Sampled-Data Based Mean Square Bipartite Consensus of Double-Integrator Multi-Agent Systems with Measurement Noises

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Abstract A distributed sampled-data based bipartite consensus protocol is proposed for double-integrator multi-agent systems with measurement noises under signed digraph. A time-varying consensus gain and the agents' states feedback are adopted to counteract the noise effect and achieve bipartite consensus. By determining the state transition matrix of the multi-agent system, we describe the dynamic behaviour of the system. Under the proposed protocol, the states of some agents converge in mean square to one random vector while the rest of agents' states are convergent to another random vector. It is noted that these two vector are at the same amplitude, however their signs are different. It is proved that sufficient conditions for achieving the mean square bipartite consensus are: (1) the topology graph is weighted balanced, structurally balanced and has a spanning tree; and (2) the time-varying consensus gain satisfies the stochastic approximation conditions. We verify the validity of the proposed protocol by numerical simulations.

Keywords Bipartite Consensus · Multi-agent Systems (MASs)
Measurement Noises · Double-integrator · Sampled-data

1 Introduction

As the most important and key problem in distributed coordination of multi-agent systems (MASs), consensus plays a fundamental role of control protocol design and distributed optimization. Most researchers assume that MASs work in an ideal communication environment, however, measurement noises always exist in reality, which means agents cannot get accurate information of each other.

To reduce the effect of noise, there has been lots of available publications. A stochastic approximation type consensus protocol for discrete-time first-order inte-

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gral MASs with communication noises was proposed and the concepts of mean square and almost sure consensus were introduced in [1]. The necessary and sufficient conditions for ensuring the mean square and almost sure consensus were proved and it was shown that the stochastic approximation type gain was necessary in [2, 3]. Both communication noises and delays were considered for the mean square consensus of first-order integral MASs in [4]. Leader-following consensus of first-order integral MASs with measurement noises was studied in [5]. It is found that for the continuous-time mean square leader-following consensus problem, the necessary and sufficient conditions of time-varying consensus gains can be relaxed [6]. Most publications concerning communication noises focus on the first-order integral MASs, more realistic, many control systems can be modeled by the second-order integral dynamics, such as the multi-robot system. Necessary and sufficient conditions were given for the mean square consensus of continuous-time second-order integral MASs under fixed topology in [7]. And a sampled-data based consensus protocol was proposed for second-order integral MASs in [8]. For the generic linear MASs, protocols for solving the mean square consensus with measurement noises were proposed under the fixed topology [9] and under the switching topology [10], respectively. And Cheng modified the consensus protocol and gave each agent its own noise-attenuation gain [11].

It is noted that those publications mentioned above are about cooperative multi-agent systems, however competition and opposition always exist in our world, for example military and politics [12]. For the conventional consensus problem, agents achieve average consensus through collaboration. However in a more realistic situation, some agents may compete and therefore the states of all agents converge to the same value except their signs and achieve bipartite consensus instead of the conventional average consensus [13].

Attempts were made to modify existing protocols for conventional consensus to solve the bipartite consensus problem. There were also many control strategies adopted. State feedback and output feedback control laws were designed to stabilize generic linear time-invariant (LTI) MASs and achieve bipartite consensus in [14]. Switching topologies were taken into consideration in [15] and a input saturation strategy for bipartite consensus on generic linear MASs was put forward in [16]. For bipartite consensus most studies did not take noises into consideration. There are very few available papers considering measurement noises, necessary and sufficient conditions were given for the mean square bipartite consensus of first-order integral MASs [17] and the mean square bipartite consensus of second-order integral MASs [18]. The proof of sufficient conditions for the high-order MASs was given in [19].

However because of the wide application of digital equipment, only sampled data can be used in the real control engineering. In this paper, a sampled-data based bipartite consensus protocol is proposed for double-integrator MASs with measurement noises under signed digraph, the bipartite consensus can be achieved by using a time-varying consensus gain and agents' states feedback to counteract the noise effect. The sufficient conditions for achieving the mean square bipartite consensus are given in this paper.

