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# Reduction and axiomization of covering generalized rough sets

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## Abstract

This paper investigates some basic properties of covering generalized rough sets, and their comparison with the corresponding ones of Pawlak's rough sets, a tool for data mining. The focus here is on the concepts and conditions for two coverings to generate the same covering lower approximation or the same covering upper approximation. The concept of reducts of coverings is introduced and the procedure to find a reduct for a covering is given. It has been proved that the reduct of a covering is the minimal covering that generates the same covering lower approximation or the same covering upper approximation, so this concept is also a technique to get rid of redundancy in data mining. Furthermore, it has been shown that covering lower and upper approximations determine each other. Finally, a set of axioms is constructed to characterize the covering lower approximation operation.

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*Keywords:* Computing with words; Covering; Covering lower and upper approximations; Fuzzy sets; Reduct; Rough sets

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## 1. Introduction

Various theories and methods have been proposed to deal with incomplete and insufficient information in classification, concept formation, and data analysis in data mining. For example, fuzzy set theory [15], rough sets [6], computing with words [12,16,17], linguistic dynamic systems [11,12], and many others, have been developed and applied to real-world problems. The focus of this paper is on the rough set theory, a tool originated by Pawlak [6] for data mining, with the particular intention to generalize it for the possible applications in computing with words and linguistic dynamic systems for modeling and analyzing complex systems and for data mining.

Pawlak's rough sets provide a systematic approach for classification of objects through an indiscernibility relation. For example, when a universe of objects is described by a family of attributes, the indiscernibility of the objects can be based on the attribute values of these objects. When two objects have the same value over a certain group of attributes, we say they are indiscernible with respect to this group of attributes, or have the same description with respect to the indiscernibility relation. Objects of the same description consist of an equivalent class and all equivalent classes form a partition of the universe. With this partition, the rough set theory approximates any subset of objects of the universe by two sets, called the lower and upper approximations.

Partition or equivalent relation, as the indiscernibility relation in Pawlak's original rough set theory, is still restrictive for many applications. To address this issue, several interesting and meaningful extensions to equivalent relation have been proposed in the past, such as tolerance relations [4,8], similarity relations [9], and others [10,13,14]. Particularly, Zakowski has used coverings of a universe for establishing the covering generalized rough set theory [18] and an extensive body of research works has been developed [1–3,7]. The covering generalized rough set theory is a model with promising potential for applications to data mining. In order to apply this theory to data mining, we address some basic problems in this theory.

Given two coverings of a universe, two covering lower approximations, as well as two upper approximations, will be induced. The issues to be addressed in this paper are (1) for a particular covering, what would be the corresponding "smallest" covering that produces the same covering lower or upper approximation? In data mining, it is an important issue to reduce the redundant information. (2) If two coverings induce the same covering lower or upper approximation, what would be the relationship between the two coverings? (3) If two coverings induce the same covering lower approximation, must they induce the same covering upper approximation? And finally, (4) is there a set of axioms to characterize the covering generalized rough sets?

A partition is no longer a partition by dropping any of its members, thus, there is no redundancy problem for a partition. As for a covering, it could still

be a covering by dropping some of its members. Furthermore, the resulting new covering might still produce the same covering lower and/or upper approximation. Hence, a covering may have “redundant” members and a procedure is needed to find its “smallest” covering that induces the same covering lower and upper approximations. This technique can be used to reduce the redundant information in data mining.

In Pawlak’s rough set theory, the main concepts are the lower and upper approximations. But, different partitions of a universe generate different lower and upper approximations. In covering generalized rough set theory, however, different coverings could generate the same covering lower or upper approximations. Therefore, we need to know under what conditions two coverings of a universe generate the same covering lower or upper approximation.

Furthermore, in Pawlak’s rough set theory, the lower and upper approximations are dual to each other and therefore determine each other. However, in the covering generalized rough set theory, the covering lower and upper approximations are no longer dual, thus, the next question would be: can the covering lower approximation still determine the covering upper approximation, and vice versa.

Finally, the problem of what constitutes the essential properties for the covering lower and upper approximations is investigated through the establishment of a set of axioms that characterize the covering lower approximation operations. However, the corresponding problem for the upper approximation operations is still an open question.

