Robust D-Stability Test of LTI General Fractional Order Control Systems

Reza Mohsenipour and Xinzhi Liu

***Abstract*—This work deals with the robust *D*-stability test of linear time-invariant (LTI) general fractional order control systems in a closed loop where the system and/or the controller may be of fractional order. The concept of general implies that the characteristic equation of the LTI closed loop control system may be of both commensurate and non-commensurate orders, both the coefficients and the orders of the characteristic equation may be nonlinear functions of uncertain parameters, and the coefficients may be complex numbers. Some new specific areas for the roots of the characteristic equation are found so that they reduce the computational burden of testing the robust *D*-stability. Based on the value set of the characteristic equation, a necessary and sufficient condition for testing the robust *D*-stability of these systems is derived. Moreover, in the case that the coefficients are linear functions of the uncertain parameters and the orders do not have any uncertainties, the condition is adjusted for further computational burden reduction. Various numerical examples are given to illustrate the merits of the achieved theorems.**

***Index Terms*—Fractional order control system, LTI system, robust *D*-stability, value set.**

I. Introduction

FRACTIONAL order calculus, is nowadays a well-known theory which deals with integrals and derivatives of arbitrary orders [1]. The superiority of fractional order calculus with respect to its integer counterpart is its ability to more accurate model the behaviour of many systems in the real world, such as viscoelastic materials [2], chaotic systems [3], waves propagation [4], biological systems [5], multi-agent systems [6], and human operator behaviors [7]. Further applications can be found in [8] and references therein. Moreover, because of the robustness and fast performance of fractional order controllers, their implementation are spreading widely [9]–[12]. Furthermore, modeling real world systems often leads to uncertain mathematical models. Hence, the robust stability and performance analysis of fractional order control systems, where the controller and/or the system may be of fractional order, have attracted much interest from many researchers.

Manuscript received September 6, 2019; revised January 14, 2020; accepted February 21, 2020. Recommended by Associate Editor Qinglong Han. *(Corresponding author: Reza Mohsenipour.)*

Citation: R. Mohsenipour and X. Liu, "Robust D-stability test of LTI general fractional order control systems," *IEEE/CAA J. Autom. Sinica*, vol. 7, no. 3, pp. 853–864, May 2020.

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Digital Object Identifier 10.1109/JAS.2020.1003159

Over the last decade, many papers have been published to present criteria for the robust stability analysis of linear time-invariant (LTI) fractional order systems by different methods. Some sufficient criteria using the linear matrix inequality (LMI) approach were introduced in [13]–[15]. For the robust stability of fractional order systems of commensurate orders between 0 and 2, some necessary and sufficient criteria were produced in [16], [17]. However, finding a solution for the LMI may lead to conservatism [18], [19]. Moreover, the LMI approach can be applied only to systems of commensurate order, and not to the ones of non-commensurate order. The robust stability of fractional order systems of commensurate order was also studied in [20] in the roots space of the characteristic equation. The aforementioned papers studied systems having uncertainties only in the coefficients of their characteristic equations. Some criteria for the robust stability of fractional order systems of both commensurate and non-commensurate orders with uncertainties in both the coefficients and the orders were reported in [21] using Young and Jensen inequalities, but these criteria were sufficient; not necessary and sufficient.

Another efficient tool to check the robust stability of LTI fractional order systems, which can be less conservative and also employed for systems of both commensurate and non-commensurate orders, is the value set. Initially, a sufficient criterion was introduced in [22] for the robust stability of fractional order systems with uncertainties only in the coefficients by using the value set tools and extending the zero exclusion condition to fractional order systems. Then, in works [23]–[25], some necessary and sufficient criteria were presented. Sufficient criteria were also presented in [26] for fractional order systems having multi-linear uncertainties in their coefficients. Moreover, the robust stability of fractional order systems having uncertainties in both their orders and coefficients was studied in [27]–[29].

All the above-surveyed works focused on stability of the system; not performance. *D*-stability analysis is a method by which one can analyze both the stability and the performance. The *D*-stability of an LTI system implies that all roots of its characteristic equation lie in a desired area of the complex plane [30]. This desired area can be chosen from the open left half-plane; therefore, *D*-stability can embrace performance in addition to the stability because the poles of a stable factional order system lie in the open left half-plane [31, Theorem 5]. By extending some results on the *D*-stability of integer order systems to fractional order ones, the robust *D*-stability of fractional order systems was first investigated in [32], and some sufficient criteria were produced. Then, necessary and sufficient criteria for the robust D-stability of fractional order systems with linear and real uncertainties only in the coefficients were reported in [33], [34]. Based on [35], if the uncertainties exist anywhere other than in the coefficients, the value set of the characteristic equation gets a nonconvex shape, and consequently the results presented in [32]–[34] are not applicable anymore.

