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Boundary Control for a Class of Reaction-diffusion Systems

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Abstract: Boundary control for a class of partial integro-differential systems with space and time dependent coefficients is considered. A control law is derived via the partial differential equation (PDE) backstepping. The existence of kernel equations is proved. Exponential stability of the closed-loop system is achieved. Simulation results are presented through figures.

Keywords: Stability, reaction-diffusion system, boundary control, backstepping, partial differential equation.

1 Introduction

A chemical reaction or biological fermentation needs the stability of the temperature. The temperature is modeled by the reaction-diffusion equation

$$u_t(x,t) = u_{xx}(x,t) + h(x,t,u,u_x)$$

where $u(x,t)$ is the temperature, and h denotes the strength of heat source. For each engineering problem, the function h has its own form[1]. To stabilize the temperature, boundary control is a feasible and economical setting in engineering. However, the boundary control design for the general form of the heat source h has not been developed.

Recently, Krstic et al.[2−6] developed a boundary control design procedure, the partial differential equation (PDE) backstepping, which was applied to establish the boundary control laws for the reaction-diffusion equations that the strength of heat source h does not depend on time t. Krstic et al.[7] and Meurer et al.[8] considered cases that the function h depends on the time t with boundary control. However, the function h only had a simple form. Wang et al.[9, 10] considered the time dependent systems by tracking control method, and solved some nonlinear problems. Meurer et al.[8, 11] considered a new method to deal with this type of problems, and Yu[12] provided another method to solve a class of nonlinear systems as well.

Motivated by the research of Krstic et al.[7] and Meurer et al.[8], in this paper the control design and stabilization of the following system is considered

$$\bar{u}_t(x,t) = \bar{u}_{xx}(x,t) + \bar{h}(x,t,\bar{u},\bar{u}_x),\ 0 < x < 1 \tag{1}$$

$$\bar{u}_x(0,t) = 0 \tag{2}$$

$$\bar{u}(1,t) = \bar{U}(t) \tag{3}$$

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where

$$\bar{h}(x,t,\bar{u},\bar{u}_x) = b(x,t)\bar{u}_x(x,t) + \bar{\lambda}(x,t)\bar{u}(x,t) + \int_0^x \bar{g}(x,y,t)\bar{u}(y,t)\mathrm{d}y$$

the coefficient $b(x,t)$ is bounded with respect to x, t, functions $\bar{\lambda}(x,t)$ and $\bar{g}(x,y,t)$ are smooth with respect to x, y and t, $\bar{u}(x,t)$ is the system signal, $\bar{U}(t)$ is the control input to be determined, the Neumann boundary condition (2) comes from the Fourier law of heat transformation. This system models the physical phenomenons like burning process with a chemical reaction[3]. The integral term means that the system has memory function with spatial variable. It is known that the open-loop is unstable (see Fig. 1).

So, the control objective is to design a control law $\bar{U}(t)$ such that the closed-loop is exponentially stable.

The system (1)−(3) is more general than that considered by Krstic et al.[3, 5−7] and Meurer et al.[8] The functions b, $\bar{\lambda}$ and $\bar{g}$ depend on the time t, which result in difficulties in mathematical computation. By a proper assumption on the coefficients about the increase order of t, a control law is established via the PDE backstepping. The existence of kernel is proved, and the stability of the closed-loop is shown, which are the contributions of this paper. Simulation results are presented by the knowledge of numerical solution of partial differential equation[13].

By the change of variables

$$\lambda(x,t) = \bar{\lambda}(x,t) - \frac{1}{4}b^2(x,t) - \frac{1}{2}b_x(x,t) + \frac{1}{2}\int_0^x b_t(s,t)\mathrm{d}s$$

$$g(x,y,t) = \bar{g}(x,y,t)\mathrm{e}^{\frac{1}{2}\int_y^x b(s,t)\mathrm{d}s}$$

$$u(x,t) = \bar{u}(x,t)\mathrm{e}^{-\frac{1}{2}\int_0^x b(s,t)\mathrm{d}s}$$

$$U(t) = \bar{U}(t)\mathrm{e}^{-\frac{1}{2}\int_0^1 b(s,t)\mathrm{d}s}$$

then system (1)−(3) is transformed into

$$u_t(x,t) = u_{xx}(x,t) + h(x,t,u),\ 0 < x < 1 \tag{4}$$

$$u_x(0,t) = 0 \tag{5}$$

$$u(1,t) = U(t) \tag{6}$$

where

$$h(x,t,u) = \lambda(x,t)u(x,t) + \int_0^x g(x,y,t)u(y,t)\mathrm{d}y.$$

Therefore, it is enough to consider the stability of system (4)−(6).

This paper is organized as follows. In Section 2, kernel equation and control law are obtained. The kernel equation is converted to an integral equation, and the existence of the integral equation is proved. In Section 3, the inverse transformation is established. The stability of the closed-loop system is proved. In Section 4, simulations are presented.

2 Control law

Motivated by the assumption of Meurer et al.[8], in this paper, it is assumed that the functions λ and g satisfy the following assumption.

Assumption 1. For the system (4)−(6), assume that the functions $\lambda(x,t)$ and $g(x,y,t)$ are differentiable with respect to t up to any order, and there exist positive constants ρ and θ such that for $i = 0, 1, 2, \cdots$, it holds that

$$\sup_{t>0} \left|\partial_t^i \lambda(x,t)\right| \le \rho^{i+1} i!$$
$$\sup_{t>0} \left|\partial_t^i g(x,y,t)\right| \le \theta^{i+1} i! \tag{7}$$

where ∂_t^i denotes $\frac{\partial^i}{\partial t^i}$.

This assumption is important while discussing the existence of the kernel (13)−(15). The condition (7) restricts the growth order of functions. This condition is also presented in the paper of Zhou[11]. The condition (7) for the functions $\lambda(x,t)$ and $g(x,y,t)$ restricts the growth order with respect to t. Whereas, with respect to x and y, continuity is enough since x and y belong to the closed interval $[0,1]$. Many functions in t, e.g., trigonometric functions, polynomial functions, satisfy this requirement.

2.1 Backstepping transformation

The main idea of PDE backstepping comes from Krstic et al.[5−6].

Firstly, choose the following target system

$$w_t(x,t) = w_{xx}(x,t) - cw(x,t),\ 0 < x < 1 \tag{8}$$
$$w_x(0,t) = 0 \tag{9}$$
$$w(1,t) = 0 \tag{10}$$

where the constant c is positive. This system is stable and the proof is given in Section 3.2.

Secondly, consider the following Volterra-type integral transformation[14]

$$w(x,t) = u(x,t) - \int_0^x k(x,y,t)u(y,t)\mathrm{d}y \tag{11}$$

where $k(x,y,t)$ is the kernel function to be determined. Choose the kernel function $k(x,y,t)$ such that if the signal $u(x,t)$ is the solution of the system (4)–(6) then the signal $w(x,t)$ is the solution of the system (8)–(10). A control law is obtained by the transformation (11) and the boundary condition (10).

