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Finite-time cascaded tracking control approach for mobile robots

Yangming Zhang [a,*], Guorong Liu [b], Biao Luo [c,a]

[a] School of Automation Science and Electrical Engineering, Beihang University (Beijing University of Aeronautics and Astronautics), Beijing 100191, PR China
[b] Hunan Institute of Engineering, Xiangtan 411104, PR China
[c] State Key Laboratory of Management and Control for Complex Systems, Institute of Automation, Chinese Academy of Sciences, Beijing 100190, PR China

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ABSTRACT

This paper develops a new control approach for trajectory tracking of mobile robots. For the purpose of tracking trajectory, the error dynamics of a mobile robot are divided into a first-order subsystem and a second-order subsystem by using a cascaded control design. Firstly, a global finite-time control law of the angular velocity is designed for the first-order system in order to stabilize the angle error of mobile robots. Subsequently, a finite-time sliding mode control law of forward velocity is synthesized, which guarantees the global stability of the second-order subsystem. Furthermore, the global uniform stability of the whole closed-loop system is analyzed by employing cascaded control theory, and some sufficient conditions are derived. Finally, the proposed control algorithm is applied to mobile robots, where simulation results demonstrate good convergence and performance.

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1. Introduction

Over the past two decades, the mobile robots have been widely used in various industrial processes. One of the key challenges in mobile robots is the control system, which is known as a class of typical nonholonomic system [10,21,40]. One important topic in control system design of mobile robots is trajectory tracking. However, the tracking error system of mobile robots is a coupled nonlinear system, which fails to meet Brockett's necessary condition [1] and complicates the problem significantly.

In recent years, the tracking control problem for nonholonomic mobile robots has attracted more attentions. Kanayama et al. [11] proposed a tracking control law by using the Taylor linearization of the corresponding error model. To extend the trajectory tracking problem in Cartesian space, Samson and Ait-Abderrahim [25] developed a global tracking control law in 1991. A smooth controller presented in [31] achieved exponential stability for any initial condition, thus improved the convergence rate of the algorithm. A time-varying feedback control method was developed by using the chain form [26].

To further investigate the tracking problem, some important research efforts have been deployed. Sliding mode control methods were proposed for mobile robots [2,22], similar problem with bounded disturbances was considered in [34]. The smooth time-varying feedback control law was introduced in [28,29], which guarantees the global exponential convergence. By combining cascaded design and backstepping approach, a tracking controller was designed in [6]. In [7], under the consideration of input torque saturation and external disturbances, the authors derived a new adaptive control scheme. On the

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* Corresponding author. Tel.: +86 188 1316 5165.
  E-mail addresses: ymz716@126.com (Y. Zhang), lgr1110@126.com (G. Liu), biao.luo@hotmail.com (B. Luo).

other hand, intelligent control theory was introduced to solve the tracking control problem of mobile robots [14,16,17,27,35,36,38]. The wide applications of these control methods promote the development of tracking control. Note that most of the aforementioned results only consider the asymptotic stability, which means that they achieve convergence in infinite settling time. In addition, the asymptotical stability [2,22,34] may not yield fast convergence for high-precision control. In fact, it is much more desirable to reach the target in finite time for practical mobile robots.

Finite-time tracking control method [19,32,33,37] is a fast control technique, which achieves the desired trajectory in finite time. A global finite-time tracking controller was given for the nonholonomic systems in [33]. For the uncertain nonlinear systems, Huang et al. [8] proved the global finite-time stabilization based on the finite-time Lyapunov stability theorem. The finite-time control techniques were employed for attitude control in [4,12]. However, the finite-time tracking control problem of mobile robots has rarely been studied till present, and only a few results have been reported [13,19,20]. Moreover, the low efficiency is still a problem of these methods, and the strong constraints on the desired velocities should be satisfied. To overcome these difficulties, in this paper, we propose a class of novel control laws based on cascaded control design. By using cascaded control design, we obtain two subsystems. One subsystem is stabilized by an improved global finite-time control law. To relax the strict constraints on the desired velocities, the other subsystem is stabilized by a finite-time sliding mode control law. By combining the finite time control technique and the sliding mode control approach, we improve the effectiveness of the converge rate of sliding mode control approach compared with [5,18,39,41].

The rest of the paper is organized as follows. The dynamical model of mobile robots and the tracking control problem are described in Section 2. In Section 3, the tracking control law is designed based on cascaded approach. Furthermore, the stability of the closed-loop system is analyzed. In Section 4, the simulation results by using the control laws are given. The conclusion is drawn in Section 5.

2. Problem description

In this section, we give the kinematic model of a mobile robot and the definition of tracking control problem.

2.1. Kinematic model

A mobile robot considered in this paper is made up of three wheels. Two driving wheels at the front of the mobile robot are parallel, driven by two independent motors. Another wheel is a driven wheel at the back of the mobile robot (see Fig. 1). Let D and r denote the length of the wheel axis and the radius of the driving wheels, respectively. The velocities v_L and v_R represent the velocities of the left wheel and the right wheel, respectively. The control variables v and w denote the forward velocity and the angular velocity of the mobile robot respectively, which can be described as [15]

$$\begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{D} & \frac{1}{D} \end{bmatrix} \begin{bmatrix} v_L \\ v_R \end{bmatrix} \tag{1}$$
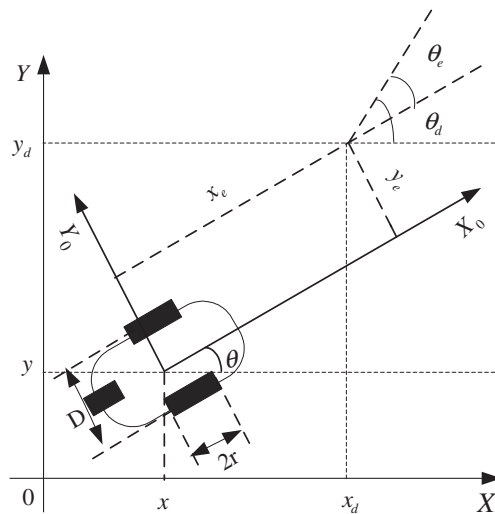

Fig. 1. Posture errors of the coordinate for a mobile robot.