References [32]–[34] considered only specific cases of LTI fractional order systems, i.e., systems with linear and real uncertainties only in the coefficients. Generally, in LTI fractional order systems, both the coefficients and the orders of the characteristic equation can include uncertainties [21], [28]. These uncertain coefficients and orders can also be nonlinear functions of the uncertainties [27], [36]. Moreover, the coefficients of an LTI system may be complex numbers [37], especially in aerospace applications where the dynamics of a system in different directions with real coefficients are compacted into one dynamical equation with complex coefficients (for more detail see, e.g., [38]). Furthermore, an LTI fractional order system can be of either commensurate or non-commensurate order [39]. LTI fractional order control systems which have all aforementioned features together will be referred to as a general fractional order control system throughout this paper. Regarding the aforementioned works, the existence of a necessary and sufficient condition for checking the robust D-stability of general fractional order control systems is an open problem which is tackled in this paper.

The main contribution of this paper is presenting a condition for checking the robust D-stability of LTI general fractional order control systems with the following merits. It is a sufficient and necessary condition, applicable to the systems of both commensurate and incommensurate orders with uncertainties in both the coefficients and the orders, and applicable to the systems whose coefficients may be complex numbers and also may be nonlinear functions of the uncertain parameters. Since the condition is based on the characteristic equation of the systems, it is applicable to systems described by any state-space and transfer function models. As further contributions, some new specific areas for the roots of the characteristic equation of general fractional order control systems are found so that they reduce the computational burden of testing the condition in some important cases, including for robust stability. Moreover, in the case that the coefficients are linear function of the uncertain parameters and the orders do not have any uncertainties, the condition is adjusted to further reduce computational burden.

The reminding sections of this work are organized as follows. In Section II, some definitions and preliminaries are given. The main results are provided in Section III. In Section IV, illustrative examples are presented. A conclusion is given in Section V.

II. Definitions and Preliminaries

Notations: Suppose that $\mathbb{N}$, $\mathbb{R}^+$, $\mathbb{R}$, and $\mathbb{C}$ denote natural, positive real, real, and complex numbers, respectively, and $W=\mathbb{N}\cup\{0\}$. For $I_1>I_2$ let $\sum_{i=I_1}^{I_2} c_i=0$ for any $c_i\in\mathbb{C}$. Considering $P\subset\mathbb{R}$, define $P^{\le i}=\{p\in P|p\le i\}$. Given $S\subset\mathbb{C}$, assume that S^C and ∂S denote respectively the complement and the boundary points of S. For any $z\in\mathbb{C}$, unless specified otherwise, $\arg(z)$ means the argument principal value of z such that $\arg(z)\in(-\pi,\pi]$ and $\arg(0)=0$.

Definition 1: Take into account the fractional order function $\delta(s)=\sum_{i=0}^{I}\alpha_i s^{\beta_i}$ where $I\in\mathbb{N}$, $\beta_i\in\mathbb{R}^+$ for any $i\in\mathbb{N}^{\le I}$, $\beta_I>\ldots>\beta_1>\beta_0=0$, and $\alpha_i\in\mathbb{C}$ for any $i\in W^{\le I}$. Then, $\delta(s)$ is called a fractional order function of commensurate order β if and only if there is a $\beta\in\mathbb{R}^+$ such that $\beta_i/\beta\in\mathbb{N}$ for any $i\in\mathbb{N}^{\le I}$. Otherwise, $\delta(s)$ is called a fractional order function of non-commensurate order. In the case where $\delta(s)$ is a fractional order function of commensurate order β, define $\delta_{\text{int}}(s)=\sum_{i=0}^{I}\alpha_i s^{\beta_i/\beta}$.

Definition 2: Consider an LTI general fractional order system described by any model with a controller in a closed loop control system where any one of the system and the controller may be of fractional order. The characteristic equation of the corresponding general fractional order control system can be written as

$$\delta(s,\boldsymbol{u})=\sum_{i=1}^{I}\alpha_i(\boldsymbol{u})\,s^{\beta_i(\boldsymbol{u})}+\alpha_0(\boldsymbol{u}) \tag{1}$$

where $\boldsymbol{u}\in U$, $U\subset\mathbb{R}^M$ is a closed, non-null, and bounded set, and $M,I\in\mathbb{N}$. Suppose the functions $\alpha_i:U\to\mathbb{C}$ and $\beta_j:U\to\mathbb{R}^+$ are continuous on U for any $i\in W^{\le I}$ and $j\in\mathbb{N}^{\le I}$. Assume $\beta_I(\boldsymbol{u})>\beta_i(\boldsymbol{u})>0$, and $\alpha_I(\boldsymbol{u})\ne 0$ for any $\boldsymbol{u}\in U$ and $i\in\mathbb{N}^{\le I-1}$. The characteristic equation in (1) is of commensurate order if and only if $\delta(s,\boldsymbol{u})$ is of commensurate order for any $\boldsymbol{u}\in U$. Otherwise, $\delta(s,\boldsymbol{u})$ is of non-commensurate order.

Definition 3: The principal branch of the characteristic equation of a general fractional order control system such as $\delta(s,\boldsymbol{u})$ in (1) is defined as $\delta_{pb}(s,\boldsymbol{u})=\delta(s_{pb},\boldsymbol{u})$ where $s_{pb}=|s|e^{j\arg(s)}$. Moreover, the value set of a fractional order function such as $\delta(s,\boldsymbol{u})$ for a $z\in\mathbb{C}$ is defined as $\delta_{vs}(z,U)=\{\delta_{pb}(z,\boldsymbol{u})|\boldsymbol{u}\in U\}$.

