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A Geometric Flow Approach for Region-based Image Segmentation-theoretical Analysis

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Abstract In this paper, we analyze the well-posedness of an image segmentation model. The main idea of that segmentation model is to minimize one energy functional by evolving a given piecewise constant image towards the image to be segmented. The evolution is controlled by a serial of mappings, which can be represented by B-spline basis functions. The evolution terminates when the energy is below a given threshold. We prove that the correspondence between two images in the segmentation model is an injective and surjective mapping under appropriate conditions. We further prove that the solution of the segmentation model exists using the direct method in the calculus of variations. These results provide the theoretical support for that segmentation model.

Keywords L^2-gradient flow; Bi-cubic B-spline; direct method; image segmentation
2000 MR Subject Classification 00A05

1 Introduction

In computer vision, image segmentation is the process of partitioning a digital image into multiple segments, with each segment having a label for every pixel in the image such that pixels with the same label share certain visual characteristics. As a very important problem in the field of image processing, image segmentation is the basis of image analysis and understanding. The goal of segmentation is to simplify and/or change the representation of an image into something that is more meaningful and easier to analyze[14].

In the past two decades, several methods have been developed for image segmentation, such as clustering method, region-growing method, watershed transformation, edge detection method, Mumford-Shah method, etc. We review two methods, the edge detection method and the Mumford-Shah method, in our paper. Edge detection methods depend on a sharp adjustment in intensity at the region boundaries. Hence the gradient $\nabla I(x)$ of the image I can be used as an edge detection operator. The classic snake(or active contour) model proposed by Kass[11] in 1987 is the most important and influential edge detection method. A snake is an energy-minimizing spline guided by external constraint forces and influenced by image forces that pull it toward features such as lines and edges. Some researchers rely on level-set formulation to define edge-function, including Caselles et al.'s mean curvature based geometric active contour model[2], Malladi et al's level-set based active contour model[3], and Caselles et al's geodesic model[15]. However, dependence on the image gradient to stop the evolution only works for objects with edges defined by gradient. This is because the discrete gradients are bounded, and the stopping function is never zero on the edges and the curve may pass through the boundary. Moreover, the edge-function contains a Gaussian function for the purpose of

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smoothing. In case of the image being very noisy, the smoothing term has to be strong, which will blur the edge features as well.

Another type of methods are Mumford-Shah methods, first proposed by Mumford and Shah on 1989[17], which do not rely on the gradient of the image and can detect objects with smooth or discontinuous boundaries. These methods belong to a variational approach which express the simplified image as the minimizer of an energy, thus invariably involve minimizing over curves in the plane, which is a challenging task from a numerical perspective. As such, there have been many research activities on how to numerically minimize the Mumford-Shah functional or its simplified versions [5,12,16,21]. For example, the very popular Chan-Vese model[5] is a level-set implementation of one special case of Mumford-Shah model. This model was further extended and generalized to segmentation of multi-channel images[4], and segmentation of an image into arbitrary regions[24]. The computational efficiency of these models has also been improved later-on[7,8,19]. Lie et al. made it possible that only one level-set function is needed to represent 2^n unique regions[13]. More recent work includes Sumengen and Manjunath's graph partitioning active contour[20] and Li et al.'s work that eliminates the need of level-set reinitialization [12].

In [9], we propose a wavelet orthonormal bases based iteration method by refining alternatively the orientations and the map using Levenberg-Marquardt algorithm and soft-thresholding, respectively. The convergence analysis of the proposed algorithm is provided and numerical experiments for simulated particle images show promising performance. In [10], a new algorithm with nonmonotone line search to solve the non-decomposable minimax optimization is proposed. We prove that the new algorithm is global convergent. Numerical results show the proposed algorithm is effective.

In this paper, our model, by no exception, minimizes the Mumford-Shah functional and segments an image into a number of piecewise-constant regions. Yet it is essentially different from popular level-set based implementations for this problem. We intends to derive a mapping $\mathbf{x}$ that deforms given contours or regions towards objects in the image. We assume the segmentation region is fixed, and the value of the piecewise-constant function $I_1(u)$ on one segmentation region is the average value of image on this segmentation region, then the sub-domains $\{\Omega_i\}$ are reshaped by the mapping $\mathbf{x}$. This iterative process is stopped until it reaches stable state. Fig 1.1 shows a result of our model.

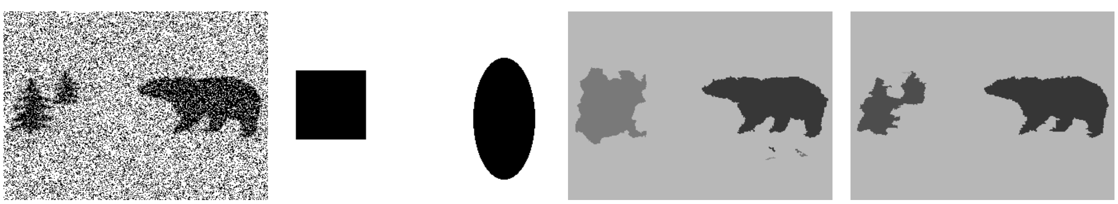

Fig.1. (a) Image to be segmented; (b) $I^{(0)}$ specified by user; (c) after 200 iterations; (d) after 400 iterations

The remaining of the paper is organized as follows: In Section 2, we review the segmentation model and algorithm, and list the new theoretical results. In Section 3, we give proof details of the regularity of the mapping $\mathbf{x}(u, v)$. Based on the analysis results of Section 3, we consider in Section 4 the existence problem of our segmentation model. We show that there exists a mapping $\mathbf{x}_0(u, v) \in X$ minimizing the new model. We conclude the paper in Section 5.

2 Algorithm Review and Theoretical Results

In this section we first review the segmentation model proposed by Ye and Xu[22]. For experimental results of this method please refer to [22]. In this paper, we give some theoretical results which are summarized in this section.

2.1 Model and Algorithm Review

Given an image $I_0(u,v)$ defined on $\Omega = [0,1]^2 \subset \mathbb{R}^2$ and let Ω_i be a decomposition of Ω, and

$$c_i = \frac{\int_{\Omega_i} I_0(\mathbf{u})d\mathbf{u}}{\text{Area}\,(\Omega_i)}, \tag{2.1}$$

thus the minimization problem can be solved by tackling two sub-problems inter-changeably:

(1) for fixed c_i, change the shape of Ω_i,

(2) for fixed Ω_i, according to (2.1), compute the new constant c_i.

We assume that the boundary of Ω_i is piecewise smooth. Since the second sub-problem is trivial to solve, how to solve the first one is the main focus.

