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The fourth type of covering-based rough sets

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ABSTRACT

As a technique for granular computing, rough sets deal with the vagueness and granularity in information systems. Covering-based rough sets have been proposed to generalize this theory for wider application. Three types of covering-based rough sets have been studied for different situations. To make the theory more complete, this paper proposes a fourth type of covering-based rough sets. Compared with the existing ones, the new type shows its special characteristic in the interdependency between its lower and upper approximations. We carry out a systematical study of this new theory. First, we discuss basic properties such as normality, contraction, and monotone. Then we investigate the conditions for this type of covering-based rough sets to satisfy the properties of Pawlak's rough sets and study the interdependency between the lower and upper approximation operations. In addition, axiomatic systems for the lower and upper approximation operations are established. Lastly, we address the relationships between this type of covering-based rough sets and the three existing ones.

1. Introduction

Data from a wide variety of fields are being collected and accumulated at a dramatic pace, facilitated by the availability of the Internet. Although much useful information is hidden in the accumulated voluminous data, it is very difficult to obtain. To mine knowledge from the rapidly growing volumes of digital data, researchers have proposed many methods besides classical logic, such as fuzzy set theory [44], rough set theory [21,22,24], computing with words [36,45], granular computing [1,14,15,42], computational theory for linguistic dynamic systems [37], and so on.

Rough set theory, originally proposed by Pawlak [20], provides a systematic approach for the classification of objects through an indiscernibility relation. An equivalence relation is the simplest formulization of an indiscernibility. Examples of applications of the rough set method in process control, economics, medical diagnosis, biochemistry, environmental science, biology, chemistry, psychology, conflict analysis, and other fields abound in [6,9,11–13,25–27,49].

However, the classical rough set theory cannot deal with some granularity problems in real information systems, and thus extensions including tolerance relations [5,31], similarity relations [33], coverings [3,46], and fuzzy rough sets [29,30,38,39] have been proposed. The work on generalized rough sets based on coverings is fruitful in both theory [2–4,19,28,34,35,43,46,48,50–52,55–60,62,63] and application [8]. A good case study can be found in [32].

Three types of covering-based rough set models already exist [3,4,34,60]. Zakowski extended Pawlak's rough set theory by using a covering of the domain, rather than a partition [46]. The new model is often referred to as the first type of covering-based rough sets. Bonikowski et al. studied this type of covering-based rough sets from the viewpoint of formal concepts [3]. Based on the mutual correspondence of the concepts of extension and intension, they formulated necessary

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and sufficient conditions for the existence of operations on rough sets, which are analogous to classical operations on sets. Zhu and Wang studied the redundancy issue in covering-based rough sets [50]. They also investigated the interdependency between the lower and upper approximation operations. An axiomatic system for the lower approximation operations was presented in the same work. Extensive research on this can be found in [7,10,17,18,51,55,60]. Pomykala studied the second type of covering rough set model [28]. His main method included interior and closure operators from topology. Extensive research on this topic can also be found in [4,51,56,60]. The third type of covering-based rough set model was first proposed in [34], with detailed properties of this type presented in [51]. The latter work also investigated the interdependency between the lower and upper approximation operations in this type of generalized rough sets. The difference between the third type of covering-based rough sets and Pawlak's rough sets was explored in [58]. This work also studied the conditions of coverings under which the common properties of classical rough sets hold for the third type of covering-based rough sets. Zhu and Wang studied the relationships between these three types of covering-based rough set models [51]. Tsang et al. [35] investigated the application aspect of covering-based rough sets. They studied an attribute reduct for covering generalized rough sets and presented an algorithm using a discernibility matrix to compute all the attribute reducts for covering generalized rough sets. Zhu and Wang also presented an example application of covering-based rough sets in [52]. A comprehensive study of these three types of covering-based rough sets was undertaken in [60].

This paper systematically studies the fourth type of covering-based rough set model proposed in [54,57]. The main contributions of the paper are 3-fold. First, we present the basic properties of this type of rough sets, and investigate the conditions under which these properties are also satisfied for this new type of generalized rough sets. Second, we introduce reducible and exact-reducible elements of a covering to investigate the conditions under which two coverings generate an identical lower or upper approximation operation. Third, we study the relationships between this type of covering-based rough sets and the three existing ones.

The remainder of this paper is organized as follows. In Section 2, we present the fundamental concepts and properties of Pawlak's rough set theory, which forms the basis of our subsequent discussion. We also define the basic concepts of coverings used in this paper. Section 3 defines the new type of covering-based rough sets. We present basic properties of this new model, and investigate the conditions under which certain properties of classical rough sets hold. In Section 4, we study the dependency of the lower and upper approximation operations by introducing reducible and exact-reducible elements of a covering. Section 5 discusses axiomatic systems for the lower and upper approximation operations and presents an axiomatic system for the lower approximation operation. We investigate the relationships between this type of covering-based rough sets and the three existing ones in Section 6. In Section 7, we present a summary of the four types of covering-based rough sets and highlight potential applications of covering rough sets and future research topics.

2. Background

2.1. Pawlak's rough sets

Let U be a finite set, the domain of discourse, and R an equivalence relation on U. R will generate a partition $U/R=\{Y_1,Y_2,\ldots,Y_m\}$ on U, where Y_1, Y_2, …, Y_m are the equivalence classes induced by the equivalence relation R. For any $X \subseteq U$, we can describe X by the equivalence classes of R, where the following two sets,

$$R_*(X) = \cup\{Y_i \in U/R | Y_i \subseteq X\},$$
$$R^*(X) = \cup\{Y_i \in U/R | Y_i \cap X \neq \emptyset\},$$

are called lower and upper approximations of X, respectively.

Let $\emptyset$ be the empty set, and $-X$ the complement of X in U. From the definitions of the approximation sets, the following hold.

Proposition 1. *The properties of Pawlak's rough sets:*

(1L) $R_*(U) = U$ *(Co − normality)*
(1H) $R^*(U) = U$ *(Co − normality)*
(2L) $R_*(\emptyset) = \emptyset$ *(Normality)*
(2H) $R^*(\emptyset) = \emptyset$ *(Normality)*
(3L) $R_*(X) \subseteq X$ *(Contraction)*
(3H) $X \subseteq R^*(X)$ *(Extension)*
(4L) $R_*(X \cap Y) = R_*(X) \cap R_*(Y)$ *(Multiplication)*
(4H) $R^*(X \cup Y) = R^*(X) \cup R^*(Y)$ *(Addition)*
(5L) $R_*(R_*(X)) = R_*(X)$ *(Idempotency)*
(5H) $R^*(R^*(X)) = R^*(X)$ *(Idempotency)*
(6L) $X \subseteq Y \Rightarrow R_*(X) \subseteq R_*(Y)$ *(Monotone)*
(6H) $X \subseteq Y \Rightarrow R^*(X) \subseteq R^*(Y)$ *(Monotone)*
(7L) $R_*(-R_*(X)) = -R_*(X)$ *(Lower complement relation)*

(7H) $R^*(-R^*(X)) = -R^*(X)$ (Upper complement relation)
(8LH) $R_*(-X) = -R^*(X)$ (Duality)
(9LH) $R_*(X) \subseteq R^*(X)$ (Appropriateness)

(3L), (4L), and (7L) are characteristic properties of the lower approximation operations [16,53] i.e., all other properties of the lower approximation can be deduced from these three properties. Similarly, (3H), (4H), and (7H) are characteristic properties of the upper approximation.