Remark 1 ([40]): Consider $\delta(s)$ in Definition 1 as a fractional order function of commensurate order β. It follows that the roots of $\delta_{\text{int}}(s)$ are the mapped roots of $\delta_{pb}(s)$ on the first Riemann sheet by the mapping s^β.

Definition 4: Assume that $D\subset\mathbb{C}$ be an open set. Then, the characteristic equation stated in (1) and its corresponding general fractional order control system are said to be D-stable for a $\boldsymbol{u}^0\in U$ if and only if $\delta_{pb}(s,\boldsymbol{u}^0)$ has no roots in D^C. Moreover, they are said to be robust D-stable if and only if $\delta_{pb}(s,\boldsymbol{u})$ has no roots in D^C for all $\boldsymbol{u}\in U$.

Definition 5: If D mentioned in Definition 4 is defined as $\{s\in\mathbb{C}|\text{Im}(s)\ge 0,\text{Re}(se^{-j\phi_1})<0,\phi_1\in[0,\pi/2)\}\cup\{s\in\mathbb{C}|\text{Im}(s)\le 0,\ \text{Re}(se^{j\phi_2})<0,\ \phi_2\in[0,\pi/2)\}$, then robust D-stability and region D are referred to as robust ϕ-stability and region Φ, respectively. Furthermore, if D is defined as $\{s\in\mathbb{C}|\text{Im}(s)\ge 0,\ \text{Re}(s)<\sigma_1\}\cup\{s\in\mathbb{C}|\text{Im}(s)\le 0,\text{Re}(s)<\sigma_2\}$, then robust D-stability and region D are referred to as robust σ-stability and region Σ, respectively. Examples of the regions Φ and Σ are shown in Fig. 1 [41]. If the coefficients of the characteristic equation are real, since the roots are symmetric with respect to the real axis, D can be chosen symmetrically, and therefore we use $\phi = \phi_1 = \phi_2$ and $\sigma = \sigma_1 = \sigma_2$. In the case where $\phi_1 = \phi_2 = 0$ or $\sigma_1 = \sigma_2 = 0$, the robust ϕ-stability or σ-stability (D-stability) is equivalent to the robust stability [31, Theorem 5].

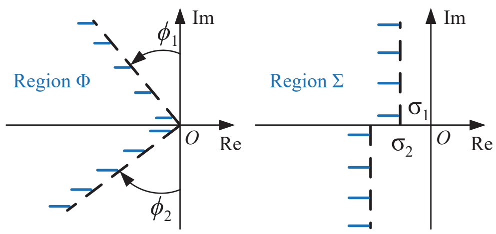

Fig. 1. Regions Φ and Σ defined in Definition 5.

Remark 2: Suppose that $\delta(s)$ stated in Definition 1 is a fractional order function of commensurate order β. According to Remark 1, it is deduced that $\delta(s)$ is ϕ-stable if and only if $\delta_{\text{int}}(s)$ has no roots in $\{s \in \mathbb{C} | \text{Im}(s) \geq 0,\ \arg(s) \leq \beta(\pi/2 + \phi_1)\}$ $\cup \{s \in \mathbb{C} | \text{Im}(s) \leq 0,\ \arg(s) \geq -\beta(\pi/2 + \phi_2)\}$.

Lemma 1 (Lemma 4 of [23]): Given $\delta(s)$ as defined in Definition 1, all roots of $\delta_{pb}(s)$ lie in $\{s \in \mathbb{C} | \hat{E}_{\min} \leq |s| \leq \hat{E}_{\max}\}$ where

$$
\begin{aligned}
\hat{E}_{\min} &= \max\left\{\hat{E}_{1\min}, \hat{E}_{2\min}\right\} \\
\hat{E}_{\max} &= \min\left\{\hat{E}_{1\max}, \hat{E}_{2\max}\right\} \\
\hat{E}_{1\min} &= \min\left\{1, \left(|\alpha_0| / \sum_{i=1}^{I} |\alpha_i|\right)^{\frac{1}{\beta_1}}\right\} \\
\hat{E}_{1\max} &= \max\left\{1, \left(\sum_{i=0}^{I-1} |\alpha_i| / |\alpha_I|\right)^{\frac{1}{\beta_I - \beta_{I-1}}}\right\} \\
\hat{E}_{2\min} &= \left(\frac{|\alpha_0|}{|\alpha_0| + \max\left\{|\alpha_i| \,\middle|\, i \in \mathbb{N}^{\leq I}\right\}}\right)^{\max\left\{\frac{i}{\beta_i} \middle| i \in \mathbb{N}^{\leq I}\right\}} \\
\hat{E}_{2\max} &= \left(1 + \frac{\max\left\{|\alpha_i| \,\middle|\, i \in W^{\leq I-1}\right\}}{|\alpha_I|}\right)^{\max\left\{\frac{I-i}{\beta_I - \beta_i} \middle| i \in W^{\leq I-1}\right\}}.
\end{aligned}
$$

Lemma 2 (Theorem 1.2 of [42]): Let $f(z)$ be analytic interior to a simple closed Jordan curve Γ and continuous and different from zero on Γ. Let K be the curve described in the w-plane by the point $w = f(z)$ and let $\Theta_\Gamma \arg(f(z))$ denote the net change in $\arg(f(z))$ as point z traverses Γ once over in the counterclockwise direction. Then the number p of zeros of $f(z)$ interior to Γ, counted with their multiplicities, is

$$
p = \frac{1}{2\pi} \Theta_\Gamma \arg(f(z)) = 1/(2\pi j) \oint_\Gamma \dot{f}(z)/f(z)\, dz.
$$

that is, p is the net number of times that K winds about the point $w = 0$.

