Adaptive Integral Operators for Signal Separation

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Abstract—The operator-based signal separation approach uses an adaptive operator to separate a signal into a set of additive subcomponents. In this paper, we show that differential operators and their initial and boundary values can be exploited to derive corresponding integral operators. Although the differential operators and the integral operators have the same null space, the latter are more robust to noisy signals. Moreover, after expanding the kernels of Frequency Modulated (FM) signals via eigen-decomposition, the operator-based approach with the integral operator can be regarded as the matched filter approach that uses eigen-functions as the matched filters. We then incorporate the integral operator into the Null Space Pursuit (NSP) algorithm to estimate the kernel and extract the subcomponent of a signal. To demonstrate the robustness and efficacy of the proposed algorithm, we compare it with several state-of-the-art approaches in separating multiple-component synthesized signals and real-life signals.

Index Terms—Integral equation, narrow band signal, null space pursuit (NSP), operator-based.

I. INTRODUCTION

IN RECENT years, several approaches [1]–[10] have been proposed to separate a single-channel signal into a mixture of several additive coherent subcomponents. The method used to separate signals depends on the definition of the subcomponents. For example, in the empirical mode decomposition (EMD) approach [2], [10]–[12], the subcomponents are Intrinsic Mode Functions (IMFs); in the Synchrosqueezed Wavelet Transform (SWT) approach, the subcomponents are Intrinsic Mode Type Functions (IMT) [7], [8], [13]; and in the operator-based approach [5], [6], a subcomponent is defined as being in the null space of an operator, which is characterized by some parameters that are estimated from the input (residual) signal.

To improve the robustness and efficacy of the operator-based approach, the Null Space Pursuit (NSP) algorithm was proposed [6]. It separates a signal S into U and R such that $U = S - R$ is

in the null space of an operator $\mathcal{T}$ by minimizing the following problem:

$$
\begin{aligned}
\hat{R} = \arg \min_R \Big\{ &\|\mathcal{T}(S - R)\|^2 \\
&+ \lambda \left(\|R\|^2 + \gamma \|S - R\|^2 \right) + F(\mathcal{T}) \Big\}.
\end{aligned}
\tag{1}
$$

The first and second terms of Eq. (1) are the same as in the operator-based approach. The parameter γ in the third term of Eq. (1) determines the amount of $S - R$ to be retained in the null space of $\mathcal{T}$. The last term is the Lagrange term for the parameters of the operator $\mathcal{T}$. Based on some assumptions, the NSP algorithm can adaptively estimate the parameters λ and γ and derive the optimal solution of Eq. (1) [6].

An attractive feature of the operator-based approach is that the operator design can be customized based on the characteristics of the signal's subcomponents. Let $f(t)$ be a subcomponent. Then, any operator $\mathcal{T}$ with $\mathcal{T}[f](t) = 0$ (i.e., $f(t)$ is in the null space of the operator) can be used in the proposed approach to "annihilate" the subcomponent signal. For instance, to annihilate a frequency modulated (FM) signal $\cos(\phi(t))$, where $\phi(t)$ is a local linear function, we can use the operator $(d^2/dt^2 + \varpi^2(t))$ (as defined in [6]). Here, $\varpi(t) = d\phi(t)/dt$ is the instantaneous frequency (IF) of the signal. In addition, the operator $d^2/dt^2 + \alpha_1(t)d/dt + \alpha_2(t)$ described in [14] can be used to annihilate an amplitude modulated and frequency modulated (AM-FM) signal $a(t)\cos(\phi(t))$, with the parameters $\alpha_1(t) = -2a'(t)/a(t)$ ($|a'(t)/a(t)|$ is the instantaneous bandwidth (IB)) and $\alpha_2(t) = \varpi^2(t) + 2(a'(t)/a(t))^2$ ($\varpi(t)$ is the IF). In [5], the general form of a differential operator is defined as

$$
\mathcal{T}_D = \sum_{k \in Z} \alpha(k) \frac{d^k}{dt^k}
\tag{2}
$$

where $\{\alpha(k)\}$ is a square summable sequence. For a mixture of narrow band signals, FM or AM-FM, the proposed differential operators can separate each subcomponent successfully; however, in some instances, particularly low SNR scenarios, the differential operators tend to amplify the noise component when estimating the parameters of the operator.