The kinematic model of the mobile robot in the *X–Y* plane is given as follows:

$$\dot{q} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & 1 \end{bmatrix} u \tag{2}$$

where $q = [x, y, \theta]^T$ denotes the posture coordinate in the center of the mobile robot mass. (x, y) is defined as the Cartesian coordinates, and θ is the orientation angle between the heading direction and the *x*-axis. The control input *u* is

$$u = [v \quad w]^T \tag{3}$$

Assume that there is no slipping effect, and the kinematic model of the mobile robot satisfies the nonholonomic constraint. For the mobile robots under above assumptions, the nonholonomic constraint equation can be expressed as

$$\dot{x}\sin\theta = \dot{y}\cos\theta \tag{4}$$

2.2. Tracking control problem

The aim of tracking control problem is to design the laws of the forward velocity v and the angular velocity *w*, such that the reference trajectory is tracked by the mobile robot. In Fig. 1, the state vector $q_d = (x_d, y_d, \theta_d)^T$ denotes the reference posture and the state vector $q = (x, y, \theta)^T$ represents the actual posture. The state vector $q_e = (x_e, y_e, \theta_e)^T$ is defined as the posture error.

Accord to the geometric relationship, an efficient global coordinate transformation is described as follows [11]:

$$q_e = \begin{bmatrix} x_e \\ y_e \\ \theta_e \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_d - x \\ y_d - y \\ \theta_d - \theta \end{bmatrix} \tag{5}$$

By taking the derivative of Eq. (5), the error dynamics of the system can be obtained as

$$\dot{x}_e = y_e w - v + v_d \cos\theta_e \tag{6}$$
$$\dot{y}_e = -x_e w + v_d \sin\theta_e \tag{7}$$
$$\dot{\theta}_e = w_d - w \tag{8}$$

Consider $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, $|v_d| \leqslant v_d^{\max}$ and $w_d \neq 0$, where $v_d^{\max}$ is appropriate constants, and $|\cdot|$ denotes the absolute value sign. In this paper, the trajectory tracking problem for mobile robots is to design a time-varying state-feedback laws of the form

$$w = w(t, x_e, y_e, \theta_e), \quad v = v(t, x_e, y_e, \theta_e) \tag{9}$$

such that three state errors x_e, y_e and θ_e converge to zero in a finite time.

3. Tracking control strategy

In this section, a control scheme combining three different strategies is proposed. Firstly, the complicated system is decomposed into two subsystems by cascaded control design. For the two subsystems, a finite-time control method and a finite-time sliding mode control are developed respectively.

3.1. Cascaded control design

Consider the cascaded system $\dot{z} = f(t, z)$ that can be expressed as

$$\dot{z}_1 = f_1(t, z_1) + g(t, z_1, z_2) \tag{10}$$
$$\dot{z}_2 = f_2(t, z_2) \tag{11}$$

where $z_1 = (x_e, y_e) \in R^2$, $z_2 = \theta_e \in R$. The function $f_1(t, z_1)$ is continuously differentiable with respect to (t, z_1), $f_2(t, z_2)$ and $g(t, z_1, z_2)$ are continuous in their arguments.

Assuming $z_2 = 0$, it follows from Eq. (10) that

$$\dot{z}_1 = f_1(t, z_1)$$

Therefore, Eq. (10) can be viewed a subsystem

$$\Sigma_1 : \dot{z}_1 = f_1(t, z_1) \tag{12}$$

which is perturbed by the following subsystem

$$\Sigma_2 : \dot{z}_2 = f_2(t, z_2) \tag{13}$$

Definition 1 [9]. *A continuous function α: $[0,\infty) \to [0,\infty)$ is said to be a class K function only if it is strictly increasing and $\alpha(0) = 0$.*

Lemma 1 *([13,30]). Consider the cascaded system (10) and (11). If the following assumptions A1–A3 hold, then the system (10) and (11) is globally uniformly stable (GUS):*

- *A1. Assume that the subsystem $\Sigma_1 : \dot{z}_1 = f_1(t,z_1)$ is GUS and that there exists an auxiliary positive definite Lyapunov function candidate $V(t,z_1)$: $R_{\geqslant 0} \times R^n \to R_{\geqslant 0}$ such that*

$$\left\|\frac{\partial V}{\partial z_1}\right\| \|z_1\| \leqslant cV(t,z_1) \quad \forall \|z_1\| \geqslant \eta$$

where $c > 0$ and $\eta > 0$ are constant. Furthermore, $\frac{\partial V}{\partial z_1}(t,z_1)$ is bounded uniformly in t for all $\|z_1\| \leqslant \eta$. In other words, there exists a constant $c_1 > 0$ and $t \geqslant t_0 \geqslant 0$ such that

$$\left\|\frac{\partial V}{\partial z_1}\right\| \leqslant c_1 \quad \forall \|z_1\| \leqslant \eta$$

- *A2. The subsystem $\Sigma_2 : \dot{z}_2 = f_2(t,z_2)$ is globally uniformly asymptotically stable.*
- *A3. For the system (10), there exists a bounded function $W(t,z_1)$: $R_{\geqslant 0} \times R^2 \to R_{\geqslant 0}$ positive definite, proper and radially unbounded, which satisfies*

$$\dot{W}(t,z_1)|_{(10)} \leqslant \gamma_1(W(t,z_1)), \quad \forall t \geqslant t_0 \geqslant 0$$

where $\dot{W}(t,z_1)|_{(\#)}$ denotes the time derivative of $W(t,z_1)$ along the solutions of the differential equation (#). γ_1: $R_{\geqslant 0} \to R_{\geqslant 0}$ is a non-decreasing function satisfying the following condition for some constant $a > 0$

$$\gamma_1(a) \geqslant 0; \quad \int_a^\infty \frac{d\eta}{\gamma_1(\eta)} = \infty$$

Proof. To prove that the cascaded system (10) and (11) is GUS, we must show that all the solutions of (10) and (11) are uniformly bounded and uniformly attractive to the origin. The similar results were given by Vidyasagar (1980, Theorem 3) except the assumption A3. From Li and Tian (2007, Theorem 1), it can be seen that assumption A3 is also obtained from Vidyasagar (1980, Theorem 3). □

Remark 1. One important idea of the cascaded control design is simplifying the design of laws [23,24], which is often significant in some cases [30]. In this paper, cascaded ideas are used to simplify the kinematic dynamics of mobile robots so as to derive a class of novel control laws.