2.2. Covering and minimal description

In this subsection, we present the basic concepts of a covering and the covering approximation space. Further details of these can be found in [3,50,51,54].

Definition 1. Let U be the domain of discourse and $\mathbf{C}$ a family of subsets of U. If none of the subsets in $\mathbf{C}$ is empty, and $\cup\mathbf{C} = U$, $\mathbf{C}$ is called a covering of U.

Since it is clear that a partition is definitely a covering, the concept of coverings is an extension of the concept of partitions.

In the following discussion, unless stated to the contrary, coverings are considered to be finite, i.e., they consist of a finite number of sets.

Definition 2 (*Covering approximation space*). Let U be a non-empty set and $\mathbf{C}$ be a covering of U. We call the ordered pair $\langle U, \mathbf{C}\rangle$ a covering approximation space.

Definition 3 (*Minimal description*). Let $\langle U, \mathbf{C}\rangle$ be a covering approximation space. If $x \in U$, the minimal description of x is defined as

$$Md(x) = \{K | x \in K \in \mathbf{C} \wedge (\forall S \in \mathbf{C} \wedge x \in S \subseteq K \Rightarrow K = S)\}.$$

Definition 4 (*Close friends*). Let $\langle U, \mathbf{C}\rangle$ be a covering approximation space. If $x \in U$, $\bigcup_{K \in Md(x)} K$ is called the close friends of x and is denoted as $CFriends(x)$.

Definition 5 (*Unary*). Let $\mathbf{C}$ be a covering of set U. $\mathbf{C}$ is called unary if $\forall x \in U$, $|Md(x)| = 1$.

Definition 6 (*Pointwise-covered*). Let $\mathbf{C}$ be a covering of U. If $\forall K \in \mathbf{C}$ and $x \in K$, $K \subseteq CFriends(x)$, we call $\mathbf{C}$ a pointwise-covered covering.

Three types of covering-based rough sets were studied in [3,4,28,34,50,60]. They are all related with respect to the minimal description. We present the definitions and basic properties of these types as follows. The lower approximations are identical for the three types of generalized rough sets based on covering, but their upper approximations differ.

Let $\langle U, \mathbf{C}\rangle$ be a covering approximation space and $X \subseteq U$.

Definition 7 (*Lower approximation*). The covering lower approximation operation $CL: P(U) \Rightarrow P(U)$ is defined as $CL(X) = \bigcup_{K \in \mathbf{C} \wedge K \subseteq X} K$.

Definition 8 (*First type of covering-based upper approximation*). The first type of covering-based upper approximation operation $FH: P(U) \Rightarrow P(U)$ is defined as $FH(X) = CL(X) \cup (\bigcup_{x \in X - CL(X)} CFriends(x))$.

The second type of covering-based rough sets was first defined in [28].

Definition 9 (*Second type of covering-based upper approximation*). Let $\mathbf{C}$ be a covering of U. The second type of covering upper approximation operation, SH, is defined as: $\forall X \subseteq U$, $SH(X) = \cup_{K \in \mathbf{C} \wedge K \cap X \neq \phi} K$.

The third type of covering-based rough sets was first defined in [34].

Definition 10 (*Third type of covering-based upper approximation*). Let $\mathbf{C}$ be a covering of U. The third type of covering upper approximation operation, TH, is defined as $TH(X) = \bigcup_{x \in X} CFriends(x)$.

In the remaining part of this section, we present the basic properties of these three types of covering-based rough sets compared with Pawlak's rough sets.

Proposition 2. [50] *CL has properties (1L), (2L), (3L), (5L), and (6L) in Proposition 1. But, properties (4L) and (7L) in Proposition 1 do not hold for CL.*

Proposition 3. [50] *FH has properties (1H), (2H), (3H), and (5H) in Proposition 1. But, properties (4H), (6H), and (7H) in Proposition 1 do not hold for FH.*

Proposition 4. [56] *SH has properties (1H), (2H), (3H), (4H), and (6H) in Proposition 1. But, properties (5H) and (7H) in Proposition 1 do not hold for SH.*

Proposition 5. [58] *TH has properties (1H), (2H), (3H), (4H), and (6H) in Proposition 1. But, properties (5H) and (7H) in Proposition 1 do not hold for TH.*

3. The fourth type of covering generalized rough sets

In this section, we define the fourth type of covering rough sets and investigate its properties.

3.1. Concepts and properties

Based on the definitions of the three types of covering rough sets in [3,46,60], we define a new type of covering rough set model as follows [54]. In this type of generalized rough set model, the lower approximation is identical to that in the three types of covering-based rough sets presented in Section 2, but the upper approximation differs from those presented.

Definition 11 (*Fourth type of covering-based upper approximation*). Let $\mathbf{C}$ be a covering of U and $P(U)$ be the power set of U. The operation $RH_{\mathbf{C}}$: $P(U) \to P(U)$ is defined as follows: $\forall X \in P(U)$,

$$RH_{\mathbf{C}}(X) = CL(X) \cup \left(\bigcup_{(K\in\mathbf{C})\wedge(K\cap(X-CL(X))\neq\emptyset)} K \right).$$

We call RH the fourth type of covering upper approximation operation, coupled with covering $\mathbf{C}$. If the covering is obvious, we omit the lowercase $\mathbf{C}$ for the two operations.

Proposition 6. *If $\mathbf{C}$ is a partition of U, RH(X) is the upper approximation operation as specified in Pawlak's original definitions.*

Proposition 7. $RH(X) = X$, *if and only if X is the union of elements in* $\mathbf{C}$.

Based on the properties of Pawlak's rough sets listed in Section 2, we begin the study on the properties of the fourth type of covering-based rough sets.

Proposition 8. *The fourth type of upper approximation has the following properties:*

(1H) $RH(U) = U$ ($Co-normality$)
(2H) $RH(\emptyset) = \emptyset$ ($Normality$)
(3H) $X \subseteq RH(X)$ ($Extension$)
(5H) $RH(RH(X)) = RH(X)$ ($Idempotency$)

However, the following properties do not generally hold for the fourth type of covering-based rough sets.

(4H) $RH(X \cup Y) = RH(X) \cup RH(Y)$ ($Addition$)
(6H) $X \subseteq Y \Rightarrow RH(X) \subseteq RH(Y)$ ($Monotone$)
(7H) $RH(-RH(X)) = -RH(X)$ ($Upper\ complement\ relation$)

Example 1 (*Counterexample for properties (4H), (6H), and (7H)*). Let $U = \{a,b,c,d\}$, $K_1 = \{a,b\}$, $K_2 = \{a,b,c\}$, $K_3 = \{c,d\}$, and $\mathbf{C} = \{K_1,K_2,K_3\}$. $\mathbf{C}$ is a covering of U.

(4H) Let $X = \{c\}$ and $Y = \{d\}$, then we have $RH(X) = \{a,b,c,d\}$ and $RH(Y) = \{c,d\}$. But, $RH(X \cup Y) = RH(\{c,d\}) = \{c,d\}$, and thus $RH(X \cup Y) \neq RH(X) \cup RH(Y)$.

(6H) Let $X = \{c\}$ and $Y = \{c,d\}$, then we have $RH(X) = \{a,b,c,d\}$ and $RH(Y) = \{c,d\}$. Although $X \subseteq Y$, $RH(X) \not\subseteq RH(Y)$.