## 2. Fundamentals of Pawlak’s rough sets

Let  $U$  be a finite set, the universe of discourse, and  $R$  an equivalent relation on  $U$ , called an indiscernibility relation in rough set theory [6].  $R$  will generate a partition  $U/R = \{Y_1, Y_2, \dots, Y_m\}$  on  $U$ , where  $Y_1, Y_2, \dots, Y_m$  are the equivalent classes, and, in rough set theory, they are also called elementary sets of  $R$ . For any  $X \subseteq U$ , we can describe  $X$  in terms of the elementary sets of  $R$ . Specifically, Pawlak [6] introduced the following two sets:

$$R_*(X) = \cup\{Y_i \in U/R \mid Y_i \subseteq X\}, \quad R^*(X) = \cup\{Y_i \in U/R \mid Y_i \cap X \neq \emptyset\}.$$

They are called the lower and upper approximations of  $X$ , respectively.

Let  $\emptyset$  be the empty set,  $\sim X$  the complement of  $X$  in  $U$ , then the following conclusions have been established for Pawlak’s rough sets [6]:

- |                                   |                |
|-----------------------------------|----------------|
| (1L) $R_*(U) = U$                 | (Co-normality) |
| (1H) $R^*(U) = U$                 | (Co-normality) |
| (2L) $R_*(\emptyset) = \emptyset$ | (Normality)    |
| (2H) $R^*(\emptyset) = \emptyset$ | (Normality)    |

(3L) $R_*(X) \subseteq X$	(Contraction)
(3H) $X \subseteq R^*(X)$	(Extension)
(4L) $R_*(X \cap Y) = R_*(X) \cap R_*(Y)$	(Multiplication)
(4H) $R^*(X \cup Y) = R^*(X) \cup R^*(Y)$	(Addition)
(5L) $R_*(R_*(X)) = R_*(X)$	(Idempotency)
(5H) $R^*(R^*(X)) = R^*(X)$	(Idempotency)
(6) $R_*(\sim X) = \sim R^*(X),$ $R^*(\sim X) = \sim R_*(X)$	(Duality)
(7L) $X \subseteq Y \Rightarrow R_*(X) \subseteq R_*(Y)$	(Monotone)
(7H) $X \subseteq Y \Rightarrow R^*(X) \subseteq R^*(Y)$	(Monotone)
(8L) $R_*(\sim R_*(X)) = \sim R_*(X)$	(Lower-complement relation)
(8H) $R^*(\sim R^*(X)) = \sim R^*(X)$	(Upper-complement relation)
(9L) $\forall K \in U/R, R_*(K) = K$	(Granularity)
(9H) $\forall K \in U/R, R^*(K) = K$	(Granularity)

It has been shown that (3L), (4L), and (8L) are the characteristic properties of the lower approximation, and, correspondingly, (3H), (4H), and (8H) are the characteristic properties of the upper approximation [5,19,20].

### 3. Concepts and properties of covering approximations

**Definition 1.** Let  $U$  be a universe of discourse,  $C$  a family of subsets of  $U$ . If none subsets in  $C$  is empty, and  $\cup C = U$ ,  $C$  is called a covering of  $U$ .

It is clear that a partition of  $U$  is certainly a covering of  $U$ , so the concept of a covering is an extension of the concept of a partition.

In the following discussion, the universe of discourse  $U$  is considered to be finite.

Now we will list some definitions and results about covering rough sets used in this paper [1,3,7].

**Definition 2.** Let  $U$  be a non-empty set,  $C$  a covering of  $U$ . We call the ordered pair  $\langle U, C \rangle$  a covering approximation space.

**Definition 3.** Let  $\langle U, C \rangle$  be a covering approximation space,  $x \in U$ , then set family

$$\text{Md}(x) = \{K \in C \mid x \in K \wedge (\forall S \in C \wedge x \in S \wedge S \subseteq K \Rightarrow K = S)\}$$

is called the minimal description of  $x$ .

**Definition 4.** For a set  $X \subseteq U$ , set family  $C_*(X) = \{K \in C \mid K \subseteq X\}$  is called the covering lower approximation set family of  $X$ .

Set  $X_* = \cup C_*(X)$  is called the covering lower approximation of  $X$ .

