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Consensus on Intervals of Communication Delay

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Abstract: This paper brings out a structured methodology for identifying intervals of communication time-delay where consensus in directed networks of multiple agents with high-order integrator dynamics is achieved. It is built upon the stability analysis of a transformed consensus problem which preserves all the nonzero eigenvalues of the Laplacian matrix of the associated communication topology graph. It is shown that networks of agents with first-order integrator dynamics can be brought to consensus independently of communication delay, on the other hand, for agents with second-order integrator dynamics, the consensus is achieved independently of communication delay only if certain conditions are satisfied. Conversely, if such conditions are not satisfied, it is shown how to compute the intervals of communication delay where multiple agents with second-order or higher-order can be brought to consensus. The paper is ended by showing an interesting example of a network of agents with second-order integrator dynamics which is consensable on the first time-delay interval, but as the time-delay increases, it loses consensability on the second time-delay interval, then it becomes consensable again on the third time-delay interval, and finally it does not achieve consensus any more on the fourth time-delay interval. This example shows the importance of analyzing consensus with time-delay in different intervals.

Keywords: Time delay systems, multi-agent systems, consensus, communication delay, roots location.

1 Introduction

One of the ongoing topics covered by the theory of multi-agent systems is consensus. The meaning of consensus problem is to make all the agents in a multi-agent system achieve an agreement on a variable of interest, assuming that each agent is able to share and/or acquire information within a subset of other agents, called neighbors. Applications of consensus are found in many practical fields, such as traffic jams in communication networks[1], formation of autonomous mobile agents[2] and underwater vehicles[3], robotics[4], etc. Many other results are summarized in [5].

In practice, time-delays are always present in multi-agent interactions. This is mainly due to computational and physical limitations in information processing, transmission channels, time-response of actuators, etc. The presence of delays has significant impact on consensus problems as it can make the system oscillate or diverge about the variable of interest[6]. Based on this fact, consensus problems are studied considering different forms of time-delays.

The class of delays due to the time spent by an agent to acquire information from another agent in the network, which can arise naturally due to physical characteristics of communication channels or sensing, is called communication delay. It essentially indicates how old is the information received from the neighboring agent. Results from [7] showed that a multi-agent system composed of individuals with a single-integrator dynamics is able to achieve consensus regardless of communication delays, as long as the information from one of the agents can reach all the other agents. This network constraint will be presented later as the existence of a directed spanning tree in the graph that describes the communication topology. From this result, most of the results in the literature have dealt only with input delays, which is the class of delays affecting the states of both local and neighbor agents, and is mainly due to the time-delay in the control inputs. Note that the term communication delay has been used interchangeably in some papers to describe input delays. In this paper, we analyze only communication delays.

In the literature, Munz et al.[8] studied leader-following consensus for multi-agent systems composed of first-order and second-order agents with delayed and intermittent communication, and presented sufficient conditions to guarantee consensus based on bilinear matrix inequalities. In [9], second-order dynamics and constant communication delays in undirected topologies were considered, and Cepeda-Gomez and Olgac presented conditions based on an introduced concept of most exigent and most critical eigenvalues. For practical applications, communication delays have also been considered in the problem of platooning of vehicles[10].

In most of these studies, as in the case of input-delays, the main purpose is to find the upper bounds of the time-delays such that consensus can still be achieved, with the usual acceptance that the system can achieve consensus for any value smaller than the upper bound. This approach is also applied for time-varying delays as in [11], which shows

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an upper bound $\bar{\tau}$ such that the system achieves consensus for any value of delay $\tau(t) \in [0, \bar{\tau}]$. A more general case is addressed in [12], which considers directed networks subject to multiple input time-varying delays belonging to an interval with lower and upper bound $[\tau_1, \tau_2]$, where $0 < \tau_1 \leq \tau_2$. In this paper, motivated by [12], we present a methodology to identify the time-delay intervals where consensus in networks of multiple agents is achieved. Nevertheless, we call attention to the main difference between [12] and the present paper: In the former, sufficient conditions are proposed for the analysis of consensus of multi-agent systems subject to input time-varying delays and switching topology. On the other hand, the present paper offers necessary and sufficient conditions for the analysis of consensus of multi-agent systems subject to communication delay. Although not as general in application as in [12], the present paper is based on an entirely different framework for assessing exactly the communication time-delay intervals where consensus in networks of multiple agents is achieved.

Contributions. In this paper, motivated by the analysis of consensus within intervals, we show that communication delays do not always degrade the system consensability, actually it may even enable the system to achieve consensus in given intervals. It is also shown that if a multi-agent system achieves consensus for a given time-delay, it may not achieve consensus for any smaller delay. The main result of consensability switches is presented for directed networks of agents described by chains of integrators with communication delay. As particular results, we present consensability conditions for networks of agents with first-order and second-order dynamics. The methodology for the analysis is similar to the ones carried out for input-delays in [13, 14], however considering communication delays. One application of the proposed results is illustrated in an example considering a network of second-order integrator agents where, based on the proposed consensability conditions, the consensus protocol gains are designed in order to address delay-independent or delay-dependent consensability. The latter case is shown to be more complex due to the fact that the system achieves consensus in two disconnected delay intervals.

Throughout the text, let $\mathbf{N}$ be the set of natural numbers, $\mathbf{R}$ be the set of real numbers, I_n be an identity matrix of size $n \in \mathbf{N}$, 0_n and 1_n be column vectors of zeros and ones of size n, respectively, $\mathbf{0}_{m\times n}$ be an $m \times n$ zero matrix with $m \in \mathbf{N}$, $\otimes$ denote the Kronecker product, and $\lambda_i\{\cdot\}$ be the i-th eigenvalue of a matrix.

2 Preliminaries

2.1 Algebraic graph theory

The information flow in a multi-agent system can be represented by a graph, following the next terminology and notation. Let the simple weighted directed graph be defined by the ordered triplet $G(V, E, A)$, where V is a set with $m \in \mathbf{N}$ vertices (nodes) arbitrarily labeled as $v_1, v_2, \cdots, v_m$, ε is a set of the edges connecting the vertices, denoted by $e_{ij} = (v_i, v_j)$, where the first element v_i is said to be the parent node (tail) and the latter v_j to be the child node (head), and $A = [a_{ij}]$ is the adjacency matrix of order $m \times m$ related to the edges, that assigns a real non-negative value a_{ij} for each e_{ji}:

$$a_{ij} \begin{cases} = 0, & \text{if } i = j \text{ or } \nexists\ e_{ji} \\ > 0, & \text{if } f\ \exists\ e_{ji}. \end{cases} \tag{1}$$

Related to A, a diagonal degree matrix is defined as $\Delta = [\Delta_{ij}]$, with elements $\Delta_{ii} = \sum_{j=1}^{m} a_{ij}$. The Laplacian matrix associated with the graph G is thus given by $L = \Delta - A$.

A directed tree is a directed graph with only one node without parents, called root, and all the other nodes with exactly one parent. Also, there is a path, a sequence of edges, connecting the root to any other node in the tree. A directed spanning tree of a graph is a directed tree that can be formed from the removal of some of the edges of such graph, with all the nodes included.

Next lemmas are used in the derivation of the further analysis.

Lemma 1.[15] The Laplacian matrix L of a given directed graph G has at least one zero eigenvalue with the associated eigenvector 1_m, and all the nonzero eigenvalues are in the open right half-plane. Furthermore, L has exactly one zero eigenvalue if and only if G has a directed spanning tree.

Lemma 2.[16] If a nonnegative matrix $M = [m_{ij}] \in \mathbf{R}^{m\times m}$ has all the row sums given as the same positive constant $\mu > 0$, then μ is an eigenvalue of M with an associated eigenvector 1_m and $\rho(M) = \mu$, where $\rho(\cdot)$ denotes the spectral radius.