III. Main Results

On the issues related to the robust stability of fractional order systems using the value set concept, the zero exclusion condition plays a key role. In this section, some areas for the roots of the characteristic equation of a general fractional order control system are obtained, and then by using these areas, the zero exclusion condition is extended for checking the robust D-stability of general fractional order control systems.

A. Areas for the Roots

The following theorem presents areas for the roots of the characteristic equation of a general fractional order control system. These areas will be used to extend and check the zero exclusion condition of the robust D-stability of general fractional order control systems.

Theorem 1: Consider the characteristic equation of a general fractional order control system, $\delta(s, \boldsymbol{u})$, as described in Definition 1. For simplicity, to the end of the theorem assume that α_i and β_j denote $\alpha_i(\boldsymbol{u})$ and $\beta_j(\boldsymbol{u})$ for any $i \in W^{\leq I}$ and $j \in \mathbb{N}^{\leq I}$. Then, for any $\boldsymbol{u} \in U$ all roots of $\delta_{pb}(s, \boldsymbol{u})$ lie in the area $\{s \in \mathbb{C} | E_{\min} \leq |s| \leq E_{\max}\}$ where

$$
\begin{aligned}
E_{\min} &= \max\{E_{1\min}, E_{2\min}\} \\
E_{\max} &= \min\{E_{1\max}, E_{2\max}\} \\
E_{1\min} &= \min\left\{1, \left(\frac{\min_{\boldsymbol{u} \in U} |\alpha_0|}{\sum_{i=1}^{I} \max_{\boldsymbol{u} \in U} |\alpha_i|}\right)^{\frac{1}{\min\left\{\min_{\boldsymbol{u} \in U} \beta_i \middle| i \in \mathbb{N}^{\leq I}\right\}}}\right\} \\
E_{1\max} &= \max\left\{1, \left(\frac{\sum_{i=0}^{I-1} \max_{\boldsymbol{u} \in U} |\alpha_i|}{\min_{\boldsymbol{u} \in U} |\alpha_I|}\right)^{\frac{1}{\min_{\boldsymbol{u} \in U} \beta_I - \gamma_1}}\right\} \\
E_{2\min} &= \left(\frac{\min_{\boldsymbol{u} \in U} |\alpha_0|}{\min_{\boldsymbol{u} \in U} |\alpha_0| + \max\left\{\max_{\boldsymbol{u} \in U} |\alpha_i| \,\middle|\, i \in \mathbb{N}^{\leq I}\right\}}\right)^{P_{2\min}} \\
E_{2\max} &= \left(1 + \frac{\max\left\{\max_{\boldsymbol{u} \in U} |\alpha_i| \,\middle|\, i \in W^{\leq I-1}\right\}}{\min_{\boldsymbol{u} \in U} |\alpha_I|}\right)^{P_{2\max}} \\
\gamma_1 &= \begin{cases} 0, & I = 1 \\ \max\left\{\max_{\boldsymbol{u} \in U} \beta_i \,\middle|\, i \in \mathbb{N}^{\leq I-1}\right\}, & I \geq 2 \end{cases} \\
P_{2\min} &= \max\left\{\frac{i}{\min_{\boldsymbol{u} \in U} \beta_i} \,\middle|\, i \in \mathbb{N}^{\leq I}\right\} \\
P_{2\max} &= \max\left\{\frac{I - i}{\min_{\boldsymbol{u} \in U} \beta_I - \max_{\boldsymbol{u} \in U} \beta_i} \,\middle|\, i \in W^{\leq I-1}\right\}.
\end{aligned}
$$

Proof: The proof is presented in the Appendix. ■

The following theorem provides areas for the roots of the characteristic equation of a general fractional order control system on the half-line $\{s \in \mathbb{C} | \arg(s) = \Lambda, \Lambda \in (-\pi, \pi]\}$. These areas reduce the computational burden of checking the robust ϕ-stability compared with those were introduced in existing works.