2 Problem Formulation

2.1 Preliminaries of Signed Graph

In the literature of bipartite consensus of multi-agent systems, the network is usually modeled by a weighted signed digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, where $\mathcal{V} = (v_1, v_2, \ldots, v_N)$ denotes the set of nodes, v_i represents the ith agent; $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ denotes the set of edges, $e_{ij} = (v_i, v_j) \in \mathcal{E}$ if and only if there is the information flow from agent j to agent i; $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N \times N}$ denotes the weighted adjacency matrix, $a_{ij} \neq 0 \Leftrightarrow e_{ij} \in \mathcal{E}$ and $a_{ij} = 0 \Leftrightarrow e_{ij} \notin \mathcal{E}$. $a_{ij} > 0$ means cooperation and $a_{ij} < 0$ means competition between agent i and agent j. In this paper, we always assume that $a_{ii} = 0$ and $a_{ij}a_{ji} \geq 0, i, j = 1, \ldots, N$. The neighborhood of node v_i is denoted by $\mathcal{N}_i = \{v_j \in \mathcal{V} | e_{ij} \in \mathcal{E}\}$. $\deg_{in}(i) = \sum_{j=1}^{N} |a_{ij}|$ is called the in-degree of node v_i and $\deg_{out}(i) = \sum_{j=1}^{N} |a_{ji}|$ is called the out-degree of node v_i. A weighted signed digraph $\mathcal{G}$ is said to be weighted balanced if $\deg_{in}(i) = \deg_{out}(i)$. The Laplacian matrix $\mathcal{L}$ of $\mathcal{G}$ is $\mathcal{L} = \mathcal{D} - \mathcal{A}$, where $\mathcal{D} = \operatorname{diag}\left(\sum_{j=1}^{N} |a_{1j}|, \ldots, \sum_{j=1}^{N} |a_{Nj}|\right)$.

Lemma 1 ([20]) *A weighted balanced digraph is connected if and only if it has a spanning tree.*

A weighted signed digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ is said structurally balanced if there exist two sets of nodes $\mathcal{V}_1$ and $\mathcal{V}_2$, $\mathcal{V}_1 \cup \mathcal{V}_2 = \mathcal{V}$, $\mathcal{V}_1 \cap \mathcal{V}_2 = \varnothing$, such that $a_{ij} \geq 0, \forall v_i, v_j \in \mathcal{V}_l (l \in \{1, 2\})$ and $a_{ij} \leq 0, \forall v_i \in \mathcal{V}_l, v_j \in \mathcal{V}_q, l \neq q (l, q \in \{1, 2\})$ [17].

Lemma 2 ([21]) *If a weighted signed digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ is structurally balanced, the Laplacian matrix of $\mathcal{G}$ has at least one zero eigenvalue and all other eigenvalues have positive real parts. Particularly, $\mathcal{G}$ has exactly one zero eigenvalue if and only if $\mathcal{G}$ has a spanning tree.*

2.2 Consensus Protocol

Consider a MAS with N agents, where the ith agent is described by the following sampled-data double-integrator dynamics

$$z_i[k+1] = Az_i[k] + Bu_i[k], \tag{1}$$

where $z_i[k] = (x_i[k], v_i[k])^T (k \in \mathbb{N}, i = 1, \ldots, N)$, $A = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}$, $\quad B = \begin{bmatrix} \frac{T^2}{2} \\ T \end{bmatrix}$, T is the sampling period, $x_i[k] \in \mathbb{R}$, $v_i[k] \in \mathbb{R}$ and $u_i[k] \in \mathbb{R}$ are the position, velocity and control input of the ith agent at the kth sampling point respectively.

Because of measurement noises, the ith agent receives the neighbor agent's information $y_{ij}[k] = (y_{xij}[k], y_{vij}[k])^T = (x_j[k] + n_{xij}[k], v_j[k] + n_{vij}[k])^T$, $j \in \mathcal{N}_i$, where $y_{xij}[k]$ and $y_{vij}[k]$ denote the measurements of the jth agent's position and velocity. $\{n_{xij}[k], n_{vij}[k], k \in \mathbb{N}, i, j = 1, \ldots, N\}$ is the independent random noise sequence with zero mean, and

$$\sup_{k\in\mathbb{N}} \max_{i,j=1,\ldots,N} E(n_{xij}^2[k]) < \sigma_{max}, \quad \sup_{k\in\mathbb{N}} \max_{i,j=1,\ldots,N} E(n_{vij}^2[k]) < \sigma_{max}.$$

The bipartite consensus problem under measurement noises is to design a distributed protocol for system (1) such that the states of some agents converge in mean square to one random vector while the rest of agents' states are convergent to another random vector with the same amplitude and opposite sign.

