

# Output-Based High-Order Bipartite Consensus under Directed Antagonistic Networks

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**Abstract**—In the presence of negative weights in communication graph, bipartite consensus is an extension of the traditional consensus problems where the communication weights are all positive. An output-based distributed control protocol is established to solve the bipartite consensus of the homogeneous multi-agent systems. Bipartite consensus problem is equivalent to a linear stabilizable and detectable problem by introducing a gauge transformation. If the multi-agent systems can reach the bipartite consensus, then the signed digraph should be structurally balanced and contains a spanning tree. Finally, the implementation provides three cases to validate the effectiveness of our developed criteria.

## I. INTRODUCTION

In recent decades, there has been tremendous interest in multi-agent systems with abundant literatures [1]–[3]. The idea of distributed or decentralized algorithm can trace back to [4] and [5] to adapt the advent of networks. In [6], Vicsek et al. proposed a novel type of phase transition in a system of self-driven particles which is the origin of nearest neighbour rules. Then Jadbabaie et al. [1] introduced nearest neighbour rules into the multi-agent systems based on Vicsek’s model. Please refer to survey papers [2], [7], [8] for more details and the references cited therein.

Bipartite graph [9] is a basic concept in graph theory which is appropriate to represent the communication topology of the bipartite consensus. It is more suitable to suppose that some of the agents are cooperative while the rest are competitive in several physical scenarios. For instance, the polarization of the community can be divided into two groups holding the opposite opinions, such as two-party political systems, competing sport teams and rival business cartels. More details about social networks can be referred to [10].

Bipartite consensus has some similar aspects that in consensus problems, thus it is essential to discuss that on what conditions bipartite consensus can be reached. In traditional consensus problems, a consensus condition on when to converge and converging to what values was given in [11]. Additionally, the consensusability of high-order linear multi-agent systems was discussed and some necessary and sufficient conditions were given under some mild conditions in [12]. Hence, the consensusability of the multi-agent systems not

only depends on the communication topology between them but also on the agent’s dynamics. Communication topology with digraph is more feasible than the undirected graph in physical implementations, thus the situation of directed antagonistic networks discussed in this paper is worth studying. Moreover, full-state observations are difficult to be obtained when considering the consensus of multi-agent systems, therefore, output-feedback control [13] is a wise choice to deal with this problem.

To the best of authors’ knowledge, some pioneering works were provided in [14], meanwhile [15] was the first to propose the concept of bipartite consensus which was also called “agreed upon dissensus [16]”. In [15], Altafini introduced the negative weights to the communication topology and demonstrated that bipartite consensus can be reached with the existence of antagonistic interactions. On one hand, it proposed that one of the most important requirements for the signed graph was structural balance [17]. On the other hand, it also proposed both the linear and nonlinear Laplacian feedback designs to solve bipartite consensus. Nevertheless, it only dealt with the simplest condition where the dynamic of each agent was just related to the distributed control without any control information of system matrix  $A$ . Consequently, Hu et al. extended the bipartite consensus to directed signed networks [18] and formation control [19] with the same dynamics. In addition, Valcher and Misra [20] discussed a more complex situation that the dynamics of multi-agent systems were in high-order with antagonistic interactions and the bipartite consensus can be achieved under the stabilizability assumption with a sort of equilibrium between two fully competing groups.

The remainder of this article is organized as follows: properties of signed graph and basic definitions of bipartite consensus are given in Section 2; output-based distributed control protocol is discussed under directed antagonistic networks in Section 3; implementation of bipartite consensus is conducted to demonstrate the validity of the established method in Section 4, while closing with the conclusion of the whole paper in Section 5.

## II. BACKGROUNDS AND PRELIMINARIES

First, we introduce some basic definitions and properties of the communication topology. A triple  $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$  is called a (weighted) signed graph if  $\mathcal{V} = \{v_1, v_2, \dots, v_N\}$  is the set of nodes,  $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$  is the set of edges, and  $\mathcal{A} = (\mathcal{A}_{ij}) \in \mathbb{R}^{N \times N}$  is the matrix of the signed weights of  $\mathcal{G}$ . Here we denote  $\mathcal{A}_{ij}$  as the element of the  $i$ th row and  $j$ th column of the matrix  $\mathcal{A}$ .