To determine the kernel function $k(x,y,t)$, by (11), it is obtained that

$$\begin{aligned}
&w_t(x,t) = u_t(x,t) - \\
&\int_0^x k_t(x,y,t)u(y,t)\mathrm{d}y - \int_0^x k(x,y,t)u_t(y,t)\mathrm{d}y = \\
&u_{xx}(x,t) + k_y(x,x,t)u(x,t) - k_y(x,0,t)u(0,t) - \\
&k(x,x,t)u_x(x,t) + k(x,0,t)u_x(0,t) + \lambda(x,t)u(x,t) - \\
&\int_0^x \Big(k_{yy}(x,y,t) + \lambda(y,t)k(x,y,t)\Big)u(y,t)\mathrm{d}y - \\
&\int_0^x k_t(x,y,t)u(y,t)\mathrm{d}y + \int_0^x g(x,y,t)u(y,t)\mathrm{d}y - \\
&\int_0^x \int_0^y k(x,y,t)g(y,z,t)u(z,t)\mathrm{d}z\mathrm{d}y
\end{aligned}$$

and

$$\begin{aligned}
w_x(x,t) = u_x(x,t) - k(x,x,t)u(x,t) - \\
\int_0^x k_x(x,y,t)u(y,t)\mathrm{d}y.
\end{aligned} \tag{12}$$

$$\begin{aligned}
&w_{xx}(x,t) = u_{xx}(x,t) - \partial_x k(x,x,t)u(x,t) - k(x,x,t)\times \\
&u_x(x,t) - k_x(x,x,t)u(x,t) - \int_0^x k_{xx}(x,y,t)u(y,t)\mathrm{d}y
\end{aligned}$$

where

$$\partial_x k(x,x,t) = k_x(x,x,t)|_{y=x} + k_y(x,x,t)|_{y=x}.$$

Thus

$$\begin{aligned}
&w_t(x,t) - w_{xx}(x,t) + cw(x,t) = \\
&\Big(\lambda(x,t) + c + 2\partial_x k(x,x,t)\Big)u(x,t) + \\
&k(x,0,t)u_x(0,t) - k_y(x,0,t)u(0,t) + \\
&\int_0^x \Big(k_{xx}(x,y,t) - k_{yy}(x,y,t) - k_t(x,y,t) - \\
&(\lambda(y,t) + c)\,k(x,y,t) + g(x,y,t) - \\
&\int_y^x g(z,y,t)k(x,z,t)\mathrm{d}z\Big)u(y,t)\mathrm{d}y.
\end{aligned}$$

Let $k(x,y,t)$ satisfy the following conditions

$$\begin{aligned}
&k_{xx}(x,y,t) - k_{yy}(x,y,t) - k_t(x,y,t) - \\
&(\lambda(y,t) + c)\,k(x,y,t) + g(x,y,t) - \\
&\int_y^x g(z,y,t)k(x,z,t)\mathrm{d}z = 0
\end{aligned} \tag{13}$$

$$k_y(x,0,t) = 0 \tag{14}$$

$$\partial_x k(x,x,t) = -\frac{1}{2}\left(\lambda(x,t) + c\right) \tag{15}$$

then (8) is satisfied. Let

$$k(0,0,t) = 0 \tag{16}$$

then, by (5) and (12), the boundary condition (9) is satisfied. Moreover, the condition (15) is equivalent to

$$k(x,x,t) = -\frac{1}{2}\int_0^x \left(\lambda(s,t) + c\right)\mathrm{d}s \tag{17}$$

since (16). By (11), take the control law as

$$U(t) = \int_0^1 k(1, y, t)u(y, t)\mathrm{d}y \tag{18}$$

then the boundary condition (10) is satisfied. Now, it remains to solve the kernel (13)–(14) and (17).

2.2 Existence of the kernel function

There exists a mathematical difficulty to solve the kernel (13)−(14) and (17) analytically. Alternately, the existence of a solution is shown.

To show the existence of the solution, there is a need to transform the PDE problem (13)−(14) and (17) into an integral equation. Let

$$\xi = x + y, \quad \eta = x - y \tag{19}$$

and write

$$\phi(\xi, \eta, t) = k\left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2}, t\right)$$

$$\tilde{\lambda}(x, t) = \lambda\left(\frac{x}{2}, t\right) + c, \ x \in [0, 1]$$

it is known that $\tilde{\lambda}(x, t)$ satisfies Assumption 1 as well.

From (13)−(14) and (17), it is obtained that

$$\begin{aligned}\phi_{\xi\eta}(\xi, \eta, t) = &\frac{1}{4}\phi_t(\xi, \eta, t) + \frac{1}{4}\tilde{\lambda}(\xi-\eta, t)\phi(\xi, \eta, t)+\\ &\frac{1}{4}\int_{\frac{\xi-\eta}{2}}^{\frac{\xi+\eta}{2}} \phi\Big(\frac{\xi+\eta}{2}+z, \frac{\xi+\eta}{2}-z, t\Big)\times\\ &g\left(z, \frac{\xi-\eta}{2}, t\right)\mathrm{d}z-\\ &\frac{1}{4}g\Big(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2}, t\Big)\end{aligned} \tag{20}$$

$$\phi(\xi, 0, t) = -\frac{1}{4}\int_0^{\xi}\tilde{\lambda}(s, t)\mathrm{d}s \tag{21}$$

$$\phi_\xi(\xi, \xi, t) = \phi_\eta(\xi, \xi, t). \tag{22}$$

Integrate both sides of (20) with respect to η from 0 to η, noting (21), then it holds

$$\begin{aligned}\phi_\xi(\xi, \eta, t) =&\frac{1}{4}\int_0^{\eta}\phi_t(\xi, s, t)\mathrm{d}s+\\ &\frac{1}{4}\int_0^{\eta}\tilde{\lambda}(\xi-s, t)\phi(\xi, s, t)\mathrm{d}s-\\ &\frac{1}{4}\tilde{\lambda}(\xi, t) - \frac{1}{4}\int_0^{\eta} g\left(\frac{\xi+s}{2}, \frac{\xi-s}{2}, t\right)\mathrm{d}s+\\ &\frac{1}{4}\int_0^{\eta}\int_{\frac{\xi-s}{2}}^{\frac{\xi+s}{2}}\phi\left(\frac{\xi+s}{2}+z, \frac{\xi+s}{2}-z, t\right)\times\\ &g\left(z, \frac{\xi-s}{2}, t\right)\mathrm{d}z\mathrm{d}s.\end{aligned} \tag{23}$$