Define the transformation matrix $T_z = \begin{bmatrix} \frac{T^2}{2} & T^2 \\ -3T & 0 \end{bmatrix}$. Let $\bar{z}_i[k] = T_z^{-1} z_i[k] = (\bar{z}_{xi}[k], \bar{z}_{vi}[k])^T$, then system (1) can be transformed into

$$\bar{z}_i[k+1] = \bar{A}\bar{z}_i[k] + \bar{B}u_i[k], \tag{2}$$

$$\bar{A} = T_z^{-1} A T_z = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}, \quad \bar{B} = T_z^{-1} B = \begin{bmatrix} -\frac{1}{3} \\ \frac{2}{3} \end{bmatrix}.$$

In this paper, we propose the following distributed bipartite consensus protocol

$$u_i[k] = K_1 \bar{z}_i[k] + a[k] K_2 \sum_{j\in\mathcal{N}_i} |a_{ij}|(\mathrm{sgn}(a_{ij})\bar{y}_{ij}[k] - \bar{z}_i[k]), \tag{3}$$

where $K_1 = \begin{bmatrix} 4.5 & 0 \end{bmatrix}$, $K_2 = \begin{bmatrix} 1.5 & 1.5 \end{bmatrix}$, $\bar{y}_{ij}[k] = T_z^{-1} y_{ij}[k]$, and $a[k] > 0$ is the non-increasing consensus gain sequence.

3 Main Result

Applying (3) into system (1) leads to the following closed-loop system

$$\begin{aligned} \bar{Z}[k+1] &= [I_N \otimes (\bar{A} + \bar{B}K_1) - a[k]\mathcal{L} \otimes (\bar{B}K_2)]\bar{Z}[k] \\ &\quad + a[k][\Sigma \otimes (\bar{B}K_2 T_z^{-1})]N[k], \end{aligned} \tag{4}$$

where $\mathcal{L}$ is the Laplacian matrix of $\mathcal{G}$, $\bar{Z}[k] = (\bar{z}_1^T[k], \ldots, \bar{z}_N^T[k])^T \in \mathbb{R}^{2N}$,
$\Sigma = \mathrm{diag}(\tilde{a}_1, \ldots, \tilde{a}_N) \in \mathbb{R}^{N\times N^2}$, $\tilde{a}_i = (a_{i1}, \ldots, a_{iN}) \in \mathbb{R}^{1\times N}$,
$N[k] = (n_{x11}[k], n_{v11}[k], \ldots, n_{x1N}[k], n_{v1N}[k], \ldots, n_{xNN}[k], n_{vNN}[k])^T \in \mathbb{R}^{2N^2}$.

For system (1) we adopt the following assumptions

(A1) $\mathcal{G}$ is weighted balanced and has a spanning tree.

(A2) $\mathcal{G}$ is structurally balanced.

(A3) $\sum_{k=0}^{\infty} a[k] = \infty$. (A4) $\sum_{k=0}^{\infty} a^2[k] < \infty$.

If assumption (A2) holds, then $\mathcal{V}$ can be divided into two subsets $\mathcal{V}_1$ and $\mathcal{V}_2$, for simplification, we assume $\mathcal{V}_1 = \{v_1, v_2, \ldots, v_{K_0}\}$, $\mathcal{V}_2 = \{v_{K_0+1}, \ldots, v_N\}$, and if assumption (A1) holds, $\mathcal{L}$ has the following eigenvector ω of eigenvalue 0 [13] $\omega = [\underbrace{1, \ldots, 1}_{K_0}, -1, \ldots, -1]^T$.