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In a directed graph (digraph), a pair of edges sharing the same nodes  $(v_i, v_j), (v_j, v_i) \in \mathcal{E}$  is called a digon [15]. We assume that  $\mathcal{A}_{ij}\mathcal{A}_{ji} \geq 0$ , which means that all digons cannot have the opposite signs. In this paper we call this property digon sign-symmetry. Otherwise, we call it digon sign-nonsymmetry. Antagonistic networks contain competing interactions between some agents, thus, the signed digraph  $\mathcal{G}(\mathcal{A})$  is a good choice to represent the competing behaviors when  $\mathcal{A}_{ij} < 0$ . The interaction between the  $i$ th and the  $j$ th agent is cooperative if  $\mathcal{A}_{ij} > 0$ , otherwise antagonistic if  $\mathcal{A}_{ij} < 0$ . The element of Laplacian matrix  $\mathcal{L}$  is defined as follows:

$$\mathcal{L}_{ij} = \begin{cases} \sum_{v_k \in \mathcal{N}_i} |\mathcal{A}_{ik}|, & \text{if } i = j, \\ -\mathcal{A}_{ij}, & \text{if } i \neq j, \end{cases} \quad (1)$$

where  $\mathcal{N}_i = \{v_j | (v_j, v_i) \in \mathcal{E}\}$  is the neighbour node set of agent  $i$ .

Before proceeding, we introduce the following definitions and lemma to help demonstrating our main theorems.

*Definition 1 (Structurally balanced, cf. [15]):* A signed graph  $\mathcal{G}(\mathcal{A})$  is said to be structurally balanced if it contains a bipartition of the nodes  $\mathcal{V}_1, \mathcal{V}_2, \mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2, \mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset$  such that  $\mathcal{A}_{ij} \geq 0, \forall v_i, v_j \in \mathcal{V}_p$  ( $p \in \{1, 2\}$ );  $\mathcal{A}_{ij} \leq 0, \forall v_i \in \mathcal{V}_p, v_j \in \mathcal{V}_q, p \neq q$  ( $p, q \in \{1, 2\}$ ). Otherwise it is called structurally unbalanced.

*Definition 2 (Gauge transformation):* A gauge transformation is a change of orthant order in  $\mathbb{R}^n$  operated by a matrix  $\mathcal{D}_1 = \text{diag}(\xi)$ , where

$$\mathcal{D} = \{\mathcal{D}_1 | \mathcal{D}_1 = \text{diag}(\xi), \xi = [\xi_1, \xi_2, \dots, \xi_n]^\top, \xi_i \in \{+1, -1\}\}$$

contains all the gauge transformations in  $\mathbb{R}^n$ .

*Lemma 1 (cf. [15]):* A strongly connected, digon sign-symmetric signed digraph  $\mathcal{G}(\mathcal{A})$  is structurally balanced if and only if any of the following equivalent conditions hold:

- 1)  $\exists \mathcal{D}_1 \in \mathcal{D}$  such that  $\mathcal{D}_1 \mathcal{A} \mathcal{D}_1$  has all nonnegative entries;
- 2) 0 is an eigenvalue of  $\mathcal{L}$ .

Refer to [15] for the corresponding proof. We assume that the signed digraph  $\mathcal{G}$  is weight balanced, digon sign-symmetric and structurally balanced throughout this paper. According to Definition 1, this is equivalent to saying that the agents can be split into two disjoint groups, the cooperative interactions between pairs of agents belonging to the same group and the antagonistic interactions between pairs of agents belonging to the different groups.

### III. OUTPUT-BASED BIPARTITE CONSENSUS

We consider a multi-agent system consisting of  $N$  agents with homogeneous continuous-time dynamic system.  $x_i(t)$  is the state of  $i$ th agent,  $i = 1, 2, \dots, N$ , described as follows:

$$\begin{aligned} \dot{x}_i(t) &= Ax_i(t) + Bu_i(t), \\ y_i(t) &= Cx_i(t), \quad i = 1, 2, \dots, N, \end{aligned} \quad (2)$$

where  $x_i(t) \in \mathbb{R}^n$ ,  $u_i(t) \in \mathbb{R}^r$  and  $y_i(t) \in \mathbb{R}^m$  are the state, control and output of the  $i$ th agent, respectively. Here we assume that  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times r}$  and  $C \in \mathbb{R}^{m \times n}$  are constant matrices. Under the antagonistic networks, Valcher

et al. [20] adopted the static state-feedback distributed control protocol:

$$\begin{aligned} u_i(t) &= -K \sum_{v_j \in \mathcal{N}_i} |\mathcal{A}_{ij}| \cdot [x_i(t) - \text{sgn}(\mathcal{A}_{ij})x_j(t)], \\ t &\geq 0, \forall i = 1, 2, \dots, N, \end{aligned} \quad (3)$$

where  $K \in \mathbb{R}^{n \times m}$  is the feedback weighted constant matrix to be designed and  $\text{sgn}(\cdot)$  is the sign function. Protocol (3) is distributed and only depends on the relative errors of the static states between the  $i$ th agent and its corresponding neighbours. In [20], Valcher and Misra mentioned that with the state-feedback distributed control protocol, the  $N$  homogeneous agents can reach the bipartite consensus in an undirected signed graph. Hence, it would be more practical to take the available output measurements into consideration.

