

Linguistic Single-Valued Neutrosophic Power Aggregation Operators and Their Applications to Group Decision-Making Problems

Harish Garg and Nancy

Abstract—Linguistic single-valued neutrosophic set (LSVNS) is a more reliable tool, which is designed to handle the uncertainties of the situations involving the qualitative data. In the present manuscript, we introduce some power aggregation operators (AOs) for the LSVNSs, whose purpose is to diminish the influence of inevitable arguments about the decision-making process. For it, first we develop some averaging power operators, namely, linguistic single-valued neutrosophic (LSVN) power averaging, weighted average, ordered weighted average, and hybrid averaging AOs along with their desirable properties. Further, we extend it to the geometric power AOs for LSVNSs. Based on the proposed work; an approach to solve the group decision-making problems is given along with the numerical example. Finally, a comparative study and the validity tests are present to discuss the reliability of the proposed operators.

Index Terms—Aggregation operator (AO), group decision making, linguistic neutrosophic numbers, neutrosophic set, power operators.

I. INTRODUCTION

MULTIPLE criteria group decision-making (MCGDM) problems seek great attention to practical fields, whose main objective is to determine the most desirable alternative to the finite alternatives according to the preference values of the criteria given by different decision makers. However, in order to process the imprecision in data, fuzzy set (FS) [1] and intuitionistic fuzzy set (IFS) [2] theories are the most successful ones, which characterize the criterion values in terms of membership degree. Numerable attempts have been made by different researchers in processing the information values using different operators under these environments [3]–[13]. It is remarked that neither the FS nor IFS theory is able to deal with indeterminate and inconsistent data. For instance, consider an expert which gives their opinion about a certain object in such a way that 0.5 being the “possibility that the statement is true”, 0.7 being the “possibility that the statement is false” and 0.2 being the “possibility that he or she is not sure”. To resolve this, Smarandache [14] introduced a new component called the “indeterminacy-membership function” and added to the “truth membership function” and “falsity member-

ship function”, all which are independent of each other and lying in $[0^-, 1^+]$, and the corresponding set is known as a neutrosophic set (NS). NS theory handles the indeterminate information, but this theory is hard to implement on the practical problems, therefore, Wang *et al.* [15] presented the single-valued neutrosophic (SVN) set (SVNS), a special case of NSs. Due to its importance, several researchers have made their efforts to enrich the concept of neutrosophic sets in the decision-making process.

In order to evaluate the given information in decision-making, the important aspect of solving the problem is to design an appropriate mathematical function which aggregates the different preference of the decision makers into the collective ones. In that direction, Ye [16] presented the operational laws of SVNSs and SVN weighted average and geometric (WAG) AOs (aggregation operators) denoted by single-valued neutrosophic weighted average (SVNWA) and single-valued neutrosophic weighted geometric (SVNWG). Peng *et al.* [17] defined the improved operations of SVN numbers (SVNNs) and developed their corresponding ordered WAG AOs. Nancy and Garg [18] developed the WAG operators by using the Frank norm operations. Later on, some different kinds of the AOs have been proposed by the authors in [19]–[22].

All these above operators, aggregate the given criterion values without considering the precedence relationship among them. To get rid of this flaw, Wu *et al.* [23] defined the prioritized WAG operators for SVNNs. Liu and Wang [24] developed the prioritized ordered WAG operators while Ji *et al.* [25] established the single-valued prioritized Bonferroni mean operator by using the Frank operations. Garg and Nancy [26] developed a nonlinear programming based TOPSIS (“Technique for order preference by similarity to ideal solution”) approach for solving the decision-making problems under the interval NS environment. Yang and Li [27] extend the power operator to NS domain. Aside from these, various authors incorporated the idea of NS theory into the different fields [28]–[32].

In the neutrosophic environment, the information which is evaluated is quantitative in nature and is expressed by the means of numeric numbers. But in real applications, the ‘decision-makers’ opinions or preferences is usually uncertain due to increase in complexities and the subjective nature of human thoughts. Thus, the exact numbers are not the best option to represent such kind of qualitative information. For this, a new

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The authors are with the School of Mathematics, Thapar Institute of Engineering & Technology, Deemed University, Patiala-147004, Punjab, India (e-mail: harishg58iitr@gmail.com; nancyverma16@gmail.com).

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concept, namely, linguistic variables (LVs) [33] has been established to access the information which cannot estimate by exact numbers. Due to the great importance of linguistic variables, it has become the hot topic of research among the researchers. Based on this idea, Li *et al.* [34] introduce the linguistic neutrosophic sets (LNSs) in which membership, indeterminacy, and non-membership is expressed as a LV instead of real numbers and also proposed some Heronian mean operators. Fang and Ye [35] introduced most basic aggregation operators, namely, Linguistic neutrosophic number WAG operators. Garg and Nancy [36] presented some neutrosophic prioritized AOs under the linguistic SVNS (LSVNS) environment.

As these basic operators are crucial tools in the aggregating process and easily gives the best choice according to the given information, but these operators skip the consideration of the relationship between the given data. In the present work, we introduce different types of power AOs in LSVNS environment. The LSVNS can easily express the uncertain qualitative information in the best way and power AO provides more versatility in the information aggregation process. To get the advantage of both LSVNS and power aggregation, we proposed power AOs for linguistic single-valued neutrosophic numbers (LSVNNs) of two types: The first type of power aggregation operators aggregates the input data by assigning the weights based on the support function and the second type of power operators considers not only the existing evaluated weights but also use the weights evaluated from support function. Further, the basic properties of these operators have been figuring out. In the end, a method for solving the MCGDM problems has been presented and then applies this approach to a practical example.

The rest of the work is summarized as: Some basic features related to NSs and the LNSs are presented in Section II. In Section III, operational laws and based on it, some series of LSVN power weighted AOs have been proposed along with their certain properties. Section IV established the group decision-making approach based on the proposed operators and validate with a numerical example. Lastly, the conclusion has been summarized in Section V.

II. PRELIMINARIES

In this section, some basic concepts on NSs over the universal set X have been reviewed.

Definition 1 [14]: A neutrosophic set (NS) α over X is given by

$$\alpha = \{(x, \theta_\alpha(x), \psi_\alpha(x), \sigma_\alpha(x)) \mid x \in X\} \quad (1)$$

where for each x , $\theta_\alpha(x)$, $\psi_\alpha(x)$, $\sigma_\alpha(x) \subseteq [0^-, 1^+]$ represents the extent of agreeance, indeterminacy and disagreeance, respectively such that $\theta_\alpha(x) + \psi_\alpha(x) + \sigma_\alpha(x) \leq 3^+$. On the other hand, if $\theta_\alpha(x)$, $\psi_\alpha(x)$, $\sigma_\alpha(x) \in [0, 1]$ such that $0 \leq \theta_\alpha(x) + \psi_\alpha(x) + \sigma_\alpha(x) \leq 3$ for all x then this set is called SVNS [15]. A pair $\alpha = (\theta_\alpha, \psi_\alpha, \sigma_\alpha)$ is called SVN number (SVNN).

Definition 2 [35]: Let $Q = \{s_0, s_1, \dots, s_t\}$ be a linguistic term set (LTS) with odd cardinality $t+1$ and $\bar{Q} = \{s_h \mid s_0 \leq s_h \leq s_t, h \in [0, t]\}$. Then, a LSVNS A in X is defined as

$$A = \{(x, s_\theta(x), s_\psi(x), s_\sigma(x)) \mid x \in X\} \quad (2)$$

where $s_\theta(x), s_\psi(x), s_\sigma(x) \in \bar{Q}$ represent the linguistic truth, indeterminacy and falsity degrees of x to A , respectively, with condition $0 \leq \theta + \psi + \sigma \leq 3t$. A triplet $(s_\theta, s_\psi, s_\sigma)$ is called linguistic SVN number (LSVNN). Further, these LTS must be satisfy the following properties:

- 1) $s_k \leq s_h \Leftrightarrow k \leq h$;
- 2) $\text{Neg}(s_k) = s_h$ such that $h = t - k$.

Definition 3 [35]: In order to compare the LSVNNs, a score S and an accuracy function H can be represented as $S(\beta) = (2t + \theta - \psi - \sigma)/3 \in [0, t]$ and $H(\beta) = (\theta + \psi + \sigma)/3$ for LSVNN $\beta = (s_\theta, s_\psi, s_\sigma)$. Thus, based on these functions, an order relation between two LSVNNs β and γ , denoted by $\beta > \gamma$, is defined if either $S(\beta) > S(\gamma)$ or $S(\beta) = S(\gamma) \wedge H(\beta) > H(\gamma)$.

Definition 4 [37]: Let $\alpha_1 = (s_{\theta_1}, s_{\psi_1}, s_{\sigma_1})$ and $\alpha_2 = (s_{\theta_2}, s_{\psi_2}, s_{\sigma_2})$ be two LSVNNs. Then, the distance between α_1 and α_2 is given below:

$$d(\alpha_1, \alpha_2) = \frac{1}{3t} (|\theta_1 - \theta_2| + |\psi_1 - \psi_2| + |\sigma_1 - \sigma_2|). \quad (3)$$

Definition 5 [38]: Let $\alpha_1, \alpha_2, \dots, \alpha_n$ be 'n' attributes then the power averaging (PA) operator is defined as

$$PA(\alpha_1, \alpha_2, \dots, \alpha_n) = \sum_{j=1}^n \frac{1 + G(\alpha_j)}{\sum_{j=1}^n (1 + G(\alpha_j))} \alpha_j \quad (4)$$

where $G(\alpha_j) = \sum_{k=1, j \neq k}^n \text{supp}(\alpha_j, \alpha_k)$ and $\text{supp}(\alpha_j, \alpha_k)$ is the support for α_j from α_k , defined as $\text{supp}(\alpha_j, \alpha_k) = 1 - d(\alpha_j, \alpha_k)$ and satisfies the following properties.

- 1) $\text{supp}(\alpha_j, \alpha_k) \in [0, 1]$;
- 2) $\text{supp}(\alpha_j, \alpha_k) = \text{supp}(\alpha_k, \alpha_j)$;
- 3) $\text{supp}(\alpha_j, \alpha_k) \geq \text{supp}(\alpha_u, \alpha_v)$ if $d(\alpha_j, \alpha_k) \leq d(\alpha_u, \alpha_v)$.

Definition 6 [35], [36]: Let $\alpha = (s_\theta, s_\psi, s_\sigma)$, $\alpha_1 = (s_{\theta_1}, s_{\psi_1}, s_{\sigma_1})$ and $\alpha_2 = (s_{\theta_2}, s_{\psi_2}, s_{\sigma_2})$ be three LSVNNs, then

- 1) $\alpha^c = (s_\sigma, s_\psi, s_\theta)$;
- 2) $\alpha_1 \cup \alpha_2 = (\max(s_{\theta_1}, s_{\theta_2}), \min(s_{\psi_1}, s_{\psi_2}), \min(s_{\sigma_1}, s_{\sigma_2}))$;
- 3) $\alpha_1 \cap \alpha_2 = (\min(s_{\theta_1}, s_{\theta_2}), \max(s_{\psi_1}, s_{\psi_2}), \max(s_{\sigma_1}, s_{\sigma_2}))$;
- 4) $\alpha_1 = \alpha_2$ if $s_{\theta_1} = s_{\theta_2}$, $s_{\psi_1} = s_{\psi_2}$ and $s_{\sigma_1} = s_{\sigma_2}$;
- 5) $\alpha_1 \geq \alpha_2$ if $s_{\theta_1} \geq s_{\theta_2}$, $s_{\psi_1} \leq s_{\psi_2}$ and $s_{\sigma_1} \leq s_{\sigma_2}$.

Definition 7 [12]: A function $\zeta : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is called t-norm if it satisfies the boundary, monotonicity, commutativity and associativity. On the other hand, a function ς defined by $\varsigma(x, y) = 1 - \zeta(1 - x, 1 - y) \forall x, y \in [0, 1]$ is called t-conorm. The t-norm ζ is generated by a decreasing function k as $\zeta(x, y) = k^{-1}(k(x) + k(y))$ with $k(1) = 0$ and the t-conorm ς is generated as $\varsigma(x, y) = h^{-1}(h(x) + h(y))$, where $h(x) = k(1 - x)$.

III. POWER AGGREGATION OPERATOR FOR LINGUISTIC NEUTROSOPHIC SETS

In this section, some new power AOs for linguistic neutrosophic sets namely, linguistic single-valued neutrosophic-power weighted average (LSVN-PWA), linguistic single-valued neutrosophic-power ordered weighted average (LSVN-POWA), linguistic single-valued neutrosophic-power weighted geometric (LSVN-PWG), Linguistic single-valued neutrosophic-power ordered weighted geometric (LSVN-POWG) etc., have been presented to aggregate the

LSVNNs.