(7H) If $X = \{a,b,c\}$, $RH(X) = \{a,b,c\}$, $-RH(X) = \{d\}$, and $RH(-RH(X)) = RH(\{d\}) = \{c,d\}$, and therefore, $RH(-RH(X)) \neq -RH(X)$.

3.2. Conditions under which covering-based rough sets satisfy certain classical properties

For the fourth type of covering-based rough sets, some of the properties of classical rough sets listed in Section 2 are no longer valid. Here we address the issues when these properties do hold for this type of covering-based rough sets. First, we investigate what conditions a covering should have so that the corresponding lower approximation operation generated by such a covering will satisfy property (4L) in Proposition 1.

Theorem 1 [60]. *CL satisfies*

$$(4L)\ CL(X \cap Y) = CL(X) \cap CL(Y),$$

if and only if **C** *satisfies the following properties:* $\forall K_1, K_2 \in \mathbf{C}$, $K_1 \cap K_2$ *is the union of a finite number of elements in* **C**.

Then we consider the similar issue for property (6H) in Proposition 1.

Proposition 9. *If RH satisfies*

$$(6H)\ X \subseteq Y \Rightarrow RH(X) \subseteq RH(Y),$$

then **C** *satisfies the following properties:* $\forall K_1, K_2 \in \mathbf{C}$, $K_1 \cap K_2$ *is the union of a finite number of elements in* **C**.

Proof. $\Rightarrow$: $RH(K_1 \cap K_2) \subseteq RH(K_1) = K_1$ and $RH(K_1 \cap K_2) \subseteq RH(K_2) = K_2$, so $RH(K_1 \cap K_2) \subseteq K_1 \cap K_2$. By property (3H) in Proposition 8, $K_1 \cap K_2 \subseteq RH(K_1 \cap K_2)$, so $K_1 \cap K_2 = RH(K_1 \cap K_2)$. By Proposition 7, $K_1 \cap K_2$ is the union of a finite number of elements in **C**. $\square$

Before we investigate the issue of property (4H) in Proposition 1, we prove a lemma to show that property (4H) is equivalent to property (6H).

Lemma 1. *RH satisfies*

$$(6H)\ X \subseteq Y \Rightarrow RH(X) \subseteq RH(Y),$$

if and only if RH satisfies

$$(4H)\ RH(X \cup Y) = RH(X) \cup RH(Y).$$

Proof. $\Rightarrow$: By (6H), $RH(X) \subseteq RH(X \cup Y)$ and $RH(Y) \subseteq RH(X \cup Y)$, so $RH(X) \cup RH(Y) \subseteq RH(X \cup Y)$. On the other hand, by property (3H) in Proposition 8, $X \cup Y \subseteq RH(X) \cup RH(Y)$. By (6H), $RH(X \cup Y) \subseteq RH(RH(X) \cup RH(Y))$. By Proposition 7, $RH(RH(X) \cup RH(Y)) = RH(X) \cup RH(Y)$, so $RH(X \cup Y) \subseteq RH(X) \cup RH(Y)$. Therefore, $RH(X \cup Y) = RH(X) \cup RH(Y)$.

$\Leftarrow$: If $X \subseteq Y$, $RH(Y) = RH(X \cup Y) = RH(X) \cup RH(Y)$, so $RH(X) \subseteq RH(Y)$. $\square$

Proposition 10. *If RH satisfies*

$$(4H)\ RH(X \cup Y) = RH(X) \cup RH(Y),$$

then **C** *satisfies the following properties:* $\forall K_1, K_2 \in \mathbf{C}$, $K_1 \cap K_2$ *is the union of a finite number of elements in* **C**.

Proof. Straightforwardly by Proposition 9 and Lemma 1. $\square$

From the above three theorems, we have the following two results.

Corollary 1. *If RH satisfies*

$$(4H)\ RH(X \cup Y) = RH(X) \cup RH(Y),$$

then CL satisfies

$$(4L)\ CL(X \cap Y) = CL(X) \cap CL(Y).$$

Corollary 2. *If RH satisfies*

$$(6H)\ X \subseteq Y \Rightarrow RH(X) \subseteq RH(Y),$$

then CL satisfies

$$(4L)\ CL(X \cap Y) = CL(X) \cap CL(Y).$$

Next, we consider properties (7L) and (7H).

Theorem 2 [60]. *CL satisfies*

$$(7L)\ CL(-CL(X)) = -CL(X),$$

if and only if $\forall K_1, \ldots, K_m \in \mathbf{C}$, $-(K_1 \cup, \ldots, \cup K_m)$ *is the union of a finite number of elements in* $\mathbf{C}$.

Theorem 3. *RH satisfies (7H)* $RH(-RH(X)) = -RH(X)$, *if and only if* $\forall K_1, \ldots, K_m \in \mathbf{C}$, $-(K_1 \cup, \ldots \cup K_m)$ *is the union of a finite number of elements in* $\mathbf{C}$.

Proof. The proof is similar to that of the above Theorem. □

Corollary 3. *CL satisfies*

$$(7L)\ CL(-CL(X)) = -CL(X),$$

if and only if RH satisfies

$$(7H)\ RH(-RH(X)) = -RH(X).$$

Proof. Directly from Theorems 2 and 3. □

As for property (8LH), we obtain only partial solutions. First, we present a necessary condition for property (8LH) to hold.

Theorem 4. *If CL and RH satisfy*

$$(8LH)\ RH(-X) = -CL(X),$$

then $\forall K_1, \ldots, K_m \in \mathbf{C}$, $-(K_1 \cup, \ldots, \cup K_m)$ *is the union of finite elements in* $\mathbf{C}$.

Proof. $\forall K_1, \ldots, K_m \in \mathbf{C}, RH(-(K_1 \cup, \ldots, \cup K_m)) = -CL(K_1 \cup, \ldots, \cup K_m) = -(K_1 \cup, \ldots, \cup K_m)$, and thus, by Proposition 7, $-(K_1 \cup, \ldots, \cup K_m)$ is the union of a finite number of elements in $\mathbf{C}$. □

Combining Theorems 2–4, we have the following corollary.

Corollary 4. *If CL and RH satisfy*

$$(8LH) RH(-X) = -CL(X),$$

then CL satisfies

$$(7L) CL(-CL(X)) = -CL(X)$$

and RH satisfies

$$(7H) RH(-RH(X)) = -RH(X).$$

4. Dependency of lower and upper approximation operations

For Pawlak's rough sets, lower and upper approximation operations are dual, so the lower approximation operation uniquely determines the upper approximation operation, and vice versa. As shown in (8LH) in Remark 1, lower and upper approximation operations of the fourth type of covering-based rough sets are not dual. It is an interesting question whether the lower approximation operation uniquely determines the upper approximation operation, or the upper approximation operation uniquely determines the lower approximation operation for this type of generalized rough sets. In this section, we explore the interdependency of the lower and upper approximation operations in this fourth type of covering-based rough sets. To achieve this goal, we also investigate the conditions under which two coverings generate the same lower approximation operation or the same upper approximation operation.

Example 2 (*Two different coverings generate an identical lower approximation operation and an identical upper approximation operation*). Let $U = \{a,b,c,d\}$, $K_1 = \{a\}$, $K_2 = \{b\}$, $K_3 = \{a,b\}$, $K_4 = \{c,d\}$, and $K_5 = \{a,b,c\}$, then $\mathbf{C} = \{K_1,K_2,K_4,K_5\}$ and $\mathbf{C}' = \{K_1,K_2,K_3,K_4,K_5\}$ are two coverings of U. It is easy to see that they generate the same lower approximation operation and the same upper approximation operation.