Theorem 2: Consider the characteristic equation of a general fractional order control system, $\delta(s, \boldsymbol{u})$, as described in Definition 1. For simplicity, to the end of the theorem assume that α_i and β_j denote $\alpha_i(\boldsymbol{u})$ and $\beta_j(\boldsymbol{u})$ for any $i \in W^{\leq I}$ and $j \in \mathbb{N}^{\leq I}$. Then, for any $\boldsymbol{u} \in U$ if $\delta_{pb}(s,\boldsymbol{u})$ has any roots on the half-line $\{s \in \mathbb{C} | \arg(s) = \Lambda, \Lambda \in (-\pi,\pi]\}$, all these roots lie in the area $\{s \in \mathbb{C} | \arg(s) = \Lambda, R_{\min} \leq |s| \leq R_{\max}\}$ where

$$R_{\min} = \begin{cases} 0, & \exists \boldsymbol{u} \in U : \alpha_0 = 0 \\ \max\{R_{1\min}, R_{2\min}\}, & \text{otherwise} \end{cases}$$

$$R_{\max} = \min\{R_{1\max}, R_{2\max}, R_{3\max}, R_{4\max}, R_{5\max}, R_{6\max}\}$$

$$R_{1\min} = \min\left\{1, \left(\frac{\min\limits_{\boldsymbol{u}\in U}|\alpha_0|}{\sum\limits_{i=1}^{I}\max\limits_{\boldsymbol{u}\in U}|\alpha_i\cos(\beta_i\Lambda)|}\right)^{\frac{1}{\min\left\{\min\limits_{\boldsymbol{u}\in U}\beta_i \middle| i\in\mathbb{N}^{\leq I}\right\}}}\right\}$$

$$R_{2\min} = \left(\frac{\min\limits_{\boldsymbol{u}\in U}|\alpha_0|}{\min\limits_{\boldsymbol{u}\in U}|\alpha_0| + \max\limits_{\boldsymbol{u}\in U, i\in\mathbb{N}^{\leq I}}|\alpha_i\cos(\beta_i\Lambda)|}\right)^{\max\left\{\frac{i}{\min\limits_{\boldsymbol{u}\in U}\beta_i} \middle| i\in\mathbb{N}^{\leq I}\right\}}$$

$$R_{1\max} = \begin{cases} \infty, \; f_c = 0 \\ \max\left\{1, \left(\frac{\sum\limits_{i=0}^{I-1}\max\limits_{\boldsymbol{u}\in U}|\alpha_i\cos(\beta_i\Lambda)|}{\min\limits_{\boldsymbol{u}\in U}|\alpha_I\cos(\beta_I\Lambda)|}\right)^{H_1}\right\}, \; f_c = 1 \end{cases}$$

$$R_{2\max} = \begin{cases} \infty, \; f_s = 0 \\ \max\left\{1, \left(\frac{\sum\limits_{i=1}^{I-1}\max\limits_{\boldsymbol{u}\in U}|\alpha_i\sin(\beta_i\Lambda)|}{\min\limits_{\boldsymbol{u}\in U}|\alpha_I\sin(\beta_I\Lambda)|}\right)^{H_2}\right\}, \; f_s = 1 \end{cases}$$

$$R_{3\max} = \max\left\{1, \left(\frac{\sum\limits_{i=0}^{I-1}\max\limits_{\boldsymbol{u}\in U}\left[|\alpha_i|\left(|\cos(\beta_i\Lambda)| + |\sin(\beta_i\Lambda)|\right)\right]}{\min\limits_{\boldsymbol{u}\in U}\left[|\alpha_I|\left(|\cos(\beta_I\Lambda)| + |\sin(\beta_I\Lambda)|\right)\right]}\right)^{H_3}\right\}$$

$$R_{4\max} = \begin{cases} \infty, \; f_c = 0 \\ \left(1 + \frac{\max\limits_{\boldsymbol{u}\in U, i\in W^{\leq I-1}}|\alpha_i\cos(\beta_i\Lambda)|}{\min\limits_{\boldsymbol{u}\in U}|\alpha_I\cos(\beta_I\Lambda)|}\right)^{H_4}, \; f_c = 1 \end{cases}$$

$$R_{5\max} = \begin{cases} \infty, \; f_s = 0 \\ 0, \; f_s = 1, I = 1 \\ \left(1 + \frac{\max\limits_{\boldsymbol{u}\in U, i\in\mathbb{N}^{\leq I-1}}|\alpha_i\sin(\beta_i\Lambda)|}{\min\limits_{\boldsymbol{u}\in U}|\alpha_I\sin(\beta_I\Lambda)|}\right)^{H_5}, \; f_s = 1, \; I \geq 2 \end{cases}$$

$$R_{6\max} = \left(1 + \frac{\max\limits_{\boldsymbol{u}\in U, i\in W^{\leq I-1}}\left[|\alpha_i|\left(|\cos(\beta_i\Lambda)| + |\sin(\beta_i\Lambda)|\right)\right]}{\min\limits_{\boldsymbol{u}\in U}\left[|\alpha_I|\left(|\cos(\beta_I\Lambda)| + |\sin(\beta_I\Lambda)|\right)\right]}\right)^{H_6}$$

$$H_1 = H_2 = H_3 = \frac{1}{\min\limits_{\boldsymbol{u}\in U}\beta_I - \gamma_2}$$

$$H_5 = \max\left\{\frac{I-i}{\min\limits_{\boldsymbol{u}\in U}\beta_I - \max\limits_{\boldsymbol{u}\in U}\beta_i} \middle| i\in\mathbb{N}^{\leq I-1}\right\}$$

$$H_4 = H_6 = \max\left\{\frac{I-i}{\min\limits_{\boldsymbol{u}\in U}\beta_I - \gamma_3(i)} \middle| i\in W^{\leq I-1}\right\}$$