*Assumption 1:* All the states are unmeasurable and the outputs of the  $N$  agents can be measured without external disturbances.

Now we focus on the output-feedback case. The distributed protocol of the  $i$ th agent is the following form:

$$\begin{aligned} u_i(t) &= -K \sum_{v_j \in \mathcal{N}_i} |\mathcal{A}_{ij}| \cdot [y_i(t) - \text{sgn}(\mathcal{A}_{ij})y_j(t)], \\ t &\geq 0, \forall i = 1, 2, \dots, N, \end{aligned} \quad (4)$$

where  $K \in \mathbb{R}^{n \times m}$  is the feedback weighted constant matrix to be designed. Note that  $\mathcal{A}_{ij} \neq 0 \Leftrightarrow (v_j, v_i) \in \mathcal{E}$ , thus the equivalent form of (4) is

$$\begin{aligned} u_i(t) &= -K \sum_{j=1}^N |\mathcal{A}_{ij}| \cdot [y_i(t) - \text{sgn}(\mathcal{A}_{ij})y_j(t)], \\ t &\geq 0, \forall i = 1, 2, \dots, N. \end{aligned} \quad (5)$$

We will study under what circumstances the multi-agent systems will reach bipartite consensus. First of all, we need to specify the definition of bipartite consensus analogous to [12]:

*Definition 3:* For the  $N$  homogeneous agents' dynamics (2), if for any initial condition  $x_i(0)$

$$\begin{cases} \lim_{t \rightarrow +\infty} \|x_j(t) - x_i(t)\| \rightarrow 0, & \forall i, j \in \mathcal{V}_1 \text{ or } \forall i, j \in \mathcal{V}_2, \\ \lim_{t \rightarrow +\infty} \|x_j(t) + x_i(t)\| \rightarrow 0, & \forall i \in \mathcal{V}_1 \text{ and } \forall j \in \mathcal{V}_2, \end{cases} \quad (6)$$

where  $\mathcal{V}_1$  and  $\mathcal{V}_2$  are the distinct node sets defined in Definition 1, then we say that the system (2) can reach **bipartite consensus**.

The following part of this section will demonstrate that the bipartite consensus of multi-agent systems (2) not only depends on the topology of the antagonistic networks but also on the properties of each agent's dynamics. Additionally, consensus of cooperative multi-agent systems can be extended to the case of non-cooperative agents and bipartite consensus.

*Theorem 1:* Consider the multi-agent systems (2) with control protocol (5), where  $K \in \mathbb{R}^{r \times m}$  is a matrix to be determined. If the agents can asymptotically reach bipartite consensus, then  $(A, B, C)$  is stabilizable and detectable, and the communication digraph  $\mathcal{G}(\mathcal{A})$  contains a spanning tree with structural balance.

*Proof:* Without loss of generality, we renumber the agents in order and assume that  $\mathcal{V}_1 = \{v_1, v_2, \dots, v_k\}$  and  $\mathcal{V}_2 = \{v_{k+1}, v_{k+2}, \dots, v_N\}$ . The agents within  $\mathcal{V}_1$  or  $\mathcal{V}_2$  are cooperative but antagonistic between the two groups. With the former assumptions and Lemma 1, we can choose a gauge transformation  $\mathcal{D}_1$  from set  $\mathcal{D}$  defined in Definition 2, such that

$$\mathcal{D}_1 = \begin{bmatrix} I_k & 0 \\ 0 & -I_{N-k} \end{bmatrix}$$

satisfies that  $\mathcal{D}_1 \mathcal{A} \mathcal{D}_1^\top$  is a nonnegative matrix. Let  $\eta = [\eta_1, \eta_2, \dots, \eta_N]^\top \in \mathbb{R}^N$  be a left eigenvector of  $\mathcal{L}$  with  $\lambda_1(\mathcal{L}) = 0$ , where  $\lambda_1(\mathcal{L})$  denotes the zero eigenvalue of  $\mathcal{L}$ .  $\mathcal{D}_1 \eta$  is the left eigenvector of the  $\mathcal{D}_1 \mathcal{L} \mathcal{D}_1$ , which corresponds to the zero eigenvalue, thus  $\mathcal{D}_1 \mathcal{L} \mathcal{D}_1$  is the Laplacian matrix related to the nonnegative matrix  $\mathcal{D}_1 \mathcal{A} \mathcal{D}_1$ . We assume that

$$\eta_i = \begin{cases} \frac{1}{N}, & \text{if } i \in \mathcal{V}_1; \\ -\frac{1}{N}, & \text{if } i \in \mathcal{V}_2. \end{cases}$$