Set  $X^* = X - X_*$  is called the covering boundary of  $X$ .

Set family  $\text{Bn}(X) = \{\text{Md}(x) \mid x \in X^*\}$  is called the covering boundary approximation set family of  $X$ .

Set family  $C^*(X) = C_*(X) \cup \text{Bn}(X)$  is called the covering upper approximation set family of  $X$ .

Set  $X^* = \cup C^*(X)$  is called the covering upper approximation of  $X$ .

If  $C^*(X) = C_*(X)$ ,  $X$  is said to be definite, otherwise indefinite.

**Definition 5.** Let  $C$  be a covering of  $U$ ,  $P(U)$  the power set of  $U$ . Operations  $L_C$  and  $H_C : P(U) \rightarrow P(U)$  are defined as follows:

$$X \in P(U), \quad L_C(X) = X_*, \quad H_C(X) = X^*.$$

We call them the covering lower approximation operation and the covering upper approximation operation, coupled with the covering  $C$ , respectively. When the covering is clear, we omit the lowercase  $C$  for the two operations.

**Proposition 1.** *The covering approximation set families  $C_*(X)$  and  $C^*(X)$  have the following properties:*

- (1)  $C^*(\emptyset) = C_*(\emptyset) = \emptyset, C^*(U) = C_*(U) = C,$
- (2)  $C_*(X) \subseteq C^*(X),$
- (3)  $C_*(X_*) = C_*(X) = C^*(X_*),$
- (4)  $X \subseteq Y \Rightarrow C_*(X) \subseteq C_*(Y).$

**Proposition 2.** *If  $C$  is a partition,  $X_*$  and  $X^*$  are the Pawlak's lower and upper approximations of  $X$ .*

**Proposition 3.**  *$C^*(X) = C_*(X)$  if and only if  $X$  is a union of some subsets in the covering  $C$ .*

**Proposition 4.**  *$X_* = X$  if and only if  $C^*(X) = C_*(X)$ .*

**Corollary 1.**  *$X_* = X$  if and only if  $X$  is a union of some elements of  $C$ .*

**Proposition 5.**  *$X_* = X^*$  if and only if  $C^*(X) = C_*(X)$ .*

**Corollary 2.**  *$X_* = X^*$  if and only if  $X_* = X$ .*

**Corollary 3.**  *$X = X^*$  if and only if  $X_* = X$ .*

**Corollary 4.**  *$X^* = X$  if and only if  $X$  is a union of some elements of  $C$ .*

Corresponding to the properties of Pawlak's rough sets listed in Section 2, we have the following results.

**Proposition 6.** For a covering  $C$ , the covering lower and upper approximations have the following properties:

- |  |                |
|--|----------------|
| (1L) $U_* = U$                                     | (Co-normality) |
| (1H) $U^* = U$                                     | (Co-normality) |
| (2L) $\emptyset_* = \emptyset$                     | (Normality)    |
| (2H) $\emptyset^* = \emptyset$                     | (Normality)    |
| (3L) $X_* \subseteq X$                             | (Contraction)  |
| (3H) $X \subseteq X^*$                             | (Extension)    |
| (5L) $(X_*)_* = X_*$                               | (Idempotency)  |
| (5H) $(X^*)^* = X^*$                               | (Idempotency)  |
| (7L) $X \subseteq Y \Rightarrow X_* \subseteq Y_*$ | (Monotone)     |
| (9L) $\forall K \in C, K_* = K$                    | (Granularity)  |
| (9H) $\forall K \in C, K^* = K$                    | (Granularity)  |

From Examples 1–4, the following six properties of Pawlak's lower and upper approximations do not hold for the covering lower and upper approximations:

- |  |                             |
|--|-----------------------------|
| (4L) $X_* \cap Y_* = (X \cap Y)_*$                 | (Multiplication)            |
| (4H) $(X \cup Y)^* = X^* \cup Y^*$                 | (Addition)                  |
| (6) $X_* = \sim (\sim X)^*, X^* = \sim (\sim X)_*$ | (Duality)                   |
| (7H) $X \subseteq Y \Rightarrow X^* \subseteq Y^*$ | (Monotone)                  |
| (8L) $(\sim X_*)_* = \sim X_*$                     | (Lower-complement relation) |
| (8H) $(\sim X^*)^* = \sim X^*$                     | (Upper-complement relation) |

**Example 1 (Multiplication and addition).** Let  $U = \{a, b, c, d\}$ ,  $K_1 = \{a, b\}$ ,  $K_2 = \{a, c\}$ ,  $K_3 = \{c, d\}$ ,  $C = \{K_1, K_2, K_3\}$ . Clearly,  $C$  is a covering of  $U$ .