Integrate both sides of (23) with respect to ξ from η to ξ, it yields

$$\begin{aligned}\phi(\xi, \eta, t) =&\phi(\eta, \eta, t) - \frac{1}{4}\int_{\eta}^{\xi}\int_0^{\eta} g\left(\frac{\sigma+s}{2}, \frac{\sigma-s}{2}, t\right)\mathrm{d}s\mathrm{d}\sigma-\\ &\frac{1}{4}\int_{\eta}^{\xi}\tilde{\lambda}(s, t)\mathrm{d}s + \frac{1}{4}\int_{\eta}^{\xi}\int_0^{\eta}\phi_t(\sigma, s, t)\mathrm{d}s\mathrm{d}\sigma+\\ &\frac{1}{4}\int_{\eta}^{\xi}\int_0^{\eta}\tilde{\lambda}(\sigma-s, t)\phi(\sigma, s, t)\mathrm{d}s\mathrm{d}\sigma+\\ &\frac{1}{4}\int_{\eta}^{\xi}\int_0^{\eta}\int_{\frac{\sigma-s}{2}}^{\frac{\sigma+s}{2}}\phi\left(\frac{\sigma+s}{2}+z, \frac{\sigma+s}{2}-z, t\right)\times\\ &g\left(z, \frac{\sigma-s}{2}, t\right)\mathrm{d}z\mathrm{d}s\mathrm{d}\sigma.\end{aligned} \tag{24}$$

Now it needs to write $\phi(\eta, \eta, t)$ into an integral of functions. First

$$\partial_\eta\phi(\eta, \eta, t) = \phi_\xi(\eta, \eta, t) + \phi_\eta(\eta, \eta, t).$$

Then by (22), it is obtained that

$$\partial_\eta\phi(\eta, \eta, t) = 2\phi_\xi(\eta, \eta, t).$$

By (21), it is obtained that $\phi(0, 0, t) = 0$. Thus

$$\phi(\eta, \eta, t) = 2\int_0^{\eta}\phi_\xi(\sigma, \sigma, t)\mathrm{d}\sigma. \tag{25}$$

Hence, by (23) and (25), it holds that

$$\begin{aligned}\phi(\eta, \eta, t) =& -\frac{1}{2}\int_0^{\eta}\tilde{\lambda}(\sigma, t)\mathrm{d}\sigma + \frac{1}{2}\int_0^{\eta}\int_0^{\sigma}\phi_t(\sigma, s, t)\mathrm{d}s\mathrm{d}\sigma-\\ &\frac{1}{2}\int_0^{\eta}\int_0^{\sigma} g\left(\frac{\sigma+s}{2}, \frac{\sigma-s}{2}, t\right)\mathrm{d}s\mathrm{d}\sigma+\\ &\frac{1}{2}\int_0^{\eta}\int_0^{\sigma}\tilde{\lambda}(\sigma-s, t)\phi(\sigma, s, t)\mathrm{d}s\mathrm{d}\sigma+\\ &\frac{1}{2}\int_0^{\eta}\int_0^{\sigma}\int_{\frac{\sigma-s}{2}}^{\frac{\sigma+s}{2}} g\left(z, \frac{\sigma-s}{2}, t\right)\times\\ &\phi\left(\frac{\sigma+s}{2}+z, \frac{\sigma+s}{2}-z, t\right)\mathrm{d}z\mathrm{d}s\mathrm{d}\sigma.\end{aligned} \tag{26}$$

Then, by (24) and (26), it is obtained that

$$\phi(\xi, \eta, t) = \phi_0(\xi, \eta, t) + \Phi(\phi)(\xi, \eta, t) \tag{27}$$

where $\phi_0(\xi, \eta, t)$ and $\Phi[\phi](\xi, \eta, t)$ are defined by

$$\begin{aligned}\phi_0(\xi, \eta, t) =& -\frac{1}{2}\int_0^{\eta}\tilde{\lambda}(\sigma, t)\mathrm{d}\sigma - \frac{1}{4}\int_{\eta}^{\xi}\tilde{\lambda}(s, t)\mathrm{d}s-\\ &\frac{1}{2}\int_0^{\eta}\int_0^{\sigma} g\left(\frac{\sigma+s}{2}, \frac{\sigma-s}{2}, t\right)\mathrm{d}s\mathrm{d}\sigma-\\ &\frac{1}{4}\int_{\eta}^{\xi}\int_0^{\eta} g\left(\frac{\sigma+s}{2}, \frac{\sigma-s}{2}, t\right)\mathrm{d}s\mathrm{d}\sigma\end{aligned} \tag{28}$$

and

$$\begin{aligned}\Phi[\phi](\xi,\eta,t) =&\frac{1}{4}\int_{\eta}^{\xi}\int_{0}^{\eta}\phi_t(\sigma,s,t)\mathrm{d}s\mathrm{d}\sigma+\\ &\frac{1}{4}\int_{\eta}^{\xi}\int_{0}^{\eta}\tilde{\lambda}(\sigma-s,t)\phi(\sigma,s,t)\mathrm{d}s\mathrm{d}\sigma+\\ &\frac{1}{2}\int_{0}^{\eta}\int_{0}^{\sigma}\tilde{\lambda}(\sigma-s,t)\phi(\sigma,s,t)\mathrm{d}s\mathrm{d}\sigma+\\ &\frac{1}{2}\int_{0}^{\eta}\int_{0}^{\sigma}\int_{\frac{\sigma-s}{2}}^{\frac{\sigma+s}{2}} g\left(z,\frac{\sigma-s}{2},t\right)\times\\ &\phi\left(\frac{\sigma+s}{2}+z,\frac{\sigma+s}{2}-z,t\right)\mathrm{d}z\mathrm{d}s\mathrm{d}\sigma+\\ &\frac{1}{4}\int_{\eta}^{\xi}\int_{0}^{\eta}\int_{\frac{\sigma-s}{2}}^{\frac{\sigma+s}{2}} g\left(z,\frac{\sigma-s}{2},t\right)\times\\ &\phi\left(\frac{\sigma+s}{2}+z,\frac{\sigma+s}{2}-z,t\right)\mathrm{d}z\mathrm{d}s\mathrm{d}\sigma+\\ &\frac{1}{2}\int_{0}^{\eta}\int_{0}^{\sigma}\phi_t(\sigma,s,t)\mathrm{d}s\mathrm{d}\sigma. \end{aligned}\tag{29}$$

Therefore, the PDE problem (13)−(14) and (17) is transformed into the integral equation (27).

It is easy to show that any solution of the integral equation (27) is also a solution of the PDE problem (13)−(14) and (17) if $\phi(\xi,\eta,t)$ is twice differentiable with respect to ξ, η, and differentiable with respect to t up to any order. Thus, there is the following lemma.

Lemma 1. Any function $\phi(\xi,\eta,t)$ which satisfies (20)–(22) also satisfies the integral equation (27), and vice versa if $\phi(\xi,\eta,t)$ is twice differentiable with respect to ξ, η and differentiable with respect to t up to any order.