Also, a kind of integral operator using a simple local meanas $\mathcal{T}_I S(t) = \int_{B_t} S(x)dx$ has been proposed in [5]. However, this kind of integral operator can only annihilate the type of narrow band signals that has only one frequency or a narrow range of frequencies varying as a function of time, as defined in [15]. Therefore, we propose the following general form for an integral operator:

$$
\mathcal{T}_I S(t) = \int_{B_t} K(x, t)S(x)dx,
\tag{3}
$$

where $K(x, t)$ is the parameterized integral kernel and B_t is the integral interval at time t. A signal $S(t)$ is in the null space of the

integral operator if there exists an integral interval B_t such that $\mathcal{T}_I S(t) = 0$. In this paper, we address two important questions about integral operators: (1) How to design the kernel of an integral operator? (2) How to estimate the parameters of an integral operator and utilize the latter to perform signal separation? For the first problem, we show that the integral operators in Eq. (3) can be derived by using a differential operator of the form in Eq. (2). For the signal separation task in the second problem, we show that the estimated kernel and the integral operator can be successfully incorporated into the Null Space Pursuit (NSP) algorithm, which was originally designed to operate with differential operators. Numerically, we assess the performance of the proposed method on several synthesized signals and real-life signals. We also compare our results with those derived by a number of other methods, and show that our algorithm is more robust to noisy signals than the compared methods.

II. DIFFERENTIAL AND INTEGRAL OPERATORS

In this section, we first discuss how to find the integral kernel $K(x, t)$ in Eq. (3) from a differential operator. Then, after analyzing the properties of the derived integral operator, we derive a matched filter interpretation for it, which cannot be derived by analyzing the corresponding differential operator.

A. Deriving Integral Operators from Differential Operators

First, we claim that the differential equation in Eq. (2) can be converted to the integral equation in Eq. (3). Without loss of generality, we use the following second order differential equation as an example to demonstrate the conversion process in detail. Assume the differential equation is expressed as follows:

$$y''(t) + a_1(t)y'(t) + a_2(t)y(t) = 0, \qquad (4)$$

where $t \in [0, l]$ and the boundary conditions $y(0) = a$ and $y(l) = b$. First, by setting $y''(t) = \varphi(t)$ and integrating with respect to t on both sides twice, we have

$$y(t) = \int_0^t (t - \xi)\,\varphi(\xi)\,d\xi + y'(0)t + y(0). \qquad (5)$$

Then, if we substitute $\varphi(t) = y''(t) = -a_1(t)y'(t) - a_2(t)y(t)$ into Eq. (5), through some integral calculations, we can derive

$$y(t) = -\int_0^t \left(-a_1(\xi) + (t - \xi)(a_1'(\xi) + a_2(\xi)) \right) y(\xi)d\xi$$
$$+ (y'(0) + a_1(0)y(0))t + y(0). \qquad (6)$$

By using $y(0) = a$ and $y(l) = b$, we can compute the value of $y'(0) + a_1(0)y(0) = \frac{b-a}{l} + \frac{1}{l}\int_0^l (-a_1(\xi) + (l - \xi)(a_1'(\xi) + a_2(\xi)))y(\xi)d\xi$. Next, we substitute the value of $y'(0) + a_1(0)y(0)$ back into Eq. (6) and derive the following integral form:

$$y(t) = \int_0^l K(\xi, t)y(\xi)d\xi + C(t), \qquad (7)$$

where $C(t) = (b - a)t/l + a$ and the kernel function

$$K(\xi, t) = \begin{cases} (1 - t/l)a_1(\xi) + \xi(1 - t/l)(a_1'(\xi) + a_2(\xi)), \\ \qquad 0 \le \xi \le t \\ (-t/l)a_1(\xi) + t\,(1 - \xi/l)\,(a_1'(\xi) + a_2(\xi)), \\ \qquad t \le \xi \le l \end{cases} \qquad (8)$$

Now, it is clear that, under appropriate boundary conditions, $y(t)$ will be in the null space of the differential operator $\frac{d^2}{dt^2} + a_1(t)\frac{d}{dt} + a_2(t)$ if and only if $y(t)$ satisfies $\int_0^l K(\xi, t)y(\xi)d\xi - y(t) + C(t) = 0$ (as in Eq. (7)). Thus, for each t, we choose an integral interval $B_t = [t_a, t_b]$ and define the integral operator as

$$(\mathcal{T}_I S)(t) = \int_{B_t} K(x, t)S(x)dx + C(t) - S(t), \qquad (9)$$

with $C(t) = S(t_a) + \frac{(S(t_b) - S(t_a))(t - t_a)}{t_b - t_a}$ and $K(x, t)$ defined in Eq. (8). Although the proposed integral operator and the corresponding differential operator have the same null space, the former is numerically more stable than the latter. One reason is that integral operator can estimate the parameters more robust than the differential operator under noisy cases; the other is that, unlike differential operator, we incorporate the estimation of boundary values into our integral operator-based signal separation algorithm as discussed in Section III.