3 System dynamics

Consider a multi-agent system composed of $m \in \mathbf{N}$ agents with state variables $x_i(t) = [x_{i,1}(t) \quad x_{i,2}(t) \quad \cdots \quad x_{i,n}(t)]^{\mathrm{T}}$, for $i = 1, 2, \cdots, m$, with $x_{i,1}(t),\ x_{i,2}(t),\ \cdots,\ x_{i,n}(t) \in \mathbf{R}$ such that $x_i(t) \in \mathbf{R}^n$, and let the dynamics be given by a chain of integrators

$$\begin{aligned} \dot{x}_{i,1}(t) &= x_{i,2}(t) \\ &\vdots \\ \dot{x}_{i,n-1}(t) &= x_{i,n}(t) \\ \dot{x}_{i,n}(t) &= u_i(t) \end{aligned} \tag{2}$$

where $u_i(t) \in \mathbf{R}$ is the control input acting directly on $\dot{x}_{i,n}(t)$. This agent dynamics can be also written as

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t) \tag{3}$$

with $A = \begin{bmatrix} 0_{n-1} & I_{n-1} \\ 0 & 0_{n-1}^{\mathrm{T}} \end{bmatrix}$ and $B = \begin{bmatrix} 0_{n-1} \\ 1 \end{bmatrix}$.

We thus define consensus for multi-agent systems.

Definition 1. The multi-agent system with state variables $x_i(t) \in \mathbf{R}^n$ asymptotically achieves consensus if, for all $i \neq j$, $\lim_{t\to\infty} x_i(t) - x_j(t) = 0$ hold for any initial state conditions. A multi-agent system that is able to achieve consensus is called consensable.

The following consensus protocol, free of delays, is considered in order to drive the agents toward consensus:

$$u_i(t) = -\sum_{j=1}^{m} a_{ij} K\big(x_i(t) - x_j(t)\big) \tag{4}$$

with $K = [\alpha_1 \quad \alpha_2 \quad \cdots \quad \alpha_n]$, where the real positive scalars $\alpha_1, \alpha_2, \cdots, \alpha_n$ are the consensus protocol gains and a_{ij} are given as in (1) by the elements of the adjacency matrix A of graph G describing the network topology.

Considering (2) with consensus protocol (4), the closed-loop dynamics of the whole multi-agent system can be written as

$$\dot{x}(t) = \Gamma x(t) \tag{5}$$

where $x(t)$ is a lumped vector with all the agents$'$ states as $x^{\mathrm{T}} = [x_1^{\mathrm{T}} \; x_2^{\mathrm{T}} \; \dots \; x_m^{\mathrm{T}}]$, and

$$\Gamma = I_m \otimes A - L \otimes (BK). \tag{6}$$

The following lemmas provide important results concerning the multi-agent system in (5).

Lemma 3.[17] Matrix Γ in (5) has at least n zero eigenvalues. It has exactly n zero eigenvalues if and only if the Laplacian L has a simple zero eigenvalue. Moreover, if L has a simple zero eigenvalue, the zero eigenvalue of Γ has geometric multiplicity equal to one.

Lemma 4.[17] The system in (5) achieves consensus asymptotically if and only if matrix Γ has exactly n zero eigenvalues and all the other eigenvalues have negative real parts.

Note that, combining Lemmas 1, 3 and 4, we have that consensus in directed networks of agents with dynamics given by high-order integrators (2) and protocol (4), and free of delays, is achieved if and only if the related graph G has a directed spanning tree and all the nonzero eigenvalues of Γ in (6) lie in the open left half-plane.

4 Communication delays

In this section, we consider the presence of a communication delay $\tau > 0$ in the consensus protocol (4), such that the information from neighboring agents is delayed. Thus, the delayed consensus protocol can be written as

$$u_i(t) = -\sum_{j=1}^{m} a_{ij} K\big(x_i(t) - x_j(t-\tau)\big) \tag{7}$$

with $K = [\alpha_1 \quad \alpha_2 \quad \cdots \quad \alpha_n]$, where the real positive scalars $\alpha_1, \alpha_2, \cdots, \alpha_n$ are the consensus protocol gains and a_{ij} are given as in (1). Moreover, the initial conditions for any agent i are arbitrary and denoted by

$$x_i(\theta) = \phi_i(\theta), \quad \forall \theta \in [-\tau, 0], \quad i = 1, 2 \cdots, n$$

where ϕ_i belongs to the set of $\mathbf{R}^n$ valued continuous functions on $[-\tau, 0]$.

The closed-loop dynamics of agents described as in (3) with the delayed consensus protocol in (7) is given by

$$\dot{x}(t) = A_M x(t) + B_M x(t-\tau) \tag{8}$$

where $A_M = (I_m \otimes A) - (\Delta \otimes BK)$ and $B_M = (\Delta - L) \otimes BK$, with Δ and L are the degree and Laplacian matrices, respectively, of graph G describing the network topology.

4.1 Transformed system

In [18], it was shown that a single-order consensus problem can be translated into a stability one using a tree-type transformation. In [12], this transformation was extended to general linear dynamics with input delay and here it is used for the case of communication delay. This is done by introducing new variables $z_i(t)$ representing the disagreement on the state variables, given by

$$z_i(t) = x_1(t) - x_{i+1}(t) \tag{9}$$

for $i = 1, 2, \cdots, m-1$. The disagreement variables can be lumped in a vector $z^{\mathrm{T}} = [z_1^{\mathrm{T}} \quad z_2^{\mathrm{T}} \quad \cdots \quad z_{m-1}^{\mathrm{T}}]$ and we can write

$$z(t) = (U \otimes I_n)x(t) \tag{10}$$

$$x(t) = 1_m \otimes x_1(t) + (W \otimes I_n)z(t) \tag{11}$$

where

$$U = [1_{m-1} \quad -I_{m-1}] \quad \text{and} \quad W = [0_{m-1} \quad -I_{m-1}]^{\mathrm{T}}. \tag{12}$$

Proposition 1. The multi-agent system achieves consensus on $x_i(t)$ if and only if the lumped disagreement vector $z(t)$ reaches the origin.

Proof. The multi-agent system is said to be in consensus when $x_i(t) = \beta(t), \forall i = 1, \cdots, m$, with arbitrary $\beta(t) \in \mathbf{R}^n$, or equivalently $x(t) = 1_m \otimes \beta(t)$. Initially, consider that the multi-agent system achieves consensus, thus, from (10),

$$\begin{aligned} z(t) = (U \otimes I_n)(1_m \otimes \beta(t)) = \\ (U 1_m) \otimes \beta(t) = 0_{m-1}. \end{aligned} \tag{13}$$

Therefore, if the multi-agent system achieves consensus on $x_i(t)$, then $z(t)$ goes to zero. Conversely, assume that $z(t)$ reaches the origin, thus, from (11),

$$\begin{aligned} x(t) = 1_m \otimes x_1(t) + (W \otimes I_n)0_{m-1} = \\ 1_m \otimes \beta(t). \end{aligned} \tag{14}$$

Then, if $z(t)$ is at the origin, the multi-agent system is in consensus. □

Based on the disagreement on the state variables, we assess consensus according to Definition 1 by studying the stability of the disagreement system $z(t)$. In order to derive the next transformation, consider Assumption 1 regarding the network topology.

Assumption 1. The network topology described by the directed graph G of the multi-agent system with communication delay is assumed to be a regular graph with $\Delta = I_m$.