A. Operations Laws for LSVNNs

In this section, based on generators ζ and ς , we defined the operational laws for LSVNNs as follows.

Definition 8: Let $\alpha_1 = (s_{\theta_1}, s_{\psi_1}, s_{\sigma_1})$, $\alpha_2 = (s_{\theta_2}, s_{\psi_2}, s_{\sigma_2})$ and $\alpha = (s_{\theta}, s_{\psi}, s_{\sigma})$ be three LSVNNs and a real $\lambda > 0$, we have

$$\begin{aligned} 1) \alpha_1 \oplus \alpha_2 &= \left(s_{t(h^{-1}(h(\frac{\theta_1}{t})+h(\frac{\theta_2}{t})))}, s_{t(k^{-1}(k(\frac{\psi_1}{t})+k(\frac{\psi_2}{t})))}, \right. \\ &\left. s_{t(k^{-1}(k(\frac{\sigma_1}{t})+k(\frac{\sigma_2}{t})))} \right); \\ 2) \alpha_1 \otimes \alpha_2 &= \left(s_{t(k^{-1}(k(\frac{\theta_1}{t})+k(\frac{\theta_2}{t})))}, s_{t(h^{-1}(h(\frac{\psi_1}{t})+h(\frac{\psi_2}{t})))}, \right. \\ &\left. s_{t(h^{-1}(h(\frac{\sigma_1}{t})+h(\frac{\sigma_2}{t})))} \right); \\ 3) \lambda \alpha &= \left(s_{t(h^{-1}(\lambda h(\frac{\theta}{t})))}, s_{t(k^{-1}(\lambda k(\frac{\psi}{t})))}, s_{t(k^{-1}(\lambda k(\frac{\sigma}{t})))} \right); \\ 4) \alpha^\lambda &= \left(s_{t(k^{-1}(\lambda k(\frac{\theta}{t})))}, s_{t(h^{-1}(\lambda h(\frac{\psi}{t})))}, s_{t(h^{-1}(\lambda h(\frac{\sigma}{t})))} \right). \end{aligned}$$

Theorem 1: The operations laws defined in Definition 8 are again LSVNNs.

Proof: Since k is a decreasing map and $h(x) = k(1-x)$, therefore, we get $0 \leq k^{-1}(k(\theta_1/t) + k(\theta_2/t)) \leq 1$, $0 \leq h^{-1}(h(\psi_1/t) + h(\psi_2/t)) \leq 1$, $0 \leq h^{-1}(h(\sigma_1/t) + h(\sigma_2/t)) \leq 1$ and $0 \leq k^{-1}(k(\theta_1/t) + k(\theta_2/t)) + h^{-1}(h(\psi_1/t) + h(\psi_2/t)) + h^{-1}(h(\sigma_1/t) + h(\sigma_2/t)) \leq 3$.

Therefore, $\alpha_1 \otimes \alpha_2$ is LSVNN. Similarly, $\alpha_1 \oplus \alpha_2$, $\lambda \alpha$ and α^λ are LSVNNs. ■

Theorem 2: Let α_1 , α_2 and α be three LSVNNs and $\lambda, \lambda_1, \lambda_2$ be three positive real numbers, then

$$\begin{aligned} 1) \alpha_1 \oplus \alpha_2 &= \alpha_2 \oplus \alpha_1; \\ 2) \alpha_1 \otimes \alpha_2 &= \alpha_2 \otimes \alpha_1; \\ 3) \lambda(\alpha_1 \oplus \alpha_2) &= \lambda \alpha_1 \oplus \lambda \alpha_2; \\ 4) (\alpha_1 \otimes \alpha_2)^\lambda &= \alpha_1^\lambda \otimes \alpha_2^\lambda; \\ 5) \lambda_1 \alpha \oplus \lambda_2 \alpha &= (\lambda_1 + \lambda_2) \alpha; \\ 6) \alpha^{\lambda_1} \otimes \alpha^{\lambda_2} &= \alpha^{\lambda_1 + \lambda_2}. \end{aligned}$$

Proof: We shall prove only the parts iii) and remaining parts done similarly.

For real number $\lambda > 0$, we have

$$\begin{aligned} &\lambda(\alpha_1 \oplus \alpha_2) \\ &= \lambda \left(s_{t(h^{-1}(h(\frac{\theta_1}{t})+h(\frac{\theta_2}{t})))}, \right. \\ &\quad \left. s_{t(k^{-1}(k(\frac{\psi_1}{t})+k(\frac{\psi_2}{t})))}, s_{t(k^{-1}(k(\frac{\sigma_1}{t})+k(\frac{\sigma_2}{t})))} \right) \\ &= \left(s_{t(h^{-1}(\lambda h(h^{-1}(h(\frac{\theta_1}{t})+h(\frac{\theta_2}{t}))))}, \right. \\ &\quad \left. s_{t(k^{-1}(\lambda k(k^{-1}(k(\frac{\psi_1}{t})+k(\frac{\psi_2}{t}))))}, \right. \\ &\quad \left. s_{t(k^{-1}(\lambda k(k^{-1}(k(\frac{\sigma_1}{t})+k(\frac{\sigma_2}{t}))))} \right) \\ &= \left(s_{t(h^{-1}(\lambda h(h^{-1}(\lambda h(\frac{\theta_1}{t}))+h^{-1}(\lambda h(\frac{\theta_2}{t}))))}, \right. \\ &\quad \left. s_{t(k^{-1}(\lambda k(k^{-1}(\lambda k(\frac{\psi_1}{t}))+k^{-1}(\lambda k(\frac{\psi_2}{t}))))}, \right. \\ &\quad \left. s_{t(k^{-1}(\lambda k(k^{-1}(\lambda k(\frac{\sigma_1}{t}))+k^{-1}(\lambda k(\frac{\sigma_2}{t}))))} \right) \\ &= \lambda \alpha_1 \oplus \lambda \alpha_2. \end{aligned}$$

Remark 1: We inspect some special cases of λ and α .

- 1) If $\alpha = (0, 0, 0)$, then $\lambda \alpha = (h^{-1}(\lambda h(0)), k^{-1}(\lambda k(0)), k^{-1}(\lambda k(0))) = (0, 0, 0)$.
- 2) If $\alpha = (1, 0, 0)$, then $\lambda \alpha = (h^{-1}(\lambda h(1)), k^{-1}(\lambda k(0)), k^{-1}(\lambda k(0))) = (1, 0, 0)$.
- 3) If $\alpha = (0, 0, 1)$, then $\lambda \alpha = (h^{-1}(\lambda h(0)), k^{-1}(\lambda k(0)), k^{-1}(\lambda k(1))) = (0, 0, 1)$.
- 4) If $\lambda \rightarrow 0$ then $\lambda \alpha = (h^{-1}(\lambda h(\theta/t)), k^{-1}(\lambda k(\psi/t)), k^{-1}(\lambda k(\sigma/t))) \rightarrow (1, 0, 0)$.
- 5) If $\lambda = 1$, then $\lambda \alpha = (h^{-1}(\lambda h(\theta/t)), k^{-1}(\lambda k(\psi/t)), k^{-1}(\lambda k(\sigma/t))) \rightarrow (s_{\theta}, s_{\psi}, s_{\sigma})$.
- 6) If $\lambda \rightarrow \infty$ then $\lambda \alpha = (h^{-1}(\lambda h(\theta/t)), k^{-1}(\lambda k(\psi/t)), k^{-1}(\lambda k(\sigma/t))) \rightarrow (1, 0, 0)$.

Next, we propose some averaging and geometric power AOs for a collection of LSVNNs $\alpha_j = (s_{\theta_j}, s_{\psi_j}, s_{\sigma_j})$ ($j = 1, 2, \dots, n$) denoted by Ω with the condition that $\theta_j, \psi_j, \sigma_j \neq 0$ for any j .

B. LSVN Power Averaging Operator

Definition 9: A linguistic single-valued neutrosophic- power averaging (LSVN-PA) aggregation operator is a mapping LSVN-PA: $\Omega^n \rightarrow \Omega$, defined by

$$\text{LSVN-PA}(\alpha_1, \alpha_2, \dots, \alpha_n) = \frac{n}{\bigoplus_{j=1}^n} \frac{1+G(\alpha_j)}{\sum_{j=1}^n (1+G(\alpha_j))} \alpha_j \quad (5)$$

where $\alpha_j \in \Omega$ and $G(\alpha_j) = \sum_{k=1, j \neq k}^n \text{supp}(\alpha_j, \alpha_k)$.

Theorem 3: The aggregated value by LSVN-PA operator is still a LSVNN and is given as

$$\text{LSVN-PA}(\alpha_1, \alpha_2, \dots, \alpha_n) = \left(s_{t \left(h^{-1} \left(\frac{\sum_{j=1}^n \frac{1+G(\alpha_j)}{\sum_{j=1}^n (1+G(\alpha_j))} h \left(\frac{\theta_j}{t} \right) \right) \right)}, \right. \\ \left. s_{t \left(k^{-1} \left(\frac{\sum_{j=1}^n \frac{1+G(\alpha_j)}{\sum_{j=1}^n (1+G(\alpha_j))} k \left(\frac{\psi_j}{t} \right) \right) \right)}, \right. \\ \left. s_{t \left(k^{-1} \left(\frac{\sum_{j=1}^n \frac{1+G(\alpha_j)}{\sum_{j=1}^n (1+G(\alpha_j))} k \left(\frac{\sigma_j}{t} \right) \right) \right)} \right) \quad (6)$$

Proof: We will verify (6), by applying the mathematical induction on n . For the sake of simplicity, we take $\xi_j = (1+G(\alpha_j)) / (\sum_{j=1}^n (1+G(\alpha_j)))$. Then, the following steps of the mathematical induction have been followed for $\alpha_j = (s_{\theta_j}, s_{\psi_j}, s_{\sigma_j})$.

Step 1: For $n = 2$, we have $\alpha_1 = (s_{\theta_1}, s_{\psi_1}, s_{\sigma_1})$, $\alpha_2 = (s_{\theta_2}, s_{\psi_2}, s_{\sigma_2})$ and real numbers $\xi_1, \xi_2 > 0$, we have $\xi_1 \alpha_1 = (s_{t(h^{-1}(\xi_1 h(\theta_1/t)))}, s_{t(k^{-1}(\xi_1 k(\psi_1/t)))}, s_{t(k^{-1}(\xi_1 k(\sigma_1/t)))})$ and $\xi_2 \alpha_2 = (s_{t(h^{-1}(\xi_2 h(\theta_2/t)))}, s_{t(k^{-1}(\xi_2 k(\psi_2/t)))}, s_{t(k^{-1}(\xi_2 k(\sigma_2/t)))})$. Therefore, ■

$$\begin{aligned} \text{LSVN-PA}(\alpha_1, \alpha_2) &= \xi_1 \alpha_1 \oplus \xi_2 \alpha_2 \\ &= \left(\begin{array}{l} S_t \left(h^{-1} \left(h \left(h^{-1} \left(\xi_1 h \left(\frac{\theta_1}{t} \right) \right) \right) + h \left(h^{-1} \left(\xi_2 h \left(\frac{\theta_2}{t} \right) \right) \right) \right) \right) \\ S_t \left(k^{-1} \left(k \left(k^{-1} \left(\xi_1 k \left(\frac{\psi_1}{t} \right) \right) \right) + k \left(k^{-1} \left(\xi_2 k \left(\frac{\psi_2}{t} \right) \right) \right) \right) \right) \\ S_t \left(k^{-1} \left(k \left(k^{-1} \left(\xi_1 k \left(\frac{\sigma_1}{t} \right) \right) \right) + k \left(k^{-1} \left(\xi_2 k \left(\frac{\sigma_2}{t} \right) \right) \right) \right) \right) \end{array} \right) \\ &= \left(\begin{array}{l} S_t \left(h^{-1} \left(\xi_1 h \left(\frac{\theta_1}{t} \right) + \xi_2 h \left(\frac{\theta_2}{t} \right) \right) \right), S_t \left(k^{-1} \left(\xi_1 k \left(\frac{\psi_1}{t} \right) + \xi_2 k \left(\frac{\psi_2}{t} \right) \right) \right) \\ S_t \left(k^{-1} \left(\xi_1 k \left(\frac{\sigma_1}{t} \right) + \xi_2 k \left(\frac{\sigma_2}{t} \right) \right) \right) \end{array} \right) \\ &= \left(\begin{array}{l} S_t \left(h^{-1} \left(\sum_{j=1}^2 \xi_j h \left(\frac{\theta_j}{t} \right) \right) \right), S_t \left(k^{-1} \left(\sum_{j=1}^2 \xi_j k \left(\frac{\psi_j}{t} \right) \right) \right) \\ S_t \left(k^{-1} \left(\sum_{j=1}^2 \xi_j k \left(\frac{\sigma_j}{t} \right) \right) \right) \end{array} \right). \end{aligned}$$

Thus, (6) is true for $n = 2$.