Now it remains to show the existence of solution of the integral equation (27).

Lemma 2. Under Assumption 1, the integral equation (27) has a solution which is twice continuously differentiable with respect to ξ, η and differentiable with respect to t up to any order.

Proof. The method of successive approximation is applied to show this lemma.

Define a sequence as

$$\phi_{n+1}(\xi,\eta,t)=\phi_0(\xi,\eta,t)+\Phi[\phi_n](\xi,\eta,t),\ n=0,1,\cdots \tag{30}$$

Denote that

$$\Gamma=\{(\xi,\eta)|0<\eta<1,\eta<\xi<2-\eta\}.$$

If this sequence $\{\phi_n(\xi,\eta,t)\}$ is uniformly convergent as n approaches to the infinity, then the limit function $\phi(\xi,\eta,t)$ is the solution of the integral (27), and is twice continuously differentiable with respect to ξ, η, and differentiable with respect to t up to any order.

To show the uniform convergence of the sequence $\{\phi_n(\xi,\eta,t)\}$, denote

$$\triangle\phi_{n+1}(\xi,\eta,t)=\phi_{n+1}(\xi,\eta,t)-\phi_n(\xi,\eta,t).$$

By (29) and (30), it holds that

$$\triangle\phi_{n+1}(\xi,\eta,t)=\Phi[\triangle\phi_n](\xi,\eta,t) \tag{31}$$

since Φ is linear. Then

$$\phi_{n+1}(\xi,\eta,t)=\sum_{j=1}^{n+1}\triangle\phi_j(\xi,\eta,t)+\phi_0(\xi,\eta,t). \tag{32}$$

Thus, the convergence of the sequence $\{\phi_n(\xi,\eta,t)\}$ is equivalent to that of the series $\sum_{n=1}^{\infty}\triangle\phi_n(\xi,\eta,t)$. So, it only needs to show that the series $\sum_{n=1}^{\infty}\triangle\phi_n(\xi,\eta,t)$ is uniformly convergent. To this end, it is to show

$$|\triangle\phi_n(\xi,\eta,t)|\le M\gamma^{2n+1}\frac{(\xi+\eta)^n}{n!} \tag{33}$$

where $\gamma=\max\{3,\rho,\theta\}$, ρ and θ are the constants in the Assumption 1, and M is defined by

$$M=\max_{(\xi,\eta)\in\Gamma}\left(\frac{3}{4}+\frac{1}{4}(1+|\eta|)(|\xi+\eta|)\right).$$

It is trivial that

$$M\gamma^{2n+1}\frac{(\xi+\eta)^n}{n!}\le M\frac{\gamma^{3n+1}}{n!}$$

since $0\le\xi\le 2, 0\le\eta\le 1$. Then, by the Weierstrass M-test, it is known that the series $\sum_{n=1}^{\infty}\triangle\phi_n(\xi,\eta,t)$ is absolutely uniformly convergent if (33) holds.

In order to construct (33), first, it is to prove the following inequality

$$\begin{aligned}\sup_{t\in\mathbf{R}^+}\left|\partial_t^i\triangle\phi_n(\xi,\eta,t)\right|\le\\ M\gamma^{2n+i+1}\frac{(n+i)!}{n!}\frac{(\xi+\eta)^n}{n!}\end{aligned} \tag{34}$$

for $i=0,1,2,\cdots$, where $\mathbf{R}^+$ denotes the interval $(0,+\infty)$, and $\partial_t^i=\frac{\partial^i}{\partial t^i}$. Then (33) is the special case that $i=0$ in (34). Now it is to establish (34) via the mathematical induction.

By (7) and (28), it is obtained that

$$\begin{aligned}&\sup_{t\in\mathbf{R}^+}\left|\partial_t^i\phi_0(\xi,\eta,t)\right|\le\\ &\frac{1}{2}\int_0^{\eta}\sup_{t\in\mathbf{R}^+}\left|\partial_t^i\tilde{\lambda}(\sigma,t)\right|\mathrm{d}\sigma+\frac{1}{4}\int_{\eta}^{\xi}\sup_{t\in\mathbf{R}^+}\left|\partial_t^i\tilde{\lambda}(\sigma,t)\right|\mathrm{d}\sigma+\\ &\frac{1}{4}\int_{\eta}^{\xi}\int_0^{\eta}\sup_{t\in\mathbf{R}^+}\left|\partial_t^i g\left(\frac{\sigma+s}{2},\frac{\sigma-s}{2},t\right)\right|\mathrm{d}s\mathrm{d}\sigma+\\ &\frac{1}{2}\int_0^{\eta}\int_0^{\sigma}\sup_{t\in\mathbf{R}^+}\left|\partial_t^i g\left(\frac{\sigma+s}{2},\frac{\sigma-s}{2},t\right)\right|\mathrm{d}s\mathrm{d}\sigma\le\\ &\gamma^{i+1}i!\left(\frac{3}{4}+\frac{1}{4}(1+|\eta|)|\xi+\eta|\right)\le\\ &M\gamma^{i+1}i!.\end{aligned} \tag{35}$$

For convenience, denote

$$\tilde{\phi}_n(\sigma,s,t)=\tilde{\lambda}(\sigma-s,t)\phi_n(\sigma,s,t)$$

and

$$\hat{\phi}_n(\sigma,s,z,t)=g\Big(z,\frac{\sigma-s}{2},t\Big)\phi_n\Big(\frac{\sigma+s}{2}+z,\frac{\sigma+s}{2}-z,t\Big).$$

Then it holds that

$$\triangle\tilde{\phi}_n(\sigma,s,t)=\tilde{\lambda}(\sigma-s,t)\triangle\phi_n(\sigma,s,t) \tag{36}$$

and

$$\triangle\hat{\phi}_n(\sigma,s,z,t)=g\Big(z,\frac{\sigma-s}{2},t\Big)\triangle\phi_n\Big(\frac{\sigma+s}{2}+z,\frac{\sigma+s}{2}-z,t\Big).$$