We define the piecewise-constant function space over partition P as:

$$S(p) = \Big\{ f(\mathbf{u}) = \sum_{i=1}^{n} c_i \psi_i(\mathbf{u}) : c_i \in \mathbb{R} \Big\}, \tag{2.2}$$

where $\psi_i(\mathbf{u})$ is the characteristic function of Ω_i:

$$\psi_i(\mathbf{u}) = \begin{cases} 1, & \mathbf{u} \in \Omega_i, \\ 0, & \text{otherwise.} \end{cases}$$

According to [23], we have

$$|\Gamma| = \sum_{i=1}^{n} |\Gamma_{\Omega_i}| = \sum_{i=1}^{n} \int_{\Omega} \|\nabla \psi_i(\mathbf{u})\| d\mathbf{u}. \tag{2.3}$$

Sub-problem (1) desires that the sub-domains Ω_i are reshaped with the function values c_i on Ω_i unchanged. The objective of reshaping Ω_i is to find a best fit to the image. To achieve this, the mapping $\mathbf{x}$ is defined and the sub-domains are reshaped by the mapping. Thus the minimal partition problem is defined as follows. For a given partition $\Omega = \bigcup_{i=1}^{n} \Omega_i \cup \Gamma$ of Ω for the image I_0 and the corresponding piecewise-constant image $I_1 \in S(P_1)$, we find $\mathbf{x}(\mathbf{u}) : \Omega \to \Omega$ satisfying

(i) $\mathbf{x}$ is a C^2 mapping;

(ii) $\mathbf{x}(0,v) = [0,v]^T$, $\mathbf{x}(1,v) = [1,v]^T$, $\mathbf{x}(u,0) = [u,0]^T$ and $\mathbf{x}(u,1) = [u,1]^T$;

(iii) For a given$0 < \gamma < 1$, $\det(\mathbf{x}_u, \mathbf{x}_v) \geq \gamma$;

such that

$$\mathfrak{E}(\mathbf{x}) = \int_{\Omega} \Big(\big(I_0(u,v) - I_1(\mathbf{x}(u,v))\big)^2 + \beta \sum_{i=1}^{n} \|\nabla \psi_i(\mathbf{x}(u,v))\| \Big) dudv \tag{2.4}$$

is minimized. Once the mapping $\mathbf{x}(u,v)$ is determined, resample the image by $I_2(u,v) = I_1(\mathbf{x}(u,v))$, then $I_2(u,v)$ is the segmented image we need on the sub-problem (2).

Remark 2.1. (iii) in the definition of $\mathbf{x}$, that is $\det(\mathbf{x}_u, \mathbf{x}_v) \geq \gamma$, can guarantee the mapping $\mathbf{x}$ to be an injection and surjection (see the proof of Theorem 2.1). Only the mapping $\mathbf{x}$ is an injection and surjection, then resample is successful. Otherwise, it will emerge the self-mapping, see Fig.2. Then the control points need to reset.

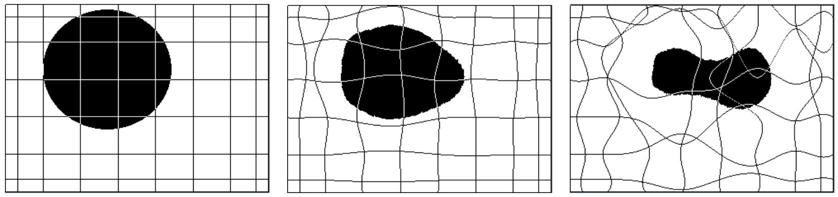

Fig.2. (a) Parameterize the domain with the self-mapping $\mathbf{x}(\mathbf{u}) = \mathbf{u}$ consisting of 80 control points; (b) evolving the control points deforms and the image on it; (c) at some point, the mapping $\mathbf{x}(\mathbf{u})$ is no longer 1-to-1, the control points need to reset otherwise divergence occurs

Now we construct an L^2-gradient flow to minimize the energy functional $\mathfrak{E}(\mathbf{x})$. Let

$$\underline{\mathbf{x}}(u, v, \varepsilon) = \mathbf{x} + \varepsilon\Phi(u, v), \qquad \Phi \in C_0^1([0,1]^2)^2,$$

then we have,

$$\delta(\mathfrak{E}(\mathbf{x}), \Phi) = \frac{d}{d\varepsilon}\mathfrak{E}(\underline{\mathbf{x}}(\cdot, \varepsilon))|_{\varepsilon=0} = \int_\Omega \Big(2|I_0(\mathbf{u}) - I_1(\mathbf{x})|\nabla I_1(\mathbf{x})\phi + \beta \sum_{i=1}^{n} \frac{\nabla^2\psi_i(\mathbf{x})\nabla\psi_i(\mathbf{x})}{\|\nabla\psi_i(\mathbf{x})\|}\phi\Big) d\mathbf{u}.$$

This yields the following weak-form L^2 gradient flow that defines the motion of $\mathbf{x}(\mathbf{u})$,

$$\int_\Omega \frac{\partial \mathbf{x}}{\partial t}\phi d\mathbf{u} = -\int_\Omega \Big(2|I_0(\mathbf{u}) - I_1(\mathbf{x})|\nabla I_1(\mathbf{x})\phi + \beta \sum_{i=1}^{n} \frac{\nabla^2\psi_i(\mathbf{x})\nabla\psi_i(\mathbf{x})}{\|\nabla\psi_i(\mathbf{x})\|}\phi\Big) d\mathbf{u}, \tag{2.5}$$

where t is a time parameter, the first and second-order partial derivatives of the characteristic functions are in the distribution sense.

Our algorithm is based on the space constructed by finite dimensional B-spline basis function. We solve (2.5) interchangeably using finite element method in the spatial discretization and explicit Euler scheme in the temporal discretization.

Let $\mathbf{x}(u, v)$ be the bi-cubic B-spline function defined on $[0,1]^2$.

$$\mathbf{x}(u, v) = [x_1(u, v), x_2(u, v)]^{\mathrm{T}} = \sum_{i=0}^{m+2}\sum_{j=0}^{n+2} \mathbf{a}_{ij} N_{i,3}(u) N_{j,3}(v),$$

where $N_{i,3}(u)$ is the cubic B-spline basis function defined on the equi-spaced knots, $\mathbf{a}_{ij}$ are control points of B, $(m+3) \times (n+3)$ is the number of all control points. An experimental result using our algorithm shows on Fig.3.