Step 2: Assume that (6) is true for $n = m$. Then, for $n = m + 1$, we have

$$\begin{aligned} \text{LSVN-PA}(\alpha_1, \alpha_2, \dots, \alpha_{m+1}) &= \bigoplus_{j=1}^m \xi_j \alpha_j \oplus \xi_{m+1} \alpha_{m+1} \\ &= \left(\begin{array}{l} S_t \left(h^{-1} \left(\sum_{j=1}^m \xi_j h \left(\frac{\theta_j}{t} \right) \right) \right) \\ S_t \left(k^{-1} \left(\sum_{j=1}^m \xi_j k \left(\frac{\psi_j}{t} \right) \right) \right) \\ S_t \left(k^{-1} \left(\sum_{j=1}^m \xi_j k \left(\frac{\sigma_j}{t} \right) \right) \right) \end{array} \right) \oplus \left(\begin{array}{l} S_t \left(h^{-1} \left(\xi_{m+1} h \left(\frac{\theta_{m+1}}{t} \right) \right) \right) \\ S_t \left(k^{-1} \left(\xi_{m+1} k \left(\frac{\psi_{m+1}}{t} \right) \right) \right) \\ S_t \left(k^{-1} \left(\xi_{m+1} k \left(\frac{\sigma_{m+1}}{t} \right) \right) \right) \end{array} \right) \\ &= \left(\begin{array}{l} S_t \left(h^{-1} \left(h \left(h^{-1} \left(\sum_{j=1}^m \xi_j h \left(\frac{\theta_j}{t} \right) \right) + h \left(h^{-1} \left(\xi_{m+1} h \left(\frac{\theta_{m+1}}{t} \right) \right) \right) \right) \right) \\ S_t \left(k^{-1} \left(k \left(k^{-1} \left(\sum_{j=1}^m \xi_j k \left(\frac{\psi_j}{t} \right) \right) + k \left(k^{-1} \left(\xi_{m+1} k \left(\frac{\psi_{m+1}}{t} \right) \right) \right) \right) \right) \\ S_t \left(k^{-1} \left(k \left(k^{-1} \left(\sum_{j=1}^m \xi_j k \left(\frac{\sigma_j}{t} \right) \right) + k \left(k^{-1} \left(\xi_{m+1} k \left(\frac{\sigma_{m+1}}{t} \right) \right) \right) \right) \right) \end{array} \right) \\ &= \left(\begin{array}{l} S_t \left(h^{-1} \left(\sum_{j=1}^{m+1} \xi_j h \left(\frac{\theta_j}{t} \right) \right) \right), S_t \left(k^{-1} \left(\sum_{j=1}^{m+1} \xi_j k \left(\frac{\psi_j}{t} \right) \right) \right) \\ S_t \left(k^{-1} \left(\sum_{j=1}^{m+1} \xi_j k \left(\frac{\sigma_j}{t} \right) \right) \right) \end{array} \right). \end{aligned}$$

Thus, the result holds for $n = m + 1$ and hence it is true for all $n \in \mathbb{Z}^+$. ■

Example 1: Let $\alpha_1 = (s_7, s_2, s_4)$, $\alpha_2 = (s_5, s_3, s_5)$ and $\alpha_3 = (s_4, s_3, s_3)$ be three LSVNNs. Based on the support function of LSVNNs, we get $G_1 = G(\alpha_1) = 1.6250$, $G_2 = G(\alpha_2) = 1.7083$ and $G_3 = G(\alpha_3) = 1.6667$ and hence $\xi_j = (1 + G_j) / (\sum_{j=1}^3 (1 + G_j))$ becomes $\xi_1 = 0.3281$, $\xi_2 = 0.3385$ and $\xi_3 = 0.3334$. Take $t = 8$ and by considering the additive generators $k(x) = -\log(x)$ if $0 < x \leq 1$ with $k(0) = \infty$ and $h(x) = k(1 - x)$ if $0 < x \leq 1$ with $h(1) = \infty$ corresponding to t-norm and t-conorm, respectively. Then, by utilizing this information we get

$$\begin{aligned} \text{LSVN-PA}(\alpha_1, \alpha_2, \alpha_3) &= \left(\begin{array}{l} S_8 \left(1 - \prod_{j=1}^3 \left(1 - \frac{\theta_j}{8} \right)^{\xi_j} \right) \\ S_8 \left(\prod_{j=1}^3 \left(\frac{\psi_j}{8} \right)^{\xi_j} \right), S_8 \left(\prod_{j=1}^3 \left(\frac{\sigma_j}{8} \right)^{\xi_j} \right) \\ S_8 \left(1 - \left(1 - \frac{7}{8} \right)^{0.3281} \times \left(1 - \frac{5}{8} \right)^{0.3385} \times \left(1 - \frac{4}{8} \right)^{0.3334} \right) \\ S_8 \left(\left(\frac{2}{8} \right)^{0.3281} \times \left(\frac{3}{8} \right)^{0.3385} \times \left(\frac{3}{8} \right)^{0.3334} \right) \\ S_8 \left(\left(\frac{4}{8} \right)^{0.3281} \times \left(\frac{5}{8} \right)^{0.3385} \times \left(\frac{3}{8} \right)^{0.3334} \right) \end{array} \right) \\ &= (s_5.6974, s_2.6263, s_3.9194) \end{aligned}$$

Further, the LSVN-PA operator satisfies certain properties which are stated as below. Here, for simplicity, we denote $\xi_j = (1 + G(\alpha_j)) / (\sum_{j=1}^n (1 + G(\alpha_j)))$.

Theorem 4 (Idempotency): If $\alpha_j = \alpha = (s_{\theta_j}, s_{\psi_j}, s_{\sigma_j}), \forall j$, then $\text{LSVN-PA}(\alpha_1, \alpha_2, \dots, \alpha_n) = \alpha$.

Proof: As $\alpha_j = \alpha, \forall j$. Thus $\xi_j = 1/n$ and therefore by LSVN-PA operator we have $\text{LSVN-PA}(\alpha_1, \alpha_2, \dots, \alpha_n) = \bigoplus_j (1/n)\alpha = \alpha$. ■

Theorem 5 (Monotonicity): Let $\alpha_j = (s_{\theta_{\alpha_j}}, s_{\psi_{\alpha_j}}, s_{\sigma_{\alpha_j}})$ and $\beta_j = (s_{\theta_{\beta_j}}, s_{\psi_{\beta_j}}, s_{\sigma_{\beta_j}})$ be two LSVNNs such that $s_{\theta_{\alpha_j}} \leq s_{\theta_{\beta_j}}, s_{\psi_{\alpha_j}} \geq s_{\psi_{\beta_j}}$ and $s_{\sigma_{\alpha_j}} \geq s_{\sigma_{\beta_j}}$ for all j , then $\text{LSVN-PA}(\alpha_1, \alpha_2, \dots, \alpha_n) \leq \text{LSVN-PA}(\beta_1, \beta_2, \dots, \beta_n)$ for a fixed ξ_j .

Proof: Since $\theta_{\alpha_j} \leq \theta_{\beta_j}, \psi_{\alpha_j} \geq \psi_{\beta_j}$ and $\sigma_{\alpha_j} \geq \sigma_{\beta_j}$. Also, k is decreasing and h is increasing map. Thus, for a fixed ξ_j , the following equations holds:

$$\begin{aligned} h^{-1} \left(\sum_{j=1}^n \xi_j h \left(\frac{\theta_{\alpha_j}}{t} \right) \right) &\leq h^{-1} \left(\sum_{j=1}^n \xi_j h \left(\frac{\theta_{\beta_j}}{t} \right) \right) \\ k^{-1} \left(\sum_{j=1}^n \xi_j k \left(\frac{\psi_{\alpha_j}}{t} \right) \right) &\geq k^{-1} \left(\sum_{j=1}^n \xi_j k \left(\frac{\psi_{\beta_j}}{t} \right) \right) \\ k^{-1} \left(\sum_{j=1}^n \xi_j k \left(\frac{\sigma_{\alpha_j}}{t} \right) \right) &\geq k^{-1} \left(\sum_{j=1}^n \xi_j k \left(\frac{\sigma_{\beta_j}}{t} \right) \right). \end{aligned}$$

By using the definition of score function of LSVNNs, we get

$$\begin{aligned} S(\text{LSVN-PA}(\alpha_1, \alpha_2, \dots, \alpha_n)) &= \left(\begin{array}{l} 2t + h^{-1} \left(\sum_{j=1}^n \xi_j h \left(\frac{\theta_{\alpha_j}}{t} \right) \right) - k^{-1} \left(\sum_{j=1}^n \xi_j k \left(\frac{\psi_{\alpha_j}}{t} \right) \right) \\ -k^{-1} \left(\sum_{j=1}^n \xi_j k \left(\frac{\sigma_{\alpha_j}}{t} \right) \right) \end{array} \right) \\ &\leq \left(\begin{array}{l} 2t + h^{-1} \left(\sum_{j=1}^n \xi_j h \left(\frac{\theta_{\beta_j}}{t} \right) \right) - k^{-1} \left(\sum_{j=1}^n \xi_j k \left(\frac{\psi_{\beta_j}}{t} \right) \right) \\ -k^{-1} \left(\sum_{j=1}^n \xi_j k \left(\frac{\sigma_{\beta_j}}{t} \right) \right) \end{array} \right) \\ &= S(\text{LSVN-PA}(\beta_1, \beta_2, \dots, \beta_n)). \end{aligned}$$

Hence, $\text{LSVN-PA}(\alpha_1, \alpha_2, \dots, \alpha_n) \leq \text{LSVN-PA}(\beta_1, \beta_2, \dots, \beta_n)$. ■

Theorem 6 (Boundedness): If $\alpha^+ = (\max_j(s_{\theta_j}), \min_j(s_{\psi_j}), \min_j(s_{\sigma_j}))$ and $\alpha^- = (\min_j(s_{\theta_j}), \max_j(s_{\psi_j}), \max_j(s_{\sigma_j}))$, then $\alpha^- \leq \text{LSVN-PA}(\alpha_1, \alpha_2, \dots, \alpha_n) \leq \alpha^+$.

Proof: For LSVNNs α_j , we have $\min_j(\theta_j) \leq \theta_j \leq \max_j(\theta_j), \min_j(\psi_j) \leq \psi_j \leq \max_j(\psi_j), \min_j(\sigma_j) \leq \sigma_j \leq \max_j(\sigma_j)$. Since the generators k, h are decreasing and increasing maps, respectively, therefore, by

$$\begin{aligned}
t \left(h^{-1} \left(\sum_{j=1}^n \xi_j h \left(\min_j \left(\frac{\theta_j}{t} \right) \right) \right) \right) &\leq t \left(h^{-1} \left(\sum_{j=1}^n \xi_j h \left(\frac{\theta_j}{t} \right) \right) \right) \\
&\leq t \left(h^{-1} \left(\sum_{j=1}^n \xi_j h \left(\max_j \left(\frac{\theta_j}{t} \right) \right) \right) \right) \\
t \left(k^{-1} \left(\sum_{j=1}^n \xi_j k \left(\min_j \left(\frac{\psi_j}{t} \right) \right) \right) \right) &\leq t \left(k^{-1} \left(\sum_{j=1}^n \xi_j k \left(\frac{\psi_j}{t} \right) \right) \right) \\
&\leq t \left(k^{-1} \left(\sum_{j=1}^n \xi_j k \left(\max_j \left(\frac{\psi_j}{t} \right) \right) \right) \right) \\
t \left(k^{-1} \left(\sum_{j=1}^n \xi_j k \left(\min_j \left(\frac{\sigma_j}{t} \right) \right) \right) \right) &\leq t \left(k^{-1} \left(\sum_{j=1}^n \xi_j k \left(\frac{\sigma_j}{t} \right) \right) \right) \\
&\leq t \left(k^{-1} \left(\sum_{j=1}^n \xi_j k \left(\max_j \left(\frac{\sigma_j}{t} \right) \right) \right) \right)
\end{aligned}$$

which implies

$$\begin{aligned}
\min_j (\theta_j) &\leq t \left(h^{-1} \left(\sum_{j=1}^n \xi_j h \left(\frac{\theta_j}{t} \right) \right) \right) \leq \max_j (\theta_j) \\
\min_j (\psi_j) &\leq t \left(k^{-1} \left(\sum_{j=1}^n \xi_j k \left(\frac{\psi_j}{t} \right) \right) \right) \leq \max_j (\psi_j) \\
\min_j (\sigma_j) &\leq t \left(k^{-1} \left(\sum_{j=1}^n \xi_j k \left(\frac{\sigma_j}{t} \right) \right) \right) \leq \max_j (\sigma_j).
\end{aligned}$$