From (31) and (35), it is obtained that

$$\begin{aligned}&\sup_{t\in\mathbf{R}^+}\Big|\partial_t^i\triangle\phi_1(\xi,\eta,t)\Big|\le\\&\quad\frac{1}{4}\int_\eta^\xi\int_0^\eta\sup_{t\in\mathbf{R}^+}\Big|\partial_t^{i+1}\phi_0(\sigma,s,t)\Big|\mathrm{d}s\mathrm{d}\sigma+\\&\quad\frac{1}{2}\int_0^\eta\int_0^\sigma\sup_{t\in\mathbf{R}^+}\Big|\partial_t^{i+1}\phi_0(\sigma,s,t)\Big|\mathrm{d}s\mathrm{d}\sigma+\\&\quad\frac{1}{4}\int_\eta^\xi\int_0^\eta\sup_{t\in\mathbf{R}^+}\Big|\partial_t^{i}\tilde{\phi}_0(\sigma,s,t)\Big|\mathrm{d}s\mathrm{d}\sigma+\\&\quad\frac{1}{2}\int_0^\eta\int_0^\sigma\sup_{t\in\mathbf{R}^+}\Big|\partial_t^{i}\tilde{\phi}_0(\sigma,s,t)\Big|\mathrm{d}s\mathrm{d}\sigma+\\&\quad\frac{1}{2}\int_0^\eta\int_0^\sigma\int_{\frac{\sigma-s}{2}}^{\frac{\sigma+s}{2}}\sup_{t\in\mathbf{R}^+}\Big|\partial_t^{i}\hat{\phi}_0(\sigma,s,z,t)\Big|\mathrm{d}z\mathrm{d}s\mathrm{d}\sigma+\\&\quad\frac{1}{4}\int_\eta^\xi\int_0^\eta\int_{\frac{\sigma-s}{2}}^{\frac{\sigma+s}{2}}\sup_{t\in\mathbf{R}^+}\Big|\partial_t^{i}\hat{\phi}_0(\sigma,s,z,t)\Big|\mathrm{d}z\mathrm{d}s\mathrm{d}\sigma.\end{aligned}\tag{37}$$

By the Leibniz′ differentiate formula in calculus tutorial, it holds that

$$\begin{aligned}&\sup_{t\in\mathbf{R}^+}\Big|\partial_t^i\tilde{\phi}_0(\sigma,s,t)\Big|=\\&\quad\sup_{t\in\mathbf{R}^+}\Big|\partial_t^i\Big(\tilde{\lambda}(\sigma-s,t)\phi_0(\sigma,s,t)\Big)\Big|\le\\&\quad\sup_{t\in\mathbf{R}^+}\Big|\sum_{j=0}^{i}C_i^j\partial_t^j\tilde{\lambda}(\sigma-s,t)\partial_t^{i-j}\phi_0(\sigma,s,t)\Big|\le\\&\quad\sum_{j=0}^{i}C_i^j\sup_{t\in\mathbf{R}^+}\Big|\partial_t^j\tilde{\lambda}(\sigma-s,t)\Big|\sup_{t\in\mathbf{R}^+}\Big|\partial_t^{i-j}\phi_0(\sigma,s,t)\Big|\le\\&\quad\sum_{j=0}^{i}C_i^j\gamma^{j+1}j!M\gamma^{i-j+1}(i-j-1)!\le\\&\quad M\gamma^{i+2}\sum_{j=0}^{i}C_i^jj!(i-j-1)!\le\\&\quad M\gamma^{i+2}(i+1)!\end{aligned}\tag{38}$$

where C_i^j denotes number of combinations, and the fact

$$\sum_{j=0}^{i}C_i^jj!(i-j-1)!=(i+1)!$$

is applied.

By the similar mathematical derivation, the following result can be established

$$\begin{aligned}&\sup_{t\in\mathbf{R}^+}\Big|\partial_t^i\hat{\phi}_0(\sigma,s,z,t)\Big|=\\&\quad\sup_{t\in\mathbf{R}^+}\Big|\partial_t^i\Big(g\Big(z,\frac{\sigma-s}{2},t\Big)\phi_0\Big(\frac{\sigma+s}{2}+z,\\&\quad\frac{\sigma+s}{2}-z,t\Big)\Big)\Big|\le M\gamma^{i+2}(i+1)!.\end{aligned}\tag{39}$$

From (35) and (37)−(39), it is obtained that

$$\begin{aligned}&\sup_{t\in\mathbf{R}^+}\big|\partial_t^i\triangle\phi_1(\xi,\eta,t)\big|\le\\&\quad\int_\eta^\xi\int_0^\eta 2\times\frac{1}{4}\times M\gamma^{i+2}(i+1)!\mathrm{d}s\mathrm{d}\sigma+\\&\quad\int_0^\eta\int_0^\sigma 2\times\frac{1}{2}\times M\gamma^{i+2}(i+1)!\mathrm{d}s\mathrm{d}\sigma+\\&\quad\int_\eta^\xi\int_0^\eta\int_{\frac{\sigma-s}{2}}^{\frac{\sigma+s}{2}}\frac{1}{2}\times M\gamma^{i+2}(i+1)!\mathrm{d}z\mathrm{d}s\mathrm{d}\sigma+\\&\quad\int_0^\eta\int_0^\sigma\int_{\frac{\sigma-s}{2}}^{\frac{\sigma+s}{2}}\frac{1}{4}\times M\gamma^{i+2}(i+1)!\mathrm{d}z\mathrm{d}s\mathrm{d}\sigma\le\\&\quad M\times3\gamma^{i+2}(i+1)!(\xi+\eta)=\\&\quad M\gamma^{i+3}(i+1)!(\xi+\eta)\end{aligned}\tag{40}$$

since $0\le\xi\le2, 0\le\eta\le1$, which shows that (34) holds for $n=1$.

The next step is to prove that if (34) holds for $1,2,\cdots,n$, then (34) also holds for $n+1$. Through the process which is similar to that to obtain (37), it is established that

$$\begin{aligned}&\sup_{t\in\mathbf{R}^+}\big|\partial_t^i\triangle\phi_{n+1}(\xi,\eta,t)\big|\le\\&\quad\frac{1}{4}\int_\eta^\xi\int_0^\eta\sup_{t\in\mathbf{R}^+}\big|\partial_t^{i+1}\triangle\phi_n(\sigma,s,t)\big|\mathrm{d}s\mathrm{d}\sigma+\\&\quad\frac{1}{2}\int_0^\eta\int_0^\sigma\sup_{t\in\mathbf{R}^+}\big|\partial_t^{i+1}\triangle\phi_n(\sigma,s,t)\big|\mathrm{d}s\mathrm{d}\sigma+\\&\quad\frac{1}{4}\int_\eta^\xi\int_0^\eta\sup_{t\in\mathbf{R}^+}\big|\partial_t^{i}\triangle\tilde{\phi}_n(\sigma,s,t)\big|\mathrm{d}s\mathrm{d}\sigma+\\&\quad\frac{1}{2}\int_0^\eta\int_0^\sigma\sup_{t\in\mathbf{R}^+}\big|\partial_t^{i}\triangle\tilde{\phi}_n(\sigma,s,t)\big|\mathrm{d}s\mathrm{d}\sigma+\\&\quad\frac{1}{2}\int_0^\eta\int_0^\sigma\int_{\frac{\sigma-s}{2}}^{\frac{\sigma+s}{2}}\sup_{t\in\mathbf{R}^+}\Big|\partial_t^{i}\triangle\hat{\phi}_n(\sigma,s,z,t)\Big|\mathrm{d}z\mathrm{d}s\mathrm{d}\sigma+\\&\quad\frac{1}{4}\int_\eta^\xi\int_0^\eta\int_{\frac{\sigma-s}{2}}^{\frac{\sigma+s}{2}}\sup_{t\in\mathbf{R}^+}\Big|\partial_t^{i}\triangle\hat{\phi}_n(\sigma,s,z,t)\Big|\mathrm{d}z\mathrm{d}s\mathrm{d}\sigma.\end{aligned}\tag{41}$$