3.2. Design of angular control law based on finite-time control technique

According to the ideas of the cascaded system, we can view (8) as a first-order subsystem Σ_2:

$$\dot{\theta}_e = w_d - w \tag{14}$$

Theorem 1. *Consider a first-order linear system*

$$\dot{x} = u \tag{15}$$

which can be stabilized by the following control law in a finite time

$$u = -x - \alpha x - \beta x^{q_0/p_0} \tag{16}$$

where $x \in R$ denotes state variables, $\alpha > 0$, $\beta > 0$, $p_0 > 0$ and $q_0 > 0$ are odd integers, $q_0/p_0 < 1$.

Proof. Substituting (16) into (15), we obtain

$$\dot{x} = -x - \alpha x - \beta x^{q_0/p_0} \tag{17}$$

From (17), we have

$$x^{-q_0/p_0}\frac{dx}{dt} + (1+\alpha)x^{1-q_0/p_0} = -\beta \tag{18}$$

Let $y = x^{1-q_0/p_0}$. We have $\frac{dy}{dt} = \frac{p_0-q_0}{p_0} x^{-q_0/p_0} \frac{dx}{dt}$, such that

$$\frac{dy}{dt} + \frac{p_0 - q_0}{p_0}(1 + \alpha)y = -\frac{p_0 - q_0}{p_0}\beta \tag{19}$$

Note that the solution of $\frac{dy}{dx} + P(x)y = Q(x)$ is given by

$$y = e^{-\int P(x)dx}\left(\int Q(x)e^{\int P(x)\,dx}dx + C_0\right) \tag{20}$$

Applying (20), the solution of (19) can be written as

$$y = e^{-\int_0^t \frac{p_0-q_0}{p_0}(1+\alpha)dt}\left(\int_0^t -\frac{p_0 - q_0}{p_0}\beta e^{\int_0^t \frac{p_0-q_0}{p_0}(1+\alpha)dt}\,dt + C_0\right) \tag{21}$$

Let $C_0 = y(0)$, we have

$$y = e^{-\frac{p_0-q_0}{p_0}(1+\alpha)t}\left(\int_0^t -\frac{p_0 - q_0}{p_0}\beta e^{\frac{p_0-q_0}{p_0}(1+\alpha)t}\,dt + y(0)\right) = -\frac{\beta}{1+\alpha} + \frac{\beta}{1+\alpha}e^{-\frac{p_0-q_0}{p_0}(1+\alpha)t} + y(0)e^{-\frac{p_0-q_0}{p_0}(1+\alpha)t} \tag{22}$$

Assume that $t = t_s$, then $x = 0$. Note that $y = 0$ when $x = 0$. From (22), we obtain

$$e^{\frac{p_0-q_0}{p_0}(1+\alpha)t_s} = \frac{\beta + (1 + \alpha)y(0)}{\beta} \tag{23}$$

From an initial state $x(0) \neq 0$ to $x = 0$, the convergence time can be expressed by

$$t_s = \frac{p_0}{(1 + \alpha)(p_0 - q_0)}\ln\frac{(1 + \alpha)x(0)^{(p_0-q_0)/p_0} + \beta}{\beta} \tag{24}$$

The proof is complete. $\square$

Consider the first-order subsystem (14), we design a finite-time control law by using Theorem 1. The control law can be designed as

$$w = w_d + \theta_e + \alpha_1\theta_e + \beta_1\theta_e^{q_1/p_1} \tag{25}$$

where $\theta_e \in R$ denotes state variables, $\alpha_1 > 0$, $\beta_1 > 0$, $p_1 > 0$ and $q_1 > 0$ are odd integers, $q_1/p_1 < 1$. From (16), the control law of angular velocity is designed in the following form:

$$\dot{\theta}_e = -\theta_e - \alpha_1\theta_e - \beta_1\theta_e^{q_1/p_1} \tag{26}$$

Therefore, the system (14) will reach $\theta_e = 0$ at time t_1, which is given by

$$t_1 = \frac{p_1}{(1 + \alpha_1)(p_1 - q_1)}\ln\frac{(1 + \alpha_1)\theta_e(0)^{(p_1-q_1)/p_1} + \beta_1}{\beta_1}$$

Remark 2. If θ_e is far away from zero, the control law (26) can be approximated by its linear term $\dot{\theta}_e = -\theta_e - \alpha_1\theta_e$ whose fast convergence when it is far away from zero is well understood. If θ_e is close to zero, the control law (26) can be approximated by the nonlinear term $\dot{\theta}_e = -\beta_1\theta_e^{q_1/p_1}$.

Theorem 2. *Consider the system* (14)*, there exists the control law* (25) *such that the system* (14) *is finite-time stable.*

Proof. According to $\theta_e + \alpha_1\theta_e + \beta_1\theta_e^{q_1/p_1} = 0$, we only obtain the solution $\theta_e = 0$. On the other hand, to analyze the stability of the control system given by (14) and (25), we choose Lyapunov function described by

$$V = \frac{1}{2}\theta_e^2 \tag{27}$$

Using (26), we can have

$$\dot{V} = \dot{\theta}_e\theta_e = (-\theta_e - \alpha_1\theta_e - \beta_1\theta_e^{q_1/p_1})\theta_e = -\theta_e^2 - \alpha_1\theta_e^2 - \beta_1\theta_e^{1+q_1/p_1} = -\theta_e^2 - \alpha_1\theta_e^2 - \beta_1\theta_e^{(p_1+q_1)/p_1}$$

Noting that $p_1 > 0$ and $q_1 > 0$ are odd integers, $\alpha_1 > 0$ and $\beta_1 > 0$, we obtain

$$\dot{V} \leqslant 0(\dot{V} = 0 \text{ if and only if } \theta_e = 0)$$

According to Lemma 2, the closed-loop system is finite-time stable.