In this context, we write the dynamics of $z(t)$ by taking the time-derivative of (10), considering the closed-loop dynamics of the multi-agent system in (8) with Assumption 1 (i.e., $\Delta = I_m$), and the inverse transformation given by (11). Therefore, we obtain the following reduced-order system:

$$\begin{aligned}\dot{z}(t) = (U \otimes I_n)(I_m \otimes (A - BK))x(t) + \\ (U \otimes I_n)((I_m - L) \otimes BK)x(t - \tau).\end{aligned} \qquad (15)$$

Replacing (11) into (15), it yields:

$$\begin{aligned}\dot{z}(t) = (U \otimes I_n)(I_m \otimes (A - BK))(1_m \otimes x_1(t)) + \\ (U \otimes I_n)(I_m \otimes (A - BK))((W \otimes I_n)z(t)) + \\ (U \otimes I_n)((I_m - L) \otimes BK)(1_m \otimes x_1(t - \tau)) + \\ (U \otimes I_n)((I_m - L) \otimes BK)((W \otimes I_n)z(t - \tau)).\end{aligned}$$

Since $U1_m = 0_{m-1}$ and $L1_m = 0_m$, then the terms with $x_1(t)$ vanish. Finally, the dynamics of $z(t)$ can be written as

$$\dot{z}(t) = \bar{A}_M z(t) + \bar{B}_M z(t - \tau) \qquad (16)$$

where $\bar{A}_M = I_{m-1} \otimes (A - BK)$ and $\bar{B}_M = (I_{m-1} - \bar{L}) \otimes BK$, with $\bar{L} = ULW$.

Related to the transformed Laplacian matrix $\bar{L}$, consider Lemma 5.

Lemma 5. Consider the Laplacian matrix L of a graph that has a directed spanning tree. Then, the eigenvalues of the transformed matrix $\bar{L} = ULW$, with U and W given in (12), are the nonzero eigenvalues of L, which are all in the open right half-plane.

Therefore, the main result of this paper follows from the stability analysis of the delayed system in 16. Stability conditions for time-delay systems have been studied in the literature using a myriad of techniques. For instance, the Lyapunov-Krasovskii theory can be used in the case of time-varying delays, and many recent and interesting results have been concerned on efficiently reducing conservativeness of sufficient conditions, see [19−23] for discrete-time delayed systems. On the other hand, for constant time-delay, a method to find the zero-crossing frequencies directly based on the conjugate symmetry property of the characteristic equation was presented in [24], Therefore, in the present paper considering the inherent properties of the consensus problem and following the analytical results in [24], we present necessary and sufficient conditions for consensus of the multi-agent system in (8), with $\Delta = I_m$. It allows us to write general results for chains of integrators, and specific results for first-order and second-order dynamics. These results are given in Section 5.

4.2 Analysis

Proposition 1 allows to establish that consensus for agents with high-order integrator dynamics as in (3), subject to protocol (7) with communication delay, and network topology described by a directed graph containing a directed spanning tree with Assumption 1 being verified, can be assessed by studying the stability of the reduced-dimension transformed system in (16). Note that the stability of (16) is dictated by the location of the roots of the transcendental function

$$\Delta_\tau(s) = \det(sI_{n(m-1)} - \bar{A}_M - \bar{B}_M e^{-s\tau}). \qquad (17)$$

Next, we present a proposition that plays a central role for the further analysis. It establishes that the stability of the roots of $\Delta_\tau(s)$ in (17) is equivalent to the stability of the roots of a simple set of quasi-polynomials.

Proposition 2. Consider the multi-agent system in (3) with protocol (7). Assume a network topology described by a directed graph with Assumption 1 and containing a directed spanning tree, with associated Laplacian matrix L. Then, the multi-agent system is consensable according to Definition 1 in the presence of communication delay $\tau > 0$ if and only if all roots of

$$p_i(s) = s^n + \left(1 + (\lambda_i\{L\} - 1)\mathrm{e}^{-s\tau}\right) \sum_{p=1}^{n} s^{p-1}\alpha_p \qquad (18)$$

have negative real parts, for $i = 1, 2, \cdots, m-1$, such that $\lambda_i\{L\}$ refers to the nonzero eigenvalues of the Laplacian matrix L, and α_p are the elements of K in (4).

Proof. Based on Proposition 1, we have that the multi-agent system in (8) asymptotically achieves consensus if and only if all roots of $\Delta_\tau(s)$ have negative real parts. Note that $\Delta_\tau(s)$ in (17) can be rewritten using the Laplace expansion for computing the determinant as

$$\begin{aligned}\Delta_\tau(s) = \det\Big(s^n I_{m-1} + \sum_{p=1}^{n} s^{p-1}\alpha_p\big(I_{m-1} - (I_{m-1} - \bar{L})\mathrm{e}^{-s\tau}\big)\Big) = \\ \prod_{i=1}^{m-1}\Big(s^n + \lambda_i\Big\{\big(I_{m-1} - (I_{m-1} - \bar{L})\mathrm{e}^{-s\tau}\big)\Big\}\sum_{p=1}^{n} s^{p-1}\alpha_p\Big) = \\ \prod_{i=1}^{m-1}\Big(s^n + \big(1 - (1 - \lambda_i\{\bar{L}\})\mathrm{e}^{-s\tau}\big)\sum_{p=1}^{n} s^{p-1}\alpha_p\Big).\end{aligned}$$

Based on Lemma 5, we can directly relate the eigenvalues of $\bar{L}$ with the nonzero eigenvalues of L. Then,

$$\Delta_\tau(s) = \prod_{i=1}^{m-1}\Big(s^n + \big(1 + (\lambda_i\{L\} - 1)\mathrm{e}^{-s\tau}\big)\sum_{p=1}^{n} s^{p-1}\alpha_p\Big)$$

assuming the eigenvalues of L ordered such that the m-th eigenvalue of L is zero, i.e., $\lambda_m\{L\} = 0$.

The previous equation shows that, for each nonzero eigenvalue of L, there are n eigenvalues for the whole system dynamics, given by the roots of the quasi-polynomials in (18). □

In short, Proposition 2 allows us to write the characteristic (17) as the quasi-polynomials in (18), whose roots dictate consensability of the multi-agent system due to Proposition 1. It is important to note that the quasi-polynomials in (18) do not explicitly require any transformation of the system due to Lemma 5, this fact is given by the adoption of the tree-type transformation in (9).

5 Main results

The following result is based on the analysis of the roots location of the quasi-polynomials in (18), which according to Proposition 2 allows to show consensus if all the roots have negative real parts. When one varies the constant value τ of the communication delay, these roots move and eventually can change from the open-right half plane to the open-left half plane, or vice-versa, which may cause consensability switch. This fact is analyzed based on the direct method for the stability analysis as in [24]. This method relies on finding a finite number of zero-crossing frequencies ω_{ij}, at which the roots of $\Delta_\tau(s)$ in (17) are over the imaginary axis, i.e., $s = \mathrm{j}\omega_{ij}$. Furthermore, it is known that if a zero-crossing happens at some τ_{ij}, other pairs of roots $(\omega_{ij}, \tau_{ij}^l)$ of $\Delta_\tau(s)$ also cross the imaginary axis with the same period ω_{ij} infinitely many times, at the instants

$$\tau_{ij}^l = \tau_{ij} + 2l\omega_{ij}^{-1}\pi, \quad l = 0, \pm1, \pm2, \cdots. \tag{19}$$

Therefore, in this section, we focus on characterizing the intervals of communication delay in which the multi-agent system in (8) achieves consensus. The agents$'$ dynamics are given as a chain of integrators and network topologies are given as directed graphs containing a directed spanning tree in accordance with Assumption 1.

The main result of this paper is stated in Theorem 1.

Theorem 1. Consider the multi-agent system in (3) with protocol (7). Assume a network topology described by a directed graph with Assumption 1 and containing a directed spanning tree, with Laplacian matrix L. Let the nonzero eigenvalues of L, subtracted by 1, be written in the exponential form, i.e., $\lambda_i\{L\} - 1 = \mu_i \mathrm{e}^{\mathrm{j}\phi_i}$. Compute:

1) $N_U(\tau)$ for $\tau = 0$, i.e., the number of unstable roots of $\Delta_\tau(s)$ in (17) with $\tau = 0$. Note that $N_U(0)$ can be determined by the nonzero eigenvalues of Γ in (6).