Example 3 (*Two coverings generate the same lower approximation of the fourth type and two different upper approximations of the fourth type*). Let $U = \{a,b,c\}$, $K_1 = \{a\}$, $K_2 = \{b,c\}$, $K_3 = \{a,b,c\}$, and $\mathbf{C}_1 = \{K_1,K_2\}, \mathbf{C}_2 = \{K_1,K_2,K_3\}$. $\mathbf{C}_1$ and $\mathbf{C}_2$ generate the same lower approximation, but they generate different upper approximations. In fact, under $\mathbf{C}_1$:

$RH(\{a\})=\{a\}$, $RH(\{b\})=\{b,c\}$, $RH(\{c\})=\{b,c\}$, $RH(\{a,b\})=\{a,b,c\}$, $RH(\{a,c\})=\{a,b,c\}$, $RH(\{b,c\})=\{a,b,c\}$, and $RH(\{a,b,c\})=\{a,b,c\}$.

However, under $\mathbf{C}_2$:

$RH(\{a\})=\{a\}$, $RH(\{b\})=\{a,b,c\}$, $RH(\{c\})=\{a,b,c\}$, $RH(\{a,b\})=\{a,b,c\}$, $RH(\{a,c\})=\{a,b,c\}$, $RH(\{b,c\})=\{a,b,c\}$, and $RH(\{a,b,c\})=\{a,b,c\}$.

4.1. Conditions for two coverings to generate the same lower approximation operation

To explore the conditions under which two coverings generate the same lower approximation operation, we introduce the key concept defined in [50] – the *reduct* of a covering.

Definition 12 (*A reducible element of a covering*). Let $\mathbf{C}$ be a covering of domain U and $K \in \mathbf{C}$. If K is the union of some sets in $\mathbf{C}-\{K\}$, we say K is reducible in $\mathbf{C}$, otherwise K is irreducible.

Definition 13 (*A reducible covering 50*). Let $\mathbf{C}$ be a covering of U. If every element in $\mathbf{C}$ is irreducible, we say $\mathbf{C}$ is irreducible; otherwise $\mathbf{C}$ is reducible.

Proposition 11 [50]. *Let $\mathbf{C}$ be a covering of domain U. If K is reducible in $\mathbf{C}$, $\mathbf{C}-\{K\}$ is still a covering of U and $\forall x \in U$.*

Proposition 12 [50]. *Let $\mathbf{C}$ be a covering of U, $K \in \mathbf{C}$, K be reducible in $\mathbf{C}$, and $K_1 \in \mathbf{C}-\{K\}$. K_1 is reducible in $\mathbf{C}$, if and only if it is reducible in $\mathbf{C}-\{K\}$.*

Proposition 11 guarantees that after deleting a reducible subset in a covering, it is still a covering, while Proposition 12 shows that deleting a reducible subset in a covering will not generate any new reducible elements or make other previously reducible elements irreducible. Consequently, we can compute the reduct of a covering of a domain by deleting all reducible elements. The remainder still consists of a covering of the domain and is irreducible.

Definition 14 (*Reduct of a covering 50*). For a covering $\mathbf{C}$ of domain U, the new irreducible covering through the above reduction is called the reduct of $\mathbf{C}$ and is denoted by $reduct(\mathbf{C})$.

By Proposition 12 and the definition of the reduct, it is clear that $reduct(\mathbf{C})$ is unique for a covering $\mathbf{C}$ of U.

The following theorem on the condition under which two coverings generate the same lower approximation comes from Ref. [50].

Theorem 5 (Condition under which two coverings generate the same lower approximation). *$\mathbf{C}_1$ and $\mathbf{C}_2$ generate the same lower approximation, if and only if $reduct(\mathbf{C}_1)=reduct(\mathbf{C}_2)$.*

4.2. Conditions for two coverings to generate the same upper approximation operation

For upper approximation operations in the fourth type of covering rough sets, we are still faced with the question of when two coverings will generate the same upper approximation operation. We propose an alternative concept, the *exact-reduct*, as a means to solving this problem.

Definition 15 (*An exact-reducible element of a covering*). Let $\mathbf{C}$ be a covering of domain U and $K \in \mathbf{C}$. If there exist $K_1,\ldots,K_m \in \mathbf{C}-\{K\}$ such that $K=K_1\cup,\ldots,\cup K_m$, and $\forall x \in K$ and $\{x\}$ not a singleton element of $\mathbf{C}$, $K \subseteq \cup\{K'|x \in K' \in \mathbf{C}-\{K\}\}$, K is called an exact-reducible element of $\mathbf{C}$.

Example 4 (*An exact-reducible element of a covering*). Let $U=\{a,b,c,d\}$, $K_1=\{a,b\}$, $K_2=\{a,c,d\}$, $K_3=\{b,c\}$, $K_4=\{b,d\}$, $K_5=\{c,d\}$, $K_6=\{a,b,c,d\}$, and $\mathbf{C}=\{K_1,K_2,K_3,K_4,K_5,K_6\}$. K_6 is an exact-reducible element of $\mathbf{C}$.

Proposition 13. *Let $\mathbf{C}$ be a covering of domain U. If K is exact-reducible in $\mathbf{C}$, then K is reducible in $\mathbf{C}$.*

Definition 16 (*An exact-reducible covering*). Let $\mathbf{C}$ be a covering of U. If every element in $\mathbf{C}$ is exact-irreducible, we say $\mathbf{C}$ is exact-irreducible; otherwise $\mathbf{C}$ is exact-reducible.

Proposition 14. *Let $\mathbf{C}$ be a covering of domain U. If K is exact-reducible in $\mathbf{C}$, $\mathbf{C}-\{K\}$ is still a covering of U.*

Similar to Proposition 12 for the reduct of a covering, the exact-reduct has the following properties.

Proposition 15. *Let* $\mathbf{C}$ *be a covering of U,* $K \in \mathbf{C}$*, K be reducible in* $\mathbf{C}$*, and* $K_1 \in \mathbf{C} - \{K\}$*.* K_1 *is reducible in* $\mathbf{C}$*, if and only if it is reducible in* $\mathbf{C} - \{K\}$*.*

Proposition 14 guarantees that after deleting a reducible subset in a covering, it is still a covering, while Proposition 15 shows that deleting a reducible subset in a covering will not generate any new reducible elements or make other previously reducible elements irreducible. Consequently, we can compute the reduct of a covering of a domain by deleting all reducible elements. The remainder still consists of a covering of the domain and is irreducible.

Definition 17 (*Exact-reduct of a covering*). For a covering $\mathbf{C}$ of domain U, the new irreducible covering through the above reduction is called the exact-reduct of $\mathbf{C}$ and is denoted by *exact-reduct*($\mathbf{C}$).

Like *reduct*($\mathbf{C}$), *exact-reduct*($\mathbf{C}$) is unique for a covering $\mathbf{C}$ of U.