Now, by using Definition 6, we have

$$\begin{aligned}
\left(\min_j (\theta_j), \max_j (\psi_j), \max_j (\sigma_j) \right) &\leq \left(t \left(h^{-1} \left(\sum_{j=1}^n \xi_j h \left(\frac{\theta_j}{t} \right) \right) \right), \right. \\
&\left. t \left(k^{-1} \left(\sum_{j=1}^n \xi_j k \left(\frac{\psi_j}{t} \right) \right) \right), t \left(k^{-1} \left(\sum_{j=1}^n \xi_j k \left(\frac{\sigma_j}{t} \right) \right) \right) \right)
\end{aligned}$$

and

$$\begin{aligned}
\left(\max_j (\theta_j), \min_j (\psi_j), \min_j (\sigma_j) \right) &\geq \left(\left(t \left(h^{-1} \left(\sum_{j=1}^n \xi_j h \left(\frac{\theta_j}{t} \right) \right) \right) \right), \right. \\
&\left. t \left(k^{-1} \left(\sum_{j=1}^n \xi_j k \left(\frac{\psi_j}{t} \right) \right) \right), t \left(k^{-1} \left(\sum_{j=1}^n \xi_j k \left(\frac{\sigma_j}{t} \right) \right) \right) \right)
\end{aligned}$$

Hence, $\alpha^- \leq \text{LSVN-PA}(\alpha_1, \alpha_2, \dots, \alpha_n) \leq \alpha^+$. \blacksquare

Theorem 7 (Shift Invariance): For LSVNNs, $\beta = (s_{\theta\beta}, s_{\psi\beta}, s_{\sigma\beta})$ and $\alpha_j = (s_{\theta_j}, s_{\psi_j}, s_{\sigma_j})$, $j = 1, 2, \dots, n$, we have $\text{LSVN-PA}(\alpha_1 \oplus \beta, \alpha_2 \oplus \beta, \dots, \alpha_n \oplus \beta) = \text{LSVN-PA}(\alpha_1, \alpha_2, \dots, \alpha_n) \oplus \beta$.

Proof: By using the addition law for any two LSVNNs, we get

$$\begin{aligned}
\alpha_j \oplus \beta &= \left(s_{t \left(h^{-1} \left(h \left(\frac{\theta_j}{t} \right) + h \left(\frac{\theta\beta}{t} \right) \right) \right)}, s_{t \left(k^{-1} \left(k \left(\frac{\psi_j}{t} \right) + k \left(\frac{\psi\beta}{t} \right) \right) \right)}, \right. \\
&\quad \left. s_{t \left(k^{-1} \left(k \left(\frac{\sigma_j}{t} \right) + k \left(\frac{\sigma\beta}{t} \right) \right) \right)} \right)
\end{aligned}$$

Thus, by using (6), we have

$\text{LSVN-PA}(\alpha_1 \oplus \beta, \alpha_2 \oplus \beta, \dots, \alpha_n \oplus \beta)$

$$\begin{aligned}
&= \left(\begin{aligned} &S_{t \left(h^{-1} \left(\sum_{j=1}^n \xi_j h \left(h^{-1} \left(h \left(\frac{\theta_j}{t} \right) + h \left(\frac{\theta\beta}{t} \right) \right) \right) \right) \right)}, \\ &S_{t \left(k^{-1} \left(\sum_{j=1}^n \xi_j k \left(k^{-1} \left(k \left(\frac{\psi_j}{t} \right) + k \left(\frac{\psi\beta}{t} \right) \right) \right) \right) \right)}, \\ &S_{t \left(k^{-1} \left(\sum_{j=1}^n \xi_j k \left(k^{-1} \left(k \left(\frac{\sigma_j}{t} \right) + k \left(\frac{\sigma\beta}{t} \right) \right) \right) \right) \right)} \end{aligned} \right) \\
&= \left(\begin{aligned} &S_{t \left(h^{-1} \left(\sum_{j=1}^n \xi_j \left(h \left(\frac{\theta_j}{t} \right) + h \left(\frac{\theta\beta}{t} \right) \right) \right) \right)}, \\ &S_{t \left(k^{-1} \left(\sum_{j=1}^n \xi_j \left(k \left(\frac{\psi_j}{t} \right) + k \left(\frac{\psi\beta}{t} \right) \right) \right) \right)}, \\ &S_{t \left(k^{-1} \left(\sum_{j=1}^n \xi_j \left(k \left(\frac{\sigma_j}{t} \right) + k \left(\frac{\sigma\beta}{t} \right) \right) \right) \right)} \end{aligned} \right) \\
&= \left(\begin{aligned} &S_{t \left(h^{-1} \left(h \left(h^{-1} \left(\sum_{j=1}^n \xi_j h \left(\frac{\theta_j}{t} \right) \right) + h \left(\frac{\theta\beta}{t} \right) \right) \right) \right)}, \\ &S_{t \left(k^{-1} \left(k \left(k^{-1} \left(\sum_{j=1}^n \xi_j k \left(\frac{\psi_j}{t} \right) \right) + k \left(\frac{\psi\beta}{t} \right) \right) \right) \right)}, \\ &S_{t \left(k^{-1} \left(k \left(k^{-1} \left(\sum_{j=1}^n \xi_j k \left(\frac{\sigma_j}{t} \right) \right) + k \left(\frac{\sigma\beta}{t} \right) \right) \right) \right)} \end{aligned} \right) \\
&= \left(\begin{aligned} &S_{t \left(h^{-1} \left(\sum_{j=1}^n \xi_j h \left(\frac{\theta_j}{t} \right) \right) \right)}, \\ &S_{t \left(k^{-1} \left(\sum_{j=1}^n \xi_j k \left(\frac{\psi_j}{t} \right) \right) \right)}, \\ &S_{t \left(k^{-1} \left(\sum_{j=1}^n \xi_j k \left(\frac{\sigma_j}{t} \right) \right) \right)} \end{aligned} \right) \oplus (s_{\theta\beta}, s_{\psi\beta}, s_{\sigma\beta}) \\
&= \text{LSVN-PA}(\alpha_1, \alpha_2, \dots, \alpha_n) \oplus \beta.
\end{aligned}$$

Theorem 8 (Homogeneity): For any real number $\gamma > 0$, we have $\text{LSVN-PA}(\gamma\alpha_1, \gamma\alpha_2, \dots, \gamma\alpha_n) = \gamma \text{LSVN-PA}(\alpha_1, \alpha_2, \dots, \alpha_n)$.

Proof: As $\alpha_j = (s_{\theta_j}, s_{\psi_j}, s_{\sigma_j})$ be a LSVNN, $\forall j$ and $\gamma > 0$ be a real value, therefore, we get $\gamma\alpha_j = (s_{(h^{-1}(\gamma h(\theta_j/t)))}, s_{t(k^{-1}(\gamma k(\psi_j/t)))}, s_{t(k^{-1}(\gamma k(\sigma_j/t)))})$. Now,

$\text{LSVN-PA}(\gamma\alpha_1, \gamma\alpha_2, \dots, \gamma\alpha_n)$

$$\begin{aligned}
&= \left(\begin{aligned} &S_{t \left(h^{-1} \left(\sum_{j=1}^n \xi_j h \left(h^{-1} \left(\gamma h \left(\frac{\theta_j}{t} \right) \right) \right) \right) \right)}, \\ &S_{t \left(k^{-1} \left(\sum_{j=1}^n \xi_j k \left(k^{-1} \left(\gamma k \left(\frac{\psi_j}{t} \right) \right) \right) \right) \right)}, \\ &S_{t \left(k^{-1} \left(\sum_{j=1}^n \xi_j k \left(k^{-1} \left(\gamma k \left(\frac{\sigma_j}{t} \right) \right) \right) \right) \right)} \end{aligned} \right) \\
&= \left(\begin{aligned} &S_{t \left(h^{-1} \left(\sum_{j=1}^n \xi_j \left(\gamma h \left(\frac{\theta_j}{t} \right) \right) \right) \right)}, \\ &S_{t \left(k^{-1} \left(\sum_{j=1}^n \xi_j \left(\gamma k \left(\frac{\psi_j}{t} \right) \right) \right) \right)}, \\ &S_{t \left(k^{-1} \left(\sum_{j=1}^n \xi_j \left(\gamma k \left(\frac{\sigma_j}{t} \right) \right) \right) \right)} \end{aligned} \right) \\
&= \left(\begin{aligned} &S_{t \left(h^{-1} \left(\gamma h \left(h^{-1} \left(\sum_{j=1}^n \xi_j h \left(\frac{\theta_j}{t} \right) \right) \right) \right) \right)}, \\ &S_{t \left(k^{-1} \left(\gamma k \left(k^{-1} \left(\sum_{j=1}^n \xi_j k \left(\frac{\psi_j}{t} \right) \right) \right) \right) \right)}, \\ &S_{t \left(k^{-1} \left(\gamma k \left(k^{-1} \left(\sum_{j=1}^n \xi_j k \left(\frac{\sigma_j}{t} \right) \right) \right) \right) \right)} \end{aligned} \right) \\
&= \gamma \text{LSVN-PA}(\alpha_1, \alpha_2, \dots, \alpha_n).
\end{aligned}$$

Furthermore, by taking $\xi_j = (1 + G(\alpha_j))/(\sum_{j=1}^n (1 + G(\alpha_j)))$ and considering the different generating function k , we obtain some special AOs as follows:

i) If we take $k(x) = -\log(x)$, then (6) becomes

$$\text{LSVN-PA}(\alpha_1, \alpha_2, \dots, \alpha_n) = \left(\begin{matrix} S \\ t \left(1 - \prod_{j=1}^n \left(1 - \frac{\theta_j}{t} \right)^{\xi_j} \right), S \left(\prod_{j=1}^n \left(\frac{\psi_j}{t} \right)^{\xi_j} \right), S \left(\prod_{j=1}^n \left(\frac{\sigma_j}{t} \right)^{\xi_j} \right) \end{matrix} \right)$$

and called as LSVN Archimedean power averaging operator.

ii) Assume $k(x) = \log\left(\frac{2-x}{x}\right)$, then (6) becomes

$$\text{LSVN-PA}(\alpha_1, \alpha_2, \dots, \alpha_n) = \left(\begin{matrix} S \\ t \left(\frac{\prod_{j=1}^n \left(1 + \frac{\theta_j}{t} \right)^{\xi_j} - \prod_{j=1}^n \left(1 - \frac{\theta_j}{t} \right)^{\xi_j}}{\prod_{j=1}^n \left(1 + \frac{\theta_j}{t} \right)^{\xi_j} + \prod_{j=1}^n \left(1 - \frac{\theta_j}{t} \right)^{\xi_j}} \right), \\ S \left(\frac{2 \prod_{j=1}^n \left(\frac{\psi_j}{t} \right)^{\xi_j}}{\prod_{j=1}^n \left(2 - \frac{\psi_j}{t} \right)^{\xi_j} + \prod_{j=1}^n \left(\frac{\psi_j}{t} \right)^{\xi_j}} \right), \\ S \left(\frac{2 \prod_{j=1}^n \left(\frac{\sigma_j}{t} \right)^{\xi_j}}{\prod_{j=1}^n \left(2 - \frac{\sigma_j}{t} \right)^{\xi_j} + \prod_{j=1}^n \left(\frac{\sigma_j}{t} \right)^{\xi_j}} \right) \end{matrix} \right)$$

and called as LSVN Einstein power averaging operator.

iii) If $k(x) = \log\left(\frac{\gamma + (1-\gamma)x}{x}\right)$, $\gamma \in (0, \infty)$ then (6) becomes

$$\text{LSVN-PA}(\alpha_1, \alpha_2, \dots, \alpha_n) = \left(\begin{matrix} S \\ t \left(\frac{\prod_{j=1}^n \left(1 + (\gamma-1) \frac{\theta_j}{t} \right)^{\xi_j} - \prod_{j=1}^n \left(1 - \frac{\theta_j}{t} \right)^{\xi_j}}{\prod_{j=1}^n \left(1 + (\gamma-1) \frac{\theta_j}{t} \right)^{\xi_j} + (\gamma-1) \prod_{j=1}^n \left(1 - \frac{\theta_j}{t} \right)^{\xi_j}} \right), \\ S \left(\frac{\gamma \prod_{j=1}^n \left(\frac{\psi_j}{t} \right)^{\xi_j}}{\prod_{j=1}^n \left(1 + (\gamma-1) \left(1 - \frac{\psi_j}{t} \right) \right)^{\xi_j} + (\gamma-1) \prod_{j=1}^n \left(\frac{\psi_j}{t} \right)^{\xi_j}} \right), \\ S \left(\frac{\gamma \prod_{j=1}^n \left(\frac{\sigma_j}{t} \right)^{\xi_j}}{\prod_{j=1}^n \left(1 + (\gamma-1) \left(1 - \frac{\sigma_j}{t} \right) \right)^{\xi_j} + (\gamma-1) \prod_{j=1}^n \left(\frac{\sigma_j}{t} \right)^{\xi_j}} \right) \end{matrix} \right)$$

and called as LSVN Hamacher power averaging AO.