From the stability analysis of the control system (14) and (25), the Lyapunov function V is positive definite (i.e. $V(t,0) = 0$ and $V(t,\theta_e) > 0$ for all $\theta_e \neq 0$), its derivative $\dot{V}$ is negative definite (i.e. $\dot{V}(t,0) = 0$ and $\dot{V}(t,\theta_e) < 0$ for all $\theta_e \neq 0$). Therefore, the closed-loop system (14) and (25) is global finite-time stable. ☐

Remark 3. The control law of the angular velocity is designed for mobile robots based on finite-time control method. The corresponding first-order system is stabilized by this finite-time control law of the angular velocity such that the angle error tends to zero in a finite time. With this, the error dynamics (6)–(8) are converted into a second-order system based on the cascaded approach. Different from the finite-time control laws in [13,19,20], we introduce two linear terms into the design of the finite-time control laws in order to obtain faster convergence rate.

3.3. Design of the forward velocity control law based on finite-time sliding mode control

Note that the control law (25) guarantees that the angular error converges to zero. Thus the nonlinear tracking error mode (6)–(8) can be reduced to the following linear time-varying system when $\theta_e = 0$.

$$\begin{bmatrix} \dot{x}_e \\ \dot{y}_e \end{bmatrix} = \begin{bmatrix} 0 & w_d(t) \\ -w_d(t) & 0 \end{bmatrix} \begin{bmatrix} x_e \\ y_e \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} [v_d(t) - v] \tag{28}$$

Define $x_1 = \begin{bmatrix} x_e \\ y_e \end{bmatrix}, \mathbf{A}_1 = \begin{bmatrix} 0 & w_d(t) \\ -w_d(t) & 0 \end{bmatrix}, \mathbf{B}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $u_1 = v_d(t) - v$, then

$$\dot{x}_1(t) = \mathbf{A}_1(t)x_1(t) + \mathbf{B}_1 u_1(t) \tag{29}$$

We are now in the position of proving the controllability of the system (29). According to the controllability criterion, we can obtain

$$\mathbf{M}_0(t) = \mathbf{B}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\mathbf{M}_1(t) = -\mathbf{A}_1(t)\mathbf{M}_0(t) + \frac{d}{dt}\mathbf{M}_0(t) = \begin{bmatrix} 0 \\ w_d(t) \end{bmatrix}$$

Furthermore, $\text{Rank}[\mathbf{M}_0(t) \vdots \mathbf{M}_1(t)] = \text{Rank}\begin{bmatrix} 1 & 0 \\ 0 & w_d(t) \end{bmatrix} = 2$, where $w_d(t) \neq 0$. Therefore, the system (29) is controllable.

Based on the theory of cascaded control design, if $w_d(t) \neq 0$, we consider (28) as the subsystem Σ_1, and (28) can be transformed into

$$\dot{x}_e = w_d y_e - v + v_d \tag{30}$$
$$\dot{y}_e = -w_d x_e \tag{31}$$

We further consider the subsystem given by (31) and design a state feedback of x_e so as to robustly stabilize the state $y_e = 0$.

Theorem 3. *Consider system (31), where sgn(·) is the sign function. Note that $w_d \neq 0$, k_0 is a constant and $k_0 > 0$. There exists $x_e = k_0 y_e sgn(w_d)$ such that the system (31) is global uniform stable.*

Proof. To solve (31) for the equilibrium point, we get $-w_d x_e = 0$, and $x_e = 0$. Therefore, the subsystem (31) only has one equilibrium point. To analyze the stability of system (31), we consider the Lyapunov function candidate $V(y_e) = \frac{1}{2}y_e^2$. The Lyapunov derivative $\dot{V}(y_e)$ is given by

$$\dot{V}(y_e) = y_e \dot{y}_e = y_e(-w_d x_e) = -k_0 y_e^2 w_d sgn(w_d)$$

Note that $sgn(w_d) > 0$ when $w_d > 0$ and $sgn(w_d) < 0$ when $w_d < 0$, so $w_d sgn(w_d) > 0$. $\dot{V}(y_e) \leqslant 0$ ($\dot{V}(y_e) = 0$ if and only if $y_e = 0$), which implies that the zero solution $x_e = 0$ to (31) is globally asymptotically stable. ☐

Remark 4. In Theorem 3, we can use a class of continuous functions such as $\frac{w_d}{|w_d|+\delta}$, $1 - e^{-k_1 w_d}$ and $-1 + e^{k_2 w_d}$ to substitute the term $sgn(w_d)$, where δ, k_1, k_2 are constant and $\delta > 0$, $k_1 > 0$, $k_2 > 0$, Theorem 3 still holds.

The sliding manifold is defined by

$$s = x_e - k_0 y_e sgn(w_d) \tag{32}$$

Using $x_e = k_0 y_e sgn(w_d)$, we obtain

$$s = x_e - k_0 y_e sgn(w_d) = 0 \tag{33}$$

To guarantee the existence of the sliding manifold s, namely $\dot{s}s < 0$, $\dot{s}$ is specified by

$$\dot{s} = -s - \alpha_2 s - \beta_2 s^{q_2/p_2} \tag{34}$$

where $\alpha_2 > 0$, $\beta_2 > 0$, $p_2 > 0$ and $q_2 > 0$ are odd integers, $q_2/p_2 < 1$.

From Theorem 1, the system (34) will reach $s = 0$ at t_2 which is written by

$$t_2 = \frac{p_2}{(1 + \alpha_2)(p_2 - q_2)} \ln \frac{(1 + \alpha_2)s(0)^{(p_2 - q_2)/p_2} + \beta_2}{\beta_2}$$

The sliding mode control law is designed using (30)–(32), (34). We can obtain

$$\dot{s} = -s - \alpha_2 s - \beta_2 s^{q_2/p_2} = \dot{x}_e - k_0 \dot{y}_e sgn(w_d) = w_d y_e - v + v_d - k_0(-w_d x_e)sgn(w_d) \tag{35}$$

Then, it follows from (35) that

$$v = v_d + w_d y_e + k_0 w_d x_e sgn(w_d) + s + \alpha_2 s + \beta_2 s^{q_2/p_2} \tag{36}$$

Consider the subsystem (30) and (31), we obtain the control law of the forward velocity which guarantees global uniform stability of this closed-loop subsystem by exploiting the finite-time sliding mode control.