Proposition 16. *Suppose* $\mathbf{C}$ *is a covering of U,* $K \in \mathbf{C}$*. The upper approximations of the fourth type of covering generated by coverings* $\mathbf{C}$ *and* $\mathbf{C} - \{K\}$*, respectively, are the same if and only if K is exact-reducible in* $\mathbf{C}$*.*

Proof. $RH(X) = CL(X) \cup (\bigcup_{K'' \cap (X - CL(X)) \neq \emptyset} K'')$. The fact that K is the union of elements in $\mathbf{C} - \{K\}$ guarantees keeping the lower approximation $CL(X)$ of X unchanged, while the condition that $\forall x \in K$ and $\{x\} \notin \mathbf{C}$, $K \subseteq \bigcup_{x \in K' \in \mathbf{C} - \{K\}\}} K'$ keeps $\bigcup_{K'' \cap (X - CL(X)) \neq \emptyset} K''$ unchanged. As a result, $RH(X)$ remains unchanged in $\mathbf{C} - \{K\}$, if and only if K is exact-reducible in $\mathbf{C}$. □

Now we reach an important conclusion on the exact-reduct and the fourth type of the upper approximation operation.

Theorem 6 (Condition under which two coverings generate the same fourth type of upper approximation). *Two coverings* $\mathbf{C}_1$ *and* $\mathbf{C}_2$ *of U generate the same fourth type of upper approximation operation, if and only if exact-reduct(*$\mathbf{C}_1$*) = exact-reduct(*$\mathbf{C}_2$*).*

4.3. Interdependency of approximation operations

Proposition 17. *Let* $\mathbf{C_1}$*,* $\mathbf{C_2}$ *be two coverings of U. If* $\mathbf{C_1}$ *and* $\mathbf{C_2}$ *generate the same upper approximation of the fourth type of covering, then reduct(*$\mathbf{C_1}$*) = reduct(*$\mathbf{C_2}$*).*

Proof. Let RH_1 and RH_2 be upper approximations of the fourth type of covering generated by coverings $\mathbf{C_1}$ and $\mathbf{C_2}$, respectively. If RH_1 and RH_2 are identical, then for any irreducible element K of $\mathbf{C_1}$, by Proposition 8 (9H), $RH_1(K) = K$, and thus $RH_2(K) = K$. By Proposition 7, there exist $K_1, \ldots, K_m$ in $\mathbf{C_2}$ such that $K = K_1 \cup, \ldots, \cup K_m$. If $m = 1$, we have proved that $K = K_1 \in \mathbf{C_2}$. Otherwise, $K_1, \ldots, K_m \subset K$. In the same way, for any $1 \leqslant i \leqslant m$, there exist $J_1, \ldots, J_{n_i} \in \mathbf{C_1}$ such that $K_i = J_1^i \cup \ldots \cup J_{n_i}^i$. Therefore, $K = K_1 \cup \ldots \cup K_m = \cup_{i=1}^m \cup_{j=1}^{n_i} J_1^i \cup \ldots \cup J_{n_i}^i$. This means that K is a reducible element in $\mathbf{C_1}$. This contradiction proves that $K \in \mathbf{C_2}$.

In the same way, we can prove that any irreducible element K of $\mathbf{C_2}$ is an element of $\mathbf{C_1}$. Therefore, *reduct*($\mathbf{C_1}$) = *reduct*($\mathbf{C_1}$). □

From Proposition 17 and the above proposition, the resulting relationship between *reduct* and *exact-reduct* follows.

Corollary 5. *Let* $\mathbf{C_1}$*,* $\mathbf{C_2}$ *be two coverings of U. If exact-reduct(*$\mathbf{C_1}$*) = exact-reduct(*$\mathbf{C_2}$*), then reduct(*$\mathbf{C_1}$*) = reduct(*$\mathbf{C_1}$*).*

Based on Theorem 5 and Proposition 17, we reach the following conclusions about the lower and upper approximation operations.

Theorem 7. *Let* $\mathbf{C_1}$*,* $\mathbf{C_2}$ *be two coverings of U. If* $\mathbf{C_1}$ *and* $\mathbf{C_2}$ *generate the same fourth type of covering upper approximations, they generate the same fourth type of covering lower approximations.*

Proof. Straightforwardly from Proposition 17 and Theorem 5. □

The interdependency between the lower and the upper approximation operations is summarized as follows.

Theorem 8. *The fourth type of covering upper approximation uniquely determines the fourth type of covering lower approximation, but the fourth type of covering lower approximation does not uniquely determine the fourth type of covering upper approximation.*

Proof. This comes from Example 3 and Theorem 7. □

5. Axiomatic system for approximation operations

Pawlak's lower and upper approximation operations have been axiomatized [16]. Now, it is important to ascertain which are the characteristic properties for the covering lower and upper approximation operations. Below we present an axiom of the covering lower approximation operations. The axiomatization of covering upper approximation operations can be found in [48].

5.1. An axiomatic system for lower approximation operations

An axiomatic system for the covering lower approximation operations is given below.

Theorem 9. *[50] Let U be a non-empty set. If an operation $L:P(U) \to P(U)$ satisfies the following properties: for any X, $Y \subseteq U$,*

(1) $L(U) = U$,
(2) $X \subseteq Y \Rightarrow L(U) \subseteq L(U)$,
(3) $L(X) \subseteq X$,
(4) $L(L(X)) = L(X)$,

then there exists a covering $\mathbf{C}$ *of U such that the covering lower approximation operation generated by* $\mathbf{C}$ *equals L.*

The above four properties for a covering lower approximation operation are independent.

5.2. The axiomatic issue for the upper approximation operations

We have as yet not found an axiomatic system for the fourth type of upper approximation operations. In fact, the popular properties listed in Proposition 8 are not sufficient to characterize the fourth type of upper approximation operations as shown in the following example.

Example 5. Let $U = \{a,b,c\}$, $K_1 = \{a,b\}$, $K_2 = \{b,c\}$, and $\mathbf{C} = \{K_1,K_2\}$. Define $H:P(U) \to P(U)$ as

$H(\emptyset) = \emptyset$, $H(K_1) = K_1$, $H(K_2) = K_2$, $H(\{a\}) = \{a,c\}$, and $H(X) = U$ for other $X \subseteq U$.

It is apparent that H satisfies (1H), (2H), (3H), (5H), and (9H), but H is not a fourth type of upper approximation operation for any covering of U.

6. Relationships between the upper approximations of the four types of coverings

For a covering $\mathbf{C}$ of U, there are four types of covering rough sets. These have the same lower approximation operation, but different upper approximation operations. Then, the first question to be asked is: What are the relationships among these?

Since $TH(X) = \bigcup_{x\in X \wedge K\in Md(x)} K = (\bigcup_{x\in CL(X)\wedge K\in Md(x)} K) \cup (\bigcup_{x\in X-CL(X)\wedge K\in Md(x)} K)$ and $\forall x \in X$ we have $x \in Md(x)$ and $Md(x) \cap X \neq \phi$, from the definitions of the four types of upper approximation operations, the following two rules hold in general. For a covering $\mathbf{C}$ of U and $X \subseteq U$,

$$FH(X) \subseteq TH(X) \subseteq SH(X), \tag{1}$$
$$FH(X) \subseteq RH(X) \subseteq SH(X). \tag{2}$$

Generally, however, the above equalities do not hold, as can be seen from the following examples.

Example 6. Let $U = \{a,b,c,d\}$, $K_1 = \{a,b\}$, $K_2 = \{a,b,c\}$, $K_3 = \{c,d\}$, and $\mathbf{C} = \{K_1,K_2,K_3\}$. $\mathbf{C}$ is a covering of U.