On the other hand, by (36), it holds that

$$\begin{aligned}&\sup_{t\in\mathbf{R}^+}\Big|\partial_t^i\triangle\hat{\phi}_n(\sigma,s,z,t)\Big|=\sup_{t\in\mathbf{R}^+}\Big|\partial_t^i\Big(g\Big(z,\frac{\sigma-s}{2},t\Big)\times\\&\quad\triangle\phi_n\Big(\frac{\sigma+s}{2}+z,\frac{\sigma+s}{2}-z,t\Big)\Big)\Big|\le\\&\quad\sum_{j=0}^{i}C_i^j\Big(\sup_{t\in\mathbf{R}^+}\Big|\partial_t^jg\left(z,\frac{\sigma-s}{2},t\right)\Big|\times\\&\quad\sup_{t\in\mathbf{R}^+}\Big|\partial_t^{i-j}\triangle\phi_n\left(\frac{\sigma+s}{2}+z,\frac{\sigma+s}{2}-z,t\right)\Big|\Big)\le\\&\quad M\gamma^{2n+i+2}\sum_{j=0}^{i}C_i^jj!(n+i-j)!\frac{(\sigma+s)^n}{(n!)^2}\le\\&\quad M\gamma^{2n+i+2}\frac{(n+i+1)!}{(n+1)}\frac{(\sigma+s)^n}{(n!)^2}\end{aligned}\tag{42}$$

since

$$\sum_{j=0}^{i}C_i^jj!(n+i-j)!=\frac{(n+i+1)!}{(n+1)}.$$

By the same method, it is obtained that

$$\begin{aligned}&\sup_{t\in\mathbf{R}^+}\left|\partial_t^i\triangle\tilde{\phi}_n(\sigma,s,t)\right| = \\ &\quad\sup_{t\in\mathbf{R}^+}\left|\partial_t^i\left(\tilde{\lambda}(\sigma-s,t)\triangle\phi_n(\sigma,s,t)\right)\right| \le \\ &\quad M\gamma^{2n+i+2}\frac{(n+i+1)!}{(n+1)}\frac{(\sigma+s)^n}{(n!)^2}.\end{aligned}\tag{43}$$

Therefore, from (34) and (42)−(43), (41) is rewritten as

$$\begin{aligned}&\sup_{t\in\mathbf{R}^+}\left|\partial_t^i\triangle\phi_{n+1}(\xi,\eta,t)\right| \le \\ &\quad M\gamma^{2n+i+2}\frac{(n+i+1)!}{(n+1)!}\frac{(\xi+\eta)^{n+1}}{(n+1)!}\Big(\frac{2}{n}+1\Big) \le \\ &\quad M\gamma^{2(n+1)+i+1}\frac{(n+i+1)!}{(n+1)!}\frac{(\xi+\eta)^{n+1}}{(n+1)!}\end{aligned}\tag{44}$$

which shows that (34) holds for $n+1$. Thus, by mathematical induction, (34) holds for all positive integers.

Let $i=0$ in (34), (33) is obtained, which shows that the series $\sum_{n=1}^{\infty}\triangle\phi_n(\xi,\eta,t)$ is uniformly convergent, namely, the sequence $\{\phi_n(\xi,\eta,t)\}$ is convergent and the limit is the solution of (27). □

3 Stability of the closed-loop

The stability of the closed-loop system under the control law (18) is shown in this section. In order to establish the stability of the closed-loop, an inverse transformation of (11) is investigated first.

3.1 Inverse transformation

Since (11) is a Volterra-type integral transformation, it is possible that it has an inverse transformation in the following form

$$u(x,t)=w(x,t)+\int_0^x l(x,y,t)w(y,t)\mathrm{d}y\tag{45}$$

where the kernel $l(x,y,t)$ is to be determined. Suppose that $w(x,t)$ is the signal of the target system (8)–(10), then determine the kernel $l(x,y,t)$ such that $u(x,t)$, which is generated through (45), is the signal of the closed-loop system (4)–(5) under the control law (18).

From (45), it is obtained that

$$\begin{aligned}u_t(x,t)=&w_t(x,t)+\int_0^x l_t(x,y,t)w(y,t)\mathrm{d}y+\\ &\int_0^x l(x,y,t)w_t(y,t)\mathrm{d}y=\\ &w_{xx}(x,t)-cw(x,t)+l(x,x,t)w_x(x,t)-\\ &l(x,0,t)w_x(0,t)-l_y(x,x,t)w(x,t)+\\ &\int_0^x\Big(l_{yy}(x,y,t)-cl(x,y,t)\Big)w(y,t)\mathrm{d}y+\\ &l_y(x,0,t)w(0,t)+\int_0^x l_t(x,y,t)w(y,t)\mathrm{d}y\end{aligned}\tag{46}$$

and

$$\begin{aligned}u_x(x,t)=&w_x(x,t)+l(x,x,t)w(x,t)+\\ &\int_0^x l_x(x,y,t)w(y,t)\mathrm{d}y\end{aligned}\tag{47}$$

$$\begin{aligned}&u_{xx}(x,t)=\\ &\quad w_{xx}(x,t)+\partial_x l(x,x,t)w(x,t)+l(x,x,t)w_x(x,t)+\\ &\quad l_x(x,x,t)w(x,t)+\int_0^x l_{xx}(x,y,t)w(y,t)\mathrm{d}y.\end{aligned}\tag{48}$$

By (46) and (48), it holds that

$$\begin{aligned}&u_t(x,t)-u_{xx}(x,t)-h(x,t,u)=\\ &\quad-\Big(\lambda(x,t)+c+2\partial_x l(x,x,t)\Big)w(x,t)+l_y(x,0,t)\times\\ &\quad w(0,t)-\int_0^x\Big(l_{xx}(x,y,t)-l_{yy}(x,y,t)-l_t(x,y,t)+\\ &\quad\Big(\lambda(x,t)+c\Big)l(x,y,t)+g(x,y,t)+\\ &\quad\int_y^x g(x,z,t)l(z,y,t)\mathrm{d}z\Big)w(y,t)\mathrm{d}y.\end{aligned}\tag{49}$$

Let $l(x,y,t)$ satisfy the following equation

$$\begin{aligned}&l_{xx}(x,y,t)-l_{yy}(x,y,t)-l_t(x,y,t)+\\ &\quad\Big(\lambda(x,t)+c\Big)l(x,y,t)+g(x,y,t)+\\ &\quad\int_y^x g(x,z,t)l(z,y,t)\mathrm{d}z=0\end{aligned}\tag{50}$$

$$l_y(x,0,t)=0\tag{51}$$

$$\partial_x l(x,x,t)=-\frac{\lambda(x,t)+c}{2}\tag{52}$$

then $u(x,t)$ satisfies (4). By (9) and (47), it is obtained that

$$u_x(0,t)=l(0,0,t)w(0,t).$$

Let

$$l(0,0,t)=0$$

then (5) is satisfied, and (52) is rewritten as

$$l(x,x,t)=-\frac{1}{2}\int_0^x\Big(\lambda(s,t)+c\Big)\mathrm{d}s.\tag{53}$$

As to (6), it is satisfied by the control law (18).