B. Matched Filter Interpretation for Integral Operators of Frequency-Modulated (FM) Signals

Recall that the differential operator for an FM signal is $(d^2/dt^2 + \varpi^2(t))$. Then, if we substitute $a_1(t) = 0$ and $a_2(t) = \varpi^2(t)$ into Eq. (8), the corresponding integral kernel can be derived as $K(x, t) = \varpi^2(x)G(x, t)$, where $G(x, t) = \begin{cases} (1 - t/l)x, & 0 \le x \le t \\ (1 - x/l)t, & t \le x \le l \end{cases}$ is the Green function, which can be used as the integral kernel in the following integral equation:

$$y(t) - \lambda \int_0^l G(x, t)y(x)dx = 0. \qquad (10)$$

The above integral equation is equal to the following differential equation:

$$y''(t) + \lambda y(t) = 0, \qquad (11)$$

with boundaries $y(0) = y(l) = 0$. For the boundary value problem in Eq. (11), the eigen value and its corresponding eigen function are $\lambda = k^2\pi^2$ and $\sqrt{2}\sin(k\pi x)$ respectively. Then, according to the Mercer Theorem [16], the kernel function $G(x, t)$ can be expanded as:

$$G(x, t) = \frac{2}{\pi^2} \sum_{k=1}^{\infty} \frac{\sin(k\pi x)\sin(k\pi t)}{k^2}. \qquad (12)$$

In the discrete case, the kernel function $G(x, t)$ can be represented as a matrix $\boldsymbol{G}$. If we compute the Singular Value Decomposition (SVD) of $\boldsymbol{G}$ as $\boldsymbol{G} = \boldsymbol{V}^T \boldsymbol{\Sigma} \boldsymbol{V}$, we obtain $\boldsymbol{V} = [\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n]^T$ with $\boldsymbol{v}_k = \sin(k\pi t)/\|\sin(k\pi t)\|^2$; and the kth eigenvalue in $\boldsymbol{\Sigma}$ is the reciprocal of the square of the instantaneous frequency of the kth basis $\boldsymbol{v}_k$. Thus, the eigen function of the integral operator is similar to a matched filter that matches and extracts the subcomponent with the desired frequency. Moreover, for each point t, if we choose an integral interval as B_t and compute $\|\int_{B_t} G(x, t)S(t)dt\|^2$, the function of the matrix $\boldsymbol{G}$ is similar to computing the short time frequency transform (STFT) of $S(t)$ with a rectangular window and support B_t.

III. ALGORITHM WITH INTEGRAL OPERATORS

Incorporating Eq. (9) into the NSP algorithm would be greatly simplified if we could choose a neighborhood $B_\tau = [\tau_a, \tau_b]$ that satisfies $S(\tau_a) = S(\tau_b) = S(\tau)$. However, because the input signal is a multiple component signal and $S(t)$ is one of its subcomponents, it is not possible to derive such interval from the input signal. Therefore, in our implementation, we choose a fixed neighborhood B_t for all t. Then, to eliminate the boundary effects, we divide the boundary values from the optimization process by modifying the integral operator defined in Eq. (9) as follows:

$$(\mathcal{T}S)(t) = \int_a^b \left(J(x,t) + K(x,t)\right) S(x) dx, \quad (13)$$

where $K(x,t)$ is defined in Eq. (8); and $J(x,t)$ is represented as $J(x,t) = \delta(x-a) - \delta(x-t) + \frac{t-a}{b-a}\delta(x-a)(\mathcal{W}_{(b-a)} - \mathcal{I})$, where $\delta(x)$ is the Dirac function with $\delta(t) = 1$ if $t = 0$, and 0 otherwise; $\mathcal{W}_{\Delta t}$ denotes the shift operator with $(\mathcal{W}_{\Delta t}S)(t) = S(t + \Delta t)$; and $\mathcal{I}$ is the identity operator. Thus, for each τ of an input signal, we extract the signal $S(t)$ from its neighborhood B_τ so that it satisfies $\|(\mathcal{T}S)(\tau)\|^2 \approx 0$.