Let  $\mathbf{1}_n$  denote a column vector with all 1 elements and  $\otimes$  denote the Kronecker product, respectively. Construct a nonsingular matrix as follows:

$$\Phi = \left[ \begin{array}{c|ccccc} \eta_1 & \eta_2 & \cdots & \eta_k & \eta_{k+1} & \cdots & \eta_N \\ \hline -\mathbf{1}_{k-1} & & I_{k-1} & & 0 & & \\ -\mathbf{1}_{N-k} & & 0 & & -I_{N-k} & & \end{array} \right],$$

thus the corresponding coordinate transformation is

$$\begin{bmatrix} \chi(t) \\ \delta_2(t) \\ \vdots \\ \delta_k(t) \\ \vdots \\ \delta_N(t) \end{bmatrix} = (\Phi \otimes I_n)x(t). \quad (7)$$

Note that  $\chi(t)$  is a vector and  $\delta_i(t) = x_i(t) - x_1(t)$ ,  $\delta_i(t) \in \mathbb{R}^n, \forall v_i \in \mathcal{V}_1$ , while  $\delta_i(t) = -x_i(t) + x_1(t)$ ,  $\delta_i(t) \in \mathbb{R}^n, \forall v_i \in \mathcal{V}_2$ . Thus, if all the  $\delta_i(t), \forall v_i \in \mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$  converge to zero as  $t \rightarrow +\infty$ , then we can say that the bipartite consensus is reached.

If the multi-agent systems (2) can reach bipartite consensus, then there exists a matrix  $K \in \mathbb{R}^{r \times n}$  with the distributed control protocols

$$u_i(t) = -K \sum_{j=1}^N |\mathcal{A}_{ij}| \cdot [y_i(t) - \text{sgn}(\mathcal{A}_{ij})y_j(t)], \\ t \geq 0, \quad \forall i = 1, 2, \dots, N,$$

such that  $\lim_{t \rightarrow +\infty} \delta_i(t) \rightarrow 0, \forall i = 2, 3, \dots, N$ .

Due to the definition of Laplacian, 0 is an eigenvalue of  $\mathcal{L}$  referring to Lemma 1. Then we can make a linear transformation as follows:

$$\Phi^{-1} \mathcal{L} \Phi = \begin{bmatrix} 0 & 0 \\ 0 & \tilde{\mathcal{L}} \end{bmatrix}. \quad (8)$$

In addition, let

$$\tilde{u}(t) = \begin{bmatrix} \tilde{u}_1(t) \\ \tilde{u}_2(t) \\ \vdots \\ \tilde{u}_N(t) \end{bmatrix} = \Phi u(t),$$

then with  $y_i(t) = Cx_i(t), \forall i = 1, 2, \dots, N$ , we have

$$\begin{bmatrix} \dot{\delta}_2(t) \\ \vdots \\ \dot{\delta}_N(t) \end{bmatrix} = (I_{N-1} \otimes A) \begin{bmatrix} \delta_2(t) \\ \vdots \\ \delta_N(t) \end{bmatrix} + (I_{N-1} \otimes B) \begin{bmatrix} \tilde{u}_2(t) \\ \vdots \\ \tilde{u}_N(t) \end{bmatrix}, \quad (9)$$

$$\begin{bmatrix} \tilde{u}_2(t) \\ \vdots \\ \tilde{u}_N(t) \end{bmatrix} = -(\tilde{\mathcal{L}} \otimes K C) \begin{bmatrix} \delta_2(t) \\ \vdots \\ \delta_N(t) \end{bmatrix}. \quad (10)$$

For the sake of analysis convenience, we define

$$\dot{\delta}(t) \triangleq \begin{bmatrix} \dot{\delta}_2(t) \\ \vdots \\ \dot{\delta}_N(t) \end{bmatrix} = [I_{N-1} \otimes A - \tilde{\mathcal{L}} \otimes B K C] \delta(t). \quad (11)$$

Hence, all the eigenvalues of the matrix  $I_{N-1} \otimes A - \tilde{\mathcal{L}} \otimes B K C$  in (11) are in the open left half plane. Suppose that  $\lambda_1 = 0, \lambda_2, \dots, \lambda_N$  are the eigenvalues of Laplacian  $\mathcal{L}$ . Then the eigenvalues of  $\tilde{\mathcal{L}}$  are  $\lambda_2, \lambda_3, \dots, \lambda_N$ . Hence we can find an invertible matrix  $T$  such that  $\tilde{\mathcal{L}}$  is similar to a Jordan canonical matrix, i.e.,

$$T^{-1} \tilde{\mathcal{L}} T = J = \text{diag}(J_1, J_2, \dots, J_l)$$

where  $J_k, k = 1, 2, \dots, l$ , are upper triangular Jordan blocks, and its diagonal elements are  $\lambda_i, i = 2, 3, \dots, N$ .