Consider the homogeneous equation

$$\bar{Z}[k+1] = [I_N \otimes (\bar{A} + \bar{B}K_1) - a[k]\mathcal{L} \otimes (\bar{B}K_2)]\bar{Z}[k]. \tag{5}$$

Define $\Phi(k, k_0) = \prod_{i=k_0}^{k-1} [I_N \otimes (\bar{A} + \bar{B}K_1) - a[k]\mathcal{L} \otimes (\bar{B}K_2)]$ ($\Phi(k_0, k_0) = I_{2N}$), which represents the state transition matrix of (5) and $Q_N = \frac{1}{N}\omega\omega^T \otimes \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, and define $F[k] = I_N \otimes (\bar{A} + \bar{B}K_1) - a[k]\mathcal{L} \otimes (\bar{B}K_2) - Q_N$.

Then we can find it easily that $Q_N \times Q_N = Q_N \times Q_N^T = Q_N^T \times Q_N = \Phi(k, k_0) \times Q_N = Q_N \times \Phi(k, k_0) = Q_N$, it can be calculated that

$$\begin{aligned}
\|F[k]\|_2^2 = {} & \|[I_N \otimes (\bar{A} + \bar{B}K_1) - a[k]\mathcal{L} \otimes (\bar{B}K_2) - Q_N][I_N \otimes (\bar{A} + \bar{B}K_1) \\
& - a[k]\mathcal{L} \otimes (\bar{B}K_2) - Q_N]^T\|_2 \\
\le {} & \|I_N \otimes (\bar{A} + \bar{B}K_1)(\bar{A} + \bar{B}K_1)^T - a[k]\mathcal{L}^T \otimes (\bar{A} + \bar{B}K_1)(\bar{B}K_2)^T \\
& - a[k]\mathcal{L} \otimes (\bar{B}K_2)(\bar{A} + \bar{B}K_1)^T - Q_N\|_2 + a^2[k]\|\mathcal{L}\mathcal{L}^T \otimes (\bar{B}K_2)(\bar{B}K_2)^T\|_2.
\end{aligned}$$

Because $(\bar{A} + \bar{B}K_1)(\bar{B}K_2)^T = (\bar{B}K_2)(\bar{A} + \bar{B}K_1)^T = \begin{bmatrix} \frac{1}{4} & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix} \triangleq C$, then

$$\|F[k]\|_2^2 \le \|I_N \otimes \Gamma_A - a[k](\mathcal{L} + \mathcal{L}^T) \otimes C - Q_N\|_2 + a^2[k]\|\mathcal{L}\mathcal{L}^T \otimes (\bar{B}K_2)(\bar{B}K_2)^T\|_2,$$

where $\Gamma_A = (\bar{A} + \bar{B}K_1)(\bar{A} + \bar{B}K_1)^T$.

If assumptions (A1) and (A2) hold, according to Lemma 1, there exists a matrix $T \in \mathbb{C}^{N\times N}$, whose first column is $\frac{1}{\sqrt{N}}\omega$ such that $T^{-1}(\mathcal{L} + \mathcal{L}^T)T = \mathrm{diag}(0, \lambda_2, \ldots, \lambda_N)$, where $0 < \lambda_2 \le \cdots \le \lambda_N$ is the eigenvalue of $(\mathcal{L} + \mathcal{L}^T)$. Then $F[k]$ can be calculated by (6).

$$\begin{aligned}
& I_N \otimes \Gamma_A - a[k](\mathcal{L} + \mathcal{L}^T) \otimes C - Q_N = (T^{-1} \otimes I_2)\times \\
& \mathrm{diag}\,(\Gamma_A - \Gamma_1, \Gamma_A - a[k]\lambda_2 C, \ldots, \Gamma_A - a[k]\lambda_N C) \times (T \otimes I_2),
\end{aligned} \tag{6}$$

where $\Gamma_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $\|\Gamma_A - a[k]\lambda_i C\|_2 = \left\| \begin{bmatrix} \frac{1}{4} - a[k]\lambda_i & \frac{1}{2}a[k]\lambda_i \\ \frac{1}{2}a[k]\lambda_i & 1 - a[k]\lambda_i \end{bmatrix} \right\|_2$

$$= \sqrt{\tfrac{1}{4}(a[k]\lambda_i)^2 + \tfrac{17}{32}(1 - a[k]\lambda_i)^2 + \tfrac{5}{32}|1 - a[k]\lambda_i|\sqrt{(3 - 3a[k]\lambda_i)^2 + (4a[k]\lambda_i)^2}}.$$

