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Bipartite Output Consensus of Power Integrator Multi-Agent Systems with Input Noises

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***Abstract*—Underactuated and weakly coupled mechanical systems of power integrators is ubiquitous in physical implementations. Due to the presence of negative communication weights, whether bipartite output consensus can be reached remains unknown. Thus, it is of practical significance to investigate this issue. Furthermore, with input noises, an adaptive disturbance compensator, along with the technique of adding power integrators, is established for the complex multi-agent systems to improve the stability and robustness performance of the system.**

I. Introduction

In the last few years, tremendous progress has been made towards multi-agent systems [1]–[5]. A novel type of phase transition of self-driven particles was proposed by Vicsek, which was the origin of nearest neighbour rules [6]. Consequently, according to Vicsek's model, Jadbabaie [7] introduced nearest neighbour rules into the multi-agent systems. Following the above idea, bipartite consensus is a new branch in multi-agent systems.

Bipartite graph is a basic concept in graph theory [8] which is suitable to represent the communication topology of bipartite consensus. In several physical scenarios, it is more reasonable to suppose that some of the agents are competitive while the rest are cooperative. Altafini introduced the negative weights to the communication topology and demonstrated that bipartite consensus can be reached in the presence of antagonistic interactions. Consequently, bipartite consensus was extended to formation control [9] and directed signed networks [10], [11] with the same dynamics. Moreover, Valcher talked about a more complex situation that the dynamics of multi-agent systems were in high-order with antagonistic interactions and bipartite consensus can be reached under the stabilizability assumption with a sort of equilibrium between two fully competing groups. However, all the aspects mentioned above are associated with linear system, while in physical implementations, power integrator system is more ubiquitous [12].

High-order power integrator system is both practically interesting because a class of weakly coupled, unstable and underactuated mechanical systems [13], which are difficult to obtain stable control, are inherently nonlinear; and conceptually interesting because it is more complex than traditional linear systems in the aspect of analysis technique. A feedback design tool called adding a power integrator was proposed and used to deal with the problem of global robust stabilization when the nonlinear systems were in a lower-triangular form [14]. In addition, adding a power integrator was also introduced in [12] to deal with the global strong stabilization with the similar form of power integrator. Furthermore, in [15], it shed light on cooperative output-synchronisation in multi-agent systems of high-order power integrator with input noises and undirected topology. Nevertheless, we concentrate on a directed graph, particularly, where the weights among agents are partly negative. Compared with the advances in the area of consensus [5], less progress has been made in bipartite consensus [16], especially, bipartite output consensus. Therefore, it is of great practical interest to investigate that on what conditions multi-agent systems of high-order power integrators can reach bipartite output consensus.

The remainder of this paper is organized as follows. Basic definitions of bipartite output consensus and properties of signed graph are provided in Section II. In view of input noises, an adaptive noise compensator is introduced in Section III to enhance the robustness of networked multi-agent systems. In Section IV, numerical examples are conducted to demonstrate the effectiveness of the criterion established in Section IV, while closing with the concluding remarks of the whole paper in Section V.

II. Backgrounds and preliminaries

A triplet $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$ is called a (weighted) signed graph if $\mathcal{V} = \{1, 2, \ldots, N\}$ is the set of nodes, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of edges, and $\mathcal{A} = (\mathcal{A}_{ij}) \in \mathbb{R}^{N\times N}$ is the matrix of the signed weights of $\mathcal{G}$. Here we denote $\mathcal{A}_{ij}$ as the element of the ith row and jth column of the matrix $\mathcal{A}$. $\mathcal{A}$ is called the adjacency matrix of the signed graph $\mathcal{G}$ with real numbers and we use the notation $\mathcal{G}(\mathcal{A}) : \mathcal{A}_{ij} \neq 0 \Leftrightarrow (j, i) \in \mathcal{E}$ to represent the signed graph corresponding to $\mathcal{A}$. Note that self-loops will not be considered in this paper, i.e., $\mathcal{A}_{ii} = 0, \forall\, i = 1, 2, \ldots, N$. In a directed graph (digraph), a pair of edges sharing the same nodes $(i, j), (j, i) \in \mathcal{E}$ is called a digon [16]. We assume that $\mathcal{A}_{ij}\mathcal{A}_{ji} \geq 0$, which means that all digons cannot have the opposite signs. In this paper we call this property digon sign-symmetry. Otherwise, we call it digon sign-nonsymmetry. Given a signed digraph $\mathcal{G}(\mathcal{A})$, $\mathcal{C}_r$ is termed as the row

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connectivity matrix of $\mathcal{A}$ and

$$\mathcal{C}_r = \begin{bmatrix} c_{r,11} & 0 & \cdots & 0 \\ 0 & c_{r,22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c_{r,NN} \end{bmatrix}$$

with diagonal elements $c_{r,ii} = \sum_{j\in\mathcal{N}_i}|\mathcal{A}_{ij}|$, where $\mathcal{N}_i = \{j \in \mathcal{V}|(j,i) \in \mathcal{E}\}$ includes the neighbour nodes of node i. The column connectivity matrix $\mathcal{C}_c$ is defined likewise, where $c_{c,ii} = \sum_{j\in\mathcal{N}_i}|\mathcal{A}_{ji}|$. In addition, if $\mathcal{C}_r = \mathcal{C}_c$, the signed digraph is called weight balanced.