2.2. Theoretical Results

To state the results, we define the space X as following,

$$\begin{aligned} X = \Big\{ &\mathbf{x}(u, v) : \mathbf{x}(u, v) = [x_1(u, v), x_2(u, v)]^{\mathrm{T}} \\ &= \sum_{i=0}^{m+2}\sum_{j=0}^{n+2} \mathbf{a}_{ij} N_{i,3}(u) N_{j,3}(v) \ \text{ satisfying } \ \text{(i)–(iii)} \Big\}. \end{aligned}$$

We define the norm $\mathbf{x} \in X$ as:

$$\|\mathbf{x}\|_X = \Big(\int_0^1\int_0^1 |\mathbf{x}|^2 du dv\Big)^{\frac{1}{2}} = \Big(\int_0^1\int_0^1 (x_1^2 + x_2^2) du dv\Big)^{\frac{1}{2}}.$$

Then we have (see Sections 3–4)

Theorem 2.1. $\mathbf{x} : [0,1]^2 \to [0,1]^2$ *satisfying (i)–(iii) is an injection and surjection.*

Theorem 2.2. *There exists a mapping* $\mathbf{x}_0(u,v) \in X$ *such that (2.4) is minimized.*

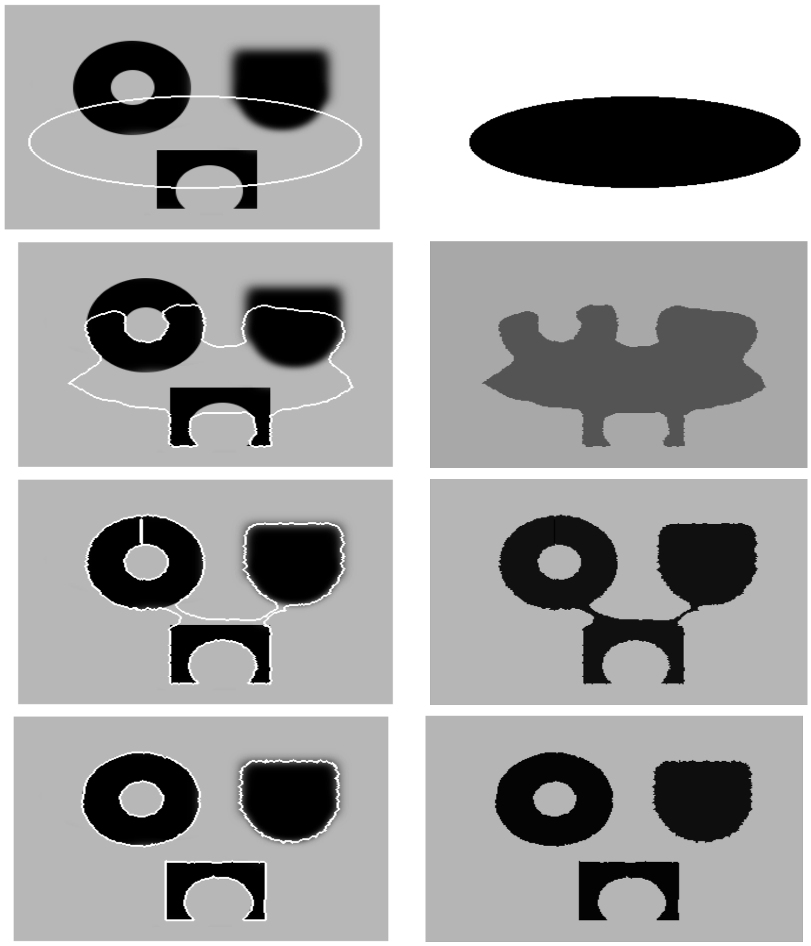

Fig.3. Detection of three geometric objects,one of which has a smooth boundary. Size: 320×240, control grid30×40, total iterations:150, CPU time: 13s

3 Regularity Analysis of Mapping $\mathbf{x}$

In this section, we first introduce the used definitions, terminologies and theorems. Then we provide the details of proof for Theorem 2.1.

3.1 Definitions and Theorems

Definition 3.1[18]**.** *Let X and Y be topological spaces; let f: $X \to Y$ be a bijection. If both the function f and the inverse function f^{-1}: $Y \to X$ are continuous, then f is called a homemorphism. f is a local homeomorphism, if for every point x in X, there exists an open set U containing x, such that $f(U)$ is open in Y and is a homeomorphism.*

Definition 3.2[6]**.** *Let $\widetilde{B}$ and B be subsets of $\mathbb{R}^2$. We say that $\pi : \widetilde{B} \to B$ is a covering mapping if*

1. *π is continuous and $\pi(\widetilde{B}) = B$.*
2. *Each point $p \in B$ has a neighborhood U in B (to be called a distinguished neighborhood of p) such that*

$$\pi^{-1}(U) = \bigcup_{\alpha} V_\alpha,$$

where the V_α's are pairwise disjoint open sets such that the restriction of π to V_α is a homeomorphism of V_α onto U.

Definition 3.3[9]**.** *Assuming that (X, ρ) is a metric space, A is a subspace of X. If every sequence in A has a convergence subsequence and the limit point of the convergence subsequence lies in A, we named that A is a self-sequentially compact set.*

Definition 3.4[6]**.** *$A \subset \mathbb{R}^n$ is arcwise connected if, given two points $p, q \in A$, there exists an arc in A joining p to q.*

Definition 3.5[6]**.** *$A \subset \mathbb{R}^n$ is connected when it is not possible to write $A = U_1 \cup U_2$, where U_1 and U_2 are nonempty open sets in A and $U_1 \cap U_2 = \emptyset$.*

Theorem 3.1[6]**.** *Let T be of class C^1 in a set D with $J(p) \neq 0$ for each $p \in D$, and let T map D one-to-one onto a set $T(D)$. Then, the inverse T^{-1} of T is of class C^1 on $T(D)$ and the differential of T^{-1} is $(dT)^{-1}$, the inverse of the differential of T. $J(p)$ denotes the Jacobian determinant of T in the point p.*

Theorem 3.2[6]**.** *Let $\pi : \widetilde{B} \to B$ be a local homeomorphism, $\widetilde{B}$ compact and B connected. Then π is a covering mapping.*

Theorem 3.3[6]**.** *Let $\pi : \widetilde{B} \to B$ be a covering mapping, $\widetilde{B}$ arcwise connected, and B simply connected. Then π is a homeomorphism.*

3.2 Theoretical Analysis of mapping x

Now, we prove the correspondence $\mathbf{x}(u, v)$ is an injection and surjection.

Remark 3.1. $[0, 1]^2$ is regarded as a topological space, i.e. $[0, 1]^2$ is a clopen set.

Lemma 3.1. $\mathbf{x} : [0, 1]^2 \to [0, 1]^2$ *satisfying (i)–(iii) is a locally one to one mapping.*

Proof. Let a point $p \in [0, 1]^2$, we determine a neighborhood B of p in which $\mathbf{x}$ is one to one. Let p' and p'' be two points near p such that the line segment jointing p' and p'' lies in $[0, 1]^2$. According to the mean value theorem, we may choose two points p_1^*, p_2^* on this line segment such that

$$\mathbf{x}(p'') - \mathbf{x}(p') = L(p'' - p'), \tag{3.1}$$

where L is the linear transformation represented by $L = (\mathbf{x}_u(p_1^*), \mathbf{x}_v(p_2^*))$.