For  $X = K_1$ ,  $Y = K_2$ , we have

$$X_* = K_1 = \{a, b\}, \quad Y_* = K_2 = \{a, c\},$$

thus,  $X_* \cap Y_* = \{a\}$ .

On the other hand,  $X \cap Y = \{a\}$ , thus

$$(X \cap Y)_* = \{a\}_* = \emptyset.$$

Therefore,  $X_* \cap Y_* \neq (X \cap Y)_*$ .

For  $X = \{a\}$ ,  $Y = \{c\}$ , we have

$$C_*(X) = C_*(Y) = \emptyset, \quad \text{Md}(a) = \{K_1, K_2\}, \quad \text{Md}(c) = \{K_2, K_3\},$$

thus,  $X^* = K_1 \cup K_2 = \{a, b, c\}$ ,  $Y^* = K_2 \cup K_3 = \{b, c, d\}$ ,

$$X^* \cup Y^* = U.$$

On the other hand,  $X \cup Y = \{a, c\}$ , thus,

$$(X \cup Y)_* = \{a, c\}.$$

It is clear that  $\text{Bn}(X \cup Y) = \emptyset$ , hence,

$$(X \cup Y)^* = (X \cup Y)_* = \{a, c\}.$$

Therefore,  $(X \cup Y)^* \neq X^* \cup Y^*$ .

**Example 2 (Duality).** Let  $U = \{a, b, c, d, e\}$ ,  $K_1 = \{a, b, c, d\}$ ,  $K_2 = \{a, b\}$ ,  $K_3 = \{e\}$ .  $C = \{K_1, K_2, K_3\}$  is a covering of  $U$ .

For  $X = K_2$ , we have  $X_* = \{a, b\}$ ,  $(\sim X)^* = (\{c, d, e\})^* = U$ , so

$$\sim (\sim X)^* = \emptyset \neq X_*.$$

For  $X = \{c, d\}$ , we have  $X^* = U$ ,  $(\sim X)_* = \{a, b\}$ , so

$$X^* \neq \sim (\sim X)_*.$$

**Example 3 (Monotone).** Let  $U = \{a, b, c\}$ ,  $K_1 = \{a, b\}$ ,  $K_2 = \{b, c\}$ ,  $C = \{K_1, K_2\}$ ,  $X = \{b\}$ ,  $Y = \{a, b\}$ , we have:

$$C_*(X) = \emptyset, \quad X_* = \emptyset,$$

$$X - X_* = \{b\}, \quad \text{Md}(b) = \{K_1, K_2\},$$

$$\text{Bn}(X) = \{K_1, K_2\}, \quad C^*(X) = \{K_1, K_2\},$$

$$X^* = K_1 \cup K_2 = \{a, b, c\}.$$

On the other hand,

$$C_*(Y) = \{K_1\}, \quad Y_* = \{a, b\},$$

$$Y - Y_* = \emptyset, \quad \text{Bn}(Y) = \emptyset,$$

$$C^*(Y) = \{K_1\}, \quad Y^* = K_1 = \{a, b\}.$$

Clearly,  $X \subseteq Y$  whereas  $X^* \subseteq Y^*$  is not valid.

**Example 4 (Lower-complement relation and upper-complement relation).** Let  $U = \{a, b, c\}$ ,  $K_1 = \{a, b\}$ ,  $K_2 = \{b, c\}$ ,  $X = \{a, b\}$ ,  $Y = \{a\}$ , we have

$$X_* = \{a, b\}, \quad (\sim X_*)_* = (\sim \{a, b\})_* = \{c\}_* = \emptyset,$$

$$Y^* = \{a, b\}, \quad (\sim Y^*)^* = (\sim \{a, b\})^* = \{c\}^* = \{b, c\}.$$

So,  $(\sim X_*)_* = \sim X_*$ ,  $(\sim Y^*)^* = \sim Y^*$  do not hold.