There are also mathematical difficulties to solve the kernel equations (50)−(51) and (53). So, it needs to show the existence of the kernel equations.

Denote

$$\psi(\xi,\eta,t)=l\left(\frac{\xi+\eta}{2},\frac{\xi-\eta}{2},t\right).$$

Then (50)−(51) and (53) are written as

$$4\psi_{\xi\eta}(\xi,\eta,t)=\psi_t(\xi,\eta,t)-\tilde{\lambda}(\xi+\eta,t)\psi(\xi,\eta,t)-\int_{\frac{\xi-\eta}{2}}^{\frac{\xi+\eta}{2}} g\Big(\frac{\xi+\eta}{2},z,t\Big)\psi\Bigg(z+\frac{\xi-\eta}{2}, z-\frac{\xi-\eta}{2},t\Bigg)\mathrm{d}z-g\left(\frac{\xi+\eta}{2},\frac{\xi-\eta}{2},t\right) \tag{54}$$

$$\psi_\xi(\xi,\xi,t)=\psi_\eta(\xi,\xi,t) \tag{55}$$

$$\psi(\xi,0,t)=-\frac{1}{4}\int_0^\xi \tilde{\lambda}(s,t)\mathrm{d}s. \tag{56}$$

Further, (54)−(56) are written as the integral equation

$$\psi(\xi,\eta,t)=\psi_0(\xi,\eta,t)+\Psi[\psi](\xi,\eta,t) \tag{57}$$

where $\psi_0(\xi,\eta,t)$ and $\Psi[\psi](\xi,\eta,t)$ are defined by

$$\psi_0(\xi,\eta,t)=-\frac{1}{4}\int_\eta^\xi \tilde{\lambda}(\sigma,t)\mathrm{d}\sigma-\frac{1}{2}\int_0^\eta \tilde{\lambda}(\sigma,t)\mathrm{d}\sigma-\frac{1}{2}\int_0^\eta\int_0^\sigma g\left(\frac{\sigma+s}{2},\frac{\sigma-s}{2},t\right)\mathrm{d}s\mathrm{d}\sigma-\frac{1}{4}\int_\eta^\xi\int_0^\eta g\left(\frac{\sigma+s}{2},\frac{\sigma-s}{2},t\right)\mathrm{d}s\mathrm{d}\sigma \tag{58}$$

$$\begin{aligned}\Psi[\psi](\xi,\eta,t)=&\frac{1}{4}\int_\eta^\xi\int_0^\eta \psi_t(\sigma,s,t)\mathrm{d}s\mathrm{d}\sigma-\\&\frac{1}{4}\int_\eta^\xi\int_0^\eta \tilde{\lambda}(\sigma+s,t)\psi(\sigma,s,t)\mathrm{d}s\mathrm{d}\sigma-\\&\frac{1}{2}\int_0^\eta\int_0^\sigma \tilde{\lambda}(\sigma+s,t)\psi(\sigma,s,t)\mathrm{d}s\mathrm{d}\sigma-\\&\frac{1}{2}\int_0^\eta\int_0^\sigma\int_{\frac{\sigma-s}{2}}^{\frac{\sigma+s}{2}} g\left(\frac{\sigma+s}{2},z,t\right)\times\\&\quad\psi\left(z+\frac{\sigma-s}{2},z-\frac{\sigma-s}{2},t\right)\mathrm{d}z\mathrm{d}s\mathrm{d}\sigma-\\&\frac{1}{4}\int_\eta^\xi\int_0^\eta\int_{\frac{\sigma-s}{2}}^{\frac{\sigma+s}{2}} g\left(\frac{\sigma+s}{2},z,t\right)\times\\&\quad\psi\left(z+\frac{\sigma-s}{2},z-\frac{\sigma-s}{2},t\right)\mathrm{d}z\mathrm{d}s\mathrm{d}\sigma+\\&\frac{1}{2}\int_0^\eta\int_0^\sigma \psi_t(\sigma,s,t)\mathrm{d}s\mathrm{d}\sigma.\end{aligned} \tag{59}$$

Equation (57) is similar to (27), thus it has the same result as Lemma 2 which is stated as follows.

Lemma 3. Under Assumption 1, the integral equation (57) has a solution which is twice continuously differentiable with respect to ξ, η and differentiable with respect to t up to any order.

3.2 Stability

The stability of the closed-loop system can be established through the boundedness of the transformation (11) and its inverse (45). In this paper, the symbol $\|\cdot\|$ denotes the L_2-norm, which is defined as follows

$$\|\vartheta\|=\left(\int_0^1 \vartheta^2(x)\mathrm{d}x\right)^{\frac{1}{2}}$$

where ϑ is a function in $L^2[0,1]$.

Theorem 1. Consider the system (4)−(5) under the control law (18). Then

$$\|u(t)\|\leq\rho\|u^0\|\mathrm{e}^{-\left(c+\frac{1}{4}\right)t} \tag{60}$$

where ρ is a positive constant defined by (67), and c is the positive constant in the target system (8)−(10), and $u^0=u(x,0)$ denotes the initial data of the system (4)−(6). Namely, the closed-loop system is exponentially stable in L_2-norm.

Proof. For the target system (8)–(10), consider the Lyapunov function[12, 15]

$$V(t)=\frac{1}{2}\int_0^1 w^2(x,t)\mathrm{d}x=\frac{1}{2}\|w(t)\|^2 \tag{61}$$

then it is desire to show its stability in the sense of L_2-norm.

Calculate the derivative of V, it is obtained that

$$\begin{aligned}\dot{V}(t)=&\int_0^1 w(x,t)w_t(x,t)\mathrm{d}x=\\&\int_0^1 w(x,t)(w_{xx}(x,t)-cw(x,t))\mathrm{d}x=\\&w(x,t)w_x(x,t)\Big|_0^1-\int_0^1 w_x^2(x,t)\mathrm{d}x-\\&c\int_0^1 w^2(x,t)\mathrm{d}x=\\&-c\|w(t)\|^2-\|w_x(t)\|^2\end{aligned}$$

where the conditions (9) and (10) are utilized. From the Poincaré inequality[5], it is obtained that

$$-\int_0^1 w_x^2(x,t)\mathrm{d}x\leq-\frac{1}{4}\int_0^1 w^2(x,t)\mathrm{d}x$$

since $w(1,t)=0$. Thus

$$\dot{V}(t)\leq-c\|w(t)\|^2-\frac{1}{4}\|w(t)\|^2=-\left(2c+\frac{1}{2}\right)V(t).$$

Therefore

$$V(t)\leq V(0)\mathrm{e}^{-\left(2c+\frac{1}{2}\right)t}. \tag{62}$$

From (61) and (62), it is obtained that

$$\|w(t)\|^2\leq\|w^0\|^2\mathrm{e}^{-\left(2c+\frac{1}{2}\right)t} \tag{63}$$

where $w^0=w(x,0)$ is the initial data of the system (8)−(10), which shows that the target system is stable.