- Let $X = \{a\}$, then we have $TH(X) = \{a,b\}$ and $SH(X) = \{a,b,c\}$, so $TH(X) \subset SH(X)$.
- Let $X = \{c,d\}$, then we have $FH(X) = \{c,d\}$ and $TH(X) = \{a,b,c\}$, so $FH(X) \subset TH(X)$.

Example 7. Let $U = \{a,b,c,d\}$, $K_1 = \{a,b\}$, $K_2 = \{a,c\}$, $K_3 = \{c,d\}$, and $X = \{a,b\}$, then $RH(X) = \{a,b\}$ and $TH(X) = \{a,b,c\}$, so

$$RH(X) \subset TH(X).$$

Example 8. Let $U = \{a,b,c,d\}$, $K_1 = \{a,b\}$, $K_2 = \{a,c\}$, $K_3 = \{c,d\}$, $K_4 = \{a,b,c\}$, and $X = \{c\}$, then $RH(X) = \{a,b,c,d\}$ and $TH(X) = \{a,c,d\}$, so

$$TH(X) \subset RH(X).$$

Example 9. Let $U = \{a,b,c,d\}$, $K_1 = \{a,b\}$, $K_2 = \{c,d\}$, $K_3 = \{a,b,c\}$, and $X = \{a\}$, then $FH(X) = \{a,b\}$ and $RH(X) = \{a,b,c\}$, so

$$FH(X) \subset RH(X).$$

Example 10. Let $U = \{a,b,c,d\}$, $K_1 = \{a,b\}$, $K_2 = \{c,d\}$, $K_3 = \{a,b,c\}$, and $X = \{a,b\}$, then $RH(X) = \{a,b\}$ and $SH(X) = \{a,b,c\}$, so

$$RH(X) \subset SH(X).$$

Now, another question arises: When are these upper approximation operations equal?

6.1. Results for FH, SH, and TH

From the above discussion, for a covering $\mathbf{C}$ of U and $X \subseteq U$, we have $FH(X) \subseteq TH(X) \subseteq SH(X)$ and the equality does not generally hold. In the following three theorems, we present sufficient and necessary conditions under which the above equalities hold. As for the proofs, please refer to paper [51].

Theorem 10. [51]

(1) *$FH = TH$, if and only if $\mathbf{C}$ is unary.*
(2) *$TH = SH$, if and only if $\mathbf{C}$ is pointwise-covered.*
(3) *$FH = SH$, if and only if $\mathbf{C}$ is a partition.*

6.2. Conditions under which FH and RH are identical

Theorem 11 (Condition under which $FH = RH$). *Let $\mathbf{C}$ be a covering of U, and FH and RH be the first and fourth types of upper approximation operations, respectively. $FH = RH$, if and only if $\mathbf{C}$ satisfies the following condition:*

$$\forall x \in U,\ \textit{if}\ \{x\} \notin \mathbf{C},\ \textit{then}\ \forall K \in \mathbf{C}\ \textit{and}\ x \in K,\ K \subseteq CFriends(x).$$

Proof. Suppose $FH = RH$. $\forall x \in U$, if $\{x\} \notin \mathbf{C}$, then $CL(\{x\}) = \phi$. Since $FH(\{x\}) = RH(\{x\})$, $\forall K \in \mathbf{C}$ and $x \in K$, $K \subseteq CFriends(x)$.

On the contrary, suppose the above condition holds. $\forall X \subseteq U$, if $K \in \mathbf{C}$ and $K \cap (X - CL(X)) \neq \phi$, there exists $x \in X - CL(X)$ such that $x \in K$. Since $x \in X - CL(X)$, so $\{x\} \notin \mathbf{C}$. By the condition, $K \subseteq CFriends(x)$, so $K \in FH(X)$. Therefore, $RH(X) \subseteq FH(X)$. Combining $FH(X) \subseteq RH(X)$, we get the result that $FH = RH$. □

6.3. Conditions under which RH and SH are identical

Theorem 12 (Condition under which $RH = SH$). *Let $\mathbf{C}$ be a covering of U, and RH and SH be the fourth and second types of upper approximation operations, respectively. $RH = SH$, if and only if $\mathbf{C}$ is a partition.*

Proof. If $\mathbf{C}$ is a partition, it is obvious that $RH = SH$.

On the contrary, suppose $RH = SH$. $\forall K, K' \in \mathbf{C}$ and $K \neq K'$, if $K \cap K' \neq \phi$. By Proposition 7, $RH(K) = K$, so $SH(K) = K$. From the definition of SH, $K' \subseteq SH(K)$, so $K' \subseteq K$. Since $K \neq K'$, $K' \subset K$. Again, by Proposition 7, $RH(K') = K'$ while by the definition of SH, $K \subseteq SH(K')$. Thus $RH(K') = K' \subset K \subseteq SH(K')$, which contradicts $RH = SH$. □

6.4. Conditions under which TH and RH are identical

Lemma 2 (*Necessary condition under which $TH = RH$*). *Let $\mathbf{C}$ be a covering of U, and TH and RH be the third and fourth types of upper approximation operations, respectively. If $TH = RH$, then $\forall K, K' \in \mathbf{C}$ and $K \neq K'$, $K \cap K'$ is the union of a finite number of elements in $\mathbf{C}$.*

Proof. Let $TH = RH$, $K \cap K' \neq \phi$, and K' is not a subset of K. By the definition, $RH(K) = K$, so $TH(K) = K$. By the definition of the third type of covering upper approximation, $\forall\ x \in K \cap K'$, $K' \notin Md(x)$; otherwise $K' \subseteq TH(K) = K$. As a result, $\forall x \in K \cap K'$, there is a $K_x \in \mathbf{C}$ such that $K_x \in Md(x)$ and $K_x \subseteq K'$. K_x must be a subset of K for $K_x \subseteq TH(K)$. This means that $K \cap K' = \bigcup_{x \in K \cap K'} K_x$. We prove that $K \cap K'$ is the union of a finite number of elements in $\mathbf{C}$. □

Corollary 6 (Necessary condition under which $TH = RH$). *Let $\mathbf{C}$ be a covering of U, and $TH = RH$, where TH and RH are the third and fourth types of upper approximation operations, respectively. If $K, K' \in \mathbf{C}$, $x \in K$, and $K' \in Md(x)$, then $K' \subseteq K$.*

Proof. If $K' \subseteq K$ does not hold, then $K \cap K' \subset K'$. According to Lemma 3, $K \cap K'$ is a reducible element in **C**, so there exists $K'' \in \mathbf{C}$ such that $x \in K'' \subseteq K \cap K' \subset K'$. This contradicts $K' \in Md(x)$. □

Corollary 7 (Necessary condition under which $TH = RH$). *Let* **C** *be a covering of U, and TH and RH be the third and fourth types of upper approximation operations, respectively. If TH = RH, then* **C** *is unary.*

Lemma 3. *Let* **C** *be a covering of U, and TH be the third type of upper approximation operation. If for* $K, K' \in \mathbf{C}$ *and* $K \neq K'$, $K \cap K'$ *is the union of a finite number of elements in* **C**, *then for any* $K_1 \in \mathbf{C}$, $TH(K_1) = K_1$.