#### 4. Reduct of coverings

From the above definitions and propositions, we know that for a universe  $U$  and a covering  $C$  of  $U$ ,  $C$  can generate covering lower and upper

approximations on  $U$ . When  $C$  is a partition, the generated covering lower and upper approximations are, respectively, the lower and upper approximations in the Pawlak's sense.

Now, we wonder if a covering  $C$  of  $U$  generates a Pawlak's lower approximation, should the covering  $C$  be a partition? That is to say, when a covering  $C$  is not a partition, can there exist a partition  $C'$  such that  $C$  and  $C'$  generate the same covering lower approximations?

In a more general sense, when should two coverings generate the same covering lower approximations or the same covering upper approximations on  $U$ .

In Pawlak's rough set theory, the lower and the upper approximations are dual, so they are dependent on each other. As to the above definitions of covering generalized rough sets, the covering lower and upper approximations on  $U$  are not dual. Now we want to ask whether the covering lower and upper approximations are dependent on each other.

In this section, we pay our attention to these questions.

**Example 5.** A non-partition covering  $C$  can generate Pawlak's lower and upper approximations. Let

$$U = \{a, b, c, d\}, \quad K_1 = \{a\}, \quad K_2 = \{b, c\}, \\ K_3 = \{d\}, \quad K_4 = \{a, d\}, \quad C = \{K_1, K_2, K_3, K_4\}.$$

The covering lower approximation generated by  $C$  is the same as the lower approximation generated by a partition  $C' = \{K_1, K_2, K_3\}$ . It is same for upper approximations.

This example also shows that two distinct coverings can generate the same covering lower and upper approximations.

**Definition 6.** Let  $C$  be a covering of a universe  $U$  and  $K \in C$ . If  $K$  is a union of some sets in  $C - \{K\}$ , we say  $K$  is a reducible element of  $C$ , otherwise  $K$  is an irreducible element of  $C$ .

**Definition 7.** Let  $C$  be a covering of  $U$ . If every element of  $C$  is an irreducible element, we say  $C$  is irreducible; otherwise  $C$  is reducible.

**Proposition 7.** Let  $C$  be a covering of a universe  $U$ . If  $K$  is a reducible element of  $C$ ,  $C - \{K\}$  is still a covering of  $U$ .

**Proposition 8.** Let  $C$  be a covering of  $U$ ,  $K \in C$ ,  $K$  is a reducible element of  $C$ , and  $K_1 \in C - \{K\}$ , then  $K_1$  is a reducible element of  $C$  if and only if it is a reducible element of  $C - \{K\}$ .



**Proof.** If  $K_1$  is a reducible element of  $C - \{K\}$ ,  $K_1$  can be expressed as a union of some sets in  $(C - \{K\}) - \{K_1\} = C - \{K, K_1\}$ . It certainly can be expressed as a union of some sets in  $C - \{K_1\}$ , so it is a reducible element of  $C$ .

On the other hand, suppose  $K_1$  is a reducible element of  $C$ . We have that  $K_1$  can be expressed as a union of some sets in  $C - \{K_1\}$ , say  $T_1, T_2, \dots, T_n$ . It is easy to see that  $T_i \subset K_1$  for every  $i$ . If all sets among  $T_1, T_2, \dots, T_n$  are not equal to  $K$ ,  $K_1$  can be expressed as the union of sets  $T_1, T_2, \dots, T_n$  in  $(C - \{K\}) - \{K_1\}$ , so  $K_1$  is a reducible element of  $C - \{K\}$ . If some one among  $T_1, T_2, \dots, T_n$  is equal to  $K$ , say  $T_1 = K$ , from that  $K$  is reducible in  $C$ , there are sets  $S_1, S_2, \dots, S_m \in C - \{K\}$ , such that  $T_1$  is the union of  $S_1, S_2, \dots, S_m$ . Because  $T_1 \subset K_1, S_1, S_2, \dots, S_m$  cannot be equal to  $K_1$ , so we have  $K_1 = T_1 \cup T_2 \cup \dots \cup T_n = S_1 \cup S_2 \cup \dots \cup S_m \cup T_2 \cup \dots \cup T_n$ , and  $S_1, S_2, \dots, S_m, T_2, \dots, T_n$  are not equal to either  $K$  or  $K_1$ . It follows that  $K_1$  is a reducible element of  $C - \{K\}$ .  $\square$