$$\gamma_2 = \begin{cases} 0, \; I = 1 \\ \max\left\{\max\limits_{\boldsymbol{u}\in U}\beta_i \middle| i\in\mathbb{N}^{\leq I-1}\right\}, \; I \geq 2 \end{cases}$$

$$\gamma_3(i) = \begin{cases} 0, \; i = 0 \\ \max\limits_{\boldsymbol{u}\in U}\beta_i, \; i \geq 1 \end{cases}$$

$$f_c = \begin{cases} 1, \; \forall \boldsymbol{u}\in U : \cos(\beta_I\Lambda) \neq 0 \\ 0, \; \text{otherwise} \end{cases}$$

$$f_s = \begin{cases} 1, \; \forall \boldsymbol{u}\in U : \sin(\beta_I\Lambda) \neq 0 \\ 0, \; \text{otherwise.} \end{cases}$$

Proof: The proof is presented in the Appendix. ■

B. Zero Exclusion Condition

The zero exclusion condition is extended for checking the robust D-stability of general fractional order control systems as a necessary and sufficient condition as follows.

Theorem 3: Consider the characteristic equation of a general fractional order control system, $\delta(s,\boldsymbol{u})$, as described in Definition 1. Suppose that U is pathwise connected. Let $D \subset \mathbb{C}$ be an open set, given for checking the robust D-stability of $\delta(s,\boldsymbol{u})$. Define $D^+ = \{s \in D | \mathrm{Im}(s) \geq 0\}$ and $D^- = \{s \in D | \mathrm{Im}(s) \leq 0\}$. Also, define every $s \in \{s \in \partial D^- | \mathrm{Im}(s) = 0, \mathrm{Re}(s) < 0\}$ as $|s|e^{-j\pi}$, rather than $-|s|$. Then, $\delta(s,\boldsymbol{u})$ is robust D-stable if and only if there exists a $\boldsymbol{u}^0 \in U$ such that $\delta(s,\boldsymbol{u}^0)$ is D-stable, and $0 \notin \delta_{vs}(z,U)$ for all $z \in S_D = \{s \in \partial D^+ \cup \partial D^- | E_{\min} \leq |s| \leq E_{\max}\}$, where $E_{\min}$ and $E_{\max}$ are calculated through Theorem 1.

Proof: The proof is presented in the Appendix. ■

Remark 3: From Theorems 1 and 3, the following special cases can be concluded:

1) If $\{s \in \mathbb{C} | E_{\min} \leq |s| \leq E_{\max}\} \subset D$, then $\delta(s,\boldsymbol{u})$ is robust D-stable.

2) If $D \cap \{s \in \mathbb{C} | E_{\min} \leq |s| \leq E_{\max}\} = \emptyset$ then $\delta(s,\boldsymbol{u})$ is not D-stable for any $\boldsymbol{u} \in U$.

Remark 4: Assume that D, mentioned in Theorem 3, can be considered as a region Φ defined in Definition 5. From Theorem 2, it follows that S_D mentioned in Theorem 3 can be replaced by $S_D = \{s \in \partial D^+ \cup \partial D^- | R_{\min} \leq |s| \leq R_{\max}\}$ where $\partial D^+ \cup \partial D^- = \partial\Phi = \{0\} \cup \{s \in \mathbb{C} | \arg(s) = \pi/2 + \phi_1, -\pi/2 - \phi_2\}$. (If one wants to avoid Theorem 2 calculations, he/she can ignore this remark). It may be noted that Theorem 2 employs Λ ($\Lambda = \pi/2 + \phi_1$ or $\Lambda = -\pi/2 - \phi_2$) for calculating $R_{\min}$ and $R_{\max}$ while Theorem 1 does not do so for calculating $E_{\min}$ and $E_{\max}$. Therefore, $R_{\max} - R_{\min} \leq E_{\max} - E_{\min}$ and accordingly S_D introduced here is smaller than the one introduced in Theorem 1 for checking the robust Φ-stability, and thereby causes a reduction in the computational burden of checking the robust D-stability.

Note that a region Φ, as described in Definition 5, corresponds with points of the complex plane whose overshoots are less than a specific value. Furthermore, considering Theorem 5 of [31], a fractional order system is stable if and only if its characteristic equation does not have any roots in the closed right half-plane. Hence, if $\phi_1 = \phi_2 = 0$, then the ϕ-stability is equivalent to the stability. Therefore, the robust ϕ-stability is an important case of the robust D-stability. It is noteworthy that the works related to the robust stability analysis of fractional order systems used theorems like Theorem 1 to check the robust stability [23]–[25], [28]. However, here as stated in Remark 4, using Theorem 2 causes a reduction of the computational burden (see Example 1).

In the case where the coefficients of the characteristic equation of a general fractional order control system are real numbers, the following remark reduces the computational burden of checking the robust D-stability.