Definition 1 (cf. [16]): $\mathcal{G}(\mathcal{A})$ is said to be structurally balanced if it contains a bipartition of the nodes $\mathcal{V}_1, \mathcal{V}_2, \mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2, \mathcal{V}_1 \cap \mathcal{V}_2 = \varnothing$ such that $\mathcal{A}_{ij} \geq 0, \forall i,j \in \mathcal{V}_p$ $(p \in \{1,2\}); \mathcal{A}_{ij} \leq 0, \forall i \in \mathcal{V}_p, j \in \mathcal{V}_q, p \neq q$ $(p,q \in \{1,2\})$. Otherwise it is called structurally unbalanced.

In this and the subsequent sections, we assume that the signed digraph $\mathcal{G}$ is digon sign-symmetric and structurally balanced. We consider the case when input channels of system (2) are contaminated with unknown disturbances $\delta = (\delta_1, \delta_2, \ldots, \delta_N)^{\mathsf{T}} \in \mathbb{R}^{N\times 1}$.

Assumption 1: There is an unknown external system

$$\begin{aligned} \dot{\theta} &= \Gamma\theta, \\ \delta &= \Phi^{\mathsf{T}}\theta, \end{aligned} \tag{1}$$

where $\theta \in \mathbb{R}^{2\times 1}, \Gamma \in \mathbb{R}^{2\times 2}, \Phi \in \mathbb{R}^{2\times N}$ and the eigenvalues of Γ are all on the imaginary axis. The marginal stability of the exosystem implies that δ_i is bounded by constant $\bar{\delta}_i$, i.e., $|\delta_i| \leq \bar{\delta}_i, \forall i$.

Suppose that the network contains N agents and the dynamic of each agent i is as follows:

$$\begin{aligned} \dot{x}_{i1} &= x_{i2}^{p_1} \\ \dot{x}_{i2} &= x_{i3}^{p_2} \\ &\vdots \\ \dot{x}_{i,n-1} &= x_{in}^{p_{n-1}} \\ \dot{x}_{in} &= u_i^{p_n} + \delta_i \\ y_i &= x_{i1} \end{aligned} \tag{2}$$

where $p_k \geq 1, \forall k \in \{1, 2, \ldots, n\}$ are odd integers and $x_i = (x_{i1}, x_{i2}, \ldots, x_{in})^{\mathsf{T}} \in \mathbb{R}^n, y_i \in \mathbb{R}, u_i \in \mathbb{R}$ are the states, output and control input of agent i, respectively. Before proceeding, we introduce the definition of **bipartite output consensus** in concert with the subsequent analyses.

Definition 2: If for any initial condition $x_i(0)$,

$$\begin{cases} \lim\limits_{t\to+\infty} \|y_j(t) - y_i(t)\| = 0, & \forall i,j \in \mathcal{V}_1 \text{ or } \forall i,j \in \mathcal{V}_2; \\ \lim\limits_{t\to+\infty} \|y_j(t) + y_i(t)\| = 0, & \forall i \in \mathcal{V}_1 \text{ and } \forall j \in \mathcal{V}_2, \end{cases} \tag{3}$$

where $\mathcal{V}_1$ and $\mathcal{V}_2$ are the distinct node sets defined in Definition 1, then we say that the multi-agent systems (2) can reach **bipartite output consensus**.

It should be noted that we suppose that the communication capability is sufficient and the communication intensity is not related to the distance between each pair of agents. Furthermore, if there is a path from agent i to agent j where $j \in \mathcal{N}_i$, then agent i can transfer its output information y_i and the information of communication weight $\mathcal{A}_{ji}$ to agent j with no data loss. Therefore, agent i can obtain the information of both $\mathcal{A}_{ij}$ and $\mathcal{A}_{ji}$ in a digraph.

Before proceeding, we give three important lemmas frequently used throughout this article.

Lemma 1 (cf. [12]): $x, y, m, n, \alpha, \beta$ are all positive real numbers, then the following inequality holds

$$\alpha x^m y^n \leq \beta x^{m+n} + \frac{n}{m+n}\left(\frac{m+n}{n}\right)^{-\frac{m}{n}} \alpha^{\frac{m+n}{n}} \beta^{-\frac{m}{n}} y^{m+n}. \tag{4}$$

Lemma 2 (cf. [12]): With $x \in \mathbb{R}, y \in \mathbb{R}$ and $p \geq 1$ is an integer, the following two inequalities hold

$$|x+y|^p \leq 2^{p-1}\left|x^p + y^p\right|, \tag{5}$$

$$(|x| + |y|)^{\frac{1}{p}} \leq |x|^{\frac{1}{p}} + |y|^{\frac{1}{p}} \leq 2^{\frac{p-1}{p}} (|x| + |y|)^{\frac{1}{p}}. \tag{6}$$