C. Weighted Power Aggregation Operator

In this section, we defined some weighted power AOs for LSVNNs.

Definition 10: A LSVN power weighted averaging (LSVN-PWA) operator is a mapping LSVN-PWA : $\Omega^n \rightarrow \Omega$ defined as

$$\text{LSVN-PWA}(\alpha_1, \alpha_2, \dots, \alpha_n) = \bigoplus_{j=1}^n \Psi_j \alpha_j \tag{7}$$

$$\text{where } \Psi_j = \frac{\omega_j(1 + G(\alpha_j))}{\sum_{j=1}^n \omega_j(1 + G(\alpha_j))}, \alpha_j \in \Omega, \omega_j \text{ is the weight vector}$$

of α_j such that $\omega_j > 0$, $\sum_{j=1}^n \omega_j = 1$ and $G(\alpha_j) = \sum_{k=1, j \neq k}^n \text{supp}(\alpha_j, \alpha_k)$.

Theorem 9: The aggregated value by LSVN-PWA operator for a collection of LSVNNs $\alpha_j = (s_{\theta_j}, s_{\psi_j}, s_{\sigma_j})$ is again LSVNN and is given by

$$\text{LSVN-PA}(\alpha_1, \alpha_2, \dots, \alpha_n) = \left(\begin{matrix} S \\ t \left(h^{-1} \left(\sum_{j=1}^n \Psi_j h \left(\frac{\theta_j}{t} \right) \right) \right), \\ S \left(t^{k-1} \left(\sum_{j=1}^n \Psi_j k \left(\frac{\psi_j}{t} \right) \right) \right), S \left(t^{k-1} \left(\sum_{j=1}^n \Psi_j k \left(\frac{\sigma_j}{t} \right) \right) \right) \end{matrix} \right) \tag{8}$$

Proof: Similar to Theorem 3. ■

Example 2: Let $\alpha_1 = (s_7, s_2, s_4)$, $\alpha_2 = (s_5, s_3, s_5)$ and $\alpha_3 = (s_4, s_3, s_3)$ be three LSVNNs and $\omega = (0.3, 0.4, 0.3)$ is the weight vector of $\alpha_j; j = 1, 2, 3$. Based on it, we get $G_1 = G(\alpha_1) = 1.6250$, $G_2 = G(\alpha_2) = 1.7083$ and $G_3 = G(\alpha_3) = 1.6667$. Take $t = 8$ and consider the additive generator as: $k(x) = -\log(x)$ if $0 < x \leq 1$ with $k(0) = \infty$, we have

$$\begin{aligned} \text{LSVN-PWA}(\alpha_1, \alpha_2, \alpha_3) &= \left(\begin{matrix} S \\ 8 \left(1 - \left(1 - \frac{7}{8} \right)^{0.2949} \times \left(1 - \frac{5}{8} \right)^{0.4056} \times \left(1 - \frac{4}{8} \right)^{0.2995} \right), \\ 8 \left(\left(\frac{2}{8} \right)^{0.2949} \times \left(\frac{3}{8} \right)^{0.4056} \times \left(\frac{3}{8} \right)^{0.2995} \right), \\ 8 \left(\left(\frac{4}{8} \right)^{0.2949} \times \left(\frac{5}{8} \right)^{0.4056} \times \left(\frac{3}{8} \right)^{0.2995} \right) \end{matrix} \right) \\ &= (s_{5.6348}, s_{2.6620}, s_{4.0174}). \end{aligned}$$

Further, as similar to LSVN-PA operator, it can be easily verified that the LSVN-PWA operator also has the properties of being boundedness, shift-invariance etc.

D. Ordered Weighted Power Averaging Operator

Definition 11: A LSVN power ordered weighted averaging (LSVN-POWA) operator is a mapping LSVN-POWA : $\Omega^n \rightarrow \Omega$, that has an associated weight vector $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$, such that $\omega_j > 0$ and $\sum_{j=1}^n \omega_j = 1$ and is defined as

$$\begin{aligned} \text{LSVN-POWA}(\alpha_1, \alpha_2, \dots, \alpha_n) &= \bigoplus_{j=1}^n \Psi_j \alpha_{\delta(j)} \\ &= \left(\begin{matrix} S \\ t \left(h^{-1} \left(\sum_{j=1}^n \Psi_j h \left(\frac{\theta_{\delta(j)}}{t} \right) \right) \right), \\ S \left(t^{k-1} \left(\sum_{j=1}^n \Psi_j k \left(\frac{\psi_{\delta(j)}}{t} \right) \right) \right), \\ S \left(t^{k-1} \left(\sum_{j=1}^n \Psi_j k \left(\frac{\sigma_{\delta(j)}}{t} \right) \right) \right) \end{matrix} \right) \tag{9} \end{aligned}$$

where δ is the permutation of $(1, 2, \dots, n)$ such that $\alpha_{\delta(j-1)} \geq \alpha_{\delta(j)}$ for $j = 2, 3, \dots, n$.

Example 3: Consider the data set as mentioned in Example 2. Then the score values of the LSVNNs are $S(\alpha_1) = 5.6667$, $S(\alpha_2) = 4.333$ and $S(\alpha_3) = 4.6667$. Since $S(\alpha_1) > S(\alpha_3) > S(\alpha_2)$, therefore, $\alpha_{\delta(1)} = \alpha_1 = (s_7, s_2, s_4)$, $\alpha_{\delta(2)} = \alpha_3 = (s_4, s_3, s_3)$ and $\alpha_{\delta(3)} = \alpha_2 = (s_5, s_3, s_5)$ which gives $G_1 = G(\alpha_1) = 1.6250$,

$G_2 = G(\alpha_2) = 1.6667$ and $G_3 = G(\alpha_3) = 1.7083$. Thus, we get

$$\begin{aligned} \text{LSVN-POWA}(\alpha_1, \alpha_2, \alpha_3) &= \left(\begin{array}{l} S_8 \left(1 - \left(1 - \frac{7}{8} \right)^{0.2949} \times \left(1 - \frac{4}{8} \right)^{0.4056} \times \left(1 - \frac{5}{8} \right)^{0.2995} \right), \\ S_8 \left(\left(\frac{2}{8} \right)^{0.2949} \times \left(\frac{3}{8} \right)^{0.4056} \times \left(\frac{3}{8} \right)^{0.2995} \right), \\ S_8 \left(\left(\frac{4}{8} \right)^{0.2949} \times \left(\frac{3}{8} \right)^{0.4056} \times \left(\frac{5}{8} \right)^{0.2995} \right) \end{array} \right) \\ &= (s_5.5615, s_2.6620, s_3.8055). \end{aligned}$$

Furthermore, LSVN-POWA operator also satisfies the properties of idempotency and boundedness.

E. Power Hybrid Averaging Operator

Definition 12: A LSVN power hybrid averaging (LSVN-PHA) operator is a mapping defined as LSVN-PHA: $\Omega^n \rightarrow \Omega$ and given by

$$\begin{aligned} \text{LSVN-PHA}(\alpha_1, \alpha_2, \dots, \alpha_n) &= \bigoplus_{j=1}^n \Psi_j \dot{\alpha}_{\delta(j)} \\ &= \left(\begin{array}{l} S_t \left(h^{-1} \left(\sum_{j=1}^n \Psi_j h \left(\frac{\dot{\alpha}_{\delta(j)}}{t} \right) \right) \right), \\ S_t \left(k^{-1} \left(\sum_{j=1}^n \Psi_j k \left(\frac{\dot{\alpha}_{\delta(j)}}{t} \right) \right) \right), \\ S_t \left(k^{-1} \left(\sum_{j=1}^n \Psi_j k \left(\frac{\sigma_{\delta(j)}}{t} \right) \right) \right) \end{array} \right) \quad (10) \end{aligned}$$

where $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ is the associated weight vector of LSVN-PHA operator satisfying $\omega_j > 0$ and $\sum_{j=1}^n \omega_j = 1$, $\dot{\alpha}_{\delta(j)}$ is the j th largest of the weighted LSVNNs $\dot{\alpha}_j = n w_j \alpha_j$, $j = 1, 2, \dots, n$, n is the number of LSVNNs, $w = (w_1, w_2, \dots, w_n)^T$ is the normalized weight vector of $\alpha_j (j = 1, 2, \dots, n)$ and $G(\alpha_j) = \sum_{k=1, j \neq k}^n \text{supp}(\alpha_j, \alpha_k)$.

Example 4: Let $\alpha_1 = (s_7, s_2, s_4)$, $\alpha_2 = (s_5, s_3, s_5)$ and $\alpha_3 = (s_4, s_3, s_3)$ be three LSVNNs, $w = (0.25, 0.55, 0.20)^T$ be corresponding weight vector, then, we evaluate $\dot{\alpha}_j = 3w_j \alpha_j$, by using the additive operator $k(x) = -\log(x)$ as below:

$$\begin{aligned} \dot{\alpha}_1 &= (S_t \left(1 - \left(1 - \frac{7}{8} \right)^{3 \times 0.25} \right), S_t \left(\left(\frac{2}{8} \right)^{3 \times 0.25} \right), S_t \left(\left(\frac{4}{8} \right)^{3 \times 0.25} \right)) \\ &= (s_3.2432, s_5.6569, s_6.7272) \\ \dot{\alpha}_2 &= (S_t \left(1 - \left(1 - \frac{5}{8} \right)^{3 \times 0.55} \right), S_t \left(\left(\frac{3}{8} \right)^{3 \times 0.55} \right), S_t \left(\left(\frac{5}{8} \right)^{3 \times 0.55} \right)) \\ &= (s_3.3355, s_4.6645, s_6.1777) \\ \dot{\alpha}_3 &= (S_t \left(1 - \left(1 - \frac{4}{8} \right)^{5 \times 0.20} \right), S_t \left(\left(\frac{3}{8} \right)^{5 \times 0.20} \right), S_t \left(\left(\frac{3}{8} \right)^{3 \times 0.20} \right)) \\ &= (s_1.0356, s_6.5750, s_6.5750). \end{aligned}$$

Then, based on the score values of $\dot{\alpha}_j$ we get $\dot{\alpha}_{\delta(1)} = (s_3.3355, s_4.6645, s_6.1777)$, $\dot{\alpha}_{\delta(2)} = (s_3.2432, s_5.6569, s_6.7272)$, and $\dot{\alpha}_{\delta(3)} = (s_1.0356, s_6.5750, s_6.5750)$. Assume that $\omega = (0.3, 0.4, 0.3)^T$ is the importance associated with LSVN-PHA operator and by using support we have, $G_1 = 1.6250$, $G_2 = 1.7083$ and $G_3 = 1.6667$ and therefore $\Psi_j = (\omega_j(1 + G_j)) / (\sum_{j=1}^3 \omega_j(1 + G_j))$ becomes $\Psi = (0.2949, 0.4056, 0.2995)$. Thus, by taking additive operator $k(x) = -\log(x)$, we get

LSVN-PHA($\alpha_1, \alpha_2, \alpha_3$)

$$\begin{aligned} &= \left(\begin{array}{l} S_8 \left(1 - \prod_{j=1}^3 \left(1 - \frac{\dot{\alpha}_{\delta(j)}}{8} \right)^{\Psi_j} \right), S_8 \left(\prod_{j=1}^3 \left(\frac{\dot{\alpha}_{\delta(j)}}{8} \right)^{\Psi_j} \right), S_8 \left(\prod_{j=1}^3 \left(\frac{\sigma_{\delta(j)}}{8} \right)^{\Psi_j} \right) \end{array} \right) \\ &= \left(\begin{array}{l} S_8 \left(1 - \left(1 - \frac{3.3355}{8} \right)^{0.2949} \times \left(1 - \frac{3.2432}{8} \right)^{0.4056} \times \left(1 - \frac{1.0356}{8} \right)^{0.2995} \right), \\ S_8 \left(\left(\frac{4.6645}{8} \right)^{0.2949} \times \left(\frac{5.6569}{8} \right)^{0.4056} \times \left(\frac{6.5750}{8} \right)^{0.2995} \right), \\ S_8 \left(\left(\frac{6.1777}{8} \right)^{0.2949} \times \left(\frac{6.7272}{8} \right)^{0.4056} \times \left(\frac{6.5750}{8} \right)^{0.2995} \right) \end{array} \right) \\ &= (s_2.6985, s_5.5904, s_6.5154). \end{aligned}$$

F. LSVN Power Geometric AOs

In this section, we presents some series of LSVN power geometric aggregation operators.