In a discrete representation, for ease of presentation, we use bold upper case (e.g. A) to represent matrices and bold lower case (e.g. a) to represent vectors. We also use matrix A_x to denote a diagonal matrix in which the diagonal elements are equal to the vector x. Then, for an input signal s, we combine all the τ, and use the NSP algorithm (as shown in Eq. (1)) to search for the residual signal r and the parameter θ by minimizing the equation

$$\mathcal{F}(r,\theta) = \sum_\tau \|T_\tau(\theta)(s-r)\|^2 \\ + \lambda_1 \left(\|r\|^2 + \gamma\|s-r\|^2\right) + \lambda_2\|D_2\theta\|^2, \quad (14)$$

where $T_\tau(\theta)$ denotes the operator matrix in the support interval B_τ with parameter vector equaling to θ; the parameters θ is defined as $[\theta_1^T, \theta_2^T]^T$ with $\theta_1(t) = a_1(t)$ and $\theta_2(t) = a_1'(t) + a_2(t)$ in Eq. (8); the parameters λ_1 and γ play the same role as in the NSP algorithm and can be estimated adaptively; the parameter λ_2 is used to regulate the parameters in the integral operator by applying a second order differential matrix D_2 to ensure that θ is smooth.

By taking the partial derivative of $\mathcal{F}$ with respect to r and setting the result to zero, we can derive

$$\hat{r} = \left(\sum_\tau T'_\tau T_\tau + (1+\gamma)\lambda_1 I\right)^{-1} \left(\sum_\tau T'_\tau T_\tau s + \lambda_1\gamma s\right), \quad (15)$$

where the prime denotes the transposition of the matrix and vector; and I denotes the identity matrix. To compute the partial derivative of F with respect to θ, we rewrite the first term in Eq. (14) as follows:

$$T_\tau(\theta)(s-r) = \left(J_\tau - I + [K_{1,\tau}, K_{2,\tau}]\begin{bmatrix} A_{\theta_1} \\ A_{\theta_2} \end{bmatrix}\right)(s-r) \\ = (J_\tau - I)(s-r) + K_\tau\theta, \quad (16)$$

where $K_\tau = [K_{1,\tau}A_{s-r}, K_{2,\tau}A_{s-r}]$. The kernel functions $K_{1,\tau}(x,t)$ and $K_{2,\tau}(x,t)$, which correspond to the matrices $K_{1,\tau}$ and $K_{2,\tau}$ in the interval $B_\tau = [a,b]$, are defined as $K_{1,\tau}(x,t) = \begin{cases} 1 - \frac{x-a}{b-a}, & a \leq t \leq x \\ \frac{a-x}{b-a}, & x \leq t \leq b \end{cases}$ and

$K_{2,\tau}(x,t) = \begin{cases} (t-a)(1-\frac{x-a}{b-a}), & a \leq t \leq x \\ (x-a)(1-\frac{t-a}{b-a}), & x \leq t \leq b \end{cases}$ respectively.

By taking the partial derivative of $\mathcal{F}$ with respect to θ and setting the result to zero, we can derive

$$\hat{\theta} = \left(\sum_\tau K'_\tau K_\tau + \lambda_2 D'_2 D_2\right)^{-1} \left(\sum_\tau K'_\tau(J_\tau - I)(s-r)\right) \quad (17)$$

where the prime denotes the transposition of the matrix and vector. Since the parameter λ_2 is less sensitive to the separation result, we choose a fixed value for it manually. According to [6], the parameters λ_1 and γ can be calculated as follows:

$$\lambda_1 = \frac{1}{1+\hat{\gamma}} \frac{s'M(\lambda_1, \hat{\gamma}, \hat{T})'s}{s'M(\lambda_1, \hat{\gamma}, \hat{T})'M(\lambda_1, \hat{\gamma}, \hat{T})s}. \quad (18)$$

where $M(\lambda_1, \hat{\gamma}, \hat{T}) = (\hat{T} + (1+\hat{\gamma})\lambda_1 I)^{-1}$ with $\hat{T} = \sum_\tau T'_\tau(\hat{\theta})T_\tau(\hat{\theta})$; and

$$\gamma = \frac{(s-\hat{r})^T s}{\|s-\hat{r}\|^2} - 1. \quad (19)$$

Based on Eqs. (15, 17–19), we propose an adaptive integral operator signal separation algorithm called NSP-I (**Algorithm 1**), which enables us to separate a mono-component signal u_1 from signal s and obtain the residual $s - u_1$. By repeating the NSP-I algorithm M times, the input signal s can be decomposed into the sum of M mono-component signals.