Furthermore, the Kronecker product contains the similar properties that

$$(T \otimes I_n)^{-1} [I_{N-1} \otimes A - \tilde{\mathcal{L}} \otimes B K C] \times (T \otimes I_n) \\ = I_{N-1} \otimes A - J \otimes B K C \quad (12)$$

is also an upper triangular block matrix implying that the eigenvalues of  $I_{N-1} \otimes A - \tilde{\mathcal{L}} \otimes B K C$  are given by the eigenvalues of  $A - \lambda_i B K C, i = 2, 3, \dots, N$ . Hence, the eigenvalues of  $A - \lambda_i B K C$  are all in the open left half plane.

Now we are in the position to demonstrate  $(A, B, C)$  is stabilizable and detectable. Suppose that at least one of  $\lambda_i, i = 2, 3, \dots, N$ , is real, for example,  $\lambda_2$ , then  $(A, B, C)$  is stabilizable and detectable because all the eigenvalues of  $A - \lambda_2 B K C$  are in the open left half plane. In addition, if  $\lambda_i, i = 2, 3, \dots, N$ , are all complex numbers, i.e., none of their imaginary parts are zeros, then note that since  $\tilde{\mathcal{L}}$  is a real matrix, the eigenvalues will appear in conjugate pair form. Without loss of generality, we suppose that  $\lambda_2$  and  $\lambda_3$  are one

of the corresponding conjugate pair roots with  $\lambda_2 = \sigma + j\omega$  and  $\lambda_3 = \sigma - j\omega$  where  $j^2 = -1$ . Then  $\forall \lambda \in \mathbb{C}$ ,

$$\begin{vmatrix} \lambda I_n - (A - \sigma BKC) & -\omega BKC \\ \omega BKC & \lambda I_n - (A - \sigma BKC) \end{vmatrix} = |\lambda I_n - (A - \lambda_2 BKC)| \cdot |\lambda I_n - (A - \lambda_3 BKC)|.$$

Considering the fact that all the eigenvalues of the matrices  $A - \lambda_2 BKC$  and  $A - \lambda_3 BKC$  are in the open left half plane, we obtain that all the eigenvalues of the matrix

$$\begin{bmatrix} A - \sigma BKC & \omega BKC \\ -\omega BKC & A - \sigma BKC \end{bmatrix}$$

are also in the open left half plane. Then we can say that

$$\begin{aligned} & \begin{bmatrix} A - \sigma BKC & \omega BKC \\ -\omega BKC & A - \sigma BKC \end{bmatrix} \\ &= \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} + \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} -\sigma KC & \omega KC \\ -\omega KC & -\sigma KC \end{bmatrix} \\ &= \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} + \begin{bmatrix} -\sigma BK & \omega BK \\ -\omega BK & -\sigma BK \end{bmatrix} \begin{bmatrix} C & 0 \\ 0 & C \end{bmatrix} \end{aligned}$$

is equivalent to that

$$\left( \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}, \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix} \right)$$

is stabilizable and

$$\left( \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}, \begin{bmatrix} C & 0 \\ 0 & C \end{bmatrix} \right)$$

is detectable. Therefore, we can know that

$$\text{rank} \begin{bmatrix} sI_n - A & 0 \\ 0 & sI_n - A \\ C & 0 \\ 0 & C \end{bmatrix} = 2\text{rank} \begin{bmatrix} sI_n - A \\ C \end{bmatrix} = 2n,$$

$\forall s \in \mathbb{C}$  and  $\text{Re}(s) \geq 0$ , where  $\text{Re}(s)$  represents the real part of complex number  $s$ . Thus,

$$\text{rank} \begin{bmatrix} sI_n - A \\ C \end{bmatrix} = n, \forall s \in \mathbb{C} \text{ and } \text{Re}(s) \geq 0, \text{ i.e., } (A, C) \text{ is detectable.}$$

Following the similar steps, we can verify that  $(A, B)$  is stabilizable. In summary, we have the conclusion that if the multi-agent systems (2) can reach bipartite consensus, then  $(A, B, C)$  is stabilizable and detectable.

