Maximizing Time-discounted Influential Sustainability in Social Networks

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***Abstract*—In social marketing practice, it is usually important to anticipate the long-term impact of the target application to maintain a long-lasting marketing effect, whereas a new product or technology should spread as quickly as possible to establish a competitive advantage. To find a balance between them, we tackle this challenge by modelling the problem as an issue of time-discounted influential sustainability. Given a threshold μ, the goal of the problem is finding a small subset of nodes as seeds and deciding the optimal timing to activate each seed that could maximize the time-discounted number of iterations, each of which actives more than μ nodes.**

We prove that solving the problem is NP-hard and the objective function is non-negative, non-monotonic, and non-submodular. Therefore we propose a greedy approach to approximately solve this problem. Our experimental results demonstrate that our solution outperforms two baseline algorithms. In order to provide meaningful advices for advertisers on selecting proper initial seed users, we further analyze and compare the performance of four seeding strategies on three typical types of social networks.

Keywords: social marketing, influence sustainability, information diffusion, seeding strategy.

I. Introduction

With the rapid development of social networking tools, and integration of online advertising and big data analysis, more and more advertisers now prefer to promote their products or services through the social marketing channels. Social marketing is a typical research field of social computing [1], which is strongly related to human social dynamics. In literature, social marketing is defined as the design, implementation, and control of programs calculated to influence the acceptability of social ideas, and it involves considerations of product planning, pricing, communication, distribution, and marketing research [2], which is a much broader idea than social advertising and even social communication. Social marketing has a positive effect on the brand image, awareness and equity, which can contribute to a significant improvement in purchasing intention [3].

As a Cyber-Physical-Social system (CPSS) [4], online social media makes it possible for tracking the spread of advertising information to infer influence diffusion mechanisms, and adjusting advertising strategy to maximum advertisers' final revenue. Generally, social marketing can be broadly modeled in terms of three components: a social network through which advertising information is propagated, a set of users that propagate the information, and a seeding strategy that activates the process by determining targeted users chosen by advertisers [5].

Many researchers have made great efforts on determining a maximum-influence seeding strategy and proposed considerably remarkable methods [5–8]. Given an influence diffusion model m and an initial seed set S, the expected final number of active nodes is denoted by $\pi(S, m)$. Most previous works mainly focus on the classical *influence maximization* problem, which is defined as finding the optimal set $S, |S| = k$ to maximize the final influenced size $\pi(S, m)$. After an initial seed set S is determined, the whole seeds belonging to S will be activated at the same time when a marketing process begins.

However, this classical problem does not cover all the full range of influence diffusion. Influence diffusion is a typical dynamic processes, which is related to temporal contexts. Analysis on message dissemination in social networks shows that although an extremely hot topic has a high final attracted size, it is generally quick to disappear from the public attention as people will soon lose interest in it [9]. Note that the inherent need of social marketing activities, such as product promotion, may expect the long-term impact and awareness of the target application to maintain a longer marketing effectiveness in social network. Moreover, a high intensity reflection may accelerate consumers' high expectations or negative effects on new products, and arouse the vigilant of competitors early.

Therefore, it is important to predict the temporal scale and think about it both in spatial and temporal aspects, such as the duration of advertising information diffusion [9] and the expected time-discounted influence spread [10]. The former is highlighted as *influential sustainability*. Let π_t be the number of active nodes of each iteration during promotion. Given a threshold μ, the influential sustainability problem is defined as deciding the optimal timing to activate each seed in S so as to maximize the number of iterations that $\pi_t > \mu$. The latter is called *time-discounted influence maximization*, which is based on an assumption that campaigns for a new product or technology should spread as quickly as possible, as early access to the market can establish a competitive advantage, that is, to establish and increase the entry of industry barriers, to prevent potential competitors to enter the market, and thus dominate the market.

To a certain extent, these two issues are contradictory. However, as a practical matter, it is necessary to find a balance between them. The work in [9] has designed several efficient algorithms to solve the influential sustainability problem. However, it assumes that the seeding strategy has been determined, whereas it is a challenging task for advertisers to determine a proper seeding strategy. Most advertisers are faced with budget limitations [11][12], which restrict them to select a finite number of users as their promotion targets. In addition to the complexity of social networks and user behaviours, advertisers usually are lack of ability to predict and control the diffusion process in social networks. Thus they are not able to estimate the marketing performance and further to select the proper seeding strategy.