Corollary 8. *Let* **C** *be a covering of U, and TH be the third type of upper approximation operation. If for* $K, K' \in \mathbf{C}$ *and* $K \neq K'$, $K \cap K'$ *is the union of a finite number of elements in* **C**, *then for any* $K_1, K_2, \ldots, K_n \in \mathbf{C}$, $TH(K_1 \cup K_2 \cup, \ldots, \cup K_n) = K_1 \cup K_2 \cup, \ldots, \cup K_n$.

Theorem 13 (Condition under which $TH = RH$). *Let* **C** *be a covering of U, and TH and RH be the third and fourth types of upper approximation operations, respectively. TH = RH, if and only if, for* $K, K' \in \mathbf{C}$, $K \neq K'$, *and* $x \in K \cap K'$, $\{x\} \in \mathbf{C}$.

Proof. Let $TH = RH$ and there exists $K, K' \in \mathbf{C}$ such that $K \neq K'$, $x \in K \cap K'$ and $\{x\} \notin \mathbf{C}$. By Lemma 3, there exists $K_1 \in \mathbf{C}$ such that $K_1 \subseteq K \cap K'$ and $x \in K_1$. By the definitions of RH, $K \subset RH(\{x\})$ and $K' \subset RH(\{x\})$. By Corollary 7, $Md(x)$ has only one element, herein K_x. Since $x \in K_1$, and **C** is unary, it is easy to see that $K_x \subseteq K_1$. Thus, by the definition of TH, $TH(\{x\}) = K_x \subseteq K \cap K' \subset RH(\{x\})$. This contradicts $TH = RH$.

Suppose that for $K, K' \in \mathbf{C}, K \neq K'$, and $x \in K \cap K', \{x\} \in \mathbf{C}$. $\forall X \subseteq U, X = CL(X) \cup (X - CL(X))$ and there exist $K_1, K_2, \ldots, K_n \in \mathbf{C}$ such that $CL(X) = K_1 \cup K_2 \cup, \ldots, \cup K_n$. Thus, by (4H) in Proposition 5 and Corollary 6, $TH(X) = TH(CL(X)) \cup TH(X - CL(X)) = TH(K_1 \cup K_2 \cup, \ldots, \cup K_n) \cup TH(X - CL(X)) = K_1 \cup K_2 \cup, \ldots, \cup K_n \cup TH(X - CL(X)) = CL(X) \cup TH(X - CL(X))$. $\forall x \in X - CL(X)$, there is only $K_x \in \mathbf{C}$ such that $x \in K_x$; otherwise, $\{x\} \in \mathbf{C}$, then $x \in CL(X)$. This contradicts $x \in X - CL(X)$. Therefore, $TH(X - CL(X)) = \bigcup_{K \in Md(x) \wedge x \in X - CL(X)} K = \bigcup_{x \in K \in \mathbf{C} \wedge x \in X - CL(X)} K$. This proves $TH(X) = CL(X) \cup TH(X - CL(X)) = CL(X) \cup (\bigcup_{x \in K \in \mathbf{C} \wedge x \in X - CL(X)} K) = RH(X)$. □

7. Conclusions and future work

We have presented a new type of covering-based generalized rough sets, discussed its properties, and proposed concepts to study the interdependency of the lower and upper approximation operations. We have also considered the axiomatic issue for this type of rough sets and its relationship with the three existing ones. As a summary, we present the properties of these four types of covering-based rough sets in Table 1.

Combined with the results given in [50,51,60], we summarize the results on the interdependency of the lower and upper approximations in Table 2.

Application of rough set theory to real problems is an important and challenging issue for rough set research. We have also worked on applying covering rough set theory to conflict analysis and the Chinese Wall security policy [52]. It seems

Table 1
Properties of upper approximation operations.

	Satisfied	Not satisfied
CL	(1L), (2L), (3L), (5L), (6L)	(4L), (7L)
FH	(1H), (2H), (3H), (5H)	(4H), (6H), (7H)
SH	(1H), (2H), (3H), (4H), (6H)	(5H), (7H)
TH	(1H), (2H), (3H), (4H), (6H)	(5H), (7H)
SH	(1H), (2H), (3H), (5H),	(4H), (6H), (7H)

Table 2
Interdependency of lower and upper approximation operations.

	Uniquely determines	Does not uniquely determine
CL	FH	
FH	CL	
CL		SH
SH		CL
CL	TH	
TH		CL
CL		RH
RH	CL	

that a covering based granular computing is more appropriate than a binary relation based one [14]. Furthermore, possibilities exist for applying the covering generalized rough set theory to computational theory for linguistic dynamic systems [36]. The relationships between covering rough sets and binary relation based rough sets [40,41,47,61] is a future topic to be explored [23,62].