Proposition 7 guarantees that after deleting a reducible element in a covering, it is still a covering, whereas Proposition 8 shows that deleting a reducible element in a covering will not generate any new reducible elements or make other originally reducible elements become irreducible elements of the new covering. So, we can compute the reduct of a covering of a universe  $U$  by deleting all reducible elements in the same time, or by deleting one reducible element in a step. The remainder still consists of a covering of the universe  $U$ , and it is irreducible. For an algorithm to compute the reduct of a covering (see [21,22]).

**Definition 8.** For a covering  $C$  of a universe  $U$ , the new irreducible covering through the above reduction is called the reduct of  $C$ , and denoted by  $\text{reduct}(C)$ .

Proposition 8 guarantees that a covering has only one reduct.

**Proposition 9.** Let  $C$  be a covering of  $U$ , and  $K$  a reducible element of  $C$ ,  $C - \{K\}$  and  $C$  have the same  $\text{Md}(x)$  for all  $x \in U$ .

**Proof.** In fact, if  $K$  is a reducible element in  $C$ ,  $K \notin \text{Md}(x)$  for all  $x \in U$  and for any element  $K_1$  of  $C$ , if  $K \subseteq K_1$ , there must be some other element  $K'$  such that  $K' \subset K \subseteq K_1$ , so  $C - \{K\}$  and  $C$  have the same  $\text{Md}(x)$  for all  $x \in U$ .  $\square$

**Corollary 5.** Suppose  $C$  is a covering of  $U$ , then  $C$  and  $\text{reduct}(C)$  have the same  $\text{Md}(x)$  for all  $x \in U$ .

**Proposition 10.** Suppose  $C$  is a covering of  $U$ ,  $K$  is a reducible element of  $C$ ,  $X \subseteq U$ , then the covering lower approximations of  $X$  generated by the covering  $C$  and the covering  $C - \{K\}$ , respectively, are same.

**Proof.** Suppose the covering lower approximations of  $X$  generated by the covering  $C$  and the covering  $C - \{K\}$  are  $X_1, X_2$  respectively. From the definition of the covering lower approximation, it is evident  $X_2 \subseteq X_1 \subseteq X$ . On the other hand, from the Proposition 6(5L) and the corollary of the Proposition 4, there exists  $K_1, K_2, \dots, K_n$  in  $C$ , such that  $X_1 = K_1 \cup K_2 \cup \dots \cup K_n$ . It is obvious that  $K_1, K_2, \dots, K_n$  are all subsets of  $X$ .

If none of  $K_1, K_2, \dots, K_n$  are equal to  $K$ , then they all belong to  $C - \{K\}$ , so  $K_1, K_2, \dots, K_n$  are all the subsets of  $X_2$ . If some one among  $K_1, K_2, \dots, K_n$  is equal to  $K$ , we suppose  $K_1 = K$ . Because  $K$  is a reducible element of  $C$ ,  $K$  can be expressed a union of some elements  $T_1, T_2, \dots, T_m$  of  $C - \{K\}$ , so,  $X_1 = K_1 \cup K_2 \cup \dots \cup K_n = T_1 \cup T_2 \cup \dots \cup T_m \cup K_2 \cup \dots \cup K_n$ . For  $T_1, T_2, \dots, T_m, K_2, \dots, K_n$  are all subsets of  $X$ , and they are elements of  $C - \{K\}$ , so they are all subsets of  $X_2$ . Now we have proved  $X_1 \subseteq X_2$ .