If $p \geq 1$ is an odd integer, then

$$|x-y|^p \leq 2^{p-1}\left|x^p - y^p\right|. \tag{7}$$

Lemma 3 (cf. [12]): Suppose that $\alpha \in \mathbb{R}, \beta \in \mathbb{R}$ are both nonnegative real numbers and $p \geq 1, q \geq 1$ are integers, then

$$\alpha^{p-1}\beta^q \leq \alpha^p + \beta^{pq}. \tag{8}$$

III. Bipartite Output Consensus with Input Noises

Theorem 1: The dynamic of each agent in the network is (2) and $x_{il}^*, l = 2, 3, \ldots, n$, can be seen as internal reference states, then with the following distributed controllers:

$$\begin{aligned} \varphi_{i1} = &\sum_{j\in\mathcal{N}_i} |\mathcal{A}_{ij}|(y_i - \operatorname{sgn}(\mathcal{A}_{ij})y_j) \\ &+ \sum_{j\in\mathcal{N}_i} |\mathcal{A}_{ji}|(y_i - \operatorname{sgn}(\mathcal{A}_{ji})y_j), \\ u_i = &-\left[(k_n\varphi_{in})^{1/p_1\cdots p_{n-1}} + \operatorname{sgn}(\varphi_{in}^{2-1/p_1\cdots p_{n-1}})\hat{\delta}_i\right]^{1/p_n}, \end{aligned}$$

$$\begin{aligned} x_{i2}^{*p_1} &= -k_1\varphi_{i1} & \varphi_{i2} &= x_{i2}^{p_1} - x_{i2}^{*p_1} \\ x_{i3}^{*p_1p_2} &= -k_2\varphi_{i2} & \varphi_{i3} &= x_{i3}^{p_1p_2} - x_{i3}^{*p_1p_2} \\ &\vdots & &\vdots \\ x_{in}^{*p_1\cdots p_{n-1}} &= -k_{n-1}\varphi_{i,n-1} & \varphi_{in} &= x_{in}^{p_1\cdots p_{n-1}} - x_{in}^{*p_1\cdots p_{n-1}} \\ & & \dot{\hat{\delta}}_i &= \kappa_i|\varphi_{in}^{2-1/p_1\cdots p_{n-1}}|, \end{aligned} \tag{9}$$

where $\hat{\delta}_i$ is the adaptive disturbance compensator, κ_i is positive gain parameter, $k_1, k_2, \ldots, k_n$ and $p_1, p_2, \ldots, p_n$ are positive constant control gains and positive odd integers, respectively, then the multi-agent systems can asymptotically achieve bipartite output consensus and $x_{i2}, x_{i3}, \ldots, x_{in}, \forall i \in \mathcal{V}$ are bounded to zero, if the signed digraph $\mathcal{G}$ is strongly connected.

Proof: $\mathcal{G}$ is a digraph, then denote $\hat{\mathcal{L}}_u = \mathcal{C}_u - \mathcal{A}_u$ as an undirected graph, where

$$\mathcal{A}_u = \frac{\mathcal{A} + \mathcal{A}^{\mathsf{T}}}{2}, \quad \mathcal{C}_u = \frac{\mathcal{C}_r + \mathcal{C}_c}{2}.$$

We first define a potential function V_1 associated with the Laplacian matrix $\hat{\mathcal{L}}_u$ as follows:

$$V_1 = \frac{1}{2}\vec{x}^{\mathsf{T}}\hat{\mathcal{L}}_u\vec{x} = \frac{1}{2}\sum_{(i,j)\in\mathcal{E}}^{N} |\mathcal{A}_{u,ij}|(x_{i1} - \mathrm{sgn}(\mathcal{A}_{u,ij})x_{j1})^2, \tag{10}$$

where $\vec{x} = (x_{11}, x_{21}, \ldots, x_{N1})^{\mathsf{T}}$ and $\mathcal{A}_{u,ij}$ is the element of matrix $\mathcal{A}_u$. Then

$$\begin{aligned}\dot{V}_1 =& \frac{1}{2}[(\mathcal{C}_r - \mathcal{A})\vec{x} + (\mathcal{C}_c - \mathcal{A}^{\mathsf{T}})\vec{x}]^{\mathsf{T}}\dot{\vec{x}} \\ =& \frac{1}{2}\sum_{i=1}^{N}\varphi_{i1}x_{i2}^{p_1} \\ =& \frac{1}{2}\sum_{i=1}^{N}\varphi_{i1}x_{i2}^{*p_1} + \frac{1}{2}\sum_{i=1}^{N}\varphi_{i1}(x_{i2}^{p_1} - x_{i2}^{*p_1}).\end{aligned} \tag{11}$$

Let $\varphi_{i2} = x_{i2}^{p_1} - x_{i2}^{*p_1}$, then with Lemma 1, we can imply that

$$\begin{aligned}\dot{V}_1 &= -\frac{k_1}{2}\sum_{i=1}^{N}\varphi_{i1}^2 + \frac{1}{2}\sum_{i=1}^{N}\varphi_{i1}\varphi_{i2} \\ &\le -b_{11}\sum_{i=1}^{N}\varphi_{i1}^2 + b_{12}\sum_{i=1}^{N}\varphi_{i2}^2,\end{aligned} \tag{12}$$

where b_{11} and b_{12} are two positive constants satisfied with the inequality zooming in (12).