This paper is targeted at studying advertisers' decisions on seeding strategy in social marketing for the time-discounted influential sustainability maximization problem. We present a formal definition of the problem, and give emphasis on analyzing its properties. To solve the problem, we introduce a greedy algorithm. Also, we design experiments in three typical social networks: Erdős–Rényi random graph [13], Watts-Strogatz small world graph [14] and Barabási-Albert scale-free graph [15] to make further investigations of our research using real-world census data in China. Our research will provide reliable support for advertisers to determine their seed users in social marketing.

The remainder of this paper is organized as follows. In Section II, we describe some preliminaries. Section III presents the time-discounted influential sustainability maximization problem. In Section IV, a greedy algorithm is proposed to solve the problem. Section V conducts experiments to validate the efficiency of the greedy algorithm, and also gives detailed analysis of the experimental results. Section VI concludes.

II. Preliminaries

In this section, we first model a social network as a directed graph. Next, we introduce two typical influence diffusion models: the independent cascade (IC) model and the linear threshold (LT) model. Moreover, we describe several seeding strategies.

A. Social Network

Generally, a social network is a social structure of nodes that represents individuals and the relationships between them within a certain domain[16]. Therefore a social network is modeled as a directed graph $G(V, E, \mathbb{P})$, where $V = v_1, v_2, ..., v_n$ represents the set of n nodes, $E = e_1, e_2, ..., e_m$ represents the set of m directed edges, and $\mathbb{P} : E \to (0, 1)$ is a probability function which associates an influence probability $p_i \in (0, 1)$ with each edge $e_i \in E, i = 1, 2, ..., m$. Given a node $v_u \in V$, the *degree* $d(v_u)$ is the number of neighbors of v_u.

The dynamics process of social marketing can be well represented as an information cascading process, during which decentralized nodes in a network environment act on the basis of how their neighbors act at the earlier time [17]. Therefore, a node's degree is considered to be an important factor to measure its influence. Given an edge $e_i(v_w, v_u), v_w, v_u \in V$, the corresponding influence probability p_i is the probability that v_u is activated by v_w separately after v_w is active.

B. Influence Diffusion Models

For the propagation of ideas or innovations through a social network represented by a directed graph G, we will refer to each individual node as either being active (an adopter of the ideas or innovations) or inactive. The number of total users in the social network is fixed during the whole promotion period. Before a promotion starts, we assume that all nodes are inactive at step 0.

The IC model [18] and LT model [19] are two seminal graph-based influence diffusion models. They are based on directed graphs, where each node can be activated by a monotonicity assumption that the active node can not be deactivated.

1) Independent Cascade Model: In the IC model, when a node v_w is active at step t, it will try once to activate each currently inactive neighbor v_u through the edge $e_i(v_w, v_u)$; it succeeds with the probability p_i defined on the edge. Once v_w succeeds, v will be active at step $t+1$. Whether or not v_w succeeds at step t, it cannot make any further attempts. If there are multiple newly active neighbors of an inactive node, their attempts will be executed successively in an arbitrary order.

2) Linear Threshold Model: In the LT model, there is an influence threshold $\rho(v_u) \in (0, 1)$ for each node v_u. An inactive node v_u is activated by its activated neighbors if the sum of influence probability exceeds its influence threshold. Let $\tau(v_u) = \{v_w | v_w, v_u \in V, e_i(v_w, v_u) \in E\}$ denote the set of parent nodes of v_u. There is a constraint that $\sum_{\tau(v_u)} p_i \leq 1$. If $\sum_{\tau(v_u)} p_i \geq \rho(v_u)$ at step t, v_u will be active at step $t+1$.

C. Seeding Strategies

The seeding strategy is of particular importance, since a proper strategy can help advertisers to deliver the ideas to a wide range of target users. Generally, based on the structure information of social networks, there are three typical seeding strategies for advertisers as follows[6].

1) high-degree seeding: Using well-connected hubs as initial seeding points implies a high-degree seeding strategy. Hubs are nodes with the highest degree in a given graph, which are regarded as the best information spreader.

2) low-degree seeding: Seeding the fringes refers to a low-degree seeding strategy. In contrast to hubs, fringes are nodes with the lowest degree in a given graph. It is presented that influenceable people, rather than particularly influential individuals, drive cascades of influence [8]. Actually, for the influential sustainability problem, seeding fringes outperforms seeding hubs under some circumstances, e.g. in a highly-connected network, as the newly active hubs will bring a large number of active neighbors at a single step.

3) high-betweenness seeding: Seeding bridges is called the high-betweenness seeding strategy. A bridge is a node of a graph whose deletion increases its number of connected components. Seeding bridges may spread information to different parts of the network and prevent information from being simply looped through a highly clustered subnet.