Let

$$F(p_1, p_2) = \det(\mathbf{x}_u(p_1), \mathbf{x}_v(p_2)),$$

then $F(p_1^*, p_2^*) = \det(L)$. Moreover, since $\mathbf{x}$ is C^2, then F is C^1 continuous, and $F(p, p) \geq \gamma$. There exists a circular neighborhood B of p lying in $[0, 1]^2$, such that $F(p_1, p_2) \geq \gamma$ for all choices of the points p_1, p_2 in B. We shall prove that $\mathbf{x}$ is a one to one mapping in B. Assuming that p' and p'' lie in B and $\mathbf{x}(p') = \mathbf{x}(p'')$, we will prove $p' = p''$. Since p' and p'' lie in B and B is convex, the entire line segment joining p' to p'' also lies in B, hence both p_1^* and p_2^* are points of B. Using the property of B, we have $F(p_1^*, p_2^*) = \det(L) \neq 0$. The linear transformation L is therefore nonsingular. According to (3.1) and using the assumption that $\mathbf{x}(p') = \mathbf{x}(p'')$, we have $L(p'' - p') = 0$. Since L is nonsingular, therefore we deduce that $p' = p''$, i.e., $\mathbf{x}$ is a one to one mapping in B. □

Lemma 3.2. $\mathbf{x} : [0,1]^2 \to [0,1]^2$ *satisfying (i)–(iii) is a locally homeomorphism.*

Proof. According to Lemma 3.1, $\mathbf{x}$ is a locally one to one mapping in $[0,1]^2$. Given a point $p \in [0,1]^2$, assuming that $\mathbf{x}$ is one to one in a neighborhood B of p, then it is obvious that $\mathbf{x} : B \to \mathbf{x}(B)$ is surjective, where $\mathbf{x}(B)$ denotes the range of $\mathbf{x}$ in B. And from Theorem 3.3, we know that $\mathbf{x}^{-1}$ is continuous in $\mathbf{x}(B)$. Thus, $\mathbf{x}$ is homeomorphic in B, i.e., $\mathbf{x}$ is a local homeomorphism.

Proof of Theorem 2.1. It is obvious that $[0,1]^2$ is connected and compact. From Lemma 3.2 and Theorem 3.2, we deduce that $\mathbf{x}$ is a covering mapping. And since $[0,1]^2$ is arcwise connected and simply connected, according to Theorem 3.3, $\mathbf{x}$ is a homeomorphism. Thus, we obtain that $\mathbf{x}$ is an injection and surjection. The result is deduced. □

4 The Existence of the Solution to the Energy Minimizer

This section devotes to the proof of Theorem 2.2.

Lemma 4.1. (X, ρ) *is a closed set.*

Proof. Suppose that $\{\mathbf{x}_k\}$ is a fundamental sequence in the space (X, ρ). This sequence can be written as $\mathbf{x}_k(u,v) = \sum_{i=0}^{m+2}\sum_{j=0}^{n+2}(\mathbf{a}_{ij})_k N_{i,3}(u)N_{j,3}(v)$. It is easy to deduce that $(\mathbf{a}_{ij})_k$ are bounded, hence there exists a subsequence $(\mathbf{a}_{ij})_{k_l}$ converging to $(\mathbf{a}_{ij})_0$. Because $\{\mathbf{x}_k\}$ is a fundamental sequence, we obtain that,

$$\begin{aligned}
\lim_{k\to\infty} \mathbf{x}_k(u,v) &= \lim_{l\to\infty} \mathbf{x}_{k_l}(u,v) \\
&= \lim_{l\to\infty} \sum_{i=0}^{m+2}\sum_{j=0}^{n+2}(\mathbf{a}_{ij})_{k_l} N_{i,3}(u)N_{j,3}(v) \\
&= \sum_{i=0}^{m+2}\sum_{j=0}^{n+2}(\mathbf{a}_{ij})_0 N_{i,3}(u)N_{j,3}(v).
\end{aligned}$$

Let $\mathbf{x}_0(u,v) = \sum_{i=0}^{m+2}\sum_{j=0}^{n+2}(\mathbf{a}_{ij})_0 N_{i,3}(u)N_{j,3}(v)$. Because the range of $\mathbf{x}_k$ is $[0,1]^2$ which is a closed set, we obtain that $\mathbf{x}_0(u,v) \in [0,1]^2$.

On the other hand, it is obvious that $\mathbf{x}_0(0,v) = [0,v]^T$, $\mathbf{x}_0(1,v) = [1,v]^T$, $\mathbf{x}_0(u,0) = [u,0]^T$ and $\mathbf{x}_0(u,1) = [u,1]^T$. Moreover, since $\det((\mathrm{x_k})_\mathrm{u}, (\mathrm{x_k})_\mathrm{v}) \geq \gamma$, hence $\det((\mathbf{x}_0)_u, (\mathbf{x}_0)_v) = \lim_{k\to\infty} \det((\mathbf{x}_k)_u, (\mathbf{x}_k)_v) \geq \gamma$, i.e. $\mathbf{x}_0(u,v) \in X$. Therefore, X is a closed set. □

Lemma 4.2. X *is a self-sequentially compact set.*

Proof. Because the number of the bicubic B-spline bases is finite, X is a finite dimensional space. On the other hand, $\mathbf{x}(u,v) : [0,1]^2 \to [0,1]^2$, hence $\|\mathbf{x}(u,v)\| \leq 2$. According to Lemma 4.1, X is a closed set. Then we conclude that X is a self-sequentially compact set. □

To prove the existence of the solution of energy Model (2.4), we need to prove that the model is continuous about $\mathbf{x}$.