2) The triplets $\Psi_{ij} = (\omega_{ij}, \tau_{ij}, \Phi_{ij})$, for $i = 1, 2, \cdots, m-1$ and $j = 1, 2, \cdots, r_i$, with r_i is the number of positive roots of ω in the equation below for a given i. Thus, ω_{ij}, for each μ_i, are the positive roots of

$$\rho_i(\omega) = \Big|(\mathrm{j}\omega)^n + \sum_{p=1}^{n}(\mathrm{j}\omega)^{p-1}\alpha_p\Big|^2 - \mu_i^2\Big|\sum_{p=1}^{n}(\mathrm{j}\omega)^{p-1}\alpha_p\Big|^2. \tag{20}$$

Moreover, each τ_{ij} is any value of τ for a given ω_{ij} that satisfies the system of equations

$$\begin{cases} \sin(\omega_{ij}\tau - \phi_i) = \dfrac{-a_{0R}a_{iI} + a_{0I}a_{iR}}{|a_i|^2} \\ \cos(\omega_{ij}\tau - \phi_i) = \dfrac{-a_{0R}a_{iR} - a_{0I}a_{iI}}{|a_i|^2} \end{cases} \tag{21}$$

where $a_{0R}(\omega)$ and $a_{0I}(\omega)$ are the real and imaginary parts of $a_0(\omega) \equiv (\mathrm{j}\omega)^n + \sum_{p=1}^{n}(\mathrm{j}\omega)^{p-1}\alpha_p$, respectively, and similarly $a_{iR}(\omega)$ and $a_{iI}(\omega)$ are the real and imaginary parts of $a_i(\omega) \equiv \mu_i \sum_{p=1}^{n}(\mathrm{j}\omega)^{p-1}\alpha_p$, respectively. Finally, Φ_{ij} is calculated for each ω_{ij} as the sign of

$$\frac{\mathrm{d}}{\mathrm{d}\omega}\rho_i(\omega)\Big|_{\omega=\omega_{ij}}. \tag{22}$$

Now, define the set

$$\Psi = \{(\Psi_{ij}) : \ i = 1, 2, \cdots, m-1 \text{ and } j = 1, 2, \cdots, r_i\}.$$

Then, depending on the emptiness of the set Ψ consisting of all obtained triplets Ψ_{ij}, there are two possible cases.

Case 1. If $\Psi = \varnothing$, no consensability switches occur. Therefore, if $N_U(0) = 0$, the system achieves consensus for $\tau = 0$ and is still consensable for any $\tau > 0$, alternatively, if $N_U(0) > 0$, the system does not achieve consensus for $\tau = 0$ or for any $\tau > 0$.

Case 2. If $\Psi \neq \varnothing$, consensability switches may occur. Then, in order to identify the switches, form a table such that:

1) The first column entries are an arbitrary number of $\tau_{ij}^l > 0$, given as in (19), for all $\tau_{ij} \in \Psi$, in the ascending order.

2) The second column entries are the values of $\omega_{ij} \in \Psi$ associated with each τ_{ij}^l from the first column.

3) The third column entries are the values of $\Phi_{ij} \in \Psi$ associated with each τ_{ij}^l from the first column.

4) The fourth column entries are given by the number of unstable roots for a specific value of time-delay τ, $N_U(\tau)$. Before proceeding further, add new lines between each line in the table built so far, the elements in the fourth column will appear only in the new lines added. The first element of this column is $N_U(0)$, then the next ones are the number of unstable roots for $\tau = \tau_{ij}^l + \epsilon$, $0 < \epsilon \ll 1$. If $\Phi_{ij} = +1$ in the line below, then $N_U(\tau)$ increases by 2, if $\Phi_{ij} = -1$, then $N_U(\tau)$ decreases by 2.

Finally, the regions in the time-delay domain where the multi-agent system is consensable are those where $N_U(\tau) = 0$.

An example of the resulting table described in the procedure of Case 2 is shown in Table 1.

Proof. Initially, to identify the time-delay intervals of consensability, the zero-crossing frequencies ω_{ij} of the quasi-polynomials (18) are found using the direct method[24] in the following procedure.

Considering the conjugate symmetry of (18), for some $s = \mathrm{j}\omega$, the following holds:

$$\Big|(\mathrm{j}\omega)^n + \sum_{p=1}^{n}(\mathrm{j}\omega)^{p-1}\alpha_p\Big|^2 =$$
$$\Big|(\lambda_i\{L\}-1)\mathrm{e}^{-\mathrm{j}\omega\tau}\sum_{p=1}^{n}(\mathrm{j}\omega)^{p-1}\alpha_p\Big|^2$$

and writing the nonzero eigenvalues of L subtracted by 1 in the exponential form $\lambda_i\{L\} - 1 = \mu_i \mathrm{e}^{\mathrm{j}\phi_i}$ gives (20).

If there is no solution for (20), then the roots of the polynomials in (18) never cross the imaginary axis. Therefore, no consensability switches occur, which concludes Case 1.

On the other hand, if (20) has real solutions $\omega_{ij} > 0$, the associated values of delay τ_{ij} can be found. We follow a similar procedure used in [13] by separating the terms of (18), with $s = \mathrm{j}\omega$, in real and imaginary parts, as

$$p_i(\mathrm{j}\omega) = \Big(a_{0R}(\omega) + \mathrm{j}a_{0I}(\omega)\Big) + \Big(a_{iR}(\omega) + \mathrm{j}a_{iI}(\omega)\Big)\mathrm{e}^{-\mathrm{j}(\omega\tau - \phi_i)} \tag{23}$$

where $a_{0R}(\omega)$ and $a_{0I}(\omega)$ are the real and imaginary parts of

$$a_0(\omega) \equiv (\mathrm{j}\omega)^n + \sum_{p=1}^{n} (\mathrm{j}\omega)^{p-1}\alpha_p \tag{24}$$

respectively, and similarly $a_{iR}(\omega)$ and $a_{iI}(\omega)$ are the real and imaginary parts of

$$a_i(\omega) \equiv \mu_i \sum_{p=1}^{n} (\mathrm{j}\omega)^{p-1}\alpha_p \tag{25}$$

respectively.

Expanding the exponential term with Euler′s formula, a value for τ_{ij}, for each ω_{ij}, can be found solving the following system of equations:

$$\begin{cases} a_{iR}\cos(\omega\tau - \phi_i) + a_{iI}\sin(\omega\tau - \phi_i) = -a_{0R} \\ a_{iI}\cos(\omega\tau - \phi_i) - a_{iR}\sin(\omega\tau - \phi_i) = -a_{0I} \end{cases} \tag{26}$$

which yields (21).

Note that we can find an infinite number of values for τ satisfying (21), which is expected due to the periodic property of the transcendental function in (17). For each root on the imaginary axis $\mathrm{j}\omega_{ij}$, there are associated many periodically spaced delays τ_{ij}^l given by (19). Therefore, we can use (21) to identify one of these time-delay values and a number of other solutions can be obtained using (21). This is done in order to obtain all the positive values of delays where the crossings occur from zero up to a maximum value of interest.

Next, we analyze the tendency of the roots in (18). This allows to investigate consensability switches as the value of the time-delay increases. Then, define the quantity

$$\Phi_{ij} = \mathrm{sign}\left(\mathrm{Re}\left(\frac{\mathrm{d}s}{\mathrm{d}\tau}\Big|_{s=\mathrm{j}\omega_{ij}}\right)\right) \tag{27}$$

which is an indicator of the crossing direction of the imaginary root $\mathrm{j}\omega_{ij}$. If $\Phi_{ij} = +1$, a pair of roots of (18) crosses the imaginary axis at $\mathrm{j}\omega_{ij}$ from left to right. Conversely, if $\Phi_i = -1$, a pair of roots of (18) crosses the imaginary axis at $\mathrm{j}\omega_{ij}$ from right to left. From [24], this is given by the sign of (22).