III. Problem Statement

Let $P_k = \{(v_i, t_i) | v_i \in V, t_i \in [0, T]\}, |P_k| = k$ be the seed activation sequence, where (v_i, t_i) is a pair for a seed v_i which is choosen under a given seeding strategy ϕ , and t_i is the timestamp that the advertiser attempts to activate v_i.

As influence cascade is a probabilistic process, let θ_t be a random variable representing the size of active nodes at the round t as follows. So the size of newly activated nodes $R_{P_k}(t)$ is indicated as follows.

$$R_{P_k}(t) = \theta_t - \theta_{t-1} \tag{1}$$

The objective function to be optimized is

$$\omega_U(P_k) = \sum_t E(U(R_{P_k}(t), t)) \tag{2}$$

We denote by E the expectation in the above equation. U is the campaigner's utility function which decreases monotonically with time and increases monotonically with the size of newly active nodes. Now we can formally define our problem.

Definition 1. *Given a social network $G(V, E, \mathbb{P})$, a seeding strategy ϕ, the campaigner's utility function U, a positive integer k, the goal of time-discounted influential sustainability problem is to find the optimal seed activation sequence P_k^* of k nodes such that the campaigner' s expected utility is maximized. Formally,*

$$P_k^* = \operatorname*{argmax}_{P_k} \omega_U(P_k) \tag{3}$$

According to [9], the increase of newly activated nodes should exceed a threshold μ so as to maintain an acceptable degree of influence. Therefore, in this paper, U is defined as the unit step function multiplied by a time-discounted function, which is shown as follows.

$$U_\mu(R_{P_k}(t), t) = H(R_{P_k}(t) - \mu) * \gamma^t \tag{4}$$

where γ is the time-discounted factor $\gamma \in (0, 1)$, and the unit step function $H(x)$ is defined as

$$H(x) = \begin{cases} 1, x \geq 0 \\ 0, x < 0 \end{cases} \tag{5}$$

Lemma 1. *Given a sequence of seed activation P, computing its influence spread $R_P(\cdot)$ is NP-hard.*

Proof. Please refer to [9]. □

Lemma 2. *The objective function $\omega_U(P)$ is not monotone.*

Proof. Consider the case shown in the Fig. 1. Assumed all influence probabilities are set to 1. Let $P_1 = \{(v_1, 0)\}$, $P_2 = \{(v_1, 0), (v_2, 0)\}$. We have $P_1 \subset P_2$, $R_{P_1}(0) = 1$,$R_{P_1}(1) = 3$,$R_{P_1}(2) = 2$,$R_{P_1}(3) = 1$,$R_{P_1}(4) = 3$, $R_{P_2}(0) = 2$,$R_{P_2}(1) = 4$,$R_{P_2}(2) = 4$.

If $\mu = 1$, then $\omega_U(P_1) = 1 + \gamma + \gamma^2 + \gamma^3 + \gamma^4$, and $\omega_U(P_2) = 1+\gamma+\gamma^2$, so that $\omega_U(P_2) < \omega_U(P_1)$. If $\mu = 2$, then $\omega_U(P_1) = \gamma+\gamma^2+\gamma^4$, and $\omega_U(P_2) = 1+\gamma+\gamma^2$, so that $\omega_U(P_2) > \omega_U(P_1)$. □

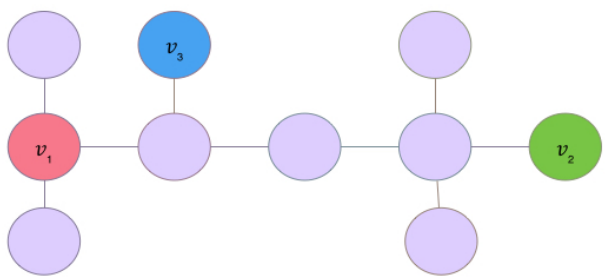

Fig. 1. An example for Lemma 2 and Lemma 3

If Ω is a finite set, a sub-modular function satisfies the following equivalent definition: For every $X, Y \subseteq \Omega$ with $X \subseteq Y$ and every $x \in \Omega \setminus Y$,we have that $f(X \cup \{x\}) - f(X) \geq f(Y \cup \{x\}) - f(Y)$.

