEEE Trans Syst Man Cybern Syst* 46(6):750–758
- Olfati-Saber R, Fax JA, Murray RM (2007) Consensus and cooperation in networked multi-agent systems. *Proc IEEE* 95(1):215–233
- Quteishat A, Lim CP, Saleh JM, Tweedale J, Jain LC (2011) A neural network-based multi-agent classifier system with a bayesian formalism for trust measurement. *Soft Comput* 15(2):221–231
- Ren W, Beard RW (2005) Consensus seeking in multiagent systems under dynamically changing interaction topologies. *IEEE Trans Autom Control* 50(5):655–661
- Scheepers C, Engelbrecht AP (2016) Training multi-agent teams from zero knowledge with the competitive coevolutionary team-based particle swarm optimiser. *Soft Comput* 20(2):607–620
- Shen J, Tan H, Wang J, Wang J, Lee S (2015) A novel routing protocol providing good transmission reliability in underwater sensor networks. *J Internet Technol* 16(1):171–178
- Smith HL (1995) Monotone dynamical systems: an introduction to the theory of competitive and cooperative systems. American Mathematical Society, Providence
- Tsitsiklis JN (1984) Problems in decentralized decision making and computation. PhD thesis, Department of EECS, MIT, Cambridge, MA, USA
- Valcher ME, Misra P (2014) On the consensus and bipartite consensus in high-order multi-agent dynamical systems with antagonistic interactions. *Syst Control Lett* 66:94–103
- Vicsek T, Czirók A, Ben-Jacob E, Cohen I, Shochet O (1995) Novel type of phase transition in a system of self-driven particles. *Phys Rev Lett* 75(6):1226–1229
- Wang D, Li C, Liu D, Mu C (2016a) Data-based robust optimal control of continuous-time affine nonlinear systems with matched uncertainties. *Inf Sci* 366:121–133
- Wang D, Liu D, Li H, Ma H, Li C (2016b) A neural-network-based online optimal control approach for nonlinear robust decentralized stabilization. *Soft Comput* 20(2):707–716
- Wang D, Ma H, Liu D (2016c) Distributed control algorithm for bipartite consensus of the nonlinear time-delayed multi-agent systems with neural networks. *Neurocomputing* 174:928–936
- Wieland P, Kim JS, Allgöwer F (2011) On topology and dynamics of consensus among linear high-order agents. *Int J Syst Sci* 42(10):1831–1842
- Xie S, Wang Y (2014) Construction of tree network with limited delivery latency in homogeneous wireless sensor networks. *Wirel Pers Commun* 78(1):231–246
- Zhang H, Qin C, Jiang B, Luo Y (2014) Online adaptive policy learning algorithm for h_∞ state feedback control of unknown affine nonlinear discrete-time systems. *IEEE Trans Cybern* 44(12):2706–2718
- Zhang H, Feng T, Yang GH, Liang H (2015a) Distributed cooperative optimal control for multiagent systems on directed graphs: an inverse optimal approach. *IEEE Trans Cybern* 45(7):1315–1326
- Zhang H, Zhang J, Yang GH, Luo Y (2015b) Leader-based optimal coordination control for the consensus problem of multi-agent differential games via fuzzy adaptive dynamic programming. *IEEE Trans Fuzzy Syst* 23(1):152–163