Finally, it remains to determine the number of roots, if any, in the right half-plane, when $\tau = 0$. It can be assessed from the eigenvalues of Γ in (6). Note that, as from [24], for an infinitesimally small τ, there will be also infinite new roots of (18) at the infinity of the left half-plane, since the degree of the polynomial $a_0(\omega)$ is strictly greater than the degree of $a_i(\omega)$.

Then, sorting this data in the ascending order of time-delays τ_{ij}^l, and considering the increase, or decrease, in the number $N_U(\tau)$ of roots in the open right half-plane, the multi-agent system is able to achieve consensus whenever $N_U(\tau) = 0$. This concludes Case 2. □

Theorem 1 brings out a structured methodology for identifying time-delay intervals where consensus in regular directed networks of multi-agent systems with high-order integrator dynamics is achieved. However, it can be further simplified when particular cases are considered.

In the following, particular results are obtained for networks of agents with first-order and second-order integrator dynamics.

Corollary 1. Agents with a single integrator dynamics, i.e., $n = 1$ in (3) and in consensus protocol (7), with a regular directed network according to Assumption 1 containing a directed spanning tree, achieve consensus independently of the communication delay.

Proof. For $n = 1$, matrix Γ in (6) becomes $-\alpha_1 L$, and Lemma 4 is satisfied according to Lemma 1 since the graph has a directed spanning tree. Thus, the system achieves consensus for $\tau = 0$, i.e., $N_U(0) = 0$ in Theorem 1. Additionally, (20) becomes

$$\omega^2 + \alpha_1^2 - \mu_i^2\alpha_1^2 = 0 \tag{28}$$

$$\omega^2 = \alpha_1^2(\mu_i^2 - 1). \tag{29}$$

Note that the eigenvalues of L are related to $\lambda_i\{L\} = \lambda_i\{\Delta - A\}$, and from Assumption 1 of regular graphs for networks with communication delay, we have that $\Delta = I_m$, yielding $\lambda_i\{L\} = \lambda_i\{I_m - A\}$, such that

$$\lambda_i\{L\} - 1 = -\lambda_i\{A\}. \tag{30}$$

Thus, the spectral radius of $\lambda_i\{L\} - 1$ gives $\mu_i^2 \leq 1$ from Lemma 2, which yields no solutions for $\omega > 0$ in (29). Therefore, this is Case 1 in Theorem 1 and no crossings occur, meaning that consensability is never lost due to uniform constant communication delay. □

The result presented by Corollary 1 agrees with the result presented in [7] for systems composed of agents with single integrator dynamics, such that consensus can be achieved regardless of communication delay, as long as the information from one of the agents can reach all the other agents, i.e., the communication graph has a directed spanning tree.

Corollary 2. Agents with second-order integrator dynamics, i.e., $n = 2$ in the dynamics (3) and in the consensus protocol (7), with a regular directed network according to Assumption 1 containing a directed spanning tree is delay-independent if all $\mu_i = 1$ or

$$\alpha_1 \leq \min_i \frac{\bar{\mu}_{i,\alpha_2}\left(1 + \left|\sqrt{(1-\mu_i^2)}\right|\right)}{\mu_i^2} \tag{31}$$

with

$$\bar{\mu}_{i,\alpha_2} = \alpha_2^2 \frac{(1-\mu_i^2)}{2}. \tag{32}$$

If not, crossings occur, and can happen in both directions. Thus, consensability switches may occur.

Proof. The proof follows from Theorem 1. For second-order integrator, (20) becomes

$$|-\omega^2 + \mathrm{j}\omega\alpha_2 + \alpha_1|^2 - \mu_i^2|\mathrm{j}\omega\alpha_2 + \alpha_1|^2 = 0 \tag{33}$$
$$\omega^4 + \omega^2(\alpha_2^2(1-\mu_i^2) - 2\alpha_1) + \alpha_1^2(1-\mu_i^2) = 0 \tag{34}$$

yielding

$$\omega_i = \left(\frac{2\alpha_1 - \alpha_2^2(1-\mu_i^2) \pm \sqrt{\Xi}}{2}\right)^{\frac{1}{2}} \tag{35}$$

where $\Xi = (\alpha_2^2(1-\mu_i^2) - 2\alpha_1)^2 - 4\alpha_1^2(1-\mu_i^2)$, Alternatively, (35) can be written as

$$\omega_i = \left(\alpha_1 - \bar{\mu}_{i,\alpha_2} \pm \sqrt{(\bar{\mu}_{i,\alpha_2} - \alpha_1)^2 - 2\left(\frac{\alpha_1}{\alpha_2}\right)^2 \bar{\mu}_{i,\alpha_2}}\right)^{\frac{1}{2}} \tag{36}$$

with $\bar{\mu}_{i,\alpha_2}$ given in (32).

First, in order to exist real $\omega_i > 0$, the term in the square-root in (36) has to be greater than or equal to zero, i.e.,

$$\begin{aligned} &(\bar{\mu}_{i,\alpha_1} - \alpha_1)^2 - 2\left(\frac{\alpha_1}{\alpha_2}\right)^2 \bar{\mu}_{i,\alpha_2} \geq 0 \\ &(\bar{\mu}_{i,\alpha_1} - \alpha_1)^2 - \alpha_1^2(1-\mu_i^2) \geq 0 \\ &(\bar{\mu}_{i,\alpha_2})^2 - 2\bar{\mu}_{i,\alpha_2}\alpha_1 + \alpha_1^2\mu_i^2 \geq 0. \end{aligned} \tag{37}$$

Thus, the roots α_1 for (37) are

$$\alpha_1' = \frac{\bar{\mu}_{i,\alpha_2}\left(1 - \left|\sqrt{(1-\mu_i^2)}\right|\right)}{\mu_i^2} \tag{38}$$
$$\alpha_1'' = \frac{\bar{\mu}_{i,\alpha_2}\left(1 + \left|\sqrt{(1-\mu_i^2)}\right|\right)}{\mu_i^2}. \tag{39}$$

Next, different hypotheses are considered depending on the location of α_1 in terms of α_1' and α_1''.

Hypothesis 1. If $\alpha_1' < \alpha_1 < \alpha_1''$, it implies that $\nexists\ \omega_i > 0$, $\omega_i \in \mathbf{R}$.

Since $\mu_i^2 > 0$ in (37), if $\alpha_1' < \alpha_1 < \alpha_1''$, there is no solution for $\omega_i > 0$, $\omega_i \in \mathbf{R}$, and thus no crossings occur.

Next condition for the existence of $\omega_i > 0$, $\omega_i \in \mathbf{R}$, in (36), is that

$$\alpha_1 - \bar{\mu}_{i,\alpha_2} \pm \left|\sqrt{(\bar{\mu}_{i,\alpha_2} - \alpha_1)^2 - \alpha_1^2(1-\mu_i^2)}\right| > 0 \tag{40}$$

thus, consider Hypotheses 2 and 3.

Hypothesis 2. If $\alpha_1 < \alpha_1'$, it implies that $\nexists\ \omega_i > 0$, $\omega_i \in \mathbf{R}$.

For an arbitrary $\check{\alpha}_1'$ in the interval $0 < \check{\alpha}_1' < \alpha_1'$, it can be shown that $\check{\alpha}_1' < \bar{\mu}_{i,\alpha_2}$. Thus, for $\alpha_1 = \check{\alpha}_1'$, it is sufficient to show

$$\check{\alpha}_1' - \bar{\mu}_{i,\alpha_2} + \left|\sqrt{(\bar{\mu}_{i,\alpha_2} - \check{\alpha}_1')^2 - \check{\alpha}_1'^2(1-\mu_i^2)}\right| > 0$$
$$\left|\sqrt{(\bar{\mu}_{i,\alpha_2} - \check{\alpha}_1')^2 - \check{\alpha}_1'^2(1-\mu_i^2)}\right| > \bar{\mu}_{i,\alpha_2} - \check{\alpha}_1'.$$

Since $\bar{\mu}_{i,\alpha_2} - \check{\alpha}_1' > 0$, then

$$\begin{aligned} &(\bar{\mu}_{i,\alpha_2} - \check{\alpha}_1')^2 - \check{\alpha}_1'^2(1-\mu_i^2) > (\bar{\mu}_{i,\alpha_2} - \check{\alpha}_1')^2 \\ &\check{\alpha}_1'^2(1-\mu_i^2) < 0. \end{aligned} \tag{41}$$

Condition (41) is a contradiction, thus (40) has no solution for $\omega_i > 0$, $\omega_i \in \mathbf{R}$, and thus no crossings occur.