Lemma 3. *The objective function $\omega_U(P)$ is not sub-modular.*

Proof. Consider the case shown in the figure 1. Let $P_3 = P_1 \cup (v_3, 0)$ and $P_4 = P_2 \cup (v_3, 0)$. We have $R_{P_3}(0) = 2$,$R_{P_3}(1) = 3$,$R_{P_3}(2) = 1$,$R_{P_3}(3) = 1$,$R_{P_3}(4) = 2$, $R_{P_4}(0) = 3$,$R_{P_4}(1) = 4$,$R_{P_4}(2) = 3$.If $\mu = 3$, then $\omega_U(P_1) = \gamma + \gamma^4$,$\omega_U(P_2) = \gamma + \gamma^2$,$\omega_U(P_3) = \gamma + \gamma^4$,$\omega_U(P_4) = 1 + \gamma + \gamma^2$, so that $\omega_U(P_4) - \omega_U(P_2) > \omega_U(P_3) - \omega_U(P_1)$. □

IV. Algorithms

As the objective function of the time-discounted influential sustainability problem is non-negative, non-monotonic, and non-submodular, we consider a greedy hill-climbing approach to approximately solve this optimization problem.

To implement this greedy algorithm, we need a method of evaluating the k-dimensional vector $\nabla\omega_U(P)$, which is defined as follows.

$$\nabla\omega_U(P) = (\omega_U(P \cup v_u, t_u))_{v_u \in S} \in \mathbb{R}^k \quad (6)$$

According to the equation 2, it is necessary to calculate $R_P(t)$ firstly, whereas it is a NP-hard problem as shown in lemma 1. Since it is unclear how to accurately evaluate $R_P(t)$ by an effective method, a good estimate is conventionally obtained by simulating the random process of each model many times.

Algorithm 1: A Greedy Algorithm

Input: A graph G, a threshold μ, a discounted factor γ, an influence diffusion model m,a seeding strategy π

Output: seed activation sequence P

$P = \emptyset$;
while $size(P)! = k$ **do**
  $S = \pi(G, k - size(P))$;
  $t' = max(0, max(t_i | (v_i, t_i) \in P))$;
  for v_u *in* S **do**
    $t_u^* = \text{argmax}_{t_u \in [t', T]}\, \omega_U(P \cup (v_u, t_u))$;
    $b_u = \max_{t \in [t_u^*, T]} R_{P \cup (v_u, t_u^*)}(t)$;
  $v_u^* = \text{argmin}_{v_u \in S}\, b_u$;
  $P = P \cup (v_u^*, t_u^*)$;
return P;

V. Experiments

In this section, we design experiments to make further investigation of our models and solutions. The agent-based propagation model will be validated in three typical network structures, namely Erdős–Rényi (ER) random graph, Watts-Strogatz (WS) small world graph and Barabasi-Albert (BA) scale-free graph.

A. Experimental Settings

First, we build a group of users according to a real-world census data of Beijing in China in 2000, in which the overall number of the sampled population is 13297. Each user is randomly mapped to a real person's profile, which contains 74 features including gender, age, career information, and so on.

Next, we construct 3 sets of randomly generated social networks.

- 10 ER random graphs
  In the case of the ER graphs, there is an adjustable parameter a representing the probability of connecting between two nodes, which is set in the range from 0.001 to 0.01 at intervals of 0.001 in our experiments. Thus, we have 10 different random graphs.
- 14 WS small world graphs
  In the WS graphs, each node is connected to b nearest neighbours. The parameter b is set to be an integer in $[2, 15]$ with the interval of 1. So we have 14 small world graphs.
- 10 BA scale-free graphs
  During the construction of the BA graphs, a new node is attached to c existing nodes. The parameter

c is also set to be an integer in $[1, 10]$ with the interval of 1 to build 10 scale-free graphs.

In all these social networks, each node refers to an agent and each edge refers to a social connection between agents. Furthermore, each edge is assigned with a random variable as the influence probability, which is computed based on the feature similarity of two connected nodes.

Then, we study four seeding strategies with $k = 20$, including those three strategies described in section II-C and a benchmark random seeding strategy, which selects random nodes for seeding.

To validate the efficiency of the proposed greedy algorithm, we implement two benchmark algorithms in addition to the greedy algorithm. The first algorithm activates all the seeds synchronously in the step 0, that is, $t_i = 0, \forall(v_i, t_i) \in P$. The second algorithm activates seeds in a random order. The former is called *the synchronous algorithm*, and the latter is called *the random algorithm*. The time-discounted factor γ is set to 0.95 and the threshold μ is set to 100. Besides, we implement two influence diffusion models, including the IC model and LT model.

For a set that including a seeding strategy π, an influence diffusion model m and an algorithm f, we implement an experiment in each social network G. Each experiment runs 10 simulations.