Algorithm 1 NSP Algorithm using an Integral Operator

1:      Input signal s and parameter λ_2. Choose a stopping threshold ε and the values of λ^0 and γ^0.

2:      Set $j \leftarrow 0$, $\hat{r}_j \leftarrow 0$, $\lambda_1^j \leftarrow \lambda_1^0$ and $\gamma^j \leftarrow \gamma^0$.

3:      **repeat**

4:          Estimate the parameters $\hat{\theta}^j$ based on Eq. (17).

5:          Compute λ_1^{j+1} based on Eq. (18).

6:          Compute $\hat{r}_{j+1}$ base on Eq. (15).

7:          Compute γ^{j+1} using Eq. (19) and set $j = j + 1$.

8:      **until** $\|\hat{r}_{j+1} - \hat{r}_j\|^2 < \epsilon$

9:      **return** Extract the mono-component $\hat{u} = (1+\gamma^j)(s-\hat{r}_j)$ and the residual signal $\hat{r} = s - \hat{u}$.

IV. EXPERIMENT RESULTS

In this section, we compare the results of applying different signal separation algorithms to both simulated and real-life signals. In the ***Simulated Signal*** experiment, we assess the accuracy and robustness of NSP-I in separating a noisy two-component AM-FM signal. The clean signal is $s(t) = s_1(t) + s_2(t)$ with $s_1(t) = (2 + \sin(0.5\pi t))\cos(10\pi t + 12\sin(0.5\pi t))$ and $s_2(t) = 0.5(2 + \cos(0.5\pi t))\cos(20\pi t + 16\sin(0.5\pi t))$. We added white Gaussian noise to $s(t)$ and acquired twelve noisy signals, whose signal-to-noise ratio (SNRs) ranged from 25 dB down to −5 dB. The noisy signals are generated by the MATLAB function *awgn(x,snr)*. In this example, the IFs of the two sub-component signals are very close and, in some time intervals, the energy of signal $s_1(t)$ is much greater than that of signal $s_2(t)$. As a result, the EMD algorithm is affected by mode mixing problem when separating the clean signal [17]. Also, since the integral operator defined in [5] depends on the extrema to determine the integral interval, it cannot separate

TABLE I
PERFORMANCE COMPARISON OF VARIOUS SEPARATION METHODS APPLIED TO NOISY SIGNALS

| Methods & SNRs(dB)		-5.0	-3.0	-1.0	1.0	3.0	5.0	7.0	9.0	13.0	17.0	21.0	25.0
NSP-I	SNR_1(dB)	7.27	10.64	12.35	14.81	19.24	20.75	22.60	24.99	26.51	27.62	29.04	29.06
	SNR_2(dB)	0.393	5.38	7.02	10.49	15.38	16.22	17.53	20.88	25.80	27.34	28.39	28.69
	Time (sec)	31.54	27.10	22.96	19.92	17.64	15.93	15.11	12.63	12.18	11.16	8.39	8.12
NSP-D	SNR_1(dB)	5.97	6.50	8.47	8.73	8.91	10.13	11.16	19.81	24.38	26.81	28.03	27.94
	SNR_2(dB)	-0.255	0.810	2.39	3.66	5.31	5.65	12.06	17.67	23.93	25.90	27.44	27.78
	Time (sec)	2.58	2.47	2.14	2.34	1.77	1.83	1.54	1.23	1.01	0.96	0.97	0.96
SWT	SNR_1(dB)	1.36	-0.688	0.251	0.099	8.96	9.08	9.32	11.54	13.60	15.68	17.70	17.79
	SNR_2(dB)	-4.58	-4.33	-4.23	-4.26	12.32	12.77	13.47	14.74	16.92	17.76	18.91	19.28
	Time (sec)	6.21	6.18	6.19	6.21	6.33	6.18	6.45	6.22	6.20	6.26	6.32	6.22
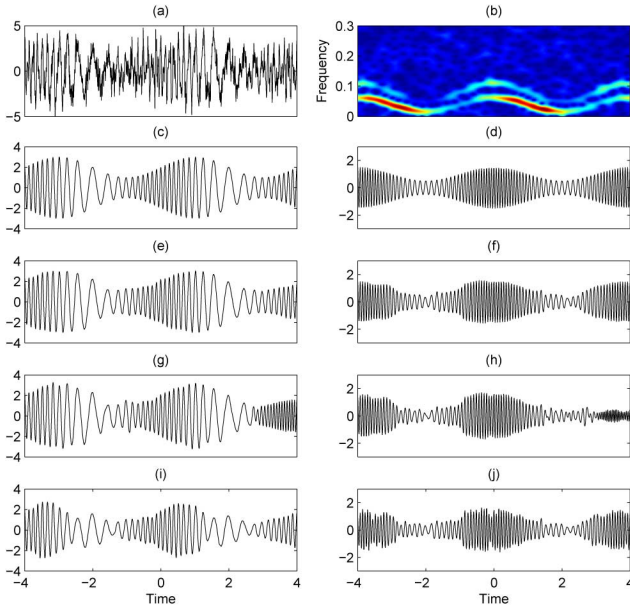
Fig. 1. (a) and (b) the input noisy signal and its Time-Frequency spectrum, respectively; (c) and (d) the original clean signal $s_1(t)$ and $s_2(t)$, respectively; (e), (g), and (i) the first subcomponents extracted by NSP-I, NSP-D and SWT respectively; (f), (h), and (j) the second subcomponents extracted by NSP-I, NSP-D and SWT respectively.
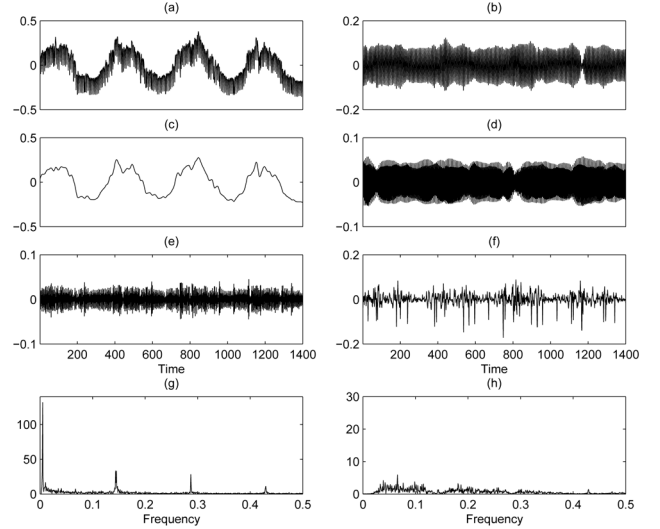