Finally, we will focus on the latter part of the theorem, i.e., the communication digraph  $\mathcal{G}(\mathcal{A})$  contains a spanning tree with structural balance.  $\mathcal{G}(\mathcal{A})$  is structurally balanced, thus, according to Lemma 1 we have that there exists  $\mathcal{D}_1 \in \mathcal{D}$  such that  $\mathcal{D}_1 \mathcal{A} \mathcal{D}_1$  has all nonnegative entries and  $\mathcal{D}_1 \mathcal{L} \mathcal{D}_1$  is the Laplacian matrix related to the nonnegative matrix  $\mathcal{D}_1 \mathcal{A} \mathcal{D}_1$ . Due to the fact that linear transformation does not affect the eigenvalues of matrix, then from Lemma 3.3 in [8],  $\lambda_i = 0$  or  $\text{Re}(\lambda_i) > 0, \forall i = 2, 3, \dots, N$ . By contradiction, because not all of the eigenvalues of  $A$  are in the open left half plane, then  $\lambda_i \neq 0, \forall i = 2, 3, \dots, N$ , otherwise there exists  $i \in \{2, 3, \dots, N\}$  such that  $\lambda_i = 0$ . Thus,  $A = A - \lambda_i BKC$  and the eigenvalues of  $A$  are all in the open left half plane, which is a contradiction. Therefore, the signed digraph  $\mathcal{G}(\mathcal{A})$  has only one zero eigenvalue, and by Lemma 3.3 in [8], it must contain a spanning tree. ■

Seeing the proof of Theorem 1, by introducing the linear transformation  $\Phi$ , we can bridge the equivalence between the stability problem of system (11) and the bipartite consensus of the multi-agent systems (2). Then the problem transforms to demonstrating whether there exists a constant gain matrix  $K \in \mathbb{R}^{r \times m}$  such that the eigenvalues of  $A - \lambda_i BKC, \forall i = 2, 3, \dots, N$  are all in the open left half plane. When the system matrix of (2) satisfies the following conditions in Theorem 2, the necessary condition in Theorem 1 is then also sufficient.

*Lemma 2* (cf. [21]): When  $(A, B)$  is stabilizable, the following Riccati equation:

$$A^\top P + PA - PBB^\top P + I_n = 0 \quad (13)$$

has a unique nonnegative definite solution  $P$  and in addition, the eigenvalues of  $A - BB^\top P$  are all in the open left half plane.

*Theorem 2*: If  $P$  is the nonnegative solution of equation (13) and assume that

$$\text{rank}[C] = \text{rank} \begin{bmatrix} C \\ B^\top P \end{bmatrix}. \quad (14)$$

Then  $(A, B)$  is stabilizable and the signed digraph  $\mathcal{G}(\mathcal{A})$  contains a spanning tree if and only if the multi-agent systems (2) can reach bipartite consensus.

*Proof*: Theorem 1 has demonstrated the necessary part and we will focus on the proof of sufficiency. Similar to the demonstration of Theorem 1, since the signed digraph  $\mathcal{G}(\mathcal{A})$  contains a spanning tree, we have that  $\lambda_i, \forall i = 2, 3, \dots, N$  are in the open right half plane, i.e.,  $\text{Re}(\lambda_i) > 0, \forall i = 2, 3, \dots, N$ . Let

$$\beta \triangleq \min_{2 \leq i \leq N} \{\text{Re}(\lambda_i)\}. \quad (15)$$

If condition (14) holds, then the solution of matrix equation

$$XC = B^\top P \quad (16)$$

exists. Without loss of generality, we choose one of these solutions denoted by  $F$ . Then let the constant gain matrix of the distributed control protocol (3) be  $K = \max\{1, \beta^{-1}\}F$ . Thus,

$$A - \lambda_i BKC = A - \lambda_i \max\{1, \beta^{-1}\}BB^\top P, \forall i = 2, 3, \dots, N.$$

Note that for any  $\sigma \geq 1$  and  $\omega \in \mathbb{R}$ , the eigenvalues of  $A - (\sigma + j\omega)BB^\top P$  are all in the open left half plane [22], then all the eigenvalues of  $A - \lambda_i BKC, \forall i = 2, 3, \dots, N$  are in the open left half plane. So  $\|\delta_i(t)\| \rightarrow 0$ , as  $t \rightarrow +\infty, \forall i = 2, 3, \dots, N$ , implying that the multi-agent systems can reach bipartite consensus. ■

#### IV. IMPLEMENTATION OF BIPARTITE CONSENSUS

In this section, we investigate the implementation on bipartite consensus to demonstrate our conclusions.