Definition 13: A LSVN power geometric (LSVN-PG) operator is a map LSVN-PG: $\Omega^n \rightarrow \Omega$, defined on a collection of LSVNN α_j as

$$\begin{aligned} \text{LSVN-PG}(\alpha_1, \alpha_2, \dots, \alpha_n) &= \bigotimes_{j=1}^n \alpha_j^{\xi_j} \\ &= \left(\begin{array}{l} S_t \left(k^{-1} \left(\sum_{j=1}^n \xi_j k \left(\frac{\alpha_j}{t} \right) \right) \right), \\ S_t \left(h^{-1} \left(\sum_{j=1}^n \xi_j h \left(\frac{\alpha_j}{t} \right) \right) \right), \\ S_t \left(h^{-1} \left(\sum_{j=1}^n \xi_j h \left(\frac{\sigma_j}{t} \right) \right) \right) \end{array} \right) \end{aligned}$$

where $\xi_j = \frac{1 + G(\alpha_j)}{\sum_{j=1}^n (1 + G(\alpha_j))}$, $G(\alpha_j) = \sum_{k=1, j \neq k}^n \text{supp}(\alpha_j, \alpha_k)$.

Definition 14: A LSVN power weighted geometric (LSVN-PWG) operator is a mapping LSVN-PWG: $\Omega^n \rightarrow \Omega$ and defined as

$$\begin{aligned} \text{LSVN-PWG}(\alpha_1, \alpha_2, \dots, \alpha_n) &= \bigotimes_{j=1}^n \alpha_j^{\Psi_j} \\ &= \left(\begin{array}{l} S_t \left(k^{-1} \left(\sum_{j=1}^n \Psi_j k \left(\frac{\alpha_j}{t} \right) \right) \right), \\ S_t \left(h^{-1} \left(\sum_{j=1}^n \Psi_j h \left(\frac{\alpha_j}{t} \right) \right) \right), \\ S_t \left(h^{-1} \left(\sum_{j=1}^n \Psi_j h \left(\frac{\sigma_j}{t} \right) \right) \right) \end{array} \right) \end{aligned}$$

where $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ is the weight vector of α_j such that $\omega_j > 0$ and $\sum_{j=1}^n \omega_j = 1$.

Definition 15: A LSVN power ordered weighted geometric (LSVN-POWG) operator is a mapping LSVN-POWG: $\Omega^n \rightarrow \Omega$, that has an associated weight vector $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$, such that $\omega_j > 0$ and $\sum_{j=1}^n \omega_j = 1$, and is defined as

$$\begin{aligned} \text{LSVN-POWG}(\alpha_1, \alpha_2, \dots, \alpha_n) &= \left(S_t \left(k^{-1} \left(\sum_{j=1}^n \Psi_j k \left(\frac{\alpha_{\delta(j)}}{t} \right) \right) \right), \right. \\ &\quad \left. S_t \left(h^{-1} \left(\sum_{j=1}^n \Psi_j h \left(\frac{\alpha_{\delta(j)}}{t} \right) \right) \right), S_t \left(h^{-1} \left(\sum_{j=1}^n \Psi_j h \left(\frac{\sigma_{\delta(j)}}{t} \right) \right) \right) \right) \end{aligned}$$

where $(\delta(1), \delta(2), \dots, \delta(n))$ is a permutation of $(1, 2, \dots, n)$ with $\alpha_{\delta(j-1)} \geq \alpha_{\delta(j)}$ for all $j = 2, 3, \dots, n$.

Definition 16: A LSVN power hybrid geometric (LSVN-PHG) operator is a mapping LSVN-PHG : $\Omega^n \rightarrow \Omega$ defined on a collection of LSVNN $\alpha_j, j = 1, 2, \dots, n$ as

$$\begin{aligned} \text{LSVN-PHG}(\alpha_1, \alpha_2, \dots, \alpha_n) &= \bigotimes_{j=1}^n \dot{\alpha}_{\delta(j)}^{\Psi_j} \\ &= \left(\begin{array}{c} S \left(r^{k-1} \left(\sum_{j=1}^n \Psi_j k \left(\frac{\theta_{\delta(j)}}{t} \right) \right) \right) \\ S \left(r^{h-1} \left(\sum_{j=1}^n \Psi_j h \left(\frac{\psi_{\delta(j)}}{t} \right) \right) \right) \\ S \left(r^{h-1} \left(\sum_{j=1}^n \Psi_j h \left(\frac{\sigma_{\delta(j)}}{t} \right) \right) \right) \end{array} \right) \end{aligned}$$

where $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ is their associated weight vector such that $\omega_j > 0$ and $\sum_{j=1}^n \omega_j = 1$, and $\dot{\alpha}_{\delta(j)}$ is the j th largest of the weighted LSVNNs $\dot{\alpha}_j$ ($\dot{\alpha}_j = \alpha_j^{n\omega_j}, j = 1, 2, \dots, n$), $w = (w_1, w_2, \dots, w_n)^T$ is the standardized weight vector of α_j .

Further, as similar to power averaging AOs, it has been observed that the above defined geometric operators also have the property of boundedness and idempotent.

IV. MCGDM APPROACH BASED ON PROPOSED OPERATORS

In this section, an approach to decision-making problems has been established by using the above AOs which is followed by a practical example.

A. Proposed Approach

Consider a decision-making problem which consists of m alternatives A_1, A_2, \dots, A_m and n criteria C_1, C_2, \dots, C_n , having the weight vector $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ such that $\omega_j > 0$ and $\sum_{j=1}^n \omega_j = 1$. Assume that a set of ‘ l ’ decision makers

D_1, D_2, \dots, D_l have evaluated these alternatives and give their preferences in terms of LSVNNs and their overall preference values are summarized in the decision matrix $R^{(q)} = (r_{ij}^{(q)})_{m \times n}$ where $r_{ij}^{(q)} = (s_{\theta_{ij}^{(q)}}, s_{\psi_{ij}^{(q)}}, s_{\sigma_{ij}^{(q)}})$, $q = 1, 2, \dots, l$. Based on these information, the following steps of the proposed approach, to find the best alternative(s), have been summarized as follows:

Step 1: Calculate $G(r_{ij}^{(q)})$, as follows:

$$G(r_{ij}^{(q)}) = \sum_{\substack{z=1 \\ z \neq q}}^l \text{supp}(r_{ij}^{(q)}, r_{ij}^{(z)}) \tag{11}$$

where $\text{supp}(r_{ij}^{(q)}, r_{ij}^{(z)}) = 1 - d(r_{ij}^{(q)}, r_{ij}^{(z)}); i = 1, 2, \dots, m; j = 1, 2, \dots, n; q, z = 1, 2, \dots, l$.

Step 2: Utilize the power average or geometric aggregation operator to aggregate the preference values of each decision maker into the collective one and represented as $r_{ij} = (s_{\theta_{ij}}, s_{\psi_{ij}}, s_{\sigma_{ij}})$ which are computed as:

i) By using LSVN-PA operator

$$\begin{aligned} r_{ij} &= \text{LSVN-PA}(r_{ij}^{(1)}, r_{ij}^{(2)}, \dots, r_{ij}^{(l)}) \\ &= \left(\begin{array}{c} S \left(r^{h-1} \left(\sum_{q=1}^l \frac{1 + G(r_{ij}^{(q)})}{\sum_{q=1}^l (1 + G(r_{ij}^{(q)}))} h \left(\frac{\theta_{ij}^{(q)}}{t} \right) \right) \right) \\ S \left(r^{k-1} \left(\sum_{q=1}^l \frac{1 + G(r_{ij}^{(q)})}{\sum_{q=1}^l (1 + G(r_{ij}^{(q)}))} k \left(\frac{\psi_{ij}^{(q)}}{t} \right) \right) \right) \\ S \left(r^{k-1} \left(\sum_{q=1}^l \frac{1 + G(r_{ij}^{(q)})}{\sum_{q=1}^l (1 + G(r_{ij}^{(q)}))} k \left(\frac{\sigma_{ij}^{(q)}}{t} \right) \right) \right) \end{array} \right) \end{aligned}$$

ii) By using LSVN-PG operator

$$\begin{aligned} r_{ij} &= \text{LSVN-PG}(r_{ij}^{(1)}, r_{ij}^{(2)}, \dots, r_{ij}^{(l)}) \\ &= \left(\begin{array}{c} S \left(r^{k-1} \left(\sum_{q=1}^l \frac{1 + G(r_{ij}^{(q)})}{\sum_{q=1}^l (1 + G(r_{ij}^{(q)}))} k \left(\frac{\theta_{ij}^{(q)}}{t} \right) \right) \right) \\ S \left(r^{h-1} \left(\sum_{q=1}^l \frac{1 + G(r_{ij}^{(q)})}{\sum_{q=1}^l (1 + G(r_{ij}^{(q)}))} h \left(\frac{\psi_{ij}^{(q)}}{t} \right) \right) \right) \\ S \left(r^{h-1} \left(\sum_{q=1}^l \frac{1 + G(r_{ij}^{(q)})}{\sum_{q=1}^l (1 + G(r_{ij}^{(q)}))} h \left(\frac{\sigma_{ij}^{(q)}}{t} \right) \right) \right) \end{array} \right) \end{aligned}$$

Step 3: Calculate $G(r_{ij}), i = 1, 2, \dots, m; j = 1, 2, \dots, n$, as $G(r_{ij}) = \sum_{u=1, j \neq u}^n \text{supp}(r_{ij}, r_{iu})$

where

$$\text{supp}(r_{ij}, r_{iu}) = 1 - d(r_{ij}, r_{iu}), i = 1, 2, \dots, m; u, j = 1, 2, \dots, n.$$

Step 4: Calculate the comprehensive evaluation value r_i of each alternative by using LSVN-PWA or LSVN-POWA or LSVN-PHA or LSVN-PWG or LSVN-POWG or LSVN-PHG operator.

Step 5: Calculate the score function $S(r_i), i = 1, 2, \dots, n$, by using Definition 3.

Step 6: Rank all the alternatives according to the descending order of score values of alternatives.

B. Illustrative Example

The above mentioned approach has been illustrated with a numerical example taken from Garg and Nancy [36] as:

Goods and services tax (GST) is one indirect tax for the whole nation, which will make India a unified common market. While GST promises to a user in an era of the unified indirect tax regime, integrating India into a single homogeneous market, it comes with certain complications inherited from the legacy tax regime. With the government gearing up to enforce the GST in Punjab from July 1, the issue of traders having limited computer knowledge and poor connectivity. In order to counter this, the

state government has planned to train more than 2000 youths as “GST Mitra” to cater the traders. Punjab GST Mitra Scheme, which has to be started as a pilot project from Patiala, proposes to assist taxpayers in furnishing the details of outward supplies, inward supplies and returns, filing claims or refunds, filing any other applications etc., in GST Regime. It aims to create a group of Tax professionals available in the locality or at the doorstep of the taxpayer, at affordable costs throughout the State of Punjab.