Let

$$I_1(\mathbf{u}) = \begin{cases} c_1, & \mathbf{u} \in \Omega_1, \\ c_2, & \mathbf{u} \in \Omega_2, \\ \vdots & \vdots \\ c_n, & \mathbf{u} \in \Omega_n, \end{cases}$$

then

$$I_1(\mathbf{x}(\mathbf{u})) = \begin{cases} c_1, & \mathbf{x}(\mathbf{u}) \in \Omega_1, \\ c_2, & \mathbf{x}(\mathbf{u}) \in \Omega_2, \\ \vdots & \vdots \\ c_n, & \mathbf{x}(\mathbf{u}) \in \Omega_n, \end{cases} \qquad I_1(\mathbf{y}(\mathbf{u})) = \begin{cases} c_1, & \mathbf{y}(\mathbf{u}) \in \Omega_1, \\ c_2, & \mathbf{y}(\mathbf{u}) \in \Omega_2, \\ \vdots & \vdots \\ c_n, & \mathbf{y}(\mathbf{u}) \in \Omega_n. \end{cases}$$

Definition 4.1.

$$\Omega_i^{\mathbf{x}} = \{\mathbf{u} = (u, v) : \mathbf{x}(\mathbf{u}) \in \Omega_i\}, \qquad i = 1, \cdots, n, \tag{4.1}$$

$$\Omega_i^{\mathbf{y}} = \{\mathbf{u} = (u, v) : \mathbf{y}(\mathbf{u}) \in \Omega_i\}, \qquad i = 1, \cdots, n. \tag{4.2}$$

Remark 4.1. Because $\mathbf{x}$ and $\mathbf{y}$ are continuous mappings, Ω_i is an open set, thus $\Omega_i^{\mathbf{x}}$ and $\Omega_i^{\mathbf{y}}$ are both open sets, $i = 1, \cdots, n$.

Lemma 4.3. *For any $\varepsilon > 0$, there exists $\delta > 0$, when $|(w, z) - (u, v)| > \varepsilon$, then $|\mathbf{x}(w, z) - \mathbf{x}(u, v)| \geq \delta$.*

Proof. Reduction to absurdity. Assuming that there exists $\varepsilon_0 > 0$, such that for any $\delta > 0$, exists the point (w_δ, z_δ) and (u_δ, v_δ), satisfying $|(w_\delta, z_\delta) - (u_\delta, v_\delta)| > \varepsilon_0$ and $|\mathbf{x}(w_\delta, z_\delta) - \mathbf{x}(u_\delta, v_\delta)| < \delta$.

Let $\delta = 1, 1/2, \cdots, 1/k, \cdots$, we get the corresponding points $\{(w_k, z_k)\}$ and the points $\{(u_k, v_k)\}$, then

$$|(w_k, z_k) - (u_k, v_k)| > \varepsilon_0 \tag{4.3}$$

and

$$|\mathbf{x}(w_k, z_k) - \mathbf{x}(u_k, v_k)| < \frac{1}{k}. \tag{4.4}$$

On the other hand, according to Theorem 2.1 and Theorem 3.3, $\mathbf{x}^{-1}$ is C^1 continuous. And because $\mathbf{x}^{-1}$: $[0, 1]^2 \to [0, 1]^2$, $\mathbf{x}^{-1}$ is uniformly continuous on $[0, 1]^2$. Thus for any $\varepsilon > 0$, there exists N, when $k > N$, $|\mathbf{x}(w_k, z_k) - \mathbf{x}(u_k, v_k)| < 1/k$, we get $|\mathbf{x}^{-1}(\mathbf{x}(w_k, z_k)) - \mathbf{x}^{-1}(\mathbf{x}(u_k, v_k))| < \varepsilon$, that is, $|(w_k, z_k) - (u_k, v_k)| < \varepsilon$, this is in contradiction with (4.3), this lemma is proved. □

Lemma 4.4. *For any $\varepsilon > 0$, there exists $\delta > 0$, when $\|\mathbf{y} - \mathbf{x}\| < \delta$, we have $\|\mathbf{y}^{-1} - \mathbf{x}^{-1}\| < \varepsilon$.*

Proof. Reduction to absurdity. Assuming that there exists $\varepsilon_0 > 0$, for any $\delta > 0$, existing $\mathbf{y}_\delta$ and $\mathbf{u}_\delta$, satisfying $\|\mathbf{y}_\delta - \mathbf{x}\| < \delta$, and

$$|\mathbf{y}_\delta^{-1}(\mathbf{u}_\delta) - \mathbf{x}^{-1}(\mathbf{u}_\delta)| \geq \varepsilon_0. \tag{4.5}$$

Let $\delta = 1,\ \frac{1}{2}, \cdots, \frac{1}{k}, \cdots$, we get the sequence $\{\mathbf{y}_k\}$ and $\{\mathbf{u}_k\}$, satisfying

$$\|\mathbf{y}_k - \mathbf{x}\| < \frac{1}{k}, \tag{4.6}$$

and

$$|\mathbf{y}_k^{-1}(\mathbf{u}_k) - \mathbf{x}^{-1}(\mathbf{u}_k)| \geq \varepsilon_0. \tag{4.7}$$

On the other hand, as the mapping $\mathbf{x}$ and $\mathbf{y}$ are surjections, hence for $\mathbf{u}_k$ in (4.7), there exist $(u_{\mathbf{x}_k}, v_{\mathbf{x}_k})$ and $(u_{\mathbf{y}_k}, v_{\mathbf{y}_k})$, satisfying $\mathbf{x}(u_{\mathbf{x}_k}, v_{\mathbf{x}_k}) = \mathbf{u}_k$ and $\mathbf{y}_k(u_{\mathbf{y}_k}, v_{\mathbf{y}_k}) = \mathbf{u}_k$. According to (4.7), we obtain

$$|(u_{\mathbf{y}_k}, v_{\mathbf{y}_k}) - (u_{\mathbf{x}_k}, v_{\mathbf{x}_k})| \geq \varepsilon_0. \tag{4.8}$$

According to Lemma 4.3, we get that there exists η_0, satisfying

$$|\mathbf{x}(u_{\mathbf{y}_k}, v_{\mathbf{y}_k}) - \mathbf{x}(u_{\mathbf{x}_k}, v_{\mathbf{x}_k})| \geq \eta_0. \tag{4.9}$$

Furthermore, according to (4.6), there exists k_0, when $k > k_0$, getting $\|\mathbf{y}_k - \mathbf{x}\| < \frac{\eta_0}{2}$, hence

$$|\mathbf{y}_k(u_{\mathbf{y}_k}, v_{\mathbf{y}_k}) - \mathbf{x}(u_{\mathbf{y}_k}, v_{\mathbf{y}_k})| < \frac{\eta_0}{2}. \tag{4.10}$$