Remark 5. By choosing the sliding manifold shown in (32), the state x_e will converge to $k_0 y_e sgn(w_d)$ in finite time by the design of the reaching law (34). Obviously, if the state $x_e = 0$, then the state $y_e = 0$. From Theorem 3, we know that y_e will converge to zero eventually with the change of convergence time. It follows that x_e will converge to zero as long as y_e converges to zero.

Remark 6. In [13,19,20], the desired angular velocity w_d for mobile robots needs to satisfy $0 < w_d^{min} \leqslant |w_d| \leqslant w_d^{max}$, where w_d^{min} and w_d^{max} are appropriate constants. In contrast, the desired angular velocity w_d only needs to satisfy $w_d \neq 0$ in this paper. Hence, the design of the control laws in this paper relaxes strict constraints on the desired velocities reported in the existing literatures.

Remark 7. From (34), it is remarkable that the design of the control law introduces a nonlinear term for the second-order system (30) and (31). In [2,3,5,18,22,39,41], the states arrive at the sliding manifold in infinite settling time. In this paper, we construct the control law of the forward velocity by using finite-time sliding mode control in order to improve the transient performance substantially. Furthermore the finite-time sliding mode control law makes the states of the error dynamics arrive at the sliding manifold in the finite time. Accordingly, it is favorable to improve the convergence rate of the closed-loop system (30), (31) and (36). In addition, the sign function $sgn(s)$ is employed to construct the reaching law $\dot{s}$ in [2,5,39,41]. However, the chattering phenomenon is caused by this form of the reaching law. It is well-known that the main disadvantage of sliding mode control method is the chattering problem. In this paper, we use the continuous function shown in (34) instead of the piecewise function form such as [2,5,39,41]. This is effective to eliminate the chattering phenomenon.

3.4. Stability analysis

In this section, we analyze the stability of the closed-loop system (6)–(8), (25) and (36) by using Lemma 1.

Theorem 4. *Consider the system* (6)–(8), *there exist the control laws* (25) *and* (36) *such that the system* (6)–(8) *is global uniform stable.*

Proof. Note that the sliding manifold s converges to zero in a finite time, we can obtain $x_e = k_0 y_e sgn(w_d)$, and (36) will be replaced by

$$v = v_d + w_d y_e + k_0 w_d x_e sgn(w_d) \tag{37}$$

Substituting (25) and (37) into (6)–(8), we have

$$\begin{bmatrix} \dot{x}_e \\ \dot{y}_e \end{bmatrix} = \begin{bmatrix} -k_0 w_d x_e sgn(w_d) \\ -x_e w_d \end{bmatrix} + \begin{bmatrix} y_e \sigma(\theta_e) + v_d(\cos\theta_e - 1) \\ v_d \sin\theta_e - x_e \sigma(\theta_e) \end{bmatrix} \tag{38}$$

$$\dot{\theta}_e = -\sigma(\theta_e) \tag{39}$$

where $\sigma(\theta_e) = \theta_e + \alpha_1 \theta_e + \beta_1 \theta_e^{q_1/p_1}$, obviously, the function $\sigma(\cdot) \in K$. Using Lemma 1, we obtain

$$f_1(t, z_1) = \begin{bmatrix} -k_0 w_d x_e sgn(w_d) \\ -x_e w_d \end{bmatrix}, \quad g(t, z_1, z_2) = \begin{bmatrix} y_e \sigma(\theta_e) + v_d(\cos\theta_e - 1) \\ v_d \sin\theta_e - x_e \sigma(\theta_e) \end{bmatrix},$$

$$f_2(t, z_2) = -\sigma(\theta_e).$$

Consider a state transformation defined by

$$z_1 = \begin{bmatrix} x_e \\ y_e \end{bmatrix}, \quad z_2 = \theta_e$$

The derivatives of z_1 and z_2 are

$$\dot{z}_1 = f_1(t, z_1) + g(t, z_1, z_2) \tag{40}$$
$$\dot{z}_2 = -\sigma(\theta_e) \tag{41}$$

• Verification of A1: Consider the system $\dot{z}_1 = f_1(t, z_1)$, we choose Lyapunov function described by

$$V = \frac{1}{2} x_e^2 + \frac{1}{2} y_e^2 \tag{42}$$

Then we compute the time derivative of (42) along the subsystem (30) and (31), we have

$$\dot{V} = \dot{x}_e x_e + \dot{y}_e y_e = (w_d y_e - v + v_d) x_e + (-w_d x_e) y_e = (-k_0 w_d x_e sgn(w_d)) x_e + (-w_d x_e) y_e$$
$$= -k_0 w_d x_e^2 sgn(w_d) - k_0 w_d y_e^2 sgn(w_d)$$

Note that $w_d sgn(w_d) > 0$ and $k_0 > 0$, we obtain

$$\dot{V} \leqslant 0 (\dot{V} = 0 \text{ if and only if } x_e = 0 \text{ and } y_e = 0)$$

Therefore, $\dot{z}_1 = f_1(t, z_1)$ is GUS and satisfies the rest of A1.
• Verification of A2: Consider the system $\dot{z}_2 = -\sigma(\theta_e)$, from the proof of Theorem 2, we can draw a conclusion that this system satisfies A2.
• Verification of A3: Consider the system (40), we choose Lyapunov function described by

$$W = \frac{1}{2} x_e^2 + \frac{1}{2} y_e^2 \tag{43}$$

which is obviously a decreasing function. Taking the derivative of (43) along the system (40), we obtain

$$\dot{W} = \dot{x}_e x_e + \dot{y}_e y_e = -k_0 w_d x_e^2 sgn(w_d) - k_0 w_d y_e^2 sgn(w_d) + v_d[x_e(\cos\theta_e - 1) + y_e \sin\theta_e] \leqslant v_d[x_e(\cos\theta_e - 1) + y_e \sin\theta_e]$$
$$\leqslant 2 v_d^{max} \|z_1\|$$

Taking $\gamma_1(\eta) = \eta$, we can verify that the system (40) satisfies A3. Therefore, using Lemma 1, we know that Theorem 4 holds. □

4. Simulation results

To demonstrate the effectiveness of the proposed control laws, we simulate the kinematic differential error model of mobile robots with (6)–(8). The following section is discussed in two different situations.