The poor internet connectivity in far-flung areas has emerged as a big stumbling block in the success of “GST Mitra” scheme. In order to provide the online services to run this scheme, the state government is planning to give contract combinedly to private mobile service provider along with state-owned BSNL. For this purpose, committee formed by the government short-listed the four internet service provider, namely, “Bharti Airtel” (A_1), “Reliance Communications” (A_2), “Vodafone India” (A_3) and “Mahanagar Telecom Nigam” (A_4) under the criteria: “Customer Services” (C_1), “Bandwidth” (C_2), “Package Deal” (C_3), “Total Cost” (C_4) with weight vector $\omega = (0.2, 0.3, 0.1, 0.4)^T$. Assume that weight vector associated with hybrid operator is $w = (0.2575, 0.3316, 0.1292, 0.2817)^T$. Government has selected three decision makers D_1, D_2 and D_3 who have the responsibilities to evaluate the given alternatives and rate their preferences in terms of linguistic term set $Q = \{s_0 = \text{“extremely poor”}, s_1 = \text{“very poor”}, s_2 = \text{“poor”}, s_3 = \text{“slightly poor”}, s_4 = \text{“fair”}, s_5 = \text{“slightly good”}, s_6 = \text{“good”}, s_7 = \text{“very good”}, s_8 = \text{“extremely good”}\}$. The rating values of these decision makers are summarized as

$$R^{(1)} = \begin{matrix} & C_1 & C_2 & C_3 & C_4 \\ \begin{matrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{matrix} & \begin{bmatrix} (s_4, s_4, s_3) & (s_3, s_1, s_5) & (s_6, s_3, s_5) & (s_2, s_3, s_6) \\ (s_4, s_3, s_2) & (s_1, s_2, s_3) & (s_5, s_1, s_3) & (s_3, s_1, s_4) \\ (s_5, s_3, s_3) & (s_2, s_3, s_4) & (s_6, s_4, s_3) & (s_1, s_4, s_3) \\ (s_3, s_5, s_1) & (s_7, s_2, s_2) & (s_3, s_2, s_3) & (s_5, s_2, s_2) \end{bmatrix} \end{matrix}$$

$$R^{(2)} = \begin{matrix} & C_1 & C_2 & C_3 & C_4 \\ \begin{matrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{matrix} & \begin{bmatrix} (s_3, s_5, s_2) & (s_4, s_4, s_7) & (s_5, s_1, s_4) & (s_2, s_1, s_3) \\ (s_5, s_1, s_2) & (s_2, s_5, s_1) & (s_3, s_5, s_4) & (s_5, s_4, s_2) \\ (s_7, s_4, s_3) & (s_2, s_3, s_5) & (s_7, s_2, s_4) & (s_3, s_1, s_2) \\ (s_2, s_6, s_5) & (s_6, s_3, s_3) & (s_5, s_1, s_2) & (s_4, s_2, s_1) \end{bmatrix} \end{matrix}$$

$$R^{(3)} = \begin{matrix} & C_1 & C_2 & C_3 & C_4 \\ \begin{matrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{matrix} & \begin{bmatrix} (s_6, s_1, s_1) & (s_5, s_2, s_1) & (s_7, s_2, s_1) & (s_6, s_1, s_2) \\ (s_5, s_2, s_2) & (s_2, s_2, s_3) & (s_6, s_2, s_6) & (s_3, s_4, s_5) \\ (s_6, s_2, s_3) & (s_2, s_6, s_4) & (s_3, s_4, s_3) & (s_4, s_2, s_1) \\ (s_3, s_5, s_4) & (s_5, s_2, s_6) & (s_4, s_5, s_2) & (s_6, s_2, s_1) \end{bmatrix} \end{matrix}$$

Without loss of generality, we have taken the additive generator $k(x) = -\log(x)$ and hence the following steps of the proposed approach have been summarized as

Step 1: By utilizing (11), we get

$$G^{(1)} = \begin{bmatrix} 1.5833 & 1.5417 & 1.3750 \\ 1.7917 & 1.8333 & 1.8750 \\ 1.7917 & 1.7500 & 1.7917 \\ 1.6250 & 1.6250 & 1.7500 \end{bmatrix}$$

$$G^{(2)} = \begin{bmatrix} 1.5417 & 1.5417 & 1.4167 \\ 1.7083 & 1.5417 & 1.7500 \\ 1.8333 & 1.7917 & 1.7083 \\ 1.7083 & 1.7917 & 1.6667 \end{bmatrix}$$

$$G^{(3)} = \begin{bmatrix} 1.5833 & 1.5833 & 1.500 \\ 1.5000 & 1.3750 & 1.4583 \\ 1.7917 & 1.6250 & 1.7500 \\ 1.6250 & 1.6250 & 1.5833 \end{bmatrix}$$

$$G^{(4)} = \begin{bmatrix} 1.3750 & 1.5833 & 1.3750 \\ 1.5417 & 1.5000 & 1.6250 \\ 1.4583 & 1.625 & 1.5833 \\ 1.8333 & 1.8333 & 1.8333 \end{bmatrix}$$

Step 2: Aggregate the values of the decision matrices by using either LSVN-PA or LSVN-PG operators and their results are summarized in Tables I and II respectively.

TABLE I
AGGREGATED MATRIX BY USING LSVN-PA

	C_1	C_2
A_1	$(s_4.5360, s_2.7813, s_1.8465)$	$(s_4.0677, s_1.5813, s_3.3363)$
A_2	$(s_4.7027, s_1.8135, s_2.0000)$	$(s_1.6786, s_2.6758, s_2.1161)$
A_3	$(s_6.1774, s_2.8798, s_3.0000)$	$(s_2.0000, s_3.7580, s_4.3105)$
A_4	$(s_2.6918, s_5.3082, s_2.7309)$	$(s_6.1857, s_2.6280, s_2.8808)$
	C_3	C_4
A_1	$(s_6.1710, s_1.8152, s_2.7440)$	$(s_3.7963, s_1.4273, s_3.2929)$
A_2	$(s_4.9101, s_2.1246, s_4.1543)$	$(s_3.7672, s_2.5262, s_3.4441)$
A_3	$(s_6.1653, s_3.2011, s_3.2906)$	$(s_2.8338, s_1.9701, s_1.8033)$
A_4	$(s_4.0856, s_2.1448, s_2.2911)$	$(s_5.1155, s_2.0000, s_1.2599)$

TABLE II
AGGREGATED MATRIX BY USING LSVN-PG

	C_1	C_2
A_1	$(s_4.1256, s_3.6681, s_2.0833)$	$(s_3.8989, s_1.6782, s_5.2836)$
A_2	$(s_4.1256, s_2.0508, s_2.0000)$	$(s_1.5817, s_3.1859, s_2.4359)$
A_3	$(s_5.9391, s_3.0624, s_3.0000)$	$(s_2.0000, s_4.2877, s_4.3675)$
A_4	$(s_2.6263, s_5.3737, s_3.6267)$	$(s_5.9497, s_2.6933, s_3.0616)$
	C_3	C_4
A_1	$(s_5.9334, s_2.0567, s_3.6427)$	$(s_2.8546, s_1.7227, s_4.0578)$
A_2	$(s_4.5047, s_2.9478, s_4.5788)$	$(s_3.5438, s_3.1846, s_3.8328)$
A_3	$(s_5.9293, s_3.4432, s_3.3461)$	$(s_2.3240, s_2.4460, s_2.0390)$
A_4	$(s_3.9144, s_2.9730, s_2.3556)$	$(s_4.9324, s_2.0000, s_1.3506)$

Step 3: Compute $G_{ij}, i = 1, 2, 3, 4; j = 1, 2, 3, 4$:

a) By using the data given in Table I, we have

$$G = \begin{bmatrix} 2.5751 & 2.7268 & 2.5942 & 2.6950 \\ 2.5930 & 2.4421 & 2.5521 & 2.6286 \\ 2.4817 & 2.2818 & 2.4827 & 2.3071 \\ 2.2282 & 2.4660 & 2.5677 & 2.4697 \end{bmatrix}$$

b) By using the data given in Table II, we have

$$G = \begin{bmatrix} 2.3506 & 2.5090 & 2.4642 & 2.5278 \\ 2.4853 & 2.4446 & 2.5485 & 2.6082 \\ 2.4808 & 2.2991 & 2.4816 & 2.3501 \\ 2.1881 & 2.4579 & 2.5426 & 2.4011 \end{bmatrix}$$

Step 4: By utilizing LSVN-PWA or LSVN-POWA or LSVN-PHA or LSVN-PWG or LSVN-POWG or LSVN-PHG operators, we get the aggregated values r_i of the alternatives

TABLE III
AGGREGATED LSVNNS BY USING DIFFERENT OPERATORS

Operators	r_1	r_2	r_3	r_4
LSVN-PWA	(\$4.3438, \$1.7169, \$2.9015)	(\$3.6184, \$2.3612, \$2.7298)	(\$4.0957, \$2.7103, \$2.7560)	(\$5.0981, \$2.6299, \$1.9928)
LSVN-POWA	(\$4.6520, \$1.8513, \$2.6680)	(\$3.6738, \$2.3002, \$2.6714)	(\$4.7434, \$3.1721, \$3.3734)	(\$4.6887, \$3.2484, \$2.3560)
LSVN-PHA	(\$4.4914, \$2.2938, \$2.9237)	(\$3.3716, \$2.5596, \$2.8466)	(\$4.4097, \$3.0225, \$3.0346)	(\$4.7872, \$2.7422, \$2.2625)
LSVN-PWG	(\$3.6218, \$2.1747, \$4.1532)	(\$3.0240, \$2.9553, \$3.2358)	(\$2.9797, \$3.2807, \$3.1541)	(\$4.5290, \$3.1437, \$2.4846)
LSVN-POWG	(\$4.1687, \$2.4317, \$4.0928)	(\$2.8835, \$2.9087, \$3.2459)	(\$3.4973, \$3.6412, \$3.6086)	(\$4.5290, \$3.9281, \$2.9560)
LSVN-PHG	(\$3.8239, \$2.4989, \$4.4578)	(\$2.4989, \$2.9087, \$2.8745)	(\$2.9550, \$3.8349, \$3.8232)	(\$3.8255, \$3.8724, \$2.7130)

$A_i, i = 1, 2, 3, 4$ and the results are summarized in Table III.

Step 5: The score values of these aggregated numbers are summarized in Table IV.

Step 6: Based on these score values, the ranking order of the alternatives is summarized in the last column of the Table IV. From this table, we can see that corresponding to the different pairs, the best alternative is either A_4 or A_1 .

C. Comparative Analysis

To compare the performance of the proposed approach with the existing operators, namely linguistic neutrosophic number (LNN) weighted averaging and geometric aggregation operators denoted by LNNWAA and LNNWGA as proposed by Fang and Ye [35], LNN weighted power Heronian average (LNWPFA) proposed by Liu *et al.* [21] and Linguistic single-valued neutrosophic Prioritized aggregation (LNPA) proposed by Garg and Nancy [36]. For it, the considered data of the problem has been taken and then the following steps of their approaches have been implemented which are summarized as follows:

Step 1: Aggregate the different preferences of decision makers by using LNNWAA or LNNWGA operators. The results of it are summarized in Tables V and VI respectively. On the other hand, if we apply LNWPFA operator (with $p = 1, q = 2$) to calculate the overall values of each alternative for each decision maker then their results are summarized in Table VII. On the other hand, the values of each alternative by LNPA operator are summarized in Table VIII.

Step 2: Based on these aggregated values, the overall rating values $r_i, i = 1, 2, 3, 4$ of the alternatives are obtained by using LNNWAA are $r_1 = (s4.3842, s1.7238, s2.8652)$, $r_2 = (s3.60.2, s2.3761, s2.6987)$, $r_3 = (s4.0426, s2.7280, s2.7629)$ and $r_4 = (s5.0746, s2.6567, s1.9992)$ while by using LNNWGA operator we get $r_1 = (s3.0983, s2.1872, s4.1251)$, $r_2 = (s2.5467, s2.9720, s3.2029)$, $r_3 = (s2.5529, s3.3087, s3.1670)$ and $r_4 = (s4.5233, s3.1853, s2.4984)$. However, by using LNWPFA operator to calculate the collective LSVN given in Table VII, we can get $r_1 = (s5.5054, s0.9201, s1.7792)$, $r_2 = (s4.7183, s1.2941, s1.5345)$, $r_3 = (s5.1725, s1.6739, s1.7312)$ and $r_4 = (s6.2089, s1.4063, s1.1454)$. By LNPA operator, these values are $r_1 = (s5.0259, s2.2665, s3.2699)$, $r_2 = (s4.2176, s2.0214, s2.6403)$, $r_3 = (s5.4767, s3.0336, s3.0597)$ and $r_4 = (s5.0871, s2.2943, s1.8710)$.

Step 3: The score values and the final ranking of the alternatives are obtained by using values of r_i and results are represented in Table IX.

It is noted from this table that the best alternative obtained by the existing approaches is still A_4 which coincides with the

proposed approach. However, the computational procedure of the proposed approach is different from the existing studies. For instance, Fang and Ye [35] weighted averaging and geometric aggregation operators were introduced based on the algebraic sum and product and do not consider the dependency factor between the attributes. Liu *et al.* [21], they considered the PA and Heronian mean operator simultaneously to interact with the different pairs of LNNs. In [36], authors have aggregated the values by using prioritized aggregation operators. On the other hand, the proposed operator takes the advantages of LSVNS and power aggregation to aggregate the process by using the support function for solving MCGDM problems.

D. Further Discussion

To justify the importance of the proposed work with respect to the existing work, an investigation is done in the form of their characteristics comparison. The results are summarized in Table X.