In the sequel, we make use of the form of $x_{i2}^{p_1}$ to define a new potential function

$$S_{i2} = \int_{x_{i2}^*}^{x_{i2}} \left(r^{p_1} - x_{i2}^{*p_1}\right)^{2-1/p_1} \mathrm{d}r, \quad \forall i \in \mathcal{V}. \tag{13}$$

Similar to Proposition B.1 in [12], $S_{i2} \ge 0$ and the corresponding partial derivatives of S_{i2} are

$$\begin{aligned}\frac{\partial S_{i2}}{\partial x_{i2}} &= \varphi_{i2}^{2-1/p_1}, \\ \frac{\partial S_{i2}}{\partial x_{i1}} &= -\left(2 - \frac{1}{p_1}\right)\frac{\partial x_{i2}^{*p_1}}{\partial x_{i1}}\int_{x_{i2}^*}^{x_{i2}}\left(r^{p_1} - x_{i2}^{*p_1}\right)^{1-1/p_1}\mathrm{d}r, \\ \frac{\partial S_{i2}}{\partial x_{j1}} &= -\left(2 - \frac{1}{p_1}\right)\frac{\partial x_{i2}^{*p_1}}{\partial x_{j1}}\int_{x_{i2}^*}^{x_{i2}}\left(r^{p_1} - x_{i2}^{*p_1}\right)^{1-1/p_1}\mathrm{d}r,\end{aligned}$$

where $j \in \mathcal{N}_i$. Similarly, define another potential function containing the information of first and second order of all the agents as follows:

$$V_2 = V_1 + \sum_{i=1}^{N} S_{i2} = V_1 + \sum_{i=1}^{N}\int_{x_{i2}^*}^{x_{i2}}\left(r^{p_1} - x_{i2}^{*p_1}\right)^{2-1/p_1}\mathrm{d}r. \tag{14}$$

Hence, the derivative of V_2 with time t is

$$\begin{aligned}\dot{V}_2 =& \dot{V}_1 + \sum_{i=1}^{N}\dot{S}_{i2} \\ =& \dot{V}_1 + \sum_{i=1}^{N}\frac{\partial S_{i2}}{\partial x_{i2}}\dot{x}_{i2} + \sum_{i=1}^{N}\frac{\partial S_{i2}}{\partial x_{i1}}\dot{x}_{i1} + \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\frac{\partial S_{i2}}{\partial x_{j1}}\dot{x}_{j1} \\ =& \dot{V}_1 + \sum_{i=1}^{N}\varphi_{i2}^{2-1/p_1}[x_{i3}^{*p_2} + (x_{i3}^{p_2} - x_{i3}^{*p_2})] + \sum_{i=1}^{N}\frac{\partial S_{i2}}{\partial x_{i1}}\dot{x}_{i1} \\ &+ \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\frac{\partial S_{i2}}{\partial x_{j1}}\dot{x}_{j1} \\ \le& -b_{11}\sum_{i=1}^{N}\varphi_{i1}^2 + b_{12}\sum_{i=1}^{N}\varphi_{i2}^2 + \sum_{i=1}^{N}\varphi_{i2}^{2-1/p_1}x_{i3}^{*p_2} \\ &+ \sum_{i=1}^{N}|\varphi_{i2}|^{2-1/p_1}|x_{i3}^{p_2} - x_{i3}^{*p_2}| \\ &+ \sum_{i=1}^{N}\left|\frac{\partial S_{i2}}{\partial x_{i1}}\right||\dot{x}_{i1}| + \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\left|\frac{\partial S_{i2}}{\partial x_{j1}}\right||\dot{x}_{j1}|.\end{aligned} \tag{15}$$

Furthermore, from (9) we can derive that $x_{i3}^{*p_2} = -k_2^{1/p_1}\varphi_{i2}^{1/p_1}$ and

$$|x_{i3}^{p_2} - x_{i3}^{*p_2}| \le 2^{\frac{p_1-1}{p_1}}|x_{i3}^{p_1p_2} - x_{i3}^{*p_1p_2}|^{1/p_1} = 2^{\frac{p_1-1}{p_1}}|\varphi_{i3}|^{1/p_1}.$$

With Lemma 1, we obtain that

$$\begin{aligned}\sum_{i=1}^{N}|\varphi_{i2}|^{2-1/p_1}|x_{i3}^{p_2} - x_{i3}^{*p_2}| &\le 2^{\frac{p_1-1}{p_1}}\sum_{i=1}^{N}|\varphi_{i2}|^{2-1/p_1}|\varphi_{i3}|^{1/p_1} \\ &\le b_{22}^{'}\sum_{i=1}^{N}\varphi_{i2}^2 + b_{23}^{'}\sum_{i=1}^{N}\varphi_{i3}^2,\end{aligned} \tag{16}$$

where $b_{22}^{'}$ and $b_{23}^{'}$ are two positive constants.