(1) Tracking of a curve line with the desired velocities given by $v_d(t) = 1$, $w_d(t) = 1$.

The desired posture $q_d = (x_d, y_d, \theta_d)^T$ is specified as

$$\begin{cases} x_d(t) = 1.5\cos(\pi t/15) \\ y_d(t) = 1.5\sin(2\pi t/15) \\ \theta_d(t) = w_d t = t \end{cases} \tag{44}$$

We employ the control laws (25) and (36), where all the parameters are given by $\alpha_1 = 4$, $\beta_1 = 8$, $p_1 = 7$, $q_1 = 5$, $p_2 = 5$, $q_2 = 3$, $k_0 = 1$, $\alpha_2 = 0.5$, $\beta_2 = 2$. In the simulation, we take the initial posture errors as $[x_e(0), y_e(0), \theta_e(0)]^T = [-1.8, 0.3, -\pi/5]^T$. The simulation results are showed in Figs. 2–4.

Furthermore, to verify that the convergence rate in this paper is faster than in [13,19], the control law of angular velocity is chosen as $w = w_d + \beta_1 \theta_e^{q_1/p_1}$ which was reported in the two references above. We then also simulate the closed loop system, where the comparative simulation results are showed in Fig. 5.

(2) Tracking of a curve line with the desired velocities given by $v_d(t) = 2t$, $w_d(t) = t$ $(|v_d(t)| \leqslant 1, |w_d(t)| \leqslant 1)$.

Fig. 2. Trajectory tracking results.

(a)

(b)   (c)

Fig. 3. Trajectory tracking errors. (a) *X* axis error; (b) *Y* axis error; (c) Angle error.

Fig. 4. Control inputs.
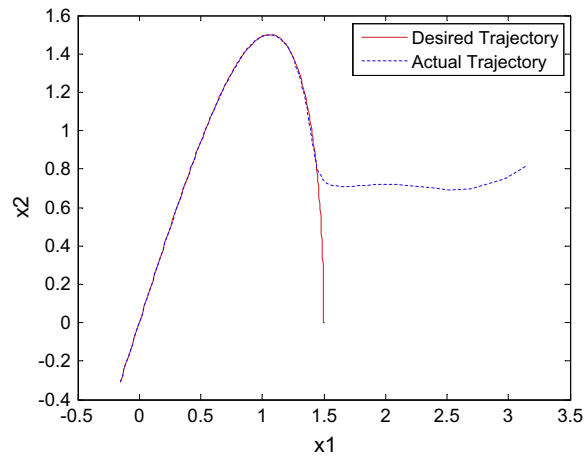

Fig. 5. Angle error.
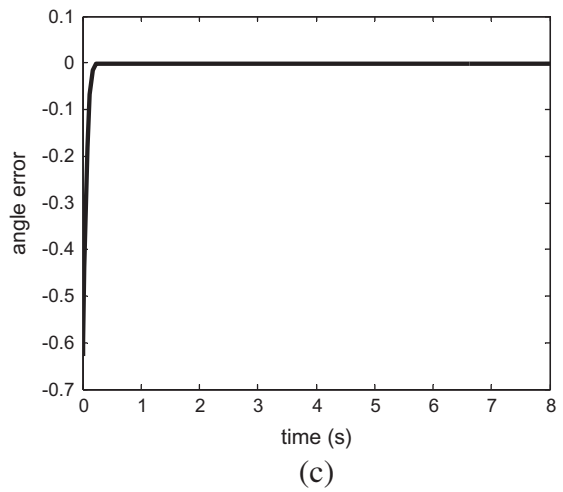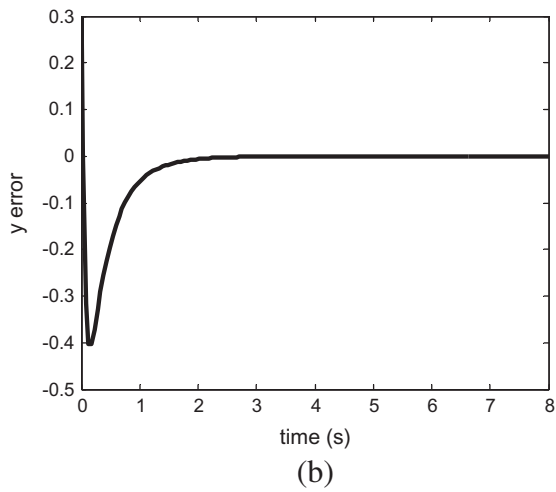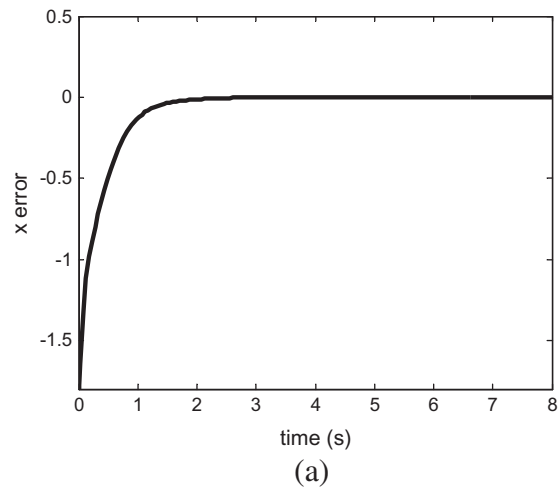

Fig. 6. Trajectory tracking results.