The approaches in [16]–[18], [25] are based on a weighted averaging operators by utilizing the quantitative information only. Also, in their approaches, they had assumed that all the input arguments are independent of each other. Therefore, their approaches are limited and unable to apply to the linguistic information. However, on the contrary, the proposed approach can capture the interrelationship among input arguments and also describe for the qualitative with linguistic information. Hence, the present approach is much better than the methods in [16]–[18], [25]. In [35], authors have considered the linguistic features during the information extract phase and hence proposed the weighted averaging and geometric operators to aggregate them. But again, during their formulation, they assumed that all the collective information is independent of each other. Thus, their approach is limited. In [21] approach, authors have proposed power Heronian mean by utilizing the linguistic information by considering the interdependency between the pairs of the input arguments. Also, in [36], the authors have presented prioritized weighted operators for the linguistic SVNNs. However, the computational process in the proposed approach is entirely different from the existing one. In the proposed approach, the information used for the analysis is based on the linguistic nature which can reflect the inherent thoughts of decision makers more accurately. Also, the information about the attribute weights are extracted from the support measures instead of the priori assigned. In addition, it considers the interaction between the pair of the input arguments during the analysis.

TABLE IV
SCORE VALUES OF AGGREGATED LSVNNS

Operators	$S(r_1)$	$S(r_2)$	$S(r_3)$	$S(r_4)$	Ranking
LSVN-PWA	5.2418	4.8425	4.8764	5.4918	$A_4 > A_1 > A_3 > A_2$
LSVN-POWA	5.3776	4.9007	4.7327	5.0281	$A_1 > A_4 > A_2 > A_3$
LSVN-PHA	5.0913	4.6551	4.7842	5.2608	$A_4 > A_1 > A_3 > A_2$
LSVN-PWG	4.4313	4.2776	4.1793	4.9669	$A_4 > A_1 > A_3 > A_2$
LSVN-POWG	4.5481	4.2430	4.0825	4.3584	$A_1 > A_4 > A_2 > A_3$
LSVN-PHG	4.2891	4.1841	3.7656	4.4134	$A_4 > A_1 > A_3 > A_2$

TABLE V
AGGREGATED MATRIX BY USING LNNWAA

	C_1	C_2
A_1	(\$4.5800, \$2.7144, \$1.8171)	(\$4.0851, \$1.5874, \$3.2711)
A_2	(\$4.6981, \$1.8171, \$2.0000)	(\$1.6836, \$2.7144, \$2.0801)
A_3	(\$6.1829, \$2.8845, \$3.0000)	(\$2.0000, \$3.7798, \$4.3089)
A_4	(\$2.6867, \$5.3133, \$2.7144)	(\$6.1829, \$2.6207, \$2.8845)
	C_3	C_4
A_1	(\$6.1829, \$1.8171, \$2.7144)	(\$3.8398, \$1.4422, \$3.3019)
A_2	(\$4.8928, \$2.1544, \$4.1602)	(\$3.7828, \$2.5198, \$3.4200)
A_3	(\$6.1829, \$3.1748, \$3.3019)	(\$2.8075, \$2.0000, \$1.8171)
A_4	(\$4.0851, \$2.1544, \$2.2894)	(\$5.1155, \$2.0000, \$1.2599)

TABLE VI
AGGREGATED MATRIX BY USING LNNWGA

	C_1	C_2
A_1	(\$1.8171, \$3.6205, \$2.0561)	(\$3.9149, \$1.6836, \$5.2411)
A_2	(\$2.0000, \$2.0561, \$2.0000)	(\$1.5874, \$3.2378, \$2.4066)
A_3	(\$3.0000, \$3.0676, \$3.0000)	(\$2.0000, \$4.3160, \$4.3089)
A_4	(\$2.7144, \$5.3793, \$3.6205)	(\$5.9439, \$2.6867, \$3.0676)
	C_3	C_4
A_1	(\$5.9439, \$2.0561, \$3.6205)	(\$2.8845, \$1.7427, \$4.0851)
A_2	(\$4.4814, \$2.9867, \$4.5800)	(\$3.5569, \$3.1797, \$3.8398)
A_3	(\$5.9439, \$3.4211, \$3.3584)	(\$2.2894, \$2.4822, \$2.0561)
A_4	(\$3.9149, \$2.9867, \$2.3538)	(\$4.9324, \$2.0000, \$1.3506)

TABLE VII
AGGREGATED VALUES OF THE ALTERNATIVES BY LNWPFA

	$R^{(1)}$	$R^{(2)}$	$R^{(3)}$
A_1	(\$3.8346, \$1.8184, \$4.3324)	(\$3.7278, \$1.8246, \$3.0596)	(\$6.4868, \$0.9092, \$0.7986)
A_2	(\$3.5145, \$1.1608, \$2.4493)	(\$4.6719, \$2.7618, \$1.2733)	(\$4.2265, \$1.9832, \$2.9664)
A_3	(\$3.5673, \$2.8156, \$2.6579)	(\$5.3880, \$1.6848, \$2.6011)	(\$4.6120, \$2.4765, \$1.8543)
A_4	(\$5.9348, \$1.8753, \$1.2836)	(\$5.2285, \$2.0137, \$1.7473)	(\$5.5571, \$2.1654, \$2.2540)

TABLE VIII
AGGREGATED VALUES OF THE ALTERNATIVES BY LNPA

	C_1	C_2
A_1	(\$4.1449, \$3.4758, \$2.2553)	(\$3.6040, \$1.6671, \$4.6022)
A_2	(\$4.5581, \$1.9737, \$2.0000)	(\$1.4475, \$2.5985, \$2.1918)
A_3	(\$6.0270, \$3.0160, \$3.0000)	(\$2.0000, \$3.2507, \$4.2559)
A_4	(\$2.7009, \$5.2991, \$1.8998)	(\$6.3733, \$2.2969, \$2.9492)
	C_3	C_4
A_1	(\$6.0332, \$2.0011, \$3.4079)	(\$2.8398, \$1.9924, \$4.3836)
A_2	(\$4.6146, \$1.9564, \$3.6782)	(\$3.7092, \$1.9633, \$3.3888)
A_3	(\$6.0265, \$3.2439, \$3.2725)	(\$2.1207, \$2.5390, \$2.2678)
A_4	(\$3.8890, \$1.9966, \$2.4444)	(\$4.9995, \$2.0000, \$1.3684)

the Table IV, the LSVN-PWA operator results A_3 as non-optimal and A_2 as the worst one. For execution of the test Criteria 1 on the proposed approach, the alternative A_3 is replaced with less desirable alternative in the original decision matrices. The changed data for the alternative A_3 is given in Table XI.

Apply the LSVN-PA operator to aggregate the modified decision-making metrics and then apply the LSVN-PWA operator to get the final aggregated values which are summarized as $r_1 = (\$4.3438, \$1.7169, \$2.9015)$, $r_2 = (\$3.6184, \$2.3612, \$2.7298)$, $r_3 = (\$2.7451, \$3.4168, \$3.5128)$ and $r_4 = (\$5.0981, \$2.6299, \$1.9928)$. The score values of these numbers are $S(r_1) = 5.2418$, $S(r_2) = 4.8425$, $S(r_3) = 3.9385$ and $S(r_4) = 5.4918$. Thus, alternatives are ranked as $A_4 > A_1 > A_2 > A_3$ which indicates A_4 is the best alternative as given by the actual problem. From this, we conclude that the given approach does not reverse the original decision after replacing the non-optimal alternative by some other alternative. Therefore, the proposed method is effective according to test Criteria 1.

2) *Validity Under Criteria 2 and 3*: For testing the validity of the above method by using the remaining two test criteria's,

E. Test Criteria for Evaluating the Decision-making Methods

To validate the proposed method, Wang and Triantaphyllou [6] gave some test criteria's for validation of the decision-making approach which are given as follows:

Test criterion 1: "A valid decision-making approach should give the same best alternative even if a non-optimal alternative is changed by another worse one."

Test criterion 2: "An valid decision-making approach should have transitive property."

Test criterion 3: "Decomposition of decision-making problem into smaller problems should not affect the ranking of the original problem."

To investigate the validity of the proposed approach, these three test criteria have been tested as follows.

1) *Validity Under Criterion 1*: As, it can be easily seen from

TABLE IX
SCORE AND RANKING VALUES BY USING THE EXISTING OPERATORS

	$S(r_1)$	$S(r_2)$	$S(r_3)$	$S(r_4)$	Ranking
LNNWAA[35]	5.2651	4.8428	4.8506	5.4729	$A_4 > A_1 > A_3 > A_2$
LNNWGA[35]	4.2620	4.1240	4.0257	4.9465	$A_4 > A_1 > A_2 > A_3$
LNWPHA[21]	6.2687	5.9633	5.9225	6.5524	$A_4 > A_1 > A_2 > A_3$
LNPA[36]	5.1631	5.1853	5.1278	5.6406	$A_4 > A_2 > A_1 > A_3$

TABLE X
THE CHARACTERISTIC COMPARISONS OF DIFFERENT METHODS

Methods	Properties			
	Whether attribute weights derived from the support measures	Whether describe information using linguistic features	Whether consider the interrelationship between input arguments	Whether consider multiple group of decision makers
Peng <i>et al.</i> [17]	×	×	×	×
Ye [16]	×	×	×	×
Nancy and Garg[18]	×	×	×	×
Ji <i>et al.</i> [25]	×	×	×	×
Fang and Ye [35]	×	√	×	×
Garg and Nancy [36]	×	√	√	√
The proposed method	√	√	√	√

TABLE XI
RATING OF A_3 FOR EACH DECISION MAKER

	C_1	C_2	C_3	C_4
D_1	(s_4, s_3, s_4)	(s_2, s_4, s_4)	(s_4, s_5, s_6)	(s_1, s_5, s_3)
D_2	(s_5, s_4, s_4)	(s_1, s_3, s_5)	(s_5, s_3, s_4)	(s_2, s_3, s_2)
D_3	(s_4, s_2, s_3)	(s_2, s_6, s_5)	(s_3, s_4, s_4)	(s_3, s_2, s_3)

we decompose the original problem into various smaller decision-making problems as $\{A_1, A_2, A_3\}$, $\{A_2, A_3, A_4\}$, and $\{A_3, A_4, A_1\}$. After applying the proposed approach, we get the ranking of decomposed decision-making problems as $A_1 > A_3 > A_2$, $A_4 > A_3 > A_2$, and $A_4 > A_1 > A_3$ respectively. If we combine the ranking of these decomposed problems, we get the final ranking as $A_4 > A_1 > A_3 > A_2$, which is same as that of the original (un-decomposed) problem and also performs the transitive property. Thus, both test criteria's verify the proposed approach.

V. CONCLUSION

In this paper, an attempt has been made to present the power aggregation operators from both arithmetic and geometric points of view, under the LSVN environment for various decision-making problems. The important characteristic of the proposed work is that they consider the supporting degree between the criteria's and ignore the effect of biased assessments on the results by giving little importance to those false and biased ones compared to other ones. The proposed operators involve the simple power aggregation, power weighted aggregation, power ordered weighted aggregation, and power hybrid aggregation operators. These operators have been used to develop the group decision-making approach and then apply to a practical problem related to decision making. Further, from the comparative analysis and the validity tests,

we can conclude that the proposed approach can be successfully utilized in evaluating the best alternative and is more reliable than simple algebraic and geometric operators as it considers the relationships among the criteria and overlooks the useless biased data. Future research will focus on introducing the various other operators [39]–[43] or to extend the approach to the linguistic interval neutrosophic environment [44]–[48].

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Harish Garg received the Ph.D. degree in applied mathematics from IIT Roorkee, India, in 2013. His research interests include soft computing, decision-making, aggregation operator, and intuitionistic fuzzy set. Dr. Garg has produced more than 215 papers published in refereed International journals including *IEEE Trans. Fuzzy Sys., Inf. Sci., Appl. Intell., Expert Syst. Appl., Cognit. Comput., Soft Comput., Int. J. Intell. Syst., Appl. Soft Comput., Comput. Ind. Eng., Appl. Math. Comput., IEEE Trans. Emerging Topics Comput. Intell., Computer Oper. Research*, and many more. Also, he has published 7 book chapters. He is the Associate Editor of *Journal of Intelligent & Fuzzy Systems, International Journal of Computational Intelligence Systems, Technological and Economic Development of Economy, Mathematical Problems in Engineering, Journal of Industrial & Management Optimization, Complexity, Complex and Intelligent Systems, CAAI Transactions on Intelligence Technology* and so on. In year 2016–2019, Dr. Garg was privileged an Awarded Outstanding Reviewer for the various Elsevier journals *ASOC, AMM, CAIE, FSS, EAAI* etc. His Google citations are more than 6450. For more details, visit <http://sites.google.com/site/harishg58iitr/>.



Nancy received the Ph.D. degree in mathematics from Thapar Institute of Engineering & Technology, India. She has received the master degree in mathematics during 2012–2014 and graduation during 2009–2012 from Punjabi University Patiala, India. Currently, her research interests include uncertainty, neutrosophic set systems, and decisionmaking theory. She has published 12 articles in different reputed SCI journals.