If assumptions (A3) and (A4) hold, there exists D_1, such that for $\forall k > D_1$, $0 < a[k]\lambda_i < 1, i = 2, \ldots, N$

$$\|(\bar{A}+\bar{B}K_1)(\bar{A}+\bar{B}K_1)^T - a[k]\lambda_i C\|_2 \leq \sqrt{\frac{1}{4}(a[k]\lambda_i)^2 + (1-a[k]\lambda_i)^2} \leq 1 - a[k]\lambda_i.$$

Then $\|F[k]\|_2^2 \leq 1 - a[k]\lambda_2 + a^2[k]\|\mathcal{L}\mathcal{L}^T \otimes (\bar{B}K_2)(\bar{B}K_2)^T\|_2$.

There exists $D_2 > D_1$, such that for $\forall k > D_2$, $\frac{1}{2}a[k]\lambda_2 > a^2[k]\|\mathcal{L}\mathcal{L}^T \otimes (\bar{B}K_2)(\bar{B}K_2)^T\|_2$, it can be obtained that

$$\begin{aligned}\|F[k]\|_2 &\leq \sqrt{1 - a[k]\lambda_2 + a^2[k]\|\mathcal{L}\mathcal{L}^T \otimes (\bar{B}K_2)(\bar{B}K_2)^T\|_2} \\ &\leq \sqrt{1 - \frac{1}{2}a[k]\lambda_2} \leq 1 - \frac{1}{4}a[k]\lambda_2. \\ \|\prod_{i=k_0}^{k} F[i]\|_2 &\leq \prod_{i=k_0}^{k}\|F[i]\|_2 \leq \prod_{i=k_0}^{D_2-1}\|F[i]\|_2 \prod_{i=D_2}^{k}(1 - \frac{1}{4}a[k]\lambda_2) \\ &\leq \prod_{i=k_0}^{D_2-1}\|F[i]\|_2 \mathrm{e}^{(-\frac{1}{4}\lambda_2\sum_{i=D_2}^{k}a[i])}. \end{aligned} \tag{7}$$

If assumption (A3) holds,

$$\lim_{k\to\infty}\|\prod_{i=k_0}^{k}F[i]\|_2 = 0, \qquad \lim_{k\to\infty}\Phi(k,k_0) = Q_N. \tag{8}$$

The stochastic difference equation (4) can be solved as follows

$$\bar{Z}[k] = \Phi(k,0)\bar{Z}[0] + \sum_{i=0}^{k-1} a[i]\Phi(k,i+1)\Sigma \otimes (\bar{B}K_2T_z^{-1})N[i]. \tag{9}$$

If assumption (A4) holds, there exists N_1, such that for $\forall\varepsilon > N_1$, $\sum_{k=N_1}^{\infty} a^2[k] < \varepsilon$. Define $Q[k] = \sum_{i=0}^{k-1} a[i]\Phi(k,i+1)\Sigma \otimes (\bar{B}K_2T_z^{-1})N[i]$.

According to (7) and (8), there exists $N_2 > N_1$, such that for $\forall m > n > N_2$, $\|\Phi(m,s) - \Phi(n,s)\|_F < \varepsilon \quad (s = 0,\ldots,N_1)$, and there exists $\Phi_{max} > 0$, such that for $\forall n_1, n_2 \in \mathbb{N}$, $\|\Phi(n_1,n_2)\|_F < \Phi_{max}$, by using the similar method in [7].

$$\begin{aligned}&\|E(Q[m]-Q[n])E(Q[m]-Q[n])^T\|_F \leq 2\|(\sum_{i=n}^{m-1} a[i]\Phi(m,i+1)\Sigma \otimes (\bar{B}K_2T_z^{-1})N[i]) \\ &(\sum_{i=n}^{m-1} a[i]\Phi(m,i+1)\Sigma \otimes (\bar{B}K_2T_z^{-1})N[i])^T + 2(\sum_{i=0}^{n-1} a[i][\Phi(m,i+1) - \Phi(n,i+1)] \\ &\Sigma \otimes (\bar{B}K_2T_z^{-1})N[i])(\sum_{i=0}^{n-1} a[i][\Phi(m,i+1) - \Phi(n,i+1)]\Sigma \otimes (\bar{B}K_2T_z^{-1})N[i])^T\|_F \\ &= 2\|\sum_{i=n}^{m-1} a^2[i]\Phi(m,i+1)\Sigma \otimes (\bar{B}K_2T_z^{-1})N[i]N^T[i]\Sigma^T \otimes (\bar{B}K_2T_z^{-1})^T\Phi(m,i+1)^T\|_F\end{aligned}$$