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References

[1] A. Bargiela, W. Pedrycz, Granular Computing: An Introduction, Kluwer Academic Publishers, Boston, 2002.
[2] Z. Bonikowski, Algebraic structures of rough sets, in: W. Ziarko (Ed.), Rough Sets, Fuzzy Sets and Knowledge Discovery, Springer, 1994, pp. 243–247.
[3] Z. Bonikowski, E. Bryniarski, U. Wybraniec-Skardowska, Extensions and intentions in the rough set theory, Information Sciences 107 (1998) 149–167.
[4] E. Bryniarski, A calculus of rough sets of the first order, Bulletin Polonaise Academy of Science 36 (16) (1989) 71–77.
[5] G. Cattaneo, Abstract approximation spaces for rough theorie, in: Rough Sets in Knowledge Discovery 1: Methodology and Applications, 1998, pp. 59–98.
[6] G. Cattaneo, D. Ciucci, Algebraic structures for rough sets, in: LNCS, vol. 3135, 2004, pp. 208–252.
[7] M. Hall, G. Holmes, Benchmarking attribute selection techniques for discrete class data mining, IEEE Transactions on Knowledge and Data Engineering 15 (6) (2003) 1437–1447.
[8] Q. Hu, D. Yu, J. Liu, C. Wu, Neighborhood rough set based heterogeneous feature subset selection, Information Sciences 178 (18) (2008) 3577–3594.
[9] R. Jensen, Q. Shen, Semantics-preserving dimensionality reduction: Rough and fuzzy-rough-based approaches, IEEE Transactions On Knowledge and Data Engineering 16 (12) (2004) 1457–1471.
[10] M. Kondo, On the structure of generalized rough sets, Information Sciences 176 (5) (2005) 589–600.
[11] M. Kryszkiewicz, Rough set approach to incomplete information systems, Information Sciences 112 (1998) 39–49.
[12] M. Kryszkiewicz, Rule in incomplete information systems, Information Sciences 113 (1998) 271–292.
[13] Y. Leung, W.-Z. Wu, W.-X. Zhang, Knowledge acquisition in incomplete information systems: a rough set approach, European Journal of Operational Research 168 (2006) 164–180.
[14] T.Y. Lin, Granular computing on binary relations-analysis of conflict and chinese wall security policy, in: Rough Sets and Current Trends in Computing, vol. 2475 of LNAI, 2002, pp. 296–299.
[15] T.Y. Lin, Granular computing – structures, representations, and applications, in: LNAI, vol. 2639, 2003, pp. 16–24.
[16] T.Y. Lin, Q. Liu, Rough approximate operators: axiomatic rough set theory, in: W. Ziarko (Ed.), Rough Sets, Fuzzy Sets and Knowledge Discovery, Springer, 1994, pp. 256–260.
[17] G. Liu, The axiomatization of the rough set upper approximation operations, Fundamenta Informaticae 69 (23) (2006) 331–342.
[18] G. Liu, Generalized rough sets over fuzzy lattices, Information Sciences 178 (6) (2008) 1651–1662.
[19] J. Mordeson, Rough set theory applied to (fuzzy) ideal theory, Fuzzy Sets and Systems 121 (2001) 315–324.
[20] Z. Pawlak, Rough sets, International Journal of Computer and Information Science 11 (1982) 341–356.
[21] Z. Pawlak, Rough Sets: Theoretical Aspects of Reasoning About Data, Kluwer Academic Publishers, Boston, 1991.
[22] Z. Pawlak, A. Skowron, Rudiments of rough sets, Information Sciences 177 (1) (2007) 3–27.
[23] Z. Pawlak, A. Skowron, Rough sets: some extensions, Information Sciences 177 (1) (2007) 28–40.
[24] Z. Pawlak, A. Skowron, Rough sets and boolean reasoning, Information Sciences 177 (1) (2007) 41–73.
[25] L. Polkowski, A. Skowron (Eds.), Rough Sets in Knowledge Discovery, vol. 1, Physica–Verlag, Heidelberg, 1998.
[26] L. Polkowski, A. Skowron (Eds.), Rough Sets in Knowledge Discovery, vol. 2, Physica–Verlag, Heidelberg, 1998.
[27] L. Polkowski, A. Skowron, Rough-neuro computing, in: Rough Sets and Current Trends in Computing, vol. 2005 of LNCS, 2000, pp. 57–64.
[28] J.A. Pomykala, Approximation operations in approximation space, Bulletin Polonaise Academy of Science 35 (9–10) (1987) 653–662.
[29] K. Qin, Z. Pei, On the topological properties of fuzzy rough sets, Fuzzy Sets and Systems 151 (3) (2005) 601–613.
[30] K. Qin, Z. Pei, W. Du, The relationship among several knowledge reduction approaches, in: FSKD 2005, vol. 3613 of LNCS, 2005, pp. 1232–1241.
[31] A. Skowron, J. Stepaniuk, Tolerance approximation spaces, Fundamenta Informaticae 27 (1996) 245–253.
[32] D. Slezak, P. Wasilewski, Granular sets – foundations and case study of tolerance spaces, in: RSFDGrC 2007, vol. 4482 of LNCS, 2007, pp. 435–442.
[33] R. Slowinski, D. Vanderpooten, A generalized definition of rough approximations based on similarity, IEEE Transactions on Knowledge and Data Engineering 12 (2) (2000) 331–336.
[34] E. Tsang, D. Cheng, J. Lee, D. Yeung, On the upper approximations of covering generalized rough sets, in: Proceedings of the 3rd International Conference on Machine Learning and Cybernetics, 2004, pp. 4200–4203.
[35] E.C. Tsang, C. Degang, D.S. Yeung, Approximations and reducts with covering generalized rough sets, Computers & Mathematics with Applications 56 (1) (2008) 279–289.
[36] F.-Y. Wang, Outline of a computational theory for linguistic dynamic systems: toward computing with words, International Journal of Intelligent Control and Systems 2 (2) (1998) 211–224.
[37] F.-Y. Wang, On the abstraction of conventional dynamic systems: from numerical analysis to linguistic analysis, Information Sciences 171 (1–3) (2005) 233–259.
[38] W.-Z. Wu, J.-S. Mi, W.-X. Zhang, Generalized fuzzy rough sets, Information Sciences 151 (2003) 263–282.
[39] W.-Z. Wu, W.-X. Zhang, Constructive and axiomatic approaches of fuzzy approximation operators, Information Sciences 159 (3–4) (2004) 233–254.
[40] Y.Y. Yao, Relational interpretations of neighborhood operators and rough set approximation operators, Information Sciences 101 (1998) 239–259.
[41] Y.Y. Yao, Constructive and algebraic methods of theory of rough sets, Information Sciences 109 (1998) 21–47.
[42] Y.Y. Yao, Granular computing: basic issues and possible solutions, in: Proceedings of the 5th Joint Conference on Information Sciences, vol. 1, 2000, pp. 186–189.
[43] Z. Yun, X. Ge, X. Bai, Axiomatization and conditions for neighborhoods in a covering to form a partition, Information Sciences 181 (9) (2011) 1735–1740.
[44] L.A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 338–353.
[45] L.A. Zadeh, Fuzzy logic=computing with words, IEEE Transactions on Fuzzy Systems 4 (1996) 103–111.
[46] W. Zakowski, Approximations in the space (u, π), Demonstratio Mathematica 16 (1983) 761–769.
[47] H. Zhang, H. Liang, D. Liu, Two new operators in rough set theory with applications to fuzzy sets, Information Sciences 166 (1–4) (2004) 147–165.
[48] Y. Zhang, J. Li, W. Wu, On minimization of axiom sets characterizing covering-based approximation operators, Information Sciences 181 (14) (2011) 3032–3042.

[49] N. Zhong, Y. Yao, M. Ohshima, Peculiarity oriented multidatabase mining, IEEE Transactions on Knowledge and Data Engineering 15 (4) (2003) 952–960.
[50] W. Zhu, F.-Y. Wang, Reduction and axiomization of covering generalized rough sets, Information Sciences 152 (2003) 217–230.
[51] W. Zhu, F.-Y. Wang, Relationships among three types of covering rough sets, in: IEEE GrC 2006, 2006, pp. 43–48.
[52] W. Zhu, F.-Y. Wang, Covering based granular computing for conflict analysis, in: IEEE ISI 2006, vol. 3975 of LNCS, 2006, pp. 566–571.
[53] W. Zhu, F.-Y. Wang, Axiomatic systems of generalized rough sets, in: RSKT 2006, vol. 4062 of LNAI, 2006, pp. 216–221.
[54] W. Zhu, F.-Y. Wang, A new type of covering rough sets, in: IEEE IS 2006, London, 4–6 September, 2006, 2006, pp. 444–449.
[55] W. Zhu, F.-Y. Wang, Properties of the first type of covering-based rough sets, in: Proceedings of DM Workshop 06, ICDM 06, Hong Kong, China, December 18, 2006, pp. 407–411.
[56] W. Zhu, Properties of the second type of covering-based rough sets, in: Workshop Proceedings of GrC& BI 06, IEEE WI 06, Hong Kong, China, December 18, 2006, 2006, pp. 494–497.
[57] W. Zhu, Properties of the fourth type of covering-based rough sets, in: HIS'06, AUT Technology Park, Auckland, New Zealand, December 13–15, 2006, 2006, pp. 43–43.
[58] W. Zhu, F.-Y. Wang, Properties of the third type of covering-based rough sets, in: ICMLC'07, Hong Kong, China, 19–22 August, 2007, pp. 3746–3751.
[59] W. Zhu, Topological approaches to covering rough sets, Information Sciences 177 (6) (2007) 1499–1508.
[60] W. Zhu, F.-Y. Wang, On three types of covering rough sets, IEEE Transactions on Knowledge and Data Engineering 19 (8) (2007) 1131–1144.
[61] W. Zhu, Generalized rough sets based on relations, Information Sciences 177 (22) (2007) 4997–5011.
[62] W. Zhu, Relationship between generalized rough sets based on binary relation and covering, Information Sciences 179 (3) (2009) 210–225.
[63] W. Zhu, Relationship among basic concepts in covering-based rough sets, Information Sciences 179 (14) (2009) 2478–2486.