##### Example: Five agents bipartite consensus

We consider the multi-agent systems containing five agents with the given homogeneous dynamics described as follows:

$$\begin{aligned} \dot{x}_i(t) &= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} x_i(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_i(t), \\ y_i(t) &= [0 \ 1] x_i(t), \quad i = 1, 2, 3, 4, 5, \end{aligned} \quad (17)$$

where  $x_i(t) \in \mathbb{R}^2$  is the state,  $u_i(t) \in \mathbb{R}$  is the control and  $y_i(t) \in \mathbb{R}$  is the output of the  $i$ th agent. According to Theorem 2, we study the following three cases.

**Case 1:** The signed digraph  $\mathcal{G}_1 = (\mathcal{V}_1, \mathcal{E}_1, \mathcal{A}_1)$  represents the topology among the five agents.  $\mathcal{V}_1 = \{1, 2, 3, 4, 5\}$ ,  $\mathcal{E}_1 = \{(1, 2), (2, 3), (3, 4), (4, 5), (5, 1)\}$ ,  $\mathcal{A}_1 = (\mathcal{A}_{ij})_{5 \times 5}$ , where

$$\mathcal{A}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & -3 \\ 3 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \end{bmatrix},$$

then

$$\mathcal{L}_1 = \begin{bmatrix} 3 & 0 & 0 & 0 & 3 \\ -3 & 3 & 0 & 0 & 0 \\ 0 & -3 & 3 & 0 & 0 \\ 0 & 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & -3 & 3 \end{bmatrix}.$$

Note that the eigenvalues of  $\mathcal{L}_1$  are  $0, 2.073 \pm 2.853j$  and  $5.427 \pm 1.763j$  which are all in the open right half plane except the zero eigenvalue. Clearly,  $\mathcal{G}_1$  is structurally balanced and weight balanced containing a spanning tree.

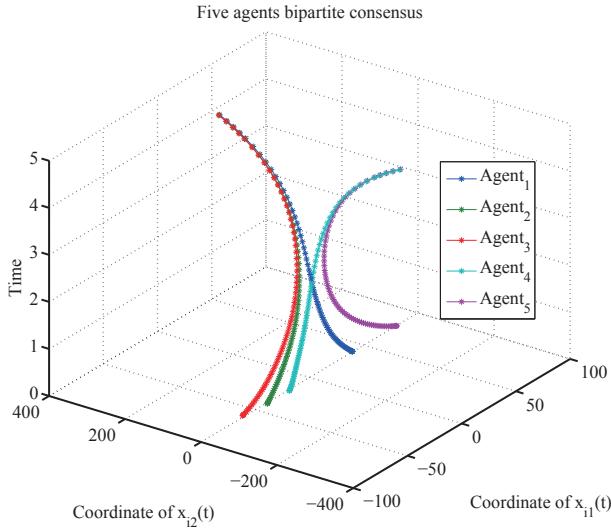


Fig. 1. Five agents bipartite consensus

Choose  $K = 1.5$ , then for any initial value  $x_i(0), \forall i = 1, 2, \dots, 5$ , the five agents can reach bipartite consensus which is shown in Fig. 1 where the x-y plane contains x-axis and y-axis and the axis perpendicular to the x-y plane is the running time.  $x_{i1}(t)$  and  $x_{i2}(t)$  are the two components of the state  $x_i(t)$ . As time goes on, agent 1, agent 2 and agent 3 reach a consensus while agent 4 and agent 5 reach another consensus with the opposite direction.

**Case 2:** The signed digraph  $\mathcal{G}_2 = (\mathcal{V}_2, \mathcal{E}_2, \mathcal{A}_2)$  represents the topology among the five agents.  $\mathcal{V}_2 = \{1, 2, 3, 4, 5\}$ ,  $\mathcal{E}_2 = \{(1, 2), (2, 3), (3, 4), (4, 5), (5, 1)\}$ ,  $\mathcal{A}_2 = (\mathcal{A}_{ij})_{5 \times 5}$ , where

$$\mathcal{A}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 3 \\ 3 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & -3 & 0 & 0 & 0 \end{bmatrix},$$

then

$$\mathcal{L}_2 = \begin{bmatrix} 3 & 0 & 0 & 0 & -3 \\ -3 & 3 & 0 & 0 & 0 \\ 0 & -3 & 3 & 0 & 0 \\ 0 & 0 & 0 & 3 & -3 \\ 0 & 3 & 0 & 0 & 3 \end{bmatrix}.$$

Note that the eigenvalues of  $\mathcal{L}_2$  are  $3, 3, 6$  and  $1.5 \pm 2.598j$  which are all in the open right half plane but  $\mathcal{G}_2$  is structurally unbalanced though containing a spanning tree.