$$+ 2\|\sum_{i=0}^{n-1} a^2[i]\Phi_{cmn}\Sigma \otimes (\bar{B}K_2T_z^{-1})N[i]N[i]^T\Sigma^T \otimes (\bar{B}K_2T_z^{-1})^T\Phi_{cmn}^T\|_F$$

$$\leq 2\varepsilon\alpha_{max}^2 N^4\sigma_{max}\|\bar{B}K_2T_z^{-1}\|_F\ (1 + 4\Phi_{max}^2 + \varepsilon \sum_{i=0}^{N_1-1} a^2[i]),$$

where $\alpha_{max} = \max_{i,j=1,\ldots,N,k\in\mathbb{N}}\{|a_{ij}[k]|\}, \quad \Phi_{cmn} = \Phi(m, i+1) - \Phi(n, i+1)$.

Then $Q[k]$ converges in mean square to a random vector Q^*, such that

$$E(Q^*Q^{*T}) = \lim_{k\to\infty} E(Q[k]Q^T[k]) = \lim_{k\to\infty}\sum_{i=0}^{k-1} a^2[k]\Phi(k, i+1)\Sigma \otimes (\bar{B}K_2T_z^{-1})\Xi$$
$$\Sigma^T \otimes (\bar{B}K_2T_z^{-1})^T\Phi^T(k, i+1).$$

If assumption (A4) holds, there exists N_3, such that for $\forall\varepsilon > 0, k > N_3$

$$\sum_{i=N_3}^{k-1} a^2[i]\|\Phi(k, i+1)\Sigma \otimes (\bar{B}K_2T_z^{-1})\Xi\ \Sigma^T \otimes (\bar{B}K_2T_z^{-1})^T\Phi^T(k, i+1)\|_F \leq \varepsilon,$$
$$\sum_{i=N_3}^{\infty} a^2[i]\|Q_N\Sigma \otimes (\bar{B}K_2T_z^{-1})\Xi\ \Sigma^T \otimes (\bar{B}K_2T_z^{-1})^T Q_N^T\|_F \leq \varepsilon,$$

and there exists $N_4 > N_3$, such that for $\forall k > N_4$

$$\sum_{i=0}^{N_3-1} a^2[i]\|\Phi(k, i+1)\Sigma \otimes (\bar{B}K_2T_z^{-1})\Xi\ \Sigma^T \otimes (\bar{B}K_2T_z^{-1})^T\Phi^T(k, i+1)$$
$$- Q_N\Sigma \otimes (\bar{B}K_2T_z^{-1})\Xi\ \Sigma^T \otimes (\bar{B}K_2T_z^{-1})^T Q_N^T\|_F \leq \varepsilon.$$

Hence it can be found that

$$\lim_{k\to\infty}\sum_{i=0}^{k-1} a^2[k]\Phi(k, i+1)\Sigma \otimes (\bar{B}K_2T_z^{-1})\Xi\ \Sigma^T \otimes (\bar{B}K_2T_z^{-1})^T\Phi^T(k, i+1)$$
$$= \sum_{i=0}^{\infty} a^2[i]Q_N\Sigma \otimes (\bar{B}K_2T_z^{-1})\Xi\ \Sigma^T \otimes (\bar{B}K_2T_z^{-1})^T Q_N^T = \omega\omega^T \otimes \begin{bmatrix} 0 & 0 \\ 0 & \Omega \end{bmatrix},$$

where $\Omega = \frac{1}{N^2}\sum_{k=0}^{\infty} a^2[k]\left(\sum_{i=0}^{N}\sum_{j\in\mathcal{N}_i} a_{ij}^2[k]\left(\frac{E(n_{xij}^2[k])}{T^4} + \frac{E(n_{vij}^2[k])}{36T^2}\right)\right)$.