Now we are in the position to concentrate on the latter two items in (15). Note that with similar proof steps in Proposition B.5 in [12], the following two inequalities hold:

$$\begin{aligned}\left|\frac{\partial S_{i2}}{\partial x_{i1}}\right||\dot{x}_{i1}| &\le 4|\varphi_{i2}|\left|\frac{\partial x_{i2}^{*p_1}}{\partial x_{i1}}\dot{x}_{i1}\right| \le 4\gamma_{21}^{i}|\varphi_{i2}|(|\varphi_{i1}| + |\varphi_{i2}|), \\ \left|\frac{\partial S_{i2}}{\partial x_{j1}}\right||\dot{x}_{j1}| &\le 4|\varphi_{i2}|\left|\frac{\partial x_{i2}^{*p_1}}{\partial x_{j1}}\dot{x}_{j1}\right| \le 4\eta_{2j}^{i}|\varphi_{i2}|(|\varphi_{j1}| + |\varphi_{j2}|),\end{aligned}$$

where $j \in \mathcal{N}_i$, γ_{21}^{i} and η_{2j}^{i} are positive constants. By virtue of Lemma 1, we obtain

$$\begin{aligned}&\sum_{i=1}^{N}\left|\frac{\partial S_{i2}}{\partial x_{i1}}\right||\dot{x}_{i1}| + \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\left|\frac{\partial S_{i2}}{\partial x_{j1}}\right||\dot{x}_{j1}| \\ \le& b_{21}^{''}\sum_{i=1}^{N}\varphi_{i1}^2 + b_{22}^{''}\sum_{i=1}^{N}\varphi_{i2}^2,\end{aligned} \tag{17}$$

where $b_{21}^{''}$ and $b_{22}^{''}$ are positive constants.

With (16) and (17), $\dot{V}_2$ can be rewritten as

$$\begin{aligned}\dot{V}_2 \le& -b_{11}\sum_{i=1}^{N}\varphi_{i1}^2 + b_{12}\sum_{i=1}^{N}\varphi_{i2}^2 - k_2^{1/p_1}\sum_{i=1}^{N}\varphi_{i2}^2 \\ &+ b_{22}^{'}\sum_{i=1}^{N}\varphi_{i2}^2 + b_{23}^{'}\sum_{i=1}^{N}\varphi_{i3}^2 + b_{21}^{''}\sum_{i=1}^{N}\varphi_{i1}^2 + b_{22}^{''}\sum_{i=1}^{N}\varphi_{i2}^2 \\ \le& -b_{21}\sum_{i=1}^{N}\varphi_{i1}^2 - b_{22}\sum_{i=1}^{N}\varphi_{i2}^2 + b_{23}\sum_{i=1}^{N}\varphi_{i3}^2,\end{aligned} \tag{18}$$

where k_2 and b_{11} are chosen properly such that $-b_{11}+b_{21}^{''}<0$ and $-k_2^{1/p_1}+b_{12}+b_{22}^{'}+b_{22}^{''}<0$, and b_{21}, b_{22}, b_{23} are positive constants.

In what follows we utilize inductive technique with similar proof steps above when $2<m\leq n-1$. Define that

$$S_{im}=\int_{x_{im}^*}^{x_{im}}\left(r^{p_1\cdots p_{m-1}}-x_{im}^{*p_1\cdots p_{m-1}}\right)^{2-1/p_1\cdots p_{m-1}}\mathrm{d}r,$$

then

$$\begin{aligned}V_m=&V_{m-1}+\sum_{i=1}^N S_{im}\\=&V_{m-1}\\&+\sum_{i=1}^N\int_{x_{im}^*}^{x_{im}}\left(r^{p_1\cdots p_{m-1}}-x_{im}^{*p_1\cdots p_{m-1}}\right)^{2-1/p_1\cdots p_{m-1}}\mathrm{d}r.\end{aligned}$$