From (11), via the Cauchy-Schwarz inequality[5], it can be obtained that

$$\|w(t)\|^2\leq 2(1+\alpha_1^2)\|u(t)\|^2 \tag{64}$$

where the positive constant α_1 is defined by

$$\alpha_1=\sup_{(x,y)\in\mathbf{D},t\in\mathbf{R}^+}|k(x,y,t)|$$

where $\mathbf{R}$ denotes whole real numbers, and

$$\boldsymbol{D}=\{(x,y)\in\mathbf{R}|0\leq x\leq 1,0\leq y\leq x\}.$$

From (45), it can be obtained that

$$\|u(t)\|^2 \leq 2(1+\beta_1^2)\|w(t)\|^2 \tag{65}$$

where positive constant β_1 is defined by

$$\beta_1 = \sup_{(x,y)\in\mathbf{D}, t\in\mathbf{R}^+} |l(x,y,t)|.$$

Therefore, by (63)–(65), it is obtained that

$$\|u(t)\|^2 \leq \rho^2\|u^0\|^2 \mathrm{e}^{-\left(2c+\frac{1}{2}\right)t} \tag{66}$$

where ρ is defined by

$$\rho = \sqrt{4(1+\alpha_1^2)(1+\beta_1^2)}. \tag{67}$$

Thus, the theorem is proved. □

4 Simulation

In this section, simulation results are presented in order to verify the design. Considering the purpose of this work is not to get the numerical solution of PDE, but verify the design, the situation that $g(x,y,t)\equiv 0$ is simulated. The other parameters of the system (1)–(3) are given as follows. The constant $c=50$, the initial state $u^0(x)=5(1-2\sin(\frac{3\pi}{2}x))$, and the coefficient $\lambda(x,t)=\cos(x)-\sin(t)+15$, which satisfies Assumption 1 with respect to time t.

Simulation results of the plant (4)–(6) are as follows. Fig. 1 shows that the open-loop is unstable. Fig. 2 shows that closed-loop is stable. Fig. 3 shows that the control input is bounded. Fig. 4 shows that the L_2-norm of $u(x,t)$ goes to zero as time increases.

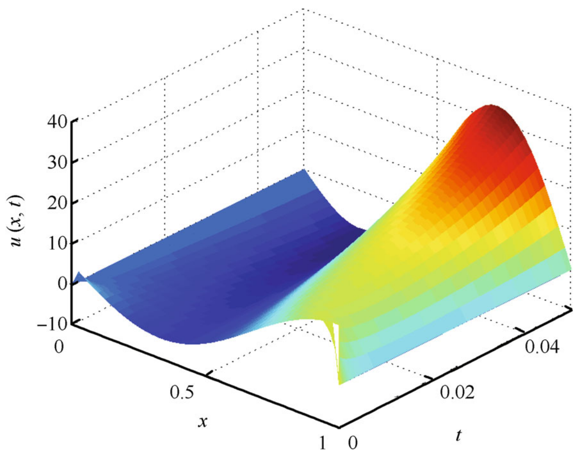

Fig. 1 Simulation result of the plant (4)–(6) of the open-loop

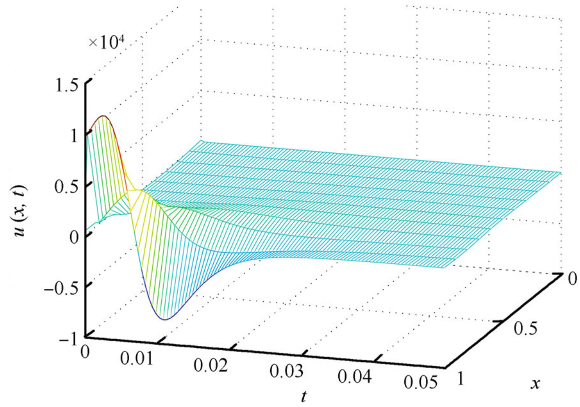

Fig. 2 Simulation result of the plant (4)–(5) with control law (18)

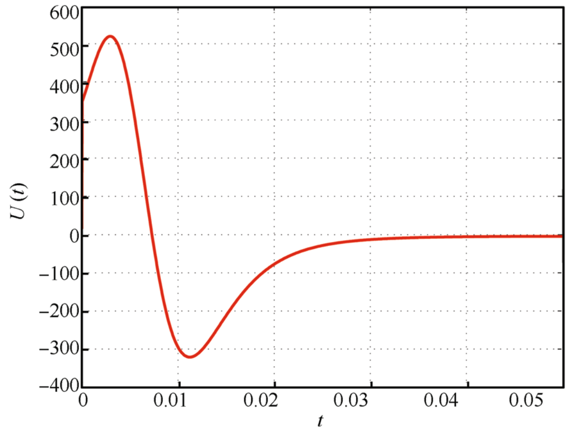

Fig. 3 The control input (18) of the system (4)–(5)

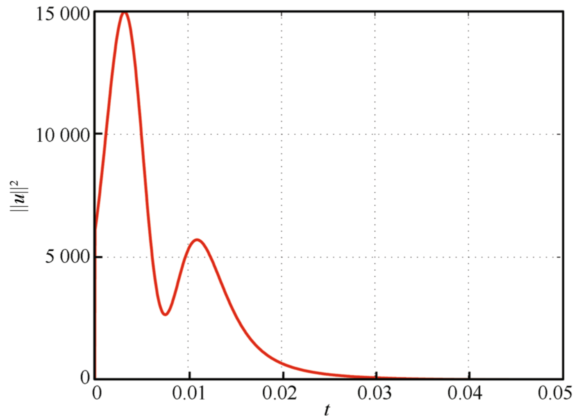

Fig. 4 The L_2-norm of closed-loop

5 Conclusions

Although the systems (4) and (5) are complicated, they can be stabilized by boundary control. Besides, a control law can established. The parameters in this paper are space- and time-dependent. As a result, the kernel functions are Volterra integral differential equations. So, Assumption 1 is required for the proof of existence of the kernel functions $k(x,y,t)$ and $l(x,y,t)$ by the method of mathematical induction. Although it is not necessary, it is difficult to establish existence of the kernel functions without this assumption.

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