Because

$$\begin{aligned} 0 = |\mathbf{u}_k - \mathbf{u}_k| =& |\mathbf{x}(u_{\mathbf{x}_k}, v_{\mathbf{x}_k}) - \mathbf{y}_k(u_{\mathbf{y}_k}, v_{\mathbf{y}_k})| \\ =& |\mathbf{x}(u_{\mathbf{x}_k}, v_{\mathbf{x}_k}) - \mathbf{x}(u_{\mathbf{y}_k}, v_{\mathbf{y}_k}) + \mathbf{x}(u_{\mathbf{y}_k}, v_{\mathbf{y}_k}) - \mathbf{y}_k(u_{\mathbf{y}_k}, v_{\mathbf{y}_k})| \\ \geq& |\mathbf{x}(u_{\mathbf{x}_k}, v_{\mathbf{x}_k}) - \mathbf{x}(u_{\mathbf{y}_k}, v_{\mathbf{y}_k})| - |\mathbf{x}(u_{\mathbf{y}_k}, v_{\mathbf{y}_k}) - \mathbf{y}_k(u_{\mathbf{y}_k}, v_{\mathbf{y}_k})| \\ \geq& \eta_0 - \frac{\eta_0}{2} \geq \frac{\eta_0}{2}, \end{aligned} \tag{4.11}$$

contradiction is gotten, this lemma is proved. □

Lemma 4.5. *For any $\varepsilon > 0$, there exists $\delta > 0$, when $\|\mathbf{y} - \mathbf{x}\| < \delta$, for any $\mathbf{u} \in \Omega_i^{\mathbf{x}} \setminus \overline{\Omega_i^{\mathbf{y}}}$, having dist$(\mathbf{u}, \partial\overline{\Omega_i^{\mathbf{y}}}) < \varepsilon$.*

Proof. For any $\mathbf{u} \in \Omega_i^{\mathbf{x}} \setminus \overline{\Omega_i^{\mathbf{y}}}$, we get $\mathbf{x}(\mathbf{u}) \in \Omega_i$. According to Lemma 4.4, we have $|\mathbf{y}^{-1}(\mathbf{x}(\mathbf{u})) - \mathbf{x}^{-1}(\mathbf{x}(\mathbf{u}))| < \varepsilon$, that is $|\mathbf{y}^{-1}(\mathbf{x}(\mathbf{u})) - \mathbf{u}| < \varepsilon$. As $\mathbf{y}(\mathbf{y}^{-1}(\mathbf{x}(\mathbf{u}))) = \mathbf{x}(\mathbf{u}) \in \Omega_i$, hence $\mathbf{y}^{-1}(\mathbf{x}(\mathbf{u})) \in \Omega_i^{\mathbf{y}}$, note that $\Omega_i^{\mathbf{y}}$ is an open set, hence dist$(\mathbf{u}, \partial\overline{\Omega_i^{\mathbf{y}}}) \leq |\mathbf{u} - \mathbf{y}^{-1}(\mathbf{x}(\mathbf{u}))| < \varepsilon$. □

Lemma 4.6. *For any $\varepsilon > 0$, there exists $\delta > 0$, when $\|\mathbf{y} - \mathbf{x}\| < \delta$, for any $\mathbf{u} \in \Omega_i^{\mathbf{y}} \setminus \overline{\Omega_i^{\mathbf{x}}}$, we have dist$(\mathbf{u}, \partial\overline{\Omega_i^{\mathbf{x}}}) < \varepsilon$.*

Proof. Proof of this lemma is similar as Lemma 4.5, here omitted. □

Lemma 4.7. *For any $\varepsilon > 0$, there exists $\delta > 0$, when $\|\mathbf{y} - \mathbf{x}\| < \delta$, $m\big((\Omega_i^{\mathbf{x}} \cup \Omega_i^{\mathbf{y}}) \setminus (\overline{\Omega_i^{\mathbf{x}} \cap \Omega_i^{\mathbf{y}}})\big) < \varepsilon$.*

Proof. Because

$$(\Omega_i^{\mathbf{x}} \cup \Omega_i^{\mathbf{y}}) \setminus \big(\overline{\Omega_i^{\mathbf{x}} \cap \Omega_i^{\mathbf{y}}}\big) = \big(\Omega_i^{\mathbf{x}} \setminus (\overline{\Omega_i^{\mathbf{x}} \cap \Omega_i^{\mathbf{y}}})\big) \cup \big(\Omega_i^{\mathbf{y}} \setminus (\overline{\Omega_i^{\mathbf{x}} \cap \Omega_i^{\mathbf{y}}})\big) = (\Omega_i^{\mathbf{x}} \setminus \overline{\Omega_i^{\mathbf{y}}}) \cup (\Omega_i^{\mathbf{y}} \setminus \overline{\Omega_i^{\mathbf{x}}}),$$

hence

$$m\big((\Omega_i^{\mathbf{x}} \cup \Omega_i^{\mathbf{y}}) \setminus (\overline{\Omega_i^{\mathbf{x}} \cap \Omega_i^{\mathbf{y}}})\big) = m\big((\Omega_i^{\mathbf{x}} \setminus \overline{\Omega_i^{\mathbf{y}}}) \cup (\Omega_i^{\mathbf{y}} \setminus \overline{\Omega_i^{\mathbf{x}}})\big) \leq m(\Omega_i^{\mathbf{x}} \setminus \overline{\Omega_i^{\mathbf{y}}}) + m(\Omega_i^{\mathbf{y}} \setminus \overline{\Omega_i^{\mathbf{x}}}).$$

Reduction to absurdity. Assuming that there exists $\varepsilon_0 > 0$, for any $\delta > 0$, existing the mapping $\mathbf{y}_\delta$ satisfying $\|\mathbf{y}_\delta - \mathbf{x}\| < \delta$, we have $m\big((\Omega_i^{\mathbf{x}} \cup \Omega_i^{\mathbf{y}}) \setminus (\overline{\Omega_i^{\mathbf{x}} \cap \Omega_i^{\mathbf{y}}})\big) > \varepsilon_0$. Assuming $m(\Omega_i^{\mathbf{x}} \setminus \overline{\Omega_i^{\mathbf{y}}}) > \frac{\varepsilon_0}{2}$, then $\Omega_i^{\mathbf{x}} \setminus \overline{\Omega_i^{\mathbf{y}}}$ is not empty, that is there exists $\mathbf{u}_0 \in \Omega_i^{\mathbf{x}} \setminus \overline{\Omega_i^{\mathbf{y}}}$. Note that $\Omega_i^{\mathbf{x}} \setminus \overline{\Omega_i^{\mathbf{y}}}$ is an open set, hence there exists r_0, satisfying $O_{\mathbf{u_o}}(r_0) \subset \Omega_i^{\mathbf{x}} \setminus \overline{\Omega_i^{\mathbf{y}}}$, hence we have dist$\big(\mathbf{u}_0, \partial(\overline{\Omega_i^{\mathbf{y}}})\big) > r_0$. This is contraction with Lemma 4.5, this lemma is proved. □

Lemma 4.8. *For any $\varepsilon > 0$, there exists $\delta > 0$, when $\|\mathbf{y} - \mathbf{x}\| < \delta$, $\big||\mathbf{y}(\Gamma_{\Omega_i})| - |\mathbf{x}(\Gamma_{\Omega_i})|\big| < \varepsilon$.*