Therefore,  $X_1 = X_2$ .  $\square$

**Corollary 6.** *Suppose  $C$  is a covering of  $U$ , then the covering lower approximations of  $X$  generated by the covering  $C$  and the covering  $\text{reduct}(C)$ , respectively, are same.*

**Proposition 11.** *Suppose  $C$  is a covering of  $U$ ,  $K$  is a reducible element of  $C$ , and  $X \subseteq U$ , then the covering upper approximations of  $X$  generated by the covering  $C$  and the covering  $C - \{K\}$ , respectively, are same.*

**Proof.** By Proposition 10, the covering lower approximations of  $X$  generated by the covering  $C$  and the covering  $C - \{K\}$ , respectively, are same, thus, from Definition 4, the corresponding covering boundary sets are same. Again, from Proposition 9, the two coverings have the same  $\text{Md}(x)$  for all  $x \in U$ , thus, by the definition of the covering upper approximation, the covering upper approximations of  $X$  generated by the two coverings respectively are same.  $\square$

**Corollary 7.** *Suppose  $C$  is a covering of  $U$ , then the covering upper approximations of  $X$  generated by the covering  $C$  and the covering  $\text{reduct}(C)$ , respectively, are same.*

Combining the corollaries of Propositions 10 and 11, we have the following conclusion.

**Theorem 1.** *Let  $C$  be a covering of  $U$ , then  $C$  and  $\text{reduct}(C)$  generate the same covering lower and upper approximations.*

**Proposition 12.** *If two irreducible coverings of  $U$  generate the same covering lower approximations for all  $X \subseteq U$ , then the two coverings are same.*

**Proof.** Let  $U$  be a universe,  $C_1, C_2$  two coverings of  $U$ , and they generate the same covering lower approximation  $L(X)$  for all  $X \subseteq U$ . Now we prove that any element of  $C_1$  is an element of  $C_2$ . Suppose  $K \in C_1$ , we have  $L(K) = K$ . From the corollary of Proposition 4, in the covering  $C_2$ ,  $K$  is a union of some elements of  $C_2$ . Let  $K_1, K_2, \dots, K_n$  be the elements of  $C_2$  such that  $K = K_1 \cup K_2 \cup \dots \cup K_n$ . By Proposition 6(9L), for any  $K_i$ ,  $L(K_i) = L(K_i)$ . Similar to the above proof, there exist elements of  $C_1$ ,  $T_{i,1}, T_{i,2}, \dots, T_{i,m(i)}$ , such that  $K_i = \bigcup_{j=1}^{m(i)} T_{i,j}$ . Thus,  $K = \bigcup_{i=1}^n \bigcup_{j=1}^{m(i)} T_{i,j}$ . Since  $C_1$  is irreducible,  $T_{i,j} = K$  for all  $i, j$ . Therefore, for all  $i$ ,  $K_i = K$ , hence,  $K$  is an element of  $C_2$ .

Similarly, any element of  $C_2$  is an element of  $C_1$ , therefore,  $C_1$  and  $C_2$  have the same elements, that is,  $C_1 = C_2$ .  $\square$

From Theorem 1 and Proposition 12, we have

**Theorem 2.** *Let  $C_1, C_2$  be two coverings of  $U$ ,  $C_1, C_2$  generate the same covering lower approximation if and only if  $\text{reduct}(C_1) = \text{reduct}(C_2)$ .*

**Corollary 8.** *Let  $C$  be a covering of  $U$ ,  $C$  generates a Pawlak's lower approximation if and only if  $\text{reduct}(C)$  is a partition.*

From Proposition 9 and the definition of covering upper approximations, we have the following result.

**Proposition 13.** *If two irreducible coverings of  $U$  generate the same covering upper approximation, the two coverings are equal.*

**Proof.** From Proposition 6(9H) and the corollary of Proposition 5, the proof of this proposition is similar to that of Proposition 11.  $\square$

From Proposition 13 and the corollary of Proposition 11, we have:

**Theorem 3.** *Let  $C_1, C_2$  be two coverings of  $U$ ,  $C_1, C_2$  generate the same covering upper approximation if and only if  $\text{reduct}(C_1) = \text{reduct}(C_2)$ .*

**Corollary 9.** *Let  $C$  be a covering of  $U$ ,  $C$  generate a Pawlak's upper approximation if and only if  $\text{reduct}(C)$  is a partition.*

**Theorem 4.** *Let  $C_1, C_2$  be two coverings of  $U$ ,  $C_1, C_2$  generate the same covering lower approximation if and only if they generate the same covering upper approximation.*

Theorem 4 shows that the covering lower approximation and the covering upper approximation determine each other.