Fig. 7. Trajectory tracking errors. (a) X axis error; (b) Y axis error; (c) Angle error.
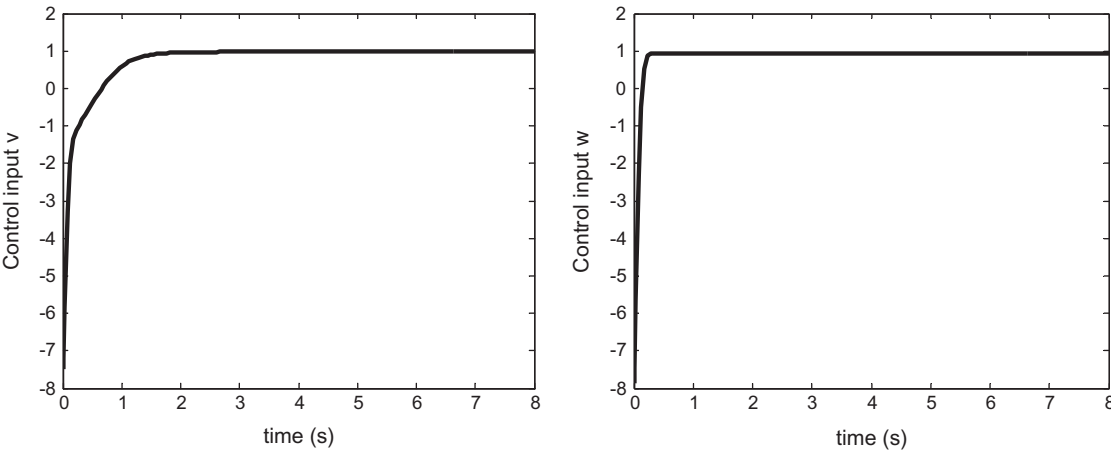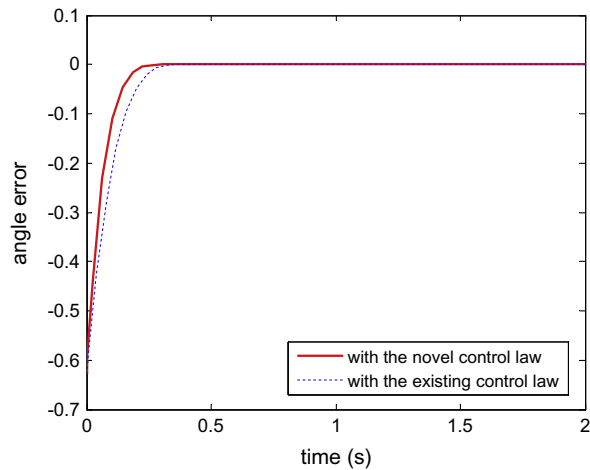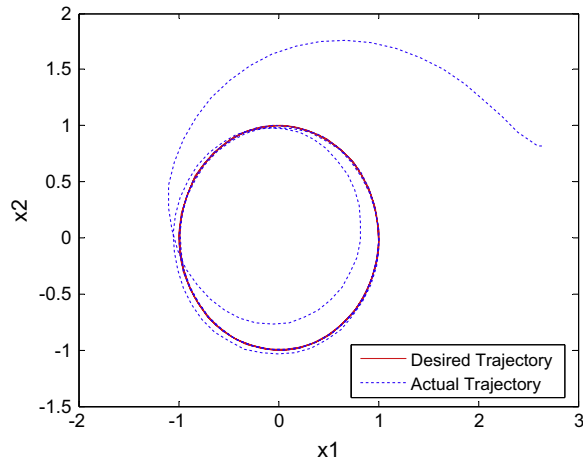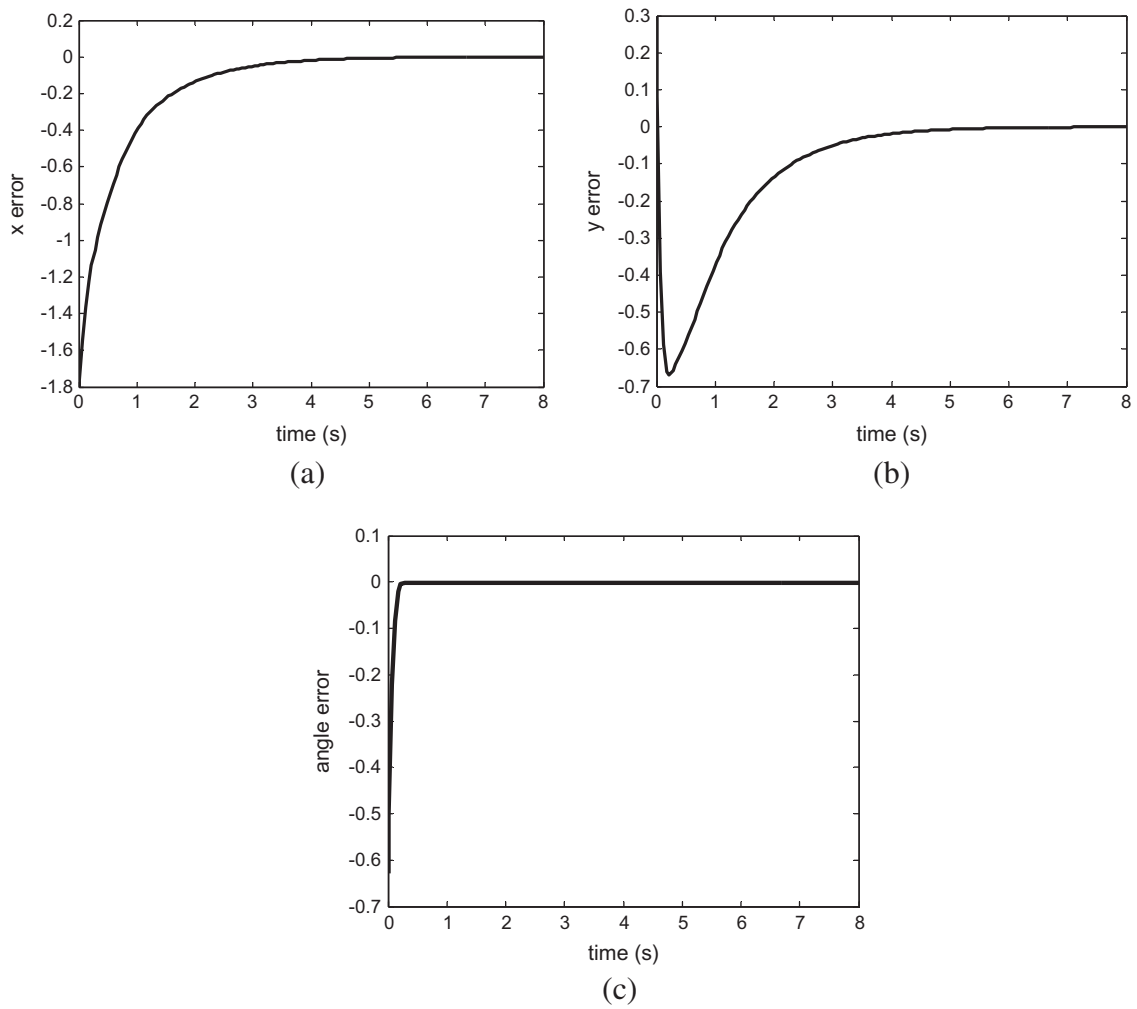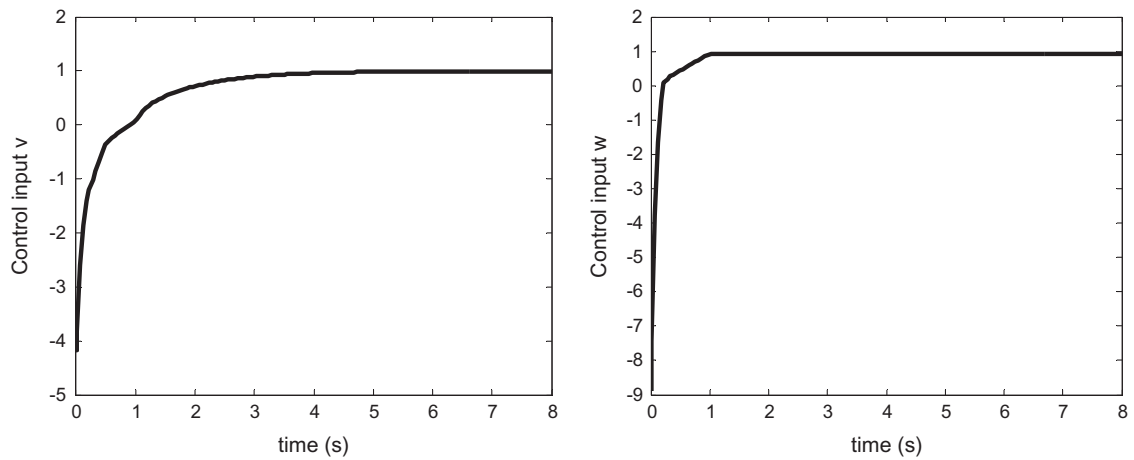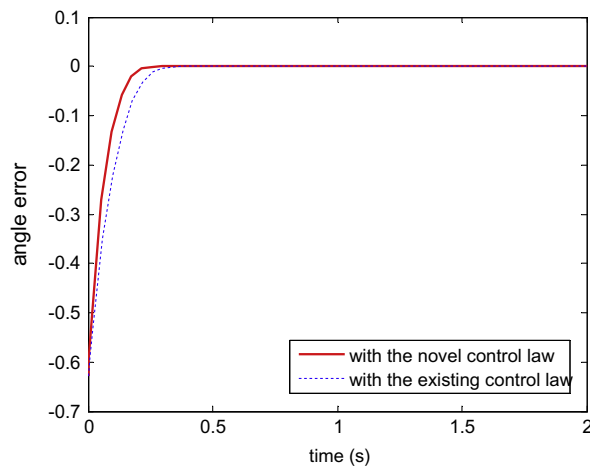