Proof. Because the boundary of Ω_i is piecewise smooth, the boundary is denoted by Γ_{Ω_i}. Assuming that Γ_k is one of sections, the parametric equation of Γ_k is $(\alpha(t), \beta(t))$, the range of

t is $[t_1, t_2]$, then arc length of Γ_k is

$$|\Gamma_k| = \int_{t_1}^{t_2} \sqrt{(\alpha'(t))^2 + (\beta'(t))^2} dt.$$

Parametric equation of the mapping acting on arc Γ_k is denoted by $\mathbf{x}(\Gamma_k) = \big(x_1(\alpha, \beta), x_2(\alpha, \beta)\big)$, the range of t is $[t_1, t_2]$, then the arc length of $\mathbf{x}(\Gamma_k)$ is

$$|\mathbf{x}(\Gamma_k)| = \int_{t_1}^{t_2} \sqrt{(\nabla x_1 \cdot \Gamma_k')^2 + (\nabla x_2 \cdot \Gamma_k')^2} dt. \tag{4.12}$$

Noting that the mapping $\mathbf{x}$ is C^2 and represented by B-spline basis function as (4.12), we deduce that $|\mathbf{x}(\Gamma_k)|$ is continuous about mapping $\mathbf{x}$, the conclusion is proved. □

Lemma 4.9. *For any $\varepsilon > 0$, there exists $\delta > 0$, when $\|\mathbf{y} - \mathbf{x}\| < \delta$, we have*

$$\begin{aligned}
&\big\|\mathfrak{E}(\mathbf{y}) - \mathfrak{E}(\mathbf{x})\big\| \\
=&\Big\| \int_\Omega \Big(\big(I_0(\mathbf{u}) - I_1(\mathbf{y}(\mathbf{u}))\big)^2 + \beta \sum_{i=1}^n \|\nabla \psi_i(\mathbf{y}(\mathbf{u}))\|\Big) d\mathbf{u} \\
&- \int_\Omega \Big(\big(I_0(\mathbf{u}) - I_1(\mathbf{x}(\mathbf{u}))\big)^2 + \beta \sum_{i=1}^n \|\nabla \psi_i(\mathbf{x}(\mathbf{u}))\|\Big) d\mathbf{u}\Big\| < \varepsilon.
\end{aligned} \tag{4.13}$$

Proof.

$$\begin{aligned}
&\big\|\mathfrak{E}(\mathbf{y}) - \mathfrak{E}(\mathbf{x})\big\| \\
=&\Big\| \int_\Omega \Big(\big(I_0(\mathbf{u}) - I_1(\mathbf{y}(\mathbf{u}))\big)^2 + \beta \sum_{i=1}^n \|\nabla \psi_i(\mathbf{y}(\mathbf{u}))\|\Big) d\mathbf{u} \\
&- \int_\Omega \Big(\big(I_0(\mathbf{u}) - I_1(\mathbf{x}(\mathbf{u}))\big)^2 + \beta \sum_{i=1}^n \|\nabla \psi_i(\mathbf{x}(\mathbf{u}))\|\Big) d\mathbf{u}\Big\| \\
\le&\Big\| \int_\Omega \big(2I_0(\mathbf{u}) - I_1(\mathbf{x}(\mathbf{u})) - I_1(\mathbf{y}(\mathbf{u}))\big) \big(I_1(\mathbf{x}(\mathbf{u})) - I_1(\mathbf{y}(\mathbf{u}))\big) d\mathbf{u}\Big\| \\
&+ \Big\| \int_\Omega \beta \sum_{i=1}^n \big(\|\nabla \psi_i(\mathbf{y}(\mathbf{u}))\| - \|\nabla \psi_i(\mathbf{x}(\mathbf{u}))\|\big) d\mathbf{u}\Big\|.
\end{aligned} \tag{4.14}$$

For simplicity, let $f(\mathbf{u}) = 2I_0(\mathbf{u}) - I_1(\mathbf{x}(\mathbf{u})) - I_1(\mathbf{y}(\mathbf{u}))$, we have

$$\begin{aligned}
&\int_\Omega \big(2I_0(\mathbf{u}) - I_1(\mathbf{x}(\mathbf{u})) - I_1(\mathbf{y}(\mathbf{u}))\big)\big(I_1(\mathbf{x}(\mathbf{u})) - I_1(\mathbf{y}(\mathbf{u}))\big) d\mathbf{u} \\
=&\int_\Omega f(\mathbf{u})\big(I_1(\mathbf{x}(\mathbf{u})) - I_1(\mathbf{y}(\mathbf{u}))\big) d\mathbf{u} \\
=&\int_\Omega \big(f(\mathbf{u}) I_1(\mathbf{x}(\mathbf{u})) - f(\mathbf{u}) I_1(\mathbf{y}(\mathbf{u}))\big)\, d\mathbf{u} \\
=&\sum_{i=1}^n c_i \int_{\Omega_i^{\mathbf{x}}} f(\mathbf{u}) d\mathbf{u} - \sum_{i=1}^n c_i \int_{\Omega_i^{\mathbf{y}}} f(\mathbf{u}) d\mathbf{u}.
\end{aligned} \tag{4.15}$$

As $f(\mathbf{u})$ is bounded function on $[0,1]^2$, the upper bound of $|f(\mathbf{u})|$ on $[0,1]^2$ denoted by M, according to the Lemma 4.7, we know

$$\begin{aligned}
&\Big\| \sum_{i=1}^{n} c_i \int_{\Omega_i^{\mathbf{x}}} f(\mathbf{u}) d\mathbf{u} - \sum_{i=1}^{n} c_i \int_{\Omega_i^{\mathbf{y}}} f(\mathbf{u}) d\mathbf{u} \Big\| \\
\leq &\sum_{i=1}^{n} c_i \int_{(\Omega_i^{\mathbf{x}} \cup \Omega_i^{\mathbf{y}}) \setminus (\overline{\Omega_i^{\mathbf{x}} \cap \Omega_i^{\mathbf{y}}})} |f(\mathbf{u})| d\mathbf{u} \leq M\varepsilon \sum_{i=1}^{n} c_i.
\end{aligned} \tag{4.16}$$

That is

$$\Big\| \int_{\Omega} \big(2I_0(\mathbf{u}) - I_1(\mathbf{x}(\mathbf{u})) - I_1(\mathbf{y}(\mathbf{u}))\big)\big(I_1(\mathbf{x}(\mathbf{u})) - I_1(\mathbf{y}(\mathbf{u}))\big) d\mathbf{u} \Big\| < M\varepsilon \sum_{i=1}^{n} c_i. \tag{4.17}$$