Hypothesis 3. If $\alpha_1 > \alpha_1''$, it implies that $\exists\ \omega_i > 0$, $\omega_i \in \mathbf{R}$, if and only if $\mu_i < 1$.

For an arbitrary $\hat{\alpha}_1''$ in the interval $\hat{\alpha}_1'' > \alpha_1''$, it can be shown that $\hat{\alpha}_1'' > \bar{\mu}_{i,\alpha_2}$. Thus, for $\alpha_1 = \hat{\alpha}_1''$, it is sufficient to show

$$\hat{\alpha}_1'' - \bar{\mu}_{i,\alpha_2} + \left|\sqrt{(\bar{\mu}_{i,\alpha_2} - \hat{\alpha}_1'')^2 - \hat{\alpha}_1''^2(1-\mu_i^2)}\right| > 0$$
$$\left|\sqrt{(\bar{\mu}_{i,\alpha_2} - \hat{\alpha}_1'')^2 - \hat{\alpha}_1''^2(1-\mu_i^2)}\right| > \bar{\mu}_{i,\alpha_2} - \hat{\alpha}_1''.$$

Since $\bar{\mu}_{i,\alpha_2} - \hat{\alpha}_1'' < 0$, then

$$\begin{aligned} &(\bar{\mu}_{i,\alpha_2} - \hat{\alpha}_1'')^2 - \hat{\alpha}_1''^2(1-\mu_i^2) < (\bar{\mu}_{i,\alpha_2} - \hat{\alpha}_1'')^2 \\ &\hat{\alpha}_1''^2(1-\mu_i^2) > 0. \end{aligned} \tag{42}$$

Condition (42) is feasible if and only if $\mu_i < 1$, yielding solutions $\omega_i > 0$, $\omega_i \in \mathbf{R}$, in (40). Therefore crossings occur.

If crossings occur, they occur in both directions, thus consensability switches may occur. For the direction in which the crossing occurs, (22) becomes

$$\Phi_{ij} = \operatorname{sgn}\left(\omega_{ij}^2 + \bar{\mu}_{i,\alpha_2} - \alpha_1\right). \tag{43}$$

Inserting (36) into (43) gives

$$\Phi_{ij} = \operatorname{sgn}\left(\pm\left|\sqrt{(\bar{\mu}_{i,\alpha_2} - \alpha_1)^2 - 2\left(\frac{\alpha_1}{\alpha_2}\right)^2 \bar{\mu}_{i,\alpha_2}}\right|\right) \tag{44}$$

which yields both positive and negative results.

From the combination of Hypotheses 1 – 3, crossings occur only when $\alpha_1 > \alpha_1''$ and $\mu_i < 1$. Therefore, there are no zero-crossing roots if

$$\alpha_1 \leq \min_i \frac{\bar{\mu}_{i,\alpha_2}\left(1 + \left|\sqrt{(1-\mu_i^2)}\right|\right)}{\mu_i^2} \tag{45}$$

which turns the system into a delay-independent system. □

6 Numerical example

Consider the multi-agent system represented by the directed network topology depicted in Fig. 1, with

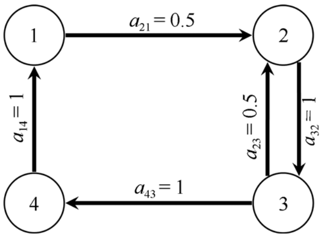


Fig. 1 Regular directed network with four agents

corresponding adjacency matrix

$$A=\begin{bmatrix}0&0&0&1\\0.5&0&0&0.5\\0&1&0&0\\0&0&1&0\end{bmatrix}\tag{46}$$

and Laplacian matrix

$$L=\begin{bmatrix}1&0&0&-1\\-0.5&1&0&-0.5\\0&-1&1&0\\0&0&-1&1\end{bmatrix}.\tag{47}$$

The agents are considered to have second-order integrator dynamics which can easily describe any mechanical system controlled by acceleration. Therefore, we have $n=2$ in (3) and (7). The example follows the application of Corollary 2. First, the eigenvalues of $\lambda_i\{L\}-1$ are written in the exponential form $\mu_i \mathrm{e}^{\mathrm{j}\phi_i}$:

$$\lambda_1\{L\}-1=0.647\,8=0.647\,8\mathrm{e}^{\mathrm{j}0}\tag{48}$$

$$\lambda_2\{L\}-1=0.176\,1+\mathrm{j}0.860\,7=0.878\,5\mathrm{e}^{\mathrm{j}1.369}\tag{49}$$

$$\lambda_3\{L\}-1=0.176\,1-\mathrm{j}0.860\,7=0.878\,5\mathrm{e}^{-\mathrm{j}1.369}\tag{50}$$

such that

$$\mu_1=0.647\,8,\ \phi_1=0\tag{51}$$

$$\mu_2=0.878\,5,\ \phi_2=1.369\tag{52}$$

$$\mu_3=0.878\,5,\ \phi_3=-1.369.\tag{53}$$

Applying Corollary 5, since $\mu_i\neq 1$, we set the consensus protocol gain $\alpha_2=1$ to compute (32). Then, the maximum value of α_1 such that consensability is delay-independent is found using (31), which yields

$$\alpha_1\leq \min\frac{\bar{\mu}_{i,\alpha_2}\left(1+\left|\sqrt{(1-\mu_i^2)}\right|\right)}{\mu_i^2}\tag{54}$$

$$\alpha_1\leq \min(1.218\,3,\ 0.218\,4,\ 0.218\,4)\tag{55}$$

$$\alpha_1\leq 0.218\,4.\tag{56}$$

It means that if α_1 is chosen to be $\alpha_1\leq 0.218\,4$, no crossings occur and the system achieves consensus independently of the communication delay.

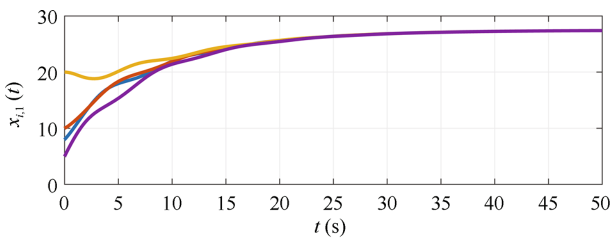
6.1 Delay-independent consensus

Consider $\alpha_2=0.21$ and, for the case of a system free of delay, consensability is checked by the nonzero eigenvalues of $\Gamma=I_m\otimes A-L\otimes(BK)$ in (6), with A and B given in (3) with $n=2$. For this example, the eigenvalues of Γ are $0,\ 0,\ -0.234\,3\pm\mathrm{j}0.029\,6,\ -0.247,\ -0.941\,8\pm\mathrm{j}0.890\,3$, and $-1.400\,8$. The two zero eigenvalues are expected since the agents are of second-order and the graph has a directed spanning tree, according to Lemma 3. Thus, the system achieves consensus asymptotically since all the nonzero eigenvalues have negative real parts, according to Lemma 4.

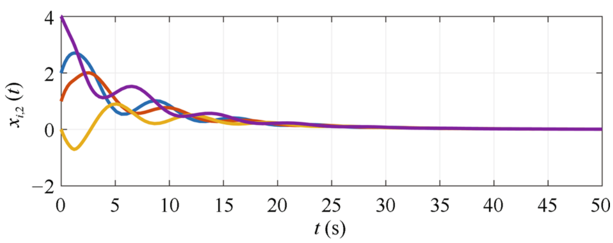
Since the system free of delay achieves consensus, and the value of α_2 is chosen such that the system is delay-independent, the multi-agent system will always be able to achieve consensus for any value of communication delay τ.