Fig. 8. Control inputs.

Fig. 9. Angle error.

The desired posture $q_d = (x_d, y_d, \theta_d)^T$ is given by

$$\begin{cases} x_d(t) = \cos(\pi t) \\ y_d(t) = \sin(\pi t) \\ \theta_d(t) = w_d t = t \end{cases} \tag{45}$$

We also use the time-varying state feedback control law (25) and (36), where all the parameters and the initial posture errors remain unchanged except $\alpha_2 = 0.3$, $\beta_2 = 1$. The simulation results are depicted in Figs. 6–8.

Similarly, the control law of angular velocity is chosen as $w = w_d + \beta_1 \theta_e^{q_1/p_1}$ which was reported in [13,19]. Then we also simulate the closed loop system in the condition of two different virtual angular velocities, the comparative simulation results are showed in Fig. 9.

As shown in Figs. 2 and 6, the red[1] solid line represents the desired trajectory; the blue dotted line represents the practical trajectory. Figs. 3 and 7 give the error of X-axis, the error of Y-axis and the error of angle for mobile robots respectively. Figs. 4 and 8 illustrate the control inputs v and w of mobile robots. In Figs. 5 and 9, the red solid line denotes the angle error with the control law designed in the present paper, and the blue dotted line denotes the angle error with the control law presented in [13,19]. Obviously, we can find that the convergence rate with respect to the error of angle is faster than the results reported in the two reported papers. According to above two different situations, as far as the desired velocities are concerned, no matter they are constant or time-varying, the mobile robot can always track the desired trajectory nicely. In the meanwhile, three state errors can be stabilized by using the proposed control laws in this paper.

5. Conclusions

In this study, a novel control method is proposed to achieve the trajectory tracking control of nonholonomic mobile robots. The error model of mobile robots is divided into two subsystems by using the cascaded design. For the first-order subsystem, this paper develops a finite-time control law in order to stabilize the angular error. To relax the strict constraints on the desired angular velocity for mobile robots, the finite-time sliding mode control is used to design the control law of the forward velocity for the second-order subsystem, which improves the convergence rate and overcomes the chattering of the sliding mode control systems. The stability of the developed control method is analyzed, and some sufficient conditions are given. Finally, the simulation results verify the feasibility of this proposed control approach. Compared to the general finite-time control methods, the proposed control method can achieve faster convergence rate.

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[1] For interpretation of color in 'Figs. 2, 5, 6, and 9', the reader is referred to the web version of this article.

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