On the other hand, according to the Lemma 4.8, we get

$$\begin{aligned}
&\Big\| \int_{\Omega} \beta \sum_{i=1}^{n} \big(\|\nabla\psi_i(\mathbf{y}(\mathbf{u}))\| - \|\nabla\psi_i(\mathbf{x}(\mathbf{u}))\|\big) d\mathbf{u} \Big\| \\
=&\beta \sum_{i=1}^{n} \big| |\mathbf{y}(\Gamma_{\Omega_i})| - |\mathbf{x}(\Gamma_{\Omega_i})| \big| < n\beta\varepsilon,
\end{aligned} \tag{4.18}$$

hence when $\|\mathbf{y} - \mathbf{x}\| < \delta$, we have

$$\big\| \mathfrak{E}(\mathbf{y}) - \mathfrak{E}(\mathbf{x}) \big\| < M\varepsilon \sum_{i=1}^{n} c_i + n\beta\varepsilon,$$

Thus the energy functional

$$\mathfrak{E}(\mathbf{x}) = \int_{\Omega} \Big(\big(I_0(u,v) - I_1(\mathbf{x}(u,v))\big)^2 + \beta \sum_{i} \|\nabla\psi_i(\mathbf{x}(u,v))\| \Big) dudv \tag{4.19}$$

is continuous on $\mathbf{x}$.

Proof of Theorem 2.2. Let $\mathbf{x}_n$ be a minimizing sequence for the Model (2.4), i.e.,

$$\lim_{n\to\infty} \mathfrak{E}(\mathbf{x}_n) = \inf_{\mathbf{x}\in X} \mathfrak{E}(\mathbf{x}).$$

Due to the Lemma 4.2, X is a self-sequentially compact set. Thus, there exists a subsequence $\mathbf{x}_{n_k}$ and $\mathbf{x}_0$ in X such that $\mathbf{x}_{n_k} \to \mathbf{x}_0$.

Finally, since the energy functional

$$\mathfrak{E}(\mathbf{x}) = \int_{\Omega} \Big(\big(I_0(u,v) - I_1(\mathbf{x}(u,v))\big)^2 + \beta \sum_{i} \|\nabla\psi_i(\mathbf{x}(u,v))\| \Big) dudv \tag{4.20}$$

is continuous on $\mathbf{x}_0$, then

$$\mathfrak{E}(\mathbf{x_0}) = \lim_{k\to\infty} \mathfrak{E}(\mathbf{x}_{n_k}) = \inf_{\mathbf{x}\in X} \mathfrak{E}(\mathbf{x}),$$

$\mathbf{x}_0$ is a minimum point of $\mathfrak{E}(\mathbf{x})$. □

5 Conclusion

An algorithm has been presented for solving image segmentation problem in [22]. This paper analyzes the validity of the model from the theoretical point of view. After a brief introduction of the segmentation model and experimental results, we proved the regularity of mapping $\mathbf{x}(u,v)$ when the mapping $\mathbf{x}(u,v)$ satisfies certain conditions. We also proved that there exists a mapping $\mathbf{x}_0(u,v) \in X$ satisfying (i)–(iii) such that the energy functional (2.4) is minimized.

References

[1] Buck, R.C. Advanced Calculus. The McGraw-Hill Book Company, New York 1956

[2] Caselles, V., Catte, F., Coll, T., Dibos, F. A geometric model for active contours in image processing. *Numerische Mathematik*, 66(1): 1–31 (1993)

[3] Caselles, V., Kimmel, R., Sapiro, R. Geodesic active contours. *International Journal of Computer Vision*, 22: 61–79 (1997)

[4] Chan, T.F., Sandberg, B.Y., Vese, L.A. Active contours without edges for vector-valued images. *Journal of Visual Communication and Image Representation*, 11: 130–141 (2000)

[5] Chan, T.F., Vese, L.A. Active contours without edges. *IEEE Transactions on Image Processing*, 10(2): 266–277 (2001)

[6] do Carmo, M.P. Differential Geometry of Curves and Surfaces. China Machine Press, Beijing, 2004

[7] Esedoglu, S., Tsai, Y.H. Threshold dynamics for the piecewise constant Mumford-Shah functional. *J. Comput. Phys.*, 211: 367–384 (2006)

[8] Gibou, F., Fedkiw, R. A fast hybrid k-means level set algorithm for segmentation. In: 4th Annual Hawaii International Conference on Statistics and Mathematics, 2005, 281–291

[9] Jing, Z.C., Li, M. A wavelet based alternative iteration method for the orientation refinement of cryo-electron microscopy 3D reconstruction. *Mathematical Modelling and Analysis*, 20(3): 396–408 (2015)

[10] Jing, Z.C., Li, M.G., Wang, C.L. A Nonmonotone Line Search Based Algorithm for Distribution Center Location Selected. *Acta Mathematicae Applicatae Sinica* (English Series), 3: 699–706 (2014)

[11] Kass, M., Witkin, A., Terzopoulos, D. Snakes: Active Contour Models. *International Journal of Computer Vision*, 1(4): 321–331 (1987)

[12] Li, C., Xu, C., Gui, C., Fox, M.D. Level set evolution without re-initialization: a new variational formulation. In: Proc. of IEEE Conference on Computer Vision and Pattern Recognition, 2005, 430–436

[13] Lie, J., Lysaker, M., Tai, X.C. A variant of the level set method and applications to image segmentation. *Mathematics of Computation*, 75: 1155–1174 (2006)

[14] Linda, G. Shapiro, George, C. Stockman. Computer Vision. Prentice-Hall, Upper Saddle River, New Jersey, 2001

[15] Malladi, R., Sethian, J.A., Vemuri, B.C. Shape modeling with front propagation: A level set approach. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 17(2): 158–175 (1995)

[16] Massari, U., Tamanini, I. Regularity properties of optimal segmentations. *J. Reine Angew. Math.*, 420: 61–84 (1991)

[17] Mumford, D., Shah, J. Optimal approximations by piecewise smooth functions and associated variational problems. *Communications on Pure and Applied Mathematics*, 42: 577–685 (1989)

[18] Munkres, J.R. Topology. China Machine Press, 2004

[19] Song, B., Chan, T. A fast algorithm for level set based optimization. Cam report 02-68, UCLA, 2002

[20] Sumengen, B., Manjunath, B. Graph partitioning active contours (GPAC) for image segmentation. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 28: 509–521 (2006)

[21] Vese, L.A., Chanm, T.F. A multiphase level set framework for image segmentation using the Mumford and Shah model. *International Journal of Computer Vision*, 50: 271–293 (2002)

[22] Ye, J., Xu, G. A geometric flow approach for region-based image segmentation. *IEEE Transactions on Image Processing*, 21(12): 4735-4745 (2012)

[23] Zhang, G.Q. Functional Analysis Lecture. Peking University Press, Beijing, 1997

[24] Ziemer, W.P. Image Processing and Analysis. Society for Industrial and Applied Mathematical, 2005