Thus, the derivatives of V_m is

$$\begin{aligned}\dot V_m=&\dot V_{m-1}+\sum_{i=1}^N\frac{\partial S_{im}}{\partial x_{im}}\dot x_{im}+\sum_{i=1}^N\sum_{l=1}^{m-1}\frac{\partial S_{im}}{\partial x_{il}}\dot x_{il}\\&+\sum_{i=1}^N\sum_{j\in\mathcal{N}_i}\frac{\partial S_{im}}{\partial x_{j1}}\dot x_{j1}\\=&\dot V_{m-1}+\sum_{i=1}^N\frac{\partial S_{im}}{\partial x_{im}}x_{i,m+1}^{*p_m}+\sum_{i=1}^N\frac{\partial S_{im}}{\partial x_{im}}(x_{i,m+1}^{p_m}-x_{i,m+1}^{*p_m})\\&+\sum_{i=1}^N\sum_{l=1}^{m-1}\frac{\partial S_{im}}{\partial x_{il}}\dot x_{il}+\sum_{i=1}^N\sum_{j\in\mathcal{N}_i}\frac{\partial S_{im}}{\partial x_{j1}}\dot x_{j1}\\\leq&\dot V_{m-1}-(k_m)^{1/p_1\cdots p_{m-1}}\sum_{i=1}^N\varphi_{im}^2\\&+\sum_{i=1}^N\left|\varphi_{im}^{2-1/p_1\cdots p_{m-1}}\right||x_{i,m+1}^{p_m}-x_{i,m+1}^{*p_m}|\\&+\sum_{i=1}^N\sum_{l=1}^{m-1}\frac{\partial S_{im}}{\partial x_{il}}\dot x_{il}+\sum_{i=1}^N\sum_{j\in\mathcal{N}_i}\frac{\partial S_{im}}{\partial x_{j1}}\dot x_{j1}\\\leq&\dot V_{m-1}-(k_m)^{1/p_1\cdots p_{m-1}}\sum_{i=1}^N\varphi_{im}^2\\&+\sum_{i=1}^N(b_{im}^{'}\varphi_{im}^2+b_{i,m+1}^{'}\varphi_{i,m+1}^2)\\&+\sum_{i=1}^N\sum_{l=1}^{m-1}\frac{\partial S_{im}}{\partial x_{il}}\dot x_{il}+\sum_{i=1}^N\sum_{j\in\mathcal{N}_i}\frac{\partial S_{im}}{\partial x_{j1}}\dot x_{j1}\\\leq&\dot V_{m-1}-(k_m)^{1/p_1\cdots p_{m-1}}\sum_{i=1}^N\varphi_{im}^2\\&+\sum_{i=1}^N(b_{im}^{'}\varphi_{im}^2+b_{i,m+1}^{'}\varphi_{i,m+1}^2)\\&+\sum_{i=1}^N\sum_{l=1}^{m-1}4|\varphi_{im}|\left|\frac{\partial x_{im}^{*p_1\cdots p_{m-1}}}{\partial x_{il}}\dot x_{il}\right|\\&+\sum_{i=1}^N\sum_{j\in\mathcal{N}_i}4|\varphi_{im}|\left|\frac{\partial x_{im}^{*p_1\cdots p_{m-1}}}{\partial x_{j1}}\dot x_{j1}\right|\\\leq&\dot V_{m-1}-(k_m)^{1/p_1\cdots p_{m-1}}\sum_{i=1}^N\varphi_{im}^2\\&+\sum_{i=1}^N(b_{im}^{'}\varphi_{im}^2+b_{i,m+1}^{'}\varphi_{i,m+1}^2)\\&+\sum_{i=1}^N\sum_{l=1}^{m-1}4\gamma_{ml}^i|\varphi_{im}|(|\varphi_{i1}|+\cdots+|\varphi_{im}|)\\&+\sum_{i=1}^N\sum_{j\in\mathcal{N}_i}4\eta_{mj}^i|\varphi_{im}|(|\varphi_{j1}|+|\varphi_{j2}|)\\\leq&-b_{m-1,1}\sum_{i=1}^N\varphi_{i1}^2-b_{m-1,2}\sum_{i=1}^N\varphi_{i2}^2-\cdots\\&-b_{m-1,m-1}\sum_{i=1}^N\varphi_{i,m-1}^2+b_{m-1,m}\sum_{i=1}^N\varphi_{i,m}^2\\&-(k_m)^{1/p_1\cdots p_{m-1}}\sum_{i=1}^N\varphi_{im}^2+\sum_{i=1}^N(b_{im}^{'}\varphi_{im}^2+b_{i,m+1}^{'}\varphi_{i,m+1}^2)\\&+\sum_{i=1}^N(b_{i1}^{''}\varphi_{i1}^2+b_{i2}^{''}\varphi_{i2}^2+\ldots+b_{im}^{''}\varphi_{im}^2).\end{aligned}\tag{19}$$

Therefore, by appropriately choosing the parameters in (19), we rewrite $\dot V_m$ as

$$\begin{aligned}\dot V_m\leq&-b_{m1}\sum_{i=1}^N\varphi_{i1}^2-b_{m2}\sum_{i=1}^N\varphi_{i2}^2-\cdots\\&-b_{mm}\sum_{i=1}^N\varphi_{im}^2+b_{m,m+1}\sum_{i=1}^N\varphi_{i,m+1}^2.\end{aligned}\tag{20}$$

Finally, we come up to demonstrating $\dot V_n\leq 0$. To that end, we define

$$V_n=V_{n-1}+\sum_{i=1}^N S_{in}+\sum_{i=1}^N\frac{1}{2\kappa_i}\tilde\delta_i^2,\tag{21}$$

where $\tilde\delta_i=\bar\delta_i-\hat\delta_i$ and $\kappa_i>0$. Then

$$\begin{aligned}\dot V_n=&\dot V_{n-1}+\sum_{i=1}^N\frac{\partial S_{in}}{\partial x_{in}}(u_i^{p_n}+\delta_i)+\sum_{i=1}^N\sum_{l=1}^{n-1}\frac{\partial S_{in}}{\partial x_{il}}\dot x_{il}\\&+\sum_{i=1}^N\sum_{j\in\mathcal{N}_i}\frac{\partial S_{in}}{\partial x_{j1}}\dot x_{j1}-\sum_{i=1}^N\frac{1}{\kappa_i}\tilde\delta_i\dot{\hat\delta}_i\\\leq&-b_{n-1,1}\sum_{i=1}^N\varphi_{i1}^2-b_{n-1,2}\sum_{i=1}^N\varphi_{i2}^2-\cdots\\&-b_{n-1,n-1}\sum_{i=1}^N\varphi_{i,n-1}^2+b_{n-1,n}\sum_{i=1}^N\varphi_{in}^2\\&+b_{n1}^{'}\sum_{i=1}^N\varphi_{i1}^2+b_{n2}^{'}\sum_{i=1}^N\varphi_{i2}^2+\cdots\end{aligned}$$

$$
\begin{aligned}
&+ b'_{nn}\sum_{i=1}^{N}\varphi_{in}^2 - k_n^{1/p_1\cdots p_{n-1}}\sum_{i=1}^{N}\varphi_{in}^2 - \sum_{i=1}^{N}\frac{1}{\kappa_i}\tilde{\delta}_i\dot{\hat{\delta}}_i \\
&+ \sum_{i=1}^{N}\varphi_{in}^{2-1/p_1\cdots p_{n-1}}(\delta_i - \mathrm{sgn}(\varphi_{in}^{2-1/p_1\cdots p_{n-1}})\hat{\delta}_i). \qquad (22)
\end{aligned}
$$