Fig. 2. (a) the input signal; (b) to (f) the first four extracted subcomponent signals and the residue derived by the NSP-I algorithm respectively; (g) and (h) the Fourier spectrum of the input signal and the residual signal respectively.

such kind of noisy data. Thus, we compare the performance of the proposed NSP-I algorithm, the original NSP algorithm (NSP-D) [6], and the SWT algorithm [18] in Table I, where SNR_i specifies the SNR of the extracted subcomponent signal $s_i(t)$ with $i = 1, 2$ and "Time" denotes the total computational time for extracting two subcomponents of each method. The results show that NSP-I yields higher SNRs than NSP-D and SWT on all the noisy signals. We show the separation results of the fifth noisy signal (SNR = 3.0) in Fig. 1. We observe that the amplitude of the signals extracted by NSP-I is better than that derived by SWT.

In the ***Real-life Signal*** experiment, we consider the signal of Poland's daily electricity consumption from 1990 to 1994 [19]. Figs. 2(a) and 2(g) show the input signal and its Fourier spectrum respectively. The Fourier spectrum shows that the input signal is comprised of three major oscillatory subcomponents plus a trend subcomponent. Fig. 2(b) to 2(f) show, respectively, the separation results derived by the NSP-I algorithm. In the extracted subcomponents, each oscillatory subcomponent contains an individual main frequency, which is consistent with the

peaks in the Fourier spectrum of the input signal. In addition, the first and third extracted subcomponents relate, respectively, to a one-week cycle and a half-week cycle, which might correlate with the work patterns of people over a week. The separation results of the EMD and NSP-D can be found in [6], [20]. As shown in [6], the NSP-D cannot extract the oscillatory subcomponent with the highest frequency. For the separation results of EMD shown in [20], we can find that there exist some mode mixing and splitting phenomenons in different extracted IMFs. Because different methods usually yield diverse results on real-life signals, finding a way to combine the results that are consistent across all the methods is an important issue that warrants further study.

V. CONCLUSION

An attractive feature of the operator-based signal separation approach is that the operator can be customized according to the subcomponent of the signal. We propose a new integral operator and show that it can be derived by the corresponding differential operator. We incorporate the proposed operator in the framework of the Null Space Pursuit algorithm to separate multi-component signals. The results of experiments demonstrate that the proposed operator can robustly separate multi-component local narrow band signals, even in a low SNR environment.

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