Remark 5: Suppose that the functions $\alpha_i(\boldsymbol{u})$ of the characteristic equation $\delta(s,\boldsymbol{u})$ mentioned in Theorem 3 are real-valued functions for any $i \in W^{\le I}$. Therefore, the roots of $\delta(s,\boldsymbol{u})$ are mirror symmetric with respect to the real axis, and consequently D can be chosen symmetrically with respect to the real axis. Given $z \in \mathbb{C}$, it can be shown easily that $\delta_{vs}(z^*, U) = \delta_{vs}^*(z, U)$ where "$*$" denotes the complex conjugate. Hence, S_D mentioned in Theorem 3 can be replaced by $S_D = \{s \in \partial D^+ | E_{\min} \le |s| \le E_{\max}\}$ for reducing the computational burthen of checking the robust D-stability.

Remark 6: If D, mentioned in Theorem 3, can be considered as a region Φ defined in Definition 5, and the functions $\alpha_i(\boldsymbol{u})$ of the characteristic equation $\delta(s,\boldsymbol{u})$ are real-valued functions for any $i \in W^{\le I}$, then S_D from Theorem 3 can be replaced by the intersection of two sets S_D introduced in Remarks 4 and 5, i.e., $S_D = \{s \in \partial D^+ | R_{\min} \le |s| \le R_{\max}\}$.

For testing whether or not $\delta(s,\boldsymbol{u}^0)$ from Theorem 3 is D-stable, the following remark is presented.

Remark 7: Take into account Theorem 3. Assume that none of the special cases stated in Remark 3 apply. Checking whether or not $\delta(s,\boldsymbol{u}^0)$ (the characteristic equation of the nominal control system) is D-stable can be performed as follows:

1) In the general case, define $\Omega^+ = \{s \in D^C | \mathrm{Im}(s) \ge 0, E_{\min} \le |s| \le E_{\max}\}$, $\Omega^- = \{s \in D^C | \mathrm{Im}(s) \le 0, E_{\min} \le |s| \le E_{\max}\}$, $\Gamma^+ = \partial\Omega^+$, and $\Gamma^- = \partial\Omega^-$. Also, define every $s \in \{s \in \Gamma^- | \mathrm{Im}(s) = 0, \mathrm{Re}(s) < 0\}$ as $|s|e^{-j\pi}$, rather than $-|s|$. Let K^+ and K^- be the curves obtained due to $\delta_{pb}(s,\boldsymbol{u}^0)$ when s traverses respectively the curves Γ^+ and Γ^- once over in the clockwise direction. Assume that p^+ and p^- are respectively the net number of times that K^+ and K^- wind about the origin in the clockwise direction. Regarding Lemma 1, $\delta_{pb}(s,\boldsymbol{u}^0)$ has no roots in $\{s \in \mathbb{C} || s| < E_{\min}, |s| > E_{\max}\}$. Therefore, from Lemma 2, it follows that $\delta(s,\boldsymbol{u}^0)$ is D-stable if and only if $\delta_{pb}(s,\boldsymbol{u}^0)$ is analytical inside Γ^+ and Γ^-, and continuous on them, K^+ and K^- do not pass the origin, and $p^+ = p^- = 0$ (or $\oint_{\Gamma^+} \dot{\delta}_{pb}(s,\boldsymbol{u}^0)/\delta_{pb}(s,\boldsymbol{u}^0) ds = \oint_{\Gamma^-} \dot{\delta}_{pb}(s,\boldsymbol{u}^0)/\delta_{pb}(s,\boldsymbol{u}^0) ds = 0$).

2) In the case where the functions $\alpha_i(\boldsymbol{u})$ are real-valued functions for any $i \in W^{\le I}$, define K^+ and p^+ similarly to the previous provision. Considering Remark 5 and Lemma 2, $\delta(s,\boldsymbol{u}^0)$ is D-stable if and only if $\delta_{pb}(s,\boldsymbol{u}^0)$ is analytical inside Γ^+ and continuous on it, K^+ does not pass the origin, and $p^+ = 0$ (or $\oint_{\Gamma^+} \dot{\delta}_{pb}(s,\boldsymbol{u}^0)/\delta_{pb}(s,\boldsymbol{u}^0) ds = 0$).

3) In the case where D can be considered as a region Φ and $\delta(s,\boldsymbol{u}^0)$ is of commensurate order, the D-stability of $\delta(s,\boldsymbol{u}^0)$ can be verified by employing Remark 2.

In Theorem 3, if $\delta(s,\boldsymbol{u})$ does not have any uncertainties in its orders, and the coefficients $\alpha_i(\boldsymbol{u})$ are linear functions for any $i \in W^{\le I}$, the following theorem is very effective for decreasing the computational burden of the robust D-stability.

Theorem 4: Assume that the characteristic equation of a fractional order control system is

$$\hat{\delta}(s,\boldsymbol{u}) = \sum_{i=1}^{I} \alpha_i(\boldsymbol{u}) s^{\beta_i} + \alpha_0(\boldsymbol{u}) \tag{2}$$

where $\boldsymbol{u} \in U$, $U \subset \mathbb{R}^M$ is a closed, non-null, bounded, path-wise connected, and convex set, and $M, I \in \mathbb{N}$. Suppose that the functions $\alpha_i : U \to \mathbb{C}$ are linear and continuous on U for any $i \in W^{\le I}$. Let $\beta_i \in \mathbb{R}^+$ for any $i \in \mathbb{N}^{\le I}$. Then, $\partial\hat{\delta}_{vs}(z,U) \subseteq \hat{\delta}_{vs}(z,U_E)$ for any $z \in \mathbb{C}$ where U_E is a set including the exposed edges of U (for more information about the edges and the exposed edges of an uncertain set see, e.g., [22]).