Simulations are carried out for $\tau=1\,\mathrm{s}$, $\tau=5\,\mathrm{s}$ and $\tau=10\,\mathrm{s}$, in order to show that the system is indeed consensable for any of these cases. See the system's state trajectories in Figs. 2 to 4, respectively. Additionally, an error metric given as the norm of the lumped disagreement vector $\|z(t)\|$ is introduced in Figs. 2(c), 3(c) and 4(c) to check stability of the transformed system as the error between the agents decreases.

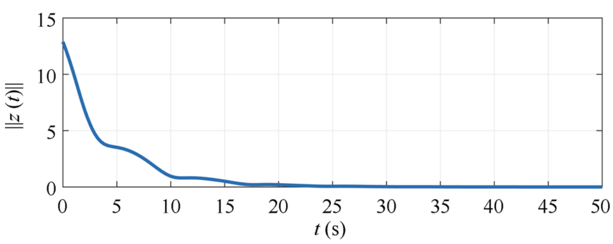
An interesting fact from Fig. 4 is that the error between the agents' states converges to zero asymptotically in Fig. 4(c), although the states $x_{i,1}(t)$ in Fig. 2 and $x_{i,2}(t)$ in Fig. 3 are varying. Note, however, that the states vary similarly.



(a) State trajectories for $x_{i,1}(t)$



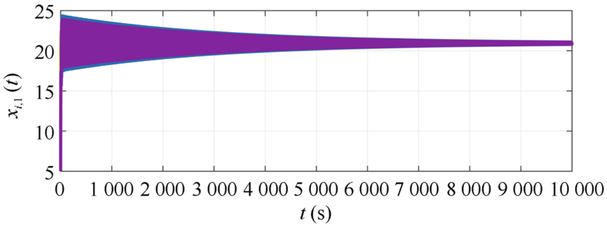
(b) State trajectories for $x_{i,2}(t)$



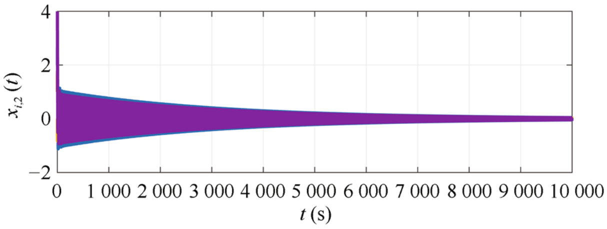
(c) Disagreement $\|z(t)\|$

Fig. 2 State trajectories and error for $\tau=1\,\mathrm{s}$

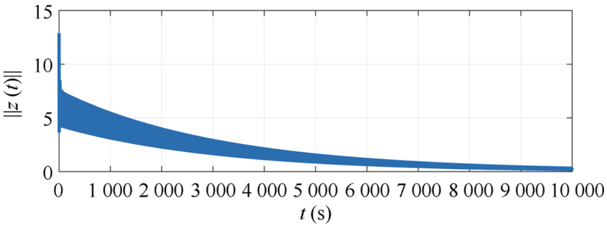
(Color versions of one or more of the figures in this paper are available online.)



(a) State trajectories for $x_{i,1}(t)$

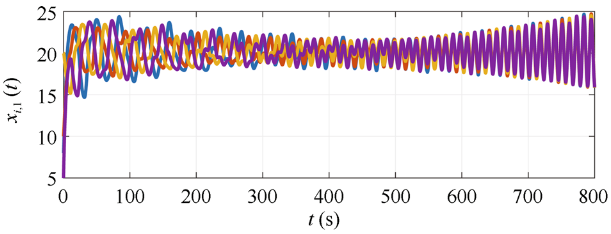


(b) State trajectories for $x_{i,2}(t)$

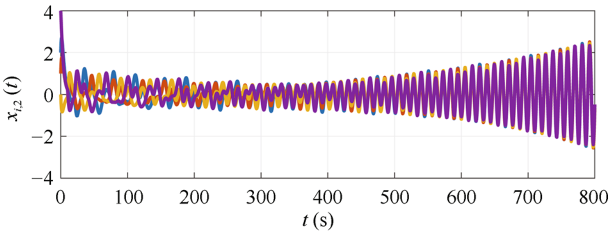


(c) Disagreement $\|z(t)\|$

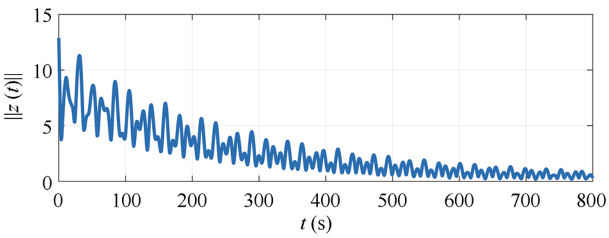
Fig. 3 State trajectories and error for $\tau = 5$ s



(a) State trajectories for $x_{i,1}(t)$



(b) State trajectories for $x_{i,2}(t)$



(c) Disagreement $\|z(t)\|$

Fig. 4 State trajectories and error for $\tau = 10$ s

6.2 Consensus on delay intervals

Finally, assume $\alpha_2 = 0.30$. For the dynamics free of delay, consensability is checked by the nonzero eigenvalues of Γ as: 0, 0, $-0.341\,6 \pm \mathrm{j}0.072\,6$, $-0.394\,4$, $-0.834\,5 \pm \mathrm{j}0.933\,3$, and $-1.253\,4$. The two zero eigenvalues are expected according to Lemma 3. Since the number of positive nonzero eigenvalues is null, then, following the procedure in Theorem 1, we have $N_U(0) = 0$.

Next, the triplets $\Psi_{ij} = (\omega_{ij}, \tau_{ij}, \Phi_{ij})$ are computed and the set Ψ is written. The elements of Ψ are summarized in Table 2. Since a nonempty set $\Psi \neq \varnothing$ is obtained, this example is the Case 2 in Theorem 1. Then, following the procedure of Theorem 1, Case 2, Table 1 is built in the ascending order of $\tau_{ij}^l > 0$.

Table 1 Consensability switches analysis

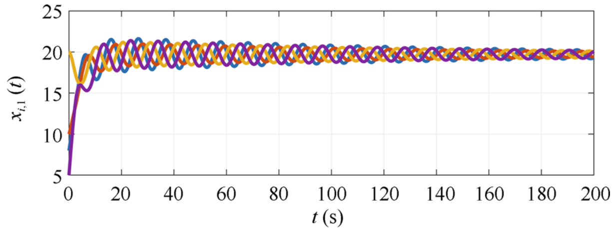
τ_{ij}^l	ω_{ij}	Φ_{ij}	$N_U(\tau)$
2.295 8	0.551 7	1	0
6.335 8	0.259 7	−1	2
7.258 5	0.551 7	1	0
13.684 4	0.551 7	1	2
16.877 3	0.259 7	−1	4
18.647 1	0.551 7	1	2
25.073 0	0.551 7	1	4
30.035 6	0.551 7	1	6
30.526 8	0.259 7	−1	8
⋮	⋮	⋮	⋮

Table 2 Elements of Ψ

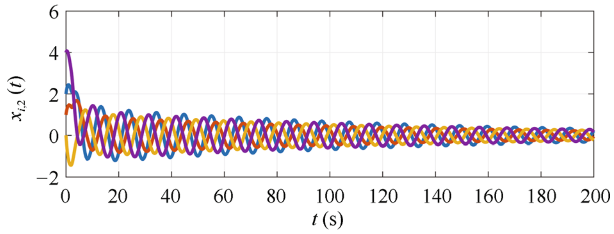
i	j	ω_{ij}	τ_{ij}	Φ_{ij}
2	1	0.259 7	16.877 3	−1
2	2	0.551 7	7.258 5	+1
3	1	0.259 7	6.335 8	−1
3	2	0.551 7	2.295 8	+1

By looking at Table 1, consensability intervals can be analyzed. Note that the system is consensable in the first time-delay interval $[0, 2.295\,8)$. As the communication time-delay increases, the system becomes unable to achieve consensus in the interval $\tau \in [2.295\,8, 6.335\,8]$. However, if the delay is even greater within $\tau \in (6.335\,8, 7.258\,5)$, the system is able to achieve consensus again, loosing consensability after $\tau \geq 7.258\,5$. Thus, the system is consensable in two disconnected intervals $\tau \in [0, 2.295\,8)$ and $\tau \in (6.335\,8, 7.258\,5)$, because after $\tau = 7.258\,5$, there will be always more roots crossing the imaginary axis from left to right ($\Phi_{ij} = +1$) than from right to left ($\Phi_{ij} = -1$), which prevents the system for achieving consensus ever again. In order to illustrate this situation, a simulation is carried out and the system state trajectories are presented for $\tau = 2$, $\tau = 4$, $\tau = 7$, and $\tau = 8$, in Figs. 5 to 8, respectively.