B. Algorithm Efficiency

Performances of the three algorithms are listed in Table I. We use BA(LT) to denote the set of experiments for the LT model and the BA graphs, and so on. Each column in the table is the expected value of final revenue under a specific algorithm. Therefore, the value of each grid is calculated as $\frac{1}{10}\sum_{\pi}\omega_U|_{G,m,f}$.

According to the results, the greedy algorithm outperforms the other two, which validates its efficiency. Besides, the differences between the greedy algorithm and the synchronous algorithm are quite small under several cases, e.g, the ER(IC) case. That is because when degrees of nodes in a given graph are generally low, it is an optimal approach to activate all the seeds synchronously.

C. Comparison between different seeding strategies

Performances of the four seeding strategies scheduled by the greedy algorithm are listed in Table II. Each column in the table is the expected value of final revenue under a specific strategy. Therefore, the value of each grid is calculated as $\frac{1}{10}\sum\omega_U|_{G,m,\pi,f=greedy}$.

TABLE I
Comparison between algorithms

	greedy	synchronous	random
ER(IC)	**230.57**	229.56	125.29
ER(LT)	**62.86**	58.8	19.32
WS(IC)	**363.87**	278.55	16.31
WS(LT)	**211.13**	190.68	11.31
BA(IC)	**185.99**	165.96	12.68
BA(LT)	**41.91**	38.56	0.41

TABLE II
Comparison between seeding strategies

	high-degree	low-degree	high-betweenness	random
ER(IC)	**61.48**	48.71	60.13	60.24
ER(LT)	**37.19**	8.45	0	17.22
WS(IC)	90.77	**92.54**	90.79	89.77
WS(LT)	**70.64**	70.42	1.99	68.07
BA(IC)	**47.14**	45.35	47.13	46.36
BA(LT)	**28.3**	0	0	13.60

According to the results, the high-degree seeding strategy outperforms the others in most cases. Even though the low-degree strategy works best in the WS(IC) case, the gap in performance between the high-degree strategy and the low-degree strategy is quite small. Therefore, it implies that choosing the most influential nodes as the initial seeds is always recommendable.

For different types of graphs, the differences of degrees among nodes are not the same. For the WS graphs, the differences are small because each node is connected to a fixed number of nearest neighbours. That's the reason why the results of the high-degree strategy and the low-degree strategy in WS graphs are similar.

When building a BA graph, a new node is attached to a fixed number c of existing nodes. Let v_m be the m-th node added to a BA graph, n be the final size of the graph and x be the temporal size of the graph. The expected degree of v_m is $\sum_m^n \frac{c}{x}$. So an earlier existed node will have a higher expected degree than a new one. The greater the parameter c is, the larger the differences are. When c is small, the results of the low-degree strategy is quite small as only a relatively small number of nodes can be active eventually. And When c is large, the results of the high-degree strategy is quite small as a large number of nodes are active in the early few steps. That's the reason why the final results of the high-degree strategy and the low-degree strategy in BA graphs are also similar.

The high-betweenness strategy does not perform well under the LT models. Particularly in both the ER(LT) case and the BA(LT) case, the number of newly activated nodes of each iteration do not exceed the threshold μ during the promotion so that the final revenue is zero.

VI. Conclusion and Future Works

In this paper, we present a problem of time-discounted influential sustainability, which extends the classical influence maximum problem. We make property analysis on the problem and propose a greedy algorithm to solve it. We evaluate the proposed approach on typical social networks with three types of structures, i.e., Erdős-Rényi random graph, Watt-Strogatz small world graph, and Barabási-Albert scale-free graph. Furthermore, we implement four seeding strategies and compare performances among them. Our experiment results show that choosing the most influential nodes as the initial seeds is always recommendable.

A greedy hill-climbing approach is proved to be within an approximation guarantee $(1-1/e)$ of the optimal solution for the influence maximum problem [7]. Note that e is the natural constant. However, as the objective function of our proposed problem is non-negative, non-monotonic, and non-submodular, the greedy approach cannot guarantee the $(1-1/e)$ approximation to the optimal solution. Therefore we plan to improve the existing greedy algorithm on the base of existing research of network science.

The SIR model [20] is one seminal non-graph based approach, which has been mainly developed to model epidemiological processes. However, due to similar patterns in the spread of epidemics and social contagion processes, we will adopt the SIR model to study on the seeding strategies for social marketing.

Besides, considering the limitations of the reported experiment, we intend to extend it with the real-world social networking data.

Acknowledgment

This work is partially supported by NSFC (71472174, 71232006, 61533019, 61233001, 71402178, 61603381) and the Early Career Development Award of SKLMCCS (Y6S9011F4E, Y6S9011F4H).

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