Note that

$$
\begin{aligned}
&\varphi_{in}^{2-1/p_1\cdots p_{n-1}}(\delta_i - \mathrm{sgn}(\varphi_{in}^{2-1/p_1\cdots p_{n-1}})\hat{\delta}_i) \\
\le &|\varphi_{in}^{2-1/p_1\cdots p_{n-1}}|\bar{\delta}_i - |\varphi_{in}^{2-1/p_1\cdots p_{n-1}}|\hat{\delta}_i \\
= &|\varphi_{in}^{2-1/p_1\cdots p_{n-1}}|\tilde{\delta}_i,
\end{aligned}
$$

therefore

$$
\begin{aligned}
\dot{V}_n = &\dot{V}_{n-1} + \sum_{i=1}^{N}\frac{\partial S_{in}}{\partial x_{in}}(u_i^{p_n} + \delta_i) + \sum_{i=1}^{N}\sum_{l=1}^{n-1}\frac{\partial S_{in}}{\partial x_{il}}\dot{x}_{il} \\
&+ \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\frac{\partial S_{in}}{\partial x_{j1}}\dot{x}_{j1} - \sum_{i=1}^{N}\frac{1}{\kappa_i}\tilde{\delta}_i\dot{\hat{\delta}}_i \\
\le &- b_{n-1,1}\sum_{i=1}^{N}\varphi_{i1}^2 - b_{n-1,2}\sum_{i=1}^{N}\varphi_{i2}^2 - \cdots \\
&- b_{n-1,n-1}\sum_{i=1}^{N}\varphi_{i,n-1}^2 + b_{n-1,n}\sum_{i=1}^{N}\varphi_{in}^2 \\
&+ b'_{n1}\sum_{i=1}^{N}\varphi_{i1}^2 + b'_{n2}\sum_{i=1}^{N}\varphi_{i2}^2 + \cdots \\
&+ b'_{nn}\sum_{i=1}^{N}\varphi_{in}^2 - k_n^{1/p_1\cdots p_{n-1}}\sum_{i=1}^{N}\varphi_{in}^2 \\
&+ \sum_{i=1}^{N}\left(|\varphi_{in}^{2-1/p_1\cdots p_{n-1}}| - \frac{1}{\kappa_i}\dot{\hat{\delta}}_i\right)\tilde{\delta}_i. \qquad (23)
\end{aligned}
$$

Let $\hat{\delta}_i$ update with

$$
\dot{\hat{\delta}}_i = \kappa_i|\varphi_{in}^{2-1/p_1\cdots p_{n-1}}|, \qquad (24)
$$

then $\dot{V}_n$ can be simplified in the following form

$$
\begin{aligned}
\dot{V}_n \le &- b_{n1}\sum_{i=1}^{N}\varphi_{i1}^2 - b_{n2}\sum_{i=1}^{N}\varphi_{i2}^2 - \cdots \\
&- b_{n,n-1}\sum_{i=1}^{N}\varphi_{i,n-1}^2 - b_{nn}\sum_{i=1}^{N}\varphi_{in}^2 \\
\le &0, \qquad (25)
\end{aligned}
$$

where $b_{n1}, b_{n2}, \ldots, b_{nn}$ are all positive constants.

Then by integrating (25) we have

$$
\begin{aligned}
V_n(t) - V_n(0) \le &- b_{n1}\sum_{i=1}^{N}\int_0^t \varphi_{i1}^2(\sigma)\mathrm{d}\sigma \\
&- b_{n2}\sum_{i=1}^{N}\int_0^t \varphi_{i2}^2(\sigma)\mathrm{d}\sigma \\
&- \cdots - b_{nn}\sum_{i=1}^{N}\int_0^t \varphi_{in}^2(\sigma)\mathrm{d}\sigma \\
\le &\ 0. \qquad (26)
\end{aligned}
$$

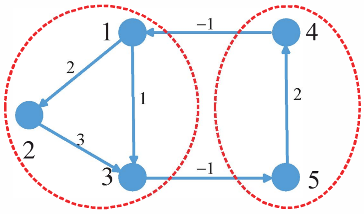

Fig. 1. Communication topology of five agents.