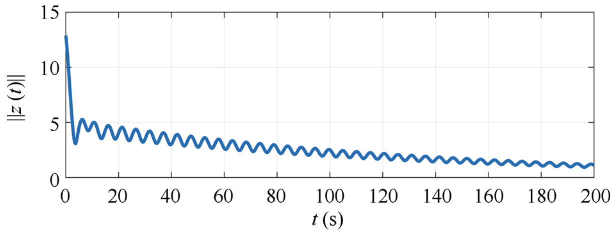
This last scenario serves as a counterexample for the usual claim that the time-delay only degrades system's performance.



(a) State trajectories for $x_{i,1}(t)$

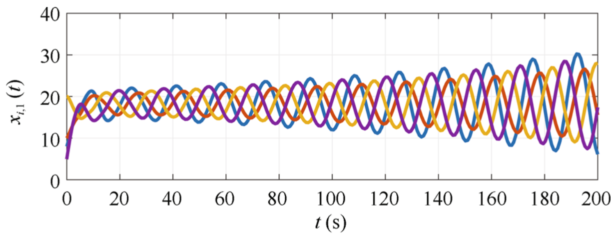


(b) State trajectories for $x_{i,2}(t)$

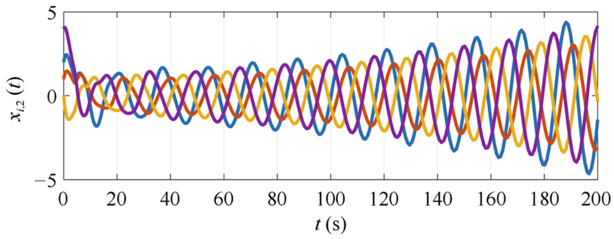


(c) Disagreement $\|z(t)\|$

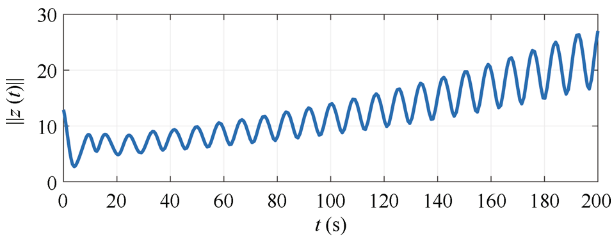
Fig. 5 State trajectories and error for $\tau = 2$ s



(a) State trajectories for $x_{i,1}(t)$

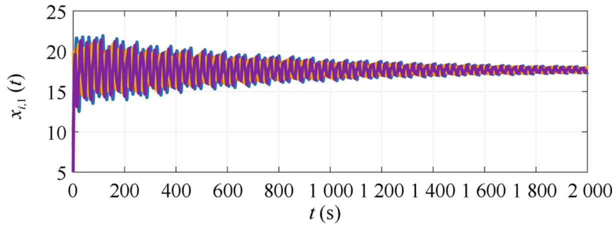


(b) State trajectories for $x_{i,2}(t)$

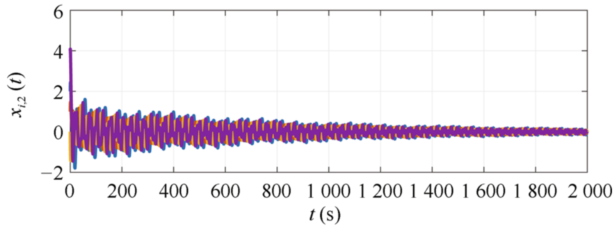


(c) Disagreement $\|z(t)\|$

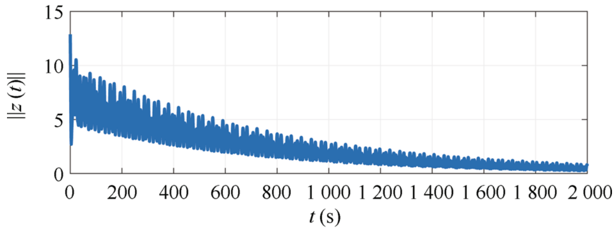
Fig. 6 State trajectories and error for $\tau = 4$ s



(a) State trajectories for $x_{i,1}(t)$

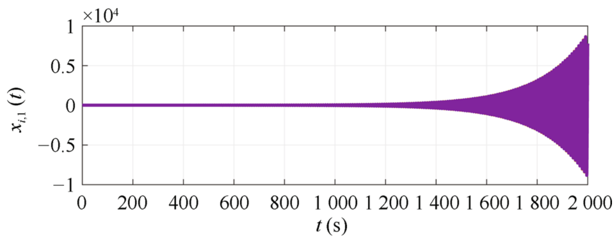


(b) State trajectories for $x_{i,2}(t)$

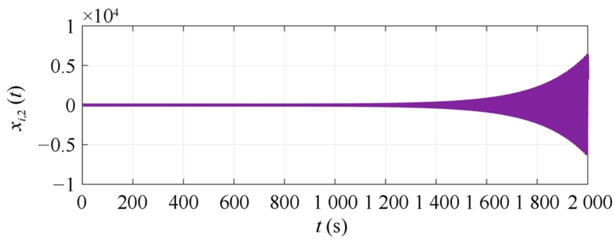


(c) Disagreement $\|z(t)\|$

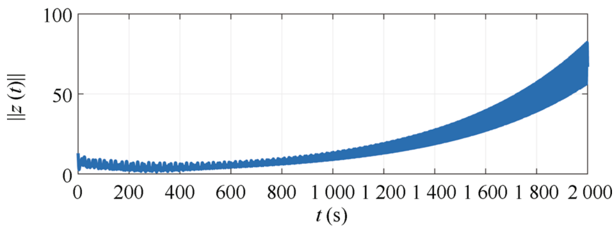
Fig. 7 State trajectories and error for $\tau = 7$ s



(a) State trajectories for $x_{i,1}(t)$



(b) State trajectories for $x_{i,2}(t)$



(c) Disagreement $\|z(t)\|$

Fig. 8 State trajectories and error for $\tau = 8$ s

7 Conclusions

This paper presented a procedure to identify the communication time-delay intervals in which consensus can be achieved, considering directed network topologies of multi-agent systems described by high-order integrator dynamics. The special case presented for agents with single integrator dynamics is consistent with previous results in the literature for communication delay, which shows that the effect of communication delay can prevent consensus only for second-order or higher-order. Besides, for second-order or higher-order, increasing the value of the communication delay does not always avoid consensus, but it can turn the system consensable in given intervals, as given in the numerical example. It serves as a counterexample for the usual acceptance that the time-delay only degrades the system′s performance. This result makes clear the importance of analyzing consensus in different intervals of the communication delay. As illustrated in the numerical example, if a multi-agent system does not achieve consensus for a given time-delay, it may achieve consensus for a greater one.

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