**Theorem 5.** *Let  $C$  is a covering of  $U$ ,  $C$  generates a Pawlak's lower approximation if and only if  $C$  generates a Pawlak's upper approximation.*

## 5. The axiomization of the lower approximation operation

As we know in Section 2, Pawlak's lower and upper approximation operations have been axiomized. Now, we want to know which are the characteristic properties for the covering lower approximation operation and the covering upper approximation operation. We get the axiomization of the covering lower approximation operation as follows. As for the axiomization of the covering upper approximation operation, it is still an open problem.

**Theorem 6.** *Let  $U$  be a non-empty set. If an operation  $L : P(U) \rightarrow P(U)$  satisfies the following properties: for any  $X, Y \subseteq U$*

$$(1) L(U) = U \quad (\text{Co-normality})$$

$$(2) X \subseteq Y \Rightarrow L(X) \subseteq L(Y) \quad (\text{Monotone})$$

$$(3) L(X) \subseteq X \quad (\text{Contraction})$$

$$(4) L(L(X)) = L(X) \quad (\text{Idempotency})$$

*then there exists a covering  $C$  of  $U$ , such that the covering lower approximation operation  $L_C$  generated by  $C$  equals to  $L$ .*

**Proof.** Let  $C = \{A \subseteq U : L(A) = A, A \neq \emptyset\}$ . By property (1) of  $L$ ,  $C$  must be a covering of  $U$ . For any  $X \subseteq U$ ,  $L_C(X) = \cup\{K \subseteq X : K \in C\}$ . For any  $K \subseteq X$  and  $K \in C$ ,  $K = L(K)$  from the definition of  $C$ . By the property (2) of  $L$ , we have  $L(K) \subseteq L(X)$ , so  $K \subseteq L(X)$ , that means  $L_C(X) \subseteq L(X)$ . On the other hand, if  $L(X) = \emptyset$ , it is obvious that  $L(X) \subseteq L_C(X)$ . If  $L(X) \neq \emptyset$ , by (4),  $L(L(X)) = L(X)$ , so  $L(X) \in C$ . From (3),  $L(X) \subseteq X$ , so  $L(X) \subseteq L_C(X)$  from the definition of  $L_C$ . Now we have proved that  $L(X) = L_C(X)$ .  $\square$

The following example shows that the above four properties for a covering lower approximation operation are necessary and independent of each other.

**Example 6.** Let  $[U = \{a, b, c\}$ , consider the following cases:

1. Let  $L(X) = \emptyset$  for any  $X \subseteq U$ . This  $L$  satisfies all the four properties except (1).
2. Let  $L(\emptyset) = \emptyset$ ,  $L(\{a\}) = \{a\}$ ,  $L(\{b\}) = \{b\}$ ,  $L(\{c\}) = \{c\}$ ,  $L(\{a, b\}) = \{a, b\}$ ,  $L(\{a, c\}) = \{a, c\}$ ,  $L(\{b, c\}) = \{b\}$ ,  $L(U) = U$ . This  $L$  satisfies all the four properties except (2). Since  $\{c\} \subseteq \{b, c\}$ , but  $L(\{c\}) = \{c\}$  and  $L(\{b, c\}) = \{b\}$ , so  $L(\{c\}) \subseteq L(\{b, c\})$  does not hold.
3. Let  $L(X) = U$  for any  $X \subseteq U$ . This  $L$  satisfies all the four properties except (3).

4. Let  $L(\{b\}) = \emptyset$ ,  $L(\{a\}) = \emptyset$ ,  $L(\{a, b\}) = \{a\}$ ,  $L(X) = X$  for all other  $X \subseteq U$ . Since  $L(L(\{a, b\})) = L(\{a\}) = \emptyset \neq L(\{a, b\})$ , this  $L$  satisfies all the four properties except (4).

## 6. Conclusions

In this paper we have shown that (a) the reduct of a covering is the minimal covering that generates the same covering lower and upper approximations, a key concept for us to reduce redundant information in data mining when using the covering generalized rough set model; and (b) the covering lower and upper approximations determine each other. In addition, a set of axioms is constructed to characterize the operations of the covering lower approximation.

However, it is still an open question regarding the axiomization of the covering upper approximations. Another issue to be investigated in the future is the applications of covering generalized rough sets in computing with words and linguistic dynamic systems.

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