Therefore, $0 \le V_n(t) \le V_n(0)$ is bounded. Since $\dot{V}_n \le 0$, we can imply that

$$
\lim_{t\to+\infty} V_n(t) = 0. \qquad (27)
$$

The preceding analysis, along with $\dot{V}_n \le 0$, yield that

$$
\lim_{t\to+\infty} \dot{V}_n(t) = 0. \qquad (28)
$$

According to the form of (25), we can infer that

$$
\lim_{t\to+\infty} \varphi_{ik}(t) = 0, \quad \forall i \in \mathcal{V}, k = 1, 2, \ldots, n.
$$

Furthermore, with regard to (9), it is clear that $x_{i2}, x_{i3}, \ldots, x_{in}$ all approach to zero when $t \to +\infty$, i.e., bounded to zero. Noting that $(1/2)\vec{x}^{\mathsf{T}}\hat{\mathcal{L}}_u\vec{x}, S_{i2}, S_{i3}, \ldots, S_{in}$ are all nonnegative items and $\mathcal{G}$ is strongly connected, along with (27) and (10), we have

$$
\lim_{t\to+\infty} V_1(t) = 0, \qquad (29)
$$

and this implies that bipartite output consensus can be asymptotically achieved, which is satisfied with (3) in Definition 2. ■

IV. Implementations and Performance Analysis

The multi-agent systems with input noises are given as follows:

$$
\begin{aligned}
\dot{x}_{i1} &= x_{i2} \\
\dot{x}_{i2} &= x_{i3}^5 \\
\dot{x}_{i3} &= u_i^5 + \delta_i, \qquad i = 1, 2, \ldots, 5,
\end{aligned} \qquad (30)
$$

and

$$
\Gamma = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},
$$

$$
\Phi = \begin{bmatrix} -0.1, & 0, & -0.1, & 0.2, & -0.3 \\ 0.2, & 0.05, & 0.5, & 0.1, & 0.05 \end{bmatrix}.
$$

The topology is shown in Figure 1 and the graph is strongly connected and structurally balanced. Agent 1, 2 and 3 are in one group, while agent 4 and 5 are in the opposite group. Given $k_1 = 0.5, k_2 = 15, k_3 = 25, \kappa_i = 0.01, i = 1, 2, \ldots, 5$, and the initial values of the five agents are chosen randomly.

It is illustrated in Figure 2 that bipartite output consensus can be achieved in the presence of input noises if the signed digraph is strongly connected. In Figure 3 and Figure 4, we can see that x_{i2} and x_{i3} are bounded to zero, which is in accordance with Theorem 1 and in turn verify the validity of our distributed control laws (9).

With several oscillations, the third dimensions of the six agents finally approach to zero, thus the developed distributed

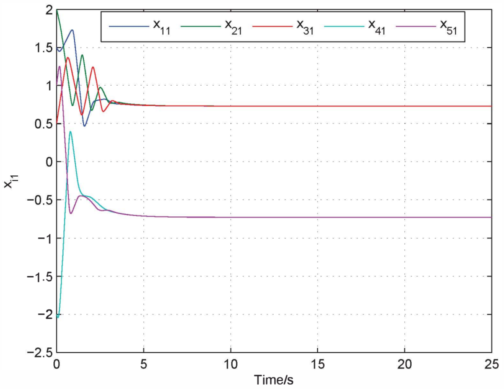

Fig. 2. Outputs trajectories $y_i = x_{i1}, i = 1, 2, \ldots, 5$.

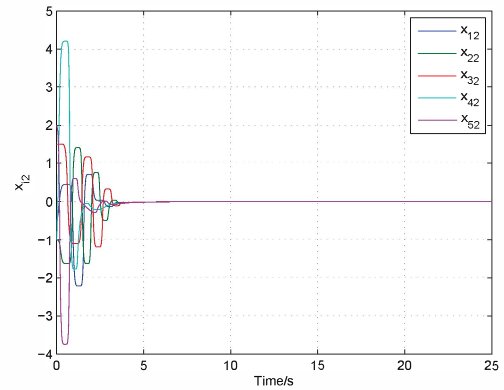

Fig. 3. Trajectories of $x_{i2}, i = 1, 2, \ldots, 5$.

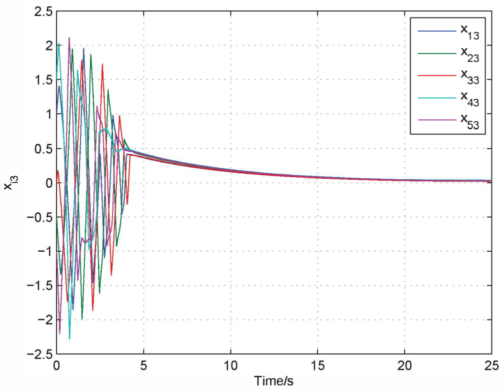

Fig. 4. Trajectories of $x_{i3}, i = 1, 2, \ldots, 5$.

control law (9) can solve the nonlinearity of high-order power integrator based merely on output information. Furthermore, due to the high nonlinearity of the third dimension in multi-agent systems, although bipartite output consensus has been achieved, x_{i3} is still varying.

V. Conclusion

We investigate bipartite output consensus in multi-agent systems of high-order power integrators with input noises and signed digraph. An adaptive disturbance compensator and the technique of adding power integrator are introduced to deal with the input noises and nonlinearity of the multi-agent systems, respectively. Moreover, in the presence of negative communication weights over the network, when the signed digraph is structurally balanced and strongly connected, then bipartite output consensus can be achieved. Our future work will focus on time delays and packet dropouts over the multi-agent systems.

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