Proof: The proof is presented in the Appendix. ■

Remark 8: Suppose that the functions $\alpha_i(\boldsymbol{u})$ of the characteristic equation $\delta(s,\boldsymbol{u})$ mentioned in Theorem 3 are linear functions for any $i \in W^{\le I}$. Moreover, assume that the orders of $\delta(s,\boldsymbol{u})$ do not have any uncertainties. According to Theorem 4, the expression $0 \notin \delta_{vs}(z,U)$ can be replaced by the expression $0 \notin \delta_{vs}(z,U_E)$ which significantly reduces the computational burden of calculating the zero exclusion condition.

Note that Theorems 1–3 are applicable to: systems of both commensurate and non-commensurate orders; systems with complex coefficients; and systems with nonlinear uncertainties in both the coefficients and the orders. Moreover, Theorem 4 is applicable to: systems of both commensurate and non-commensurate orders; systems with complex coefficients; and systems with linear uncertainties only in the coefficients.

The condition $0 \notin \delta_{vs}(z,U)$ and $0 \notin \delta_{vs}(z,U_E)$ can be checked respectively by plotting $\delta_{vs}(z,U)$ and $\delta_{vs}(z,U_E)$ graphically, and then by considering their boundary with respect to the origin.

Overall, the steps of checking the robust D-stability of a general fractional order control system with the characteristic equation described in (1) can be outlined as follows:

1) Calculate $E_{\min}$ and $E_{\max}$ through Theorem 1.

2) If one of the special cases stated in Remark 3 is applicable, determine the robust D-stability of $\delta(s,\boldsymbol{u})$ through Remark 3. Otherwise, go to the next step.

3) For a $\boldsymbol{u}^0 \in U$, investigate whether or not $\delta(s,\boldsymbol{u}^0)$ is D-stable using Remark 7. If $\delta(s,\boldsymbol{u}^0)$ is D-stable, go to the next step. Otherwise, $\delta(s,\boldsymbol{u})$ is not robust D-stable.

4) Based on Theorem 3, Remarks 4–6, determine S_D for which the zero exclusion condition should be checked.

5) According to Remark 8, determine whether the zero exclusion condition should be checked for U or U_E.

6) Plot $\delta_{vs}(z,U)$ or $\delta_{vs}(z,U_E)$, depending on the previous step, for all $z \in S_D$. In the case of plotting $\delta_{vs}(z,U)$, choose appropriate number of the vectors $\boldsymbol{u}$ from both U and its edges

such that $\partial\delta_{vs}(z,U)$ is recognizable clearly.

7) Regarding Theorem 3, determine whether $\delta(s,\boldsymbol{u})$ is robust D-stable or not.

IV. Illustrative Examples

In this section, three numerical examples are given to verify the obtained results in this paper. Example 1 investigates the robust stability of a closed loop control system whose characteristic equation is a fractional order function of commensurate order with real coefficients and linear uncertainties only in the coefficients. The efficiency of Theorem 2 is shown in this example. In Example 2, a fractional order controller of incommensurate order is suggested to σ-stabilizing a space tether system whose characteristic equation coefficients are nonlinear and real-valued functions of uncertainties. Finally, Example 3 studies the robust ϕ-stability of motion control system of a satellite whose characteristic equation has complex coefficients, and uncertainties exist in both the orders and the coefficients. The criteria introduced in the literature are challenged in Examples 2 and 3, while the presented theorems in this paper overcome the challenges well.

Example 1: In [25], the robust stability of the fractional order system

$$H(s) = \frac{[0.5,2.5]\,s^{0.4}+[1,3]}{[0.5,1.5]\,s^{0.8}+[3,5]\,s^{0.4}+[1,5]}$$

with the fractional order PI controller $C(s) = 5+0.5s^{-0.4}$ in a closed loop control system has been studied. Let the aim be to check the robust ϕ-stability of the closed loop control system for $\phi = 0$, that it is equivalent to the robust stability, by using the results presented in this paper. The characteristic equation of the closed loop control system is obtained as

$$\delta(s,\boldsymbol{u}) = u_3 s^{1.2} + (u_4+5u_1)s^{0.8} + (u_5+5u_2+0.5u_1)s^{0.4} + 0.5u_2$$

where $\boldsymbol{u} \in U$ and

$$U = \left\{ \begin{array}{l} \boldsymbol{u} = [u_1,u_2,\ldots,u_5]^T \big| u_1 \in [0.5,2.5], u_2 \in [1,3], \\ u_3 \in [0.5,1.5], u_4 \in [3,5], u_5 \in [1,5] \end{array} \right\}.$$

$\delta(s,\boldsymbol{u})$ is of commensurate order $\beta = 0.4$ and has the multi-linear uncertainties. In [25], it has been demonstrated that $\delta(s,\boldsymbol{u}^0)$ is stable for $