By using the similar method in [7], it can be proved $D(\bar{z}_{xi}^*) = 0,\ D(\bar{z}_{vi}^*) = \Omega, Cov(\bar{z}_{xi}^*, \bar{z}_{vi}^*) = 0$. For $\forall v_i, v_j \in \mathcal{V}_l\ (l \in \{1, 2\}),\ Cov(\bar{z}_{vi}^*, \bar{z}_{vj}^*) = \Omega,\ \rho(\bar{z}_{vi}^*, \bar{z}_{vj}^*) = 1$. It can be calculated that

$$\lim_{k\to\infty} E(\bar{z}_{vi}[k] - \bar{z}_{vj}^*)^2 = \lim_{k\to\infty} E(\bar{z}_{vi}[k] - \bar{z}_{vi}^* + \bar{z}_{vi}^* - \bar{z}_{vj}^*)^2$$
$$\leq \lim_{k\to\infty} E(\bar{z}_{vi}[k] - \bar{z}_{vi}^*)^2 + E(\bar{z}_{vi}^* - \bar{z}_{vj}^*)^2 + \lim_{k\to\infty} 2(E(\bar{z}_{vi}[k] - \bar{z}_{vi}^*)^2)^{\frac{1}{2}}(E(\bar{z}_{vi}^* - \bar{z}_{vj}^*)^2)^{\frac{1}{2}}$$
$$= E(\bar{z}_{vi}^*)^2 + E(\bar{z}_{vj}^*)^2 - 2E(\bar{z}_{vi}^*\bar{z}_{vj}^*) = \Omega + \Omega - 2\Omega = 0.$$

By the same process, $\lim_{k\to\infty} E(\bar{z}_{xi}[k] - \bar{z}^*_{xj})^2 = 0$.

Similarly for $\forall v_i \in \mathcal{V}_l,\ v_j \in \mathcal{V}_q,\ l \neq q\ (l, q \in \{1, 2\}),\ Cov(\bar{z}^*_{vi}, \bar{z}^*_{vj}) = -\Omega$, $\rho(\bar{z}^*_{vi}, \bar{z}^*_{vj}) = -1$,

$$\lim_{k\to\infty} E(\bar{z}_{vi}[k] + \bar{z}^*_{vj})^2 = 0, \quad \lim_{k\to\infty} E(\bar{z}_{xi}[k] + \bar{z}^*_{xj})^2 = 0.$$

Therefore for $\forall v_i \in \mathcal{V}_1,\ \bar{z}_{xi}[k]$ and $\bar{z}_{vi}[k]$ converge in mean square to the random vector $\bar{z}^*_x$ and $\bar{z}^*_v$, while for $\forall v_i \in \mathcal{V}_2,\ \bar{z}_{xi}[k]$ and $\bar{z}_{vi}[k]$ converge in mean square to the random vector $-\bar{z}^*_x$ and $-\bar{z}^*_v$.

$$\begin{aligned}
E(z^*_x) &= E(z^*_x)^2 = 0,\\
E(z^*_v) &= \frac{1}{N}\left(\sum_{i=1}^{K_0}(\frac{x_i[0]}{T^2} + \frac{v_i[0]}{6T}) - \sum_{i=K_0}^{N}(\frac{x_i[0]}{T^2} + \frac{v_i[0]}{6T})\right),\\
E(z^*_v)^2 &= \Omega + \frac{1}{N^2}\left(\sum_{i=1}^{K_0}(\frac{x_i[0]}{T^2} + \frac{v_i[0]}{6T}) - \sum_{i=K_0}^{N}(\frac{x_i[0]}{T^2} + \frac{v_i[0]}{6T})\right)^2.
\end{aligned}$$

$$\begin{aligned}
&\lim_{k\to\infty} Z[k] = (I_N \otimes T_z)Q_N(I_N \otimes T_z^{-1})Z[0] = \frac{1}{N}\omega\omega^T \otimes \begin{bmatrix} 1 & \frac{T}{6} \\ 0 & 0 \end{bmatrix} Z[0],\\
&\lim_{k\to\infty} E(x_i[k] - x^*)^2 = 0,\ \lim_{k\to\infty} E(v_i[k] - v^*)^2 = 0,\ v_i \in \mathcal{V}_1,\\
&\lim_{k\to\infty} E(x_i[k] + x^*)^2 = 0,\ \lim_{k\to\infty} E(v_i[k] + v^*)^2 = 0,\ v_i \in \mathcal{V}_2.
\end{aligned}$$

$$E(x^*) = \frac{1}{N}\left(\sum_{i=1}^{K_0}(x_i[0] + \frac{T}{6}v_i[0]) - \sum_{i=K_0}^{N}(x_i[0] + \frac{T}{6}v_i[0])\right), \quad E(v^*) = 0.$$

The above analysis gives the states of agents at the sampling instant. Using the same analysis method proposed in [8], it can be proved

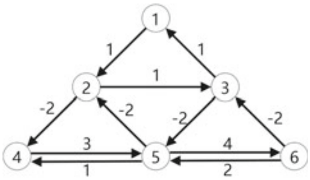
$$\lim_{t\to\infty} E(x_i(t) - x^*)^2 = 0, \quad \lim_{t\to\infty} E(v_i(t) - v^*)^2 = 0,\ v_i \in \mathcal{V}_1,$$

$$\lim_{t\to\infty} E(x_i(t) + x^*)^2 = 0, \quad \lim_{t\to\infty} E(v_i(t) + v^*)^2 = 0,\ v_i \in \mathcal{V}_2.$$

Hence the protocol (3) can solve the mean square bipartite consensus problem of double-integrator multi-agent systems (1) with measurement noises.

4 Simulation Examples

Example 1 Consider a double-integrator MAS with six agents. The communication topology $\mathcal{G}$ is shown in Fig. 1. Obviously, $\mathcal{G}$ is weighted balanced, structurally bal-



agent	1	2	3	4	5	6
$x_i[0]$	9.439	7.241	2.004	1.888	-0.158	-6.227
$v_i[0]$	5.235	-1.571	5.834	-3.289	-3.204	0.133

Fig. 1 Topology and initial states of multi-agent system in Example 1

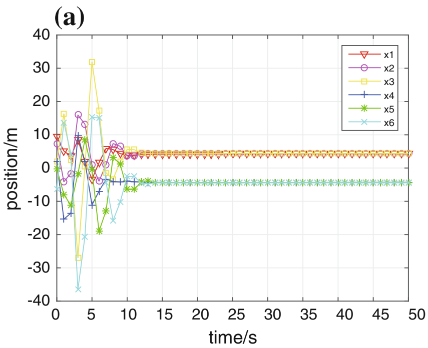


Fig. 2 Evolutions of agents in Example 1. **a** Positions; **b** velocities

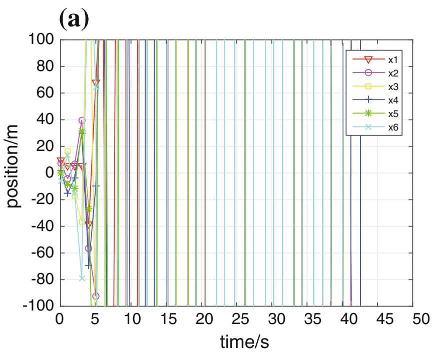


Fig. 3 Evolutions of agents in Example 2. **a** Positions; **b** velocities

anced and has a spanning tree. The sampling period is set to 1. With the effect of measurement noises, we take the time-varying consensus gain $a[k] = 1/(k+1),\ k \in \mathbb{N}$ in protocol (3) and $a[k]$ satisfies the stochastic approximation conditions. $E(x^*) = 4.304,\ E(x^*)^2 < \infty,\quad E(v^*) = E|v^*|^2 = 0$ (Fig. 2).

Example 2 If the stochastic approximation conditions are not satisfied, it should be verified whether agents can achieve bipartite consensus. Therefore we take a constant gain $a[k] = 1$, The evolutions of positions/velocities of agents are shown in Fig. 3. It is shown that assumptions (A3) and (A4) are necessary. Limited by space, we have not given proof of necessity.

5 Conclusion

In this paper, a mean square bipartite consensus protocol is proposed for double-integrator multi-agent systems with measurement noises under signed digraph. In order to counteract the noise effect and achieve bipartite consensus a time-varying consensus gain and agents' states feedback are adopted in the design of the bipartite consensus protocol. The sufficient conditions for achieving the mean square bipartite consensus is proved. In the further work, the proof of necessity of the stochastic approximation conditions will be given.

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