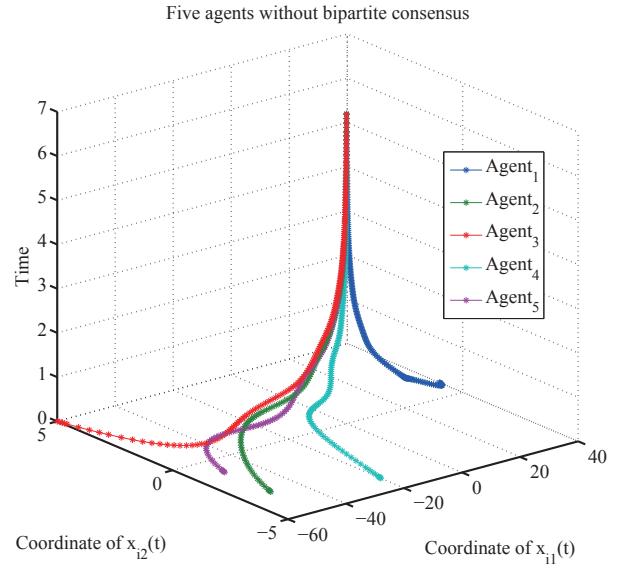


Fig. 2. Five agents without bipartite consensus

Choose  $K = 2$ , then for any initial value  $x_i(0), \forall i = 1, 2, \dots, 5$ , the five agents cannot reach bipartite consensus which is shown in Fig. 2. As time goes on, the five agents reach a consensus instead of bipartite consensus due to the structurally unbalanced signed digraph  $\mathcal{G}_2$ .

**Case 3:** The signed digraph  $\mathcal{G}_3 = (\mathcal{V}_3, \mathcal{E}_3, \mathcal{A}_3)$  represents the topology among the five agents.  $\mathcal{V}_3 = \{1, 2, 3, 4, 5\}$ ,  $\mathcal{E}_3 = \{(1, 2), (2, 3), (3, 4), (4, 3), (4, 5), (5, 1)\}$ ,  $\mathcal{A}_3 = (\mathcal{A}_{ij})_{5 \times 5}$ , where

$$\mathcal{A}_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & -2 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 2 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{bmatrix},$$

then

$$\mathcal{L}_3 = \begin{bmatrix} 2 & 0 & 0 & 0 & 2 \\ -2 & 2 & 0 & 0 & 0 \\ 0 & -2 & 4 & -2 & 0 \\ 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & -2 & 2 \end{bmatrix}.$$

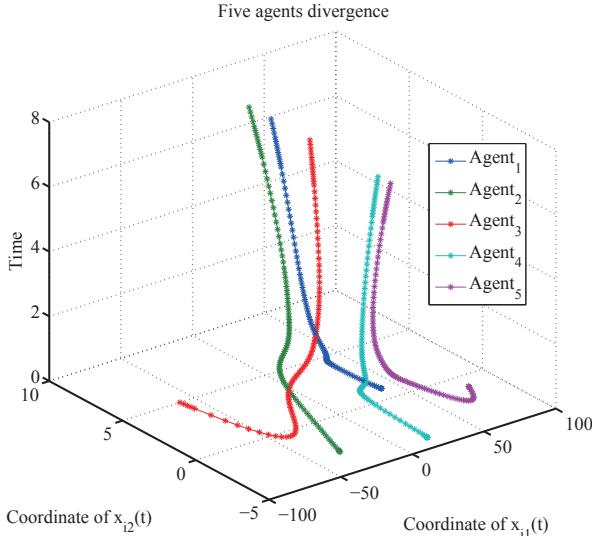


Fig. 3. Five agents with divergence

Note that the eigenvalues of  $\mathcal{L}_3$  are  $0.49, 3.755 \pm 1.490j$  and  $2 \pm 2j$  which are all in the open right half plane. Clearly,  $\mathcal{G}_3$  is digon sign-nonsymmetric but contains a spanning tree.

Choose  $K = 1.8$ , then for any initial value  $x_i(0), \forall i = 1, 2, \dots, 5$ , the five agents can neither reach bipartite consensus nor consensus which is illustrated in Fig. 3. As time goes on, the five agents diverge to different directions for the lack of digon sign-symmetry of  $\mathcal{G}_3$ .

## V. CONCLUSIONS

This paper establishes an output-based distributed feedback control protocol to solve bipartite consensus, where the multi-agent systems are in homogeneous high-order dynamics under directed antagonistic networks. When the multi-agent systems can reach bipartite consensus, the signed digraph should be structurally balanced and contains a spanning tree. Additionally, the system matrices  $(A, B, C)$  are also stabilizable and detectable. Moreover, if there exists a nonnegative matrix  $P$  satisfying the Riccati equation, the necessary condition can transfer to the necessary and sufficient condition. In the future work, we will focus on the situations with switching communication topologies and time delays which are more appropriate to the physical world.

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