Conic Fitting: New Easy Geometric Method and Revisiting Sampson Distance

Yihong Wu[1], Haoren Wang[1,2], and Fulin Tang[1,2]

[1]National Laboratory of Pattern Recognition, Institute of Automation, Chinese Academy of Sciences
[2]University of Chinese Academy of Sciences
{yhwu, haoren.wang, fulin.tang}@nlpr.ia.ac.cn

Abstract

Fitting conic from images is a preliminary step for its plentiful applications. It's a common sense that geometric distance based fitting methods are better than algebraic distance based ones. However, for a long time, there has not been a geometric distance between a data point and a general conic that allows easy computation and achieves high accuracy simultaneously. In this paper, we derive a new geometric distance between a data point and a conic by revisiting Sampson distance. The new geometric distance is accurate and simultaneously still explicit analytical representation, which is greatly easy to be implemented. Then, based on the distance, a new cost function with combining Sampson distance is constructed. The conic fitting optimization by minimizing this cost function has all the merits of the geometric distance based methods and simultaneously avoids their limitations.

1. Introduction

Conics abound in manmade objects and natural scenes. Conic fitting from images is a preliminary step for their plentiful applications in robot vision, industrial measurement, computer graphics *et al*. A specific conic in the scene is still imaged as a conic on the perspective projection but the imaged conic might be a circle, an ellipse, a hyperbola, a parabola, and even degenerate lines. If there is no prior knowledge of the scene or before recognizing a conic, the conic type in the image is not available. It follows that it is necessary to study the fitting problem for a general conic. In this paper, we study fitting a general conic without knowing its type.

One natural simple way is to use linear least square [4, 15]. But, the accuracy is not so high to meet practical tasks. Therefore, further optimizations based on some error distances are needed. Not so strictly, there are three kinds of these methods for a general conic: the statistical distance based methods, the algebraic distance based methods, and the geometric distance based methods.

The typical works of the statistical distance based methods are [5–8, 11, 14, 15]. Kanatani [5, 6] proposed a renormalization method from '*Statistical*' distance to make bias-corrected, which was improved by Zhang [15] and Wang *et al*. [14] later. The idea is very reasonable of introducing a statistical model of noise in terms of the covariance matrix to compute an unbiased estimate by adjusting to noise. Shklyar *et al*. [11] considered adjusted least squares estimators for conic fitting which are shown to be similarity invariant. Kanatani [7] gave the accuracy analyses of various techniques for conic fitting and a '*hyperaccuracy*' method of subtracting an estimated bias term from maximum likelihood solution. Furthermore, by modifying the previous renormalization works, Kanatani *et al*. derived a new scheme called hyper-renormalization which has zero bias up to high order error terms [8].

The second is the algebraic distance based methods [2, 4, 15]. The algebraic distance is minimized to optimize the conic parameters with different constraints. Bookstein [2] proposed a quadric constraint and then solved a generalized eigensystem to obtain fitting result by block decomposition. Fitzgibbon and Fisher [4] analyzed the complexity of the algebraic distance minimizations with two different constraints. Zhang [15] detailed the minimization of the algebraic distance with three different constraints.

The third is the geometric distance based methods [1, 4, 9, 10, 12, 13, 15]. The first geometric distance is the orthogonal distance from points to the conic section proposed by Nakagawa and Rosenfeld [9]. To obtain the distance of each point, a quartic equation needs to be solved. Zhang [15] derived the solution of the geometric distance as the quartic equation. A good approximation to the orthogonal distance is the Sampson distance [10] that are the weighted algebraic distance by one order differentials. Taubin [13] independently derived the approximate distance. Sturm and Gar-

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gallo [12] parameterized the fitted conic by a homography and a circle, and then proposed a new geometric distance of a data point to the conic by the parameterization.

The first kind statistical methods require noise to be Gaussian and to concentrate on a small region occurring in the tangent space. Also, the methods need to compute covariance matrix. They would become unstable when the noise level increases. The algebraic distance based methods are easy to compute. However, the performance gap between algebraic fitting and geometric fitting is wide [1,3] and the accuracy of the algebraic distance based methods is not satisfactory. It is a common sense that minimizing geometric distance is better than minimizing algebraic distance [12, 15]. Thus, we also study conic fitting as the geometric distance based methods. At present, each of the geometric distance based methods has its own limitation. The Sampson distance based method is easy to be implemented but this distance is only an approximation that cannot give high accuracy. The orthogonal distance based method can give high fitting accuracy but for each point to be fitted, a quartic equation needs to be solved. Although closed form solutions exist for the quartic equation, condition determinations are needed and numerical instability can result from the application of the analytic formula as pointed out by Fitzgibbon and Fisher [4]. Also, usually there are too many points extracted in the image and thus the complexity is not low. The homography-circle parameterized method can give high accuracy but there are many parameters to be optimized and quartic equations still need to be solved when giving the initial values.

As stated above, there has not been a geometric distance based method that allows easy computation and simultaneously for high accuracy. In this paper, we give a novel geometric distance based method, which has all the merits of the geometric distance based methods and simultaneously avoids their limitations. The contributions are: 1) The Sampson distance is revisited and its geometric meaning is exhibited. 2) A new geometric distance between a point and conic is given by analyzing the Sampson distance, which is more accurate than Sampson distance. 3) The given geometric distance is represented explicitly and analytically. During optimization, neither condition determinations nor solving equations are required. Moreover, the distance is on only five independent algebraic parameters and no any other parameters are needed. It follows that this geometric distance based fitting is greatly easy to be implemented. 4) A new constructed cost function with combining Sampson distance is constructed. The conic fitting optimization by minimizing this cost function is robust and can achieve high accuracy.

The structure of the paper is organized as follows. Section 2 provides some preliminaries. Section 3 revisits Sampson distance and gives its explicit geometric interpretation. Section 4 derives the new geometric distance. The new cost function and conic fitting algorithm are reported in Section 5. Section 6 shows experimental results and Section 7 concludes the paper.

2. Preliminaries

A bold letter denotes a vector or a matrix.

The locus of planar points with homogeneous coordinates $(x, y, w)^T$ that satisfies the equation

$$ax^2 + 2dxy + by^2 + 2exw + 2fyw + cw^2 = 0 \quad (1)$$

is a conic. We denote $\begin{pmatrix} a & d & e \\ d & b & f \\ e & f & c \end{pmatrix}$ as $\mathbf{C}$ that can represent this conic. In this paper, we study fitting a general conic without considering its type.

Given a point $\mathbf{m}$ in 2D homogeneous coordinates, let $\mathbf{Cm}$ represent a line (i.e. $\mathbf{Cm}$ is as line coordinates). Then $\mathbf{m}$ and $\mathbf{Cm}$ are of polarity relationship related to $\mathbf{C}$. The relationship is invariant under a projective transformation. $\mathbf{Cm}$ is called the polar of $\mathbf{m}$ related to $\mathbf{C}$.

3. Interpreting the Sampson distance

Let $\mathbf{m}_i = (u_i, v_i, 1)^T$, $i = 1, \cdots, N$, be the measured points in the image from a conic $\mathbf{C}$. Without noise, there are $\mathbf{m}_i^T \mathbf{C} \mathbf{m}_i = 0$, $i = 1, \cdots, N$. The polar line of a single point $\mathbf{m}$ related to $\mathbf{C}$ is $\mathbf{Cm}$. Denoted $\mathbf{Cm} = \mathbf{l} = (l_1, l_2, l_3)^T$, the distance between $\mathbf{m}$ and $\mathbf{l}$ is:

$$d(\mathbf{m}, \mathbf{l}) = \frac{|\mathbf{m}^T \mathbf{l}|}{\sqrt{l_1^2 + l_2^2}}, \quad (2)$$

where $|\bullet|$ denotes absolute value of the element included in the two bars. Substitute $\mathbf{l} = \mathbf{Cm}$ into Eq. (2), the result is:

$$d(\mathbf{m}, \mathbf{l}) = \frac{|\mathbf{m}^T \mathbf{Cm}|}{\sqrt{\mathbf{m}^T \mathbf{Gm}}}, \quad (3)$$

where $\mathbf{G} = \begin{pmatrix} a^2 + d^2, & ad + bd, & ae + df \\ ad + bd, & b^2 + d^2, & de + bf \\ ae + df, & de + bf, & e^2 + f^2 \end{pmatrix}$. The sequential principal minors of $\mathbf{G}$ are

$$a^2 + d^2 \geq 0, \ (ba - d^2)^2, \ \det(\mathbf{G}) = 0 \, .$$

$\mathbf{m}^T \mathbf{Gm}$ usually are not always zero and the reason is as follows. The denominator is zero if and only if both $l_1 = 0$ and $l_2 = 0$ (cf. Eq. (2)). This case occurs when the polar line $\mathbf{l} = \mathbf{Cm}$ is at the infinity. At the time, $\mathbf{m}$ is the center of a centric conic or at infinity for a parabola. This does not usually appear because the extracted points $\mathbf{m}$ with large noise are removed by RANSAC (RANdom SAmple Consensus) before applying a fitting method.

The well known Sampson distance for fitting a conic is:

$$d_{Sam} = \frac{|\mathbf{m}^T\mathbf{C}\mathbf{m}|}{\sqrt{\left(\frac{\partial(\mathbf{m}^T\mathbf{C}\mathbf{m})}{\partial u}\right)^2 + \left(\frac{\partial(\mathbf{m}^T\mathbf{C}\mathbf{m})}{\partial v}\right)^2}}\,. \tag{4}$$

Substituting the differential results into Eq. (3) gives:

$$d_{Sam} = \frac{|\mathbf{m}^T\mathbf{C}\mathbf{m}|}{2\sqrt{\mathbf{m}^T\mathbf{G}\mathbf{m}}}\,. \tag{5}$$

By comparing Eq. (3) with Eq. (5), we know the Sampson distance is just half of the distance of $\mathbf{m}$ to its polar line $\mathbf{l}$. The numerator of them are the usually algebraic distance and thus the Sampson distance is a weighted algebraic distance.

Taking the square sum of Eq. (3) with different points $\mathbf{m}_i$ or taking the square sum of Eq. (5) with different points $\mathbf{m}_i$ yields the cost functions for fitting the conic $\mathbf{C}$. The two cost functions are different by a fixed scalar 4 and therefore they are equivalent by optimizing to find the minimal $\mathbf{C}$.

4. New geometric distance between a point and a conic

As shown above, the Sampson distance between a point and a conic is a kind of geometric distance much simpler than the orthogonal distance. However, the Sampson distance is only a kind of approximation. Here, we derive a new geometric distance between a point and a conic. The distance is not only accurate but also explicitly analytical. It has both advantages of Sampson distance and the orthogonal distance.

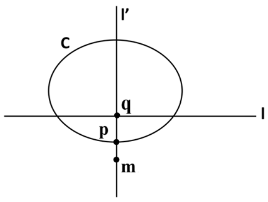


Figure 1: A geometric distance: $d\,(\mathbf{m},\,\mathbf{p})$ of $\mathbf{m}$ to $\mathbf{C}$.

As shown in Fig. 1, $\mathbf{C}$ is a conic, $\mathbf{m}$ is a point, $\mathbf{l} = \mathbf{C}\mathbf{m}$ is the polar line. Denote the line passing through $\mathbf{m}$ and orthogonal to $\mathbf{l}$ as $\mathbf{l}'$ and the intersection point of $\mathbf{l}$ with $\mathbf{l}'$ as $\mathbf{q}$. $\mathbf{l}'$ intersects $\mathbf{C}$ at two points denoted as $\mathbf{p}_+$ and $\mathbf{p}_-$. We propose a new geometric distance between $\mathbf{m}$ and $\mathbf{C}$ as:

$$d\,(\mathbf{m},\,\mathbf{C}) = \min\left\{\,d\,(\mathbf{p}_+,\,\mathbf{m})\,,\,d\,(\mathbf{p}_-,\,\mathbf{m})\,\right\}. \tag{6}$$

Let $\mathbf{p} = \arg\min_{\mathbf{p}_+,\mathbf{p}_-}\left\{\,d\,(\mathbf{p}_+,\,\mathbf{m})\,,\,d\,(\mathbf{p}_-,\,\mathbf{m})\,\right\}$. So $d\,(\mathbf{m},\,\mathbf{C}) = d\,(\mathbf{m},\,\mathbf{p})$. Clearly shown in Fig. 1, $d\,(\mathbf{m},\,\mathbf{p})$ is more accurate than $d\,(\mathbf{m},\,\mathbf{q})$ to measure the distance between $\mathbf{m}$ and $\mathbf{C}$. In order to compute $d(\mathbf{m},\mathbf{C})$, one way of obtaining $\mathbf{p}$ is to solve the following system:

$$\begin{cases} \mathbf{p}^T\mathbf{C}\mathbf{p} = 0, \\ \mathbf{p}^T\mathbf{l}' = 0, \\ \mathbf{p} = \arg\min\limits_{\mathbf{p}_+,\mathbf{p}_-}\left\{\,d\,(\mathbf{p}_+,\,\mathbf{m})\,,\,d\,(\mathbf{p}_-,\,\mathbf{m})\,\right\}. \end{cases} \tag{7}$$

Directly solving Eq. (7) is not simple, in which it is needed to determine the corresponding different solution representations according to different symbol sign assumptions. Also, the solution representations are long and complex. This process is not indeed so much easy than solving 4th order polynomials for the orthogonal distance between a point and a conic. However in the following, we can give a technique to obtain an explicitly analytical representation of Eq. (7) in a very concise way.

At first, we compute $\mathbf{q}$. $\mathbf{l} = \mathbf{C}\mathbf{m}$ is also denoted $\mathbf{l} = (l_1,\,l_2,\,l_3)^T$. $\mathbf{l}'$ is orthogonal to $\mathbf{l}$ and passes through $\mathbf{m} = (u,\,v,\,1)^T$. Thus, $\mathbf{l}' = (-l_2,\,l_1,\,l_2u - l_1v)^T$. $\mathbf{q}$ is the intersection of $\mathbf{l}$ with $\mathbf{l}'$. So it is:

$$\mathbf{l} \times \mathbf{l}' = \left(l_2^2u - l_1l_2v - l_1l_3,\; l_1^2v - l_1l_2u - l_2l_3,\; l_1^2 + l_2^2\right)^T.$$

Dehomogenizing $\mathbf{q}$ gives

$$\begin{aligned} \mathbf{q} &= \left(\frac{l_2^2u - l_1l_2v - l_1l_3}{l_1^2 + l_2^2},\; \frac{l_1^2v - l_1l_2u - l_2l_3}{l_1^2 + l_2^2},\; 1\right)^T \\ &= \mathbf{m} - \frac{\mathbf{m}^T\mathbf{C}\mathbf{m}}{\mathbf{m}^T\mathbf{G}\mathbf{m}}\bar{\mathbf{C}}\mathbf{m}\,, \end{aligned} \tag{8}$$

where $\bar{\mathbf{C}} = \begin{pmatrix} a & d & e \\ d & b & f \\ 0 & 0 & 0 \end{pmatrix}$, that is the matrix of $\mathbf{C}$ by substituting 0 for its last row.

Notice that $\mathbf{q}$ is on $\mathbf{l}$, then $\mathbf{q}^T\mathbf{l} = 0$. Since $\mathbf{l} = \mathbf{C}\mathbf{m}$, we have:

$$\mathbf{q}^T\mathbf{C}\mathbf{m} = 0. \tag{9}$$

The two solutions $\mathbf{p}_\pm$ are collinear with $\mathbf{q}$ and $\mathbf{m}$. So they can be expressed as:

$$\mathbf{p}_\pm = \lambda_1\mathbf{q} + \lambda_2\mathbf{m}, \tag{10}$$

by homogeneous coordinates, where λ_1, λ_2 are two scalars. Substituting Eq. (10) into the first equation of Eq. (7) and using Eq. (9), we obtain:

$$\left(\mathbf{q}^T\mathbf{C}\mathbf{q}\right)\lambda_1^2 + \left(\mathbf{m}^T\mathbf{C}\mathbf{m}\right)\lambda_2^2 = 0. \tag{11}$$

Usually $\mathbf{q}$ and $\mathbf{m}$ are on the different sides of $\mathbf{C}$. So, $\frac{\mathbf{q}^T\mathbf{C}\mathbf{q}}{\mathbf{m}^T\mathbf{C}\mathbf{m}} < 0$. Solving Eq. (11) gives:

$$\frac{\lambda_2}{\lambda_1} = \pm\sqrt{-\frac{\mathbf{q}^T\mathbf{C}\mathbf{q}}{\mathbf{m}^T\mathbf{C}\mathbf{m}}}. \tag{12}$$

$\frac{\mathbf{q}^T\mathbf{C}\mathbf{q}}{\mathbf{m}^T\mathbf{C}\mathbf{m}} > 0$ are called the degenerate cases which appear seldom. This does not affect the cost function construction (cf. Section 5)

Then remembering the last element of $\mathbf{m} = (u, v, 1)^T$ is 1 and that of $\mathbf{q}$ in Eq. (8) is 1, according to Eq. (10), $\mathbf{p}_{\pm}$ is dehomogenized as:

$$\frac{\lambda_1 \mathbf{q} + \lambda_2 \mathbf{m}}{\lambda_1 + \lambda_2}. \tag{13}$$

The last element is 1 too. Therefore, square of the distance of $\mathbf{p}_{\pm}$ to $\mathbf{m}$ is computed as:

$$d^2(\mathbf{p}_{\pm}, \mathbf{m}) = \frac{1}{\left(1 + \frac{\lambda_2}{\lambda_1}\right)^2} d^2(\mathbf{q}, \mathbf{m}). \tag{14}$$

Substituting Eq. (12) into Eq. (13) and choosing the smaller one give:

$$\begin{aligned} d^2(\mathbf{m}, \mathbf{C}) &= \min\{d^2(\mathbf{p}_+, \mathbf{m}), d^2(\mathbf{p}_-, \mathbf{m})\} \\ &= \frac{d^2(\mathbf{q}, \mathbf{m})}{\left(1 + \sqrt{-\frac{\mathbf{q}^T\mathbf{C}\mathbf{q}}{\mathbf{m}^T\mathbf{C}\mathbf{m}}}\right)^2} \end{aligned} \tag{15}$$

Furthermore, substituting the expression $\mathbf{q}$ of Eq. (8) into Eq. (15), we obtain the final distance representation:

$$d^2(\mathbf{m}, \mathbf{C}) = \frac{(\mathbf{m}^T\mathbf{C}\mathbf{m})^2}{\left(1 + \sqrt{\frac{(\mathbf{m}^T\mathbf{G}\mathbf{m})^2 - (\mathbf{m}^T\mathbf{C}\mathbf{m})(\mathbf{m}^T\mathbf{W}\mathbf{m})}{(\mathbf{m}^T\mathbf{G}\mathbf{m})^2}}\right)^2 (\mathbf{m}^T\mathbf{G}\mathbf{m})}, \tag{16}$$

where we use $\mathbf{C}\bar{\mathbf{C}} = (\bar{\mathbf{C}})^T\mathbf{C} = (\bar{\mathbf{C}})^T\bar{\mathbf{C}} = \mathbf{G}$ and $\mathbf{W} = (\bar{\mathbf{C}})^T\mathbf{C}\bar{\mathbf{C}}$, $\bar{\mathbf{C}}$ is as shown in Eq. (8). The denominator $\mathbf{m}^T\mathbf{G}\mathbf{m}$ cannot be zero as discussed below Eq. (3). Compared with Eq. (5), Eq. (16) is a weighted Sampson distance (or a weighted algebraic distance). The analytical presentation of $\mathbf{p}$ can also be obtained which is not given here due to the space limit.

5. Conic fitting algorithm

We now construct a cost function by using the new geometric distance derived in Section 4 and the Sampson distance. Notice that the new geometric distance requires the term in the square root to be non-negative, i.e.:

$$(\mathbf{m}_i^T\mathbf{G}\mathbf{m}_i)^2 \geq (\mathbf{m}_i^T\mathbf{C}\mathbf{m}_i)(\mathbf{m}_i^T\mathbf{W}\mathbf{m}_i). \tag{17}$$

It follows that a cost function is given as:

$$\begin{aligned} &\sum_{(\mathbf{m}_i^T\mathbf{G}\mathbf{m}_i)^2 \geq (\mathbf{m}_i^T\mathbf{C}\mathbf{m}_i)(\mathbf{m}_i^T\mathbf{W}\mathbf{m}_i)} d^2(\mathbf{m}_i, \mathbf{C}) + \\ &\frac{1}{4}\sum_{(\mathbf{m}_i^T\mathbf{G}\mathbf{m}_i)^2 < (\mathbf{m}_i^T\mathbf{C}\mathbf{m}_i)(\mathbf{m}_i^T\mathbf{W}\mathbf{m}_i)} \frac{(\mathbf{m}_i^T\mathbf{C}\mathbf{m}_i)^2}{\mathbf{m}_i^T\mathbf{G}\mathbf{m}_i} \end{aligned} \tag{18}$$

where the first summation part is from the new geometric distances Eq. (16) and the second summation part is from the Sampson distances Eq. (5). When noise is small, the second part is empty. When noise is large, there are a few points satisfying the second part condition. This cost function is consistent with the absence of noise. When noise of a point $\mathbf{m}_i$ is zero, not only both the numerators $\mathbf{m}_i^T\mathbf{C}\mathbf{m}_i = 0$ in the new geometric distance and the Sampson distance, but also the denominators of the two distances become the same. In addition, the cost function is homogenous with respect to the six parameters of $\mathbf{C}$.

Then, a conic fitting algorithm is proposed as follows.

Algorithm: Fitting image points to a conic.

Input: Extracted images points
$\mathbf{m}_i = (u_i, v_i, 1)^T, \quad i = 1, \cdots, N.$
Output: A conic $\mathbf{C}$ fitting the extracted image points.

1: Employ RANSAC to remove outliers.
2: Use linear least square method to obtain an initial estimation $\mathbf{C}_0$ from inliers. Namely, solve the linear system $\mathbf{m}_i^T\mathbf{C}\mathbf{m}_i$, $i = 1, \cdots, N$, by singular value decomposition to the coefficient matrix. The right singular vectors corresponding to the smallest singular value is the solution.
3: Minimize Eq. (18) by Levenberg-Marquardt iteration to obtain a conic $\mathbf{C}_1$ with $\mathbf{C}_0$ as the initial values. We denormalize $\mathbf{C}_1$ as $\frac{\mathbf{C}_1}{\|\mathbf{C}_1\|_F} = \mathbf{C}$ with F-norm $\|\mathbf{C}_1\|_F$.

The usual worrying problems for nonlinear optimizations are how to choose initial values and whether the iteration is convergent. Here, it is easy to obtain the initial values by linear least square method of using SVD decomposition. The obtained initial values are sufficiently reliable that always make the iterations convergent.

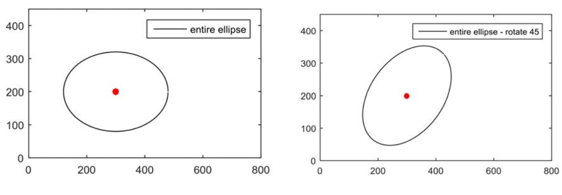
6. Experiments

Extensive simulations and experiments on real data were performed. Comparisons with typical conic fitting methods were included: Linear least square method (LLS), Sampson distance based method (Sampson) [10], H-circle method of Sturm and Gargallo (SG) [12], and orthogonal distance based method of solving 4th order equations (Orthogonal) n [15]. LLS method is the most direct one. Sampson method is the easiest one among the nonlinear geometric optimizations. SG method is the most recent geometric method for fitting a general conic. Orthogonal method minimizes the shortest distances between points and a conic that can achieve high accuracy. The results of LLS method are as the initial values and then are further optimized by Sampson method, SG method, Orthogonal method, and the proposed method in Section 5 independently on the same data. The results show that when noise levels are lower than

1% of the image sizes for hyperbolas, parabolas, and entire ellipses, all of the SG method, Orthogonal method, and the proposed method achieve nearly the same accuracies, which are better than LLS method and Sampson method at the most time. When noise levels increase to 5%, the proposed method achieves the highest accuracies among all the methods at the most time. For a section from an ellipse, the proposed method behaves all along well under all noise levels.

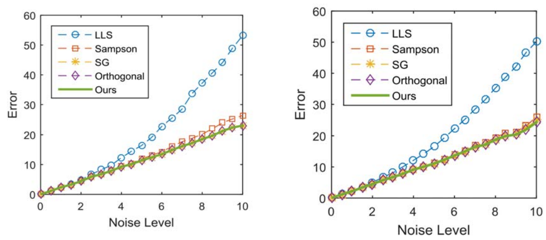
6.1. Simulations

Ellipses, hyperbolas, and parabolas are generated. Points on a conic are extracted. Then, Gaussian noise is added to each point. Under each noise level, we performed 1000 runs. Then, both of means of absolute errors and mahalanobis distance errors are computed, which have the similar results. Due to space limit, only the latter errors are shown below.



(a) (b)

Figure 2: Two used ellipses

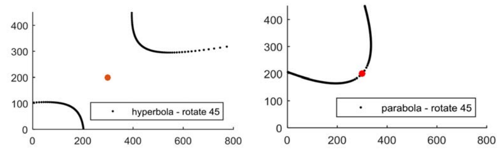


(a) (b)

Figure 3: Errors of the results from Fig. 2

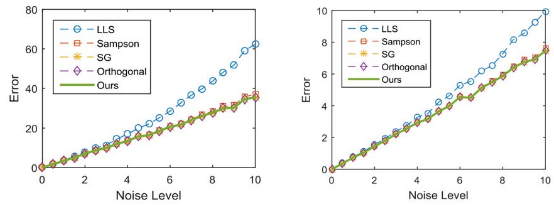
At first, Gaussian noise with 0 mean and standard deviation ranging from 0 to 10 pixels with step 0.5 pixels are added to each point on the used conic. Fig. 2 shows two used ellipses, where the right is rotated from the left by $45\,^{\circ}$. Extracted number of points are 300. The errors are shown in Fig. 3. We see that with the noise level increasing, all of SG method, Orthogonal method, and the proposed method achieve nearly the same accuracies, which are higher than LLS method or Sampson method. From the ellipses by other rotations, there are the similar results.

Other types of conics, hyperbolas and parabolas, and



(a) (b)

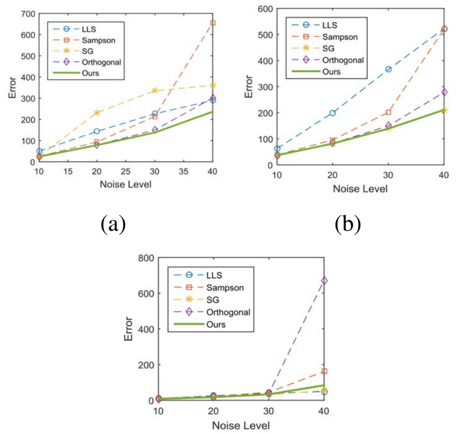
Figure 4: Two used conics



(a) (b)

Figure 5: Errors of the results from Fig. 4

their rotations are also generated. Two of them are shown in Fig. 4. Extracted number of points is 157 from Fig. 4a and is 145 from Fig. 4b. After adding Gaussian noise, the fitting results are shown in Fig. 5. We obtain the similar conclusions as before.

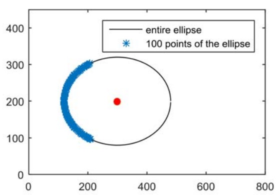


(c)

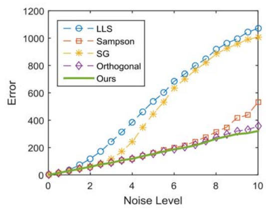
Figure 6: Errors when noise levels increase to 5%

The largest noise level added above is about 1% of the image sizes. When continually increasing the noise level to 5%, the proposed method in Section 5 achieves the highest accuracies among all the methods at the most time. The results from Fig. 2b, Fig. 4a, and Fig. 4b are shown respectively in Fig. 6.

Also, a section from an ellipse is taken as shown in Fig. 7a with blue. The results with noise shown in Fig. 7b



(a)



(b)

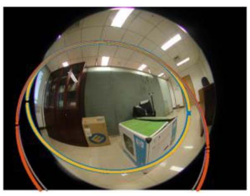
Figure 7: (a) A used section shown as the blue part on an ellipse; (b) Errors of the results from (a)

demonstrate that the proposed method in Section 5 behaves very well.

All of the simulations show that the proposed method is the most robust to noise even noise level increases to 5% of the image sizes.

Although the orthogonal distance is the shortest distance, when noise level increases, it did not achieve the best results. The reason is because solving 4-th order equations is not stable to noise. Due to the same reason, we find SG method is sensitive to the initial values. Also, we find when the number of the extracted points on an entire conic increases, SG method performs well gradually and could achieve the slightly highest accuracies when noise is small.

6.2. Experiments on real data



(a) ceiling

(b) floor

Figure 8: Fitting results of two lines of fisheye images

Images of lines under a fisheye camera are conics. Usually, we do not know the conic type. Fitting results by different methods for one line are shown in Fig. 8a as different colors. Fig. 8b shows the results for another conic. The color denotations are the same as Section 6.1. LLS and SG methods have the similar results. Sampson, Orthogonal, and the proposed methods in Section 5 have the similar results. This is consistent with the simulation results fitted from a section of a conic of Fig. 7, where Sampson method behaves better than SG method.

7. Conclusions

We give a novel geometric distance Eq. (16) between a point and a general conic. Then a new cost function Eq. (18) with combining Sampson distance is established to optimize the conic fitting. This proposed new method has merits of state of the art geometric distance based methods and simultaneously avoids their disadvantages, which allows easy computation and simultaneously achieves high accuracy and robustness. The idea can be extended to fitting other kind planar curves or higher-dimensional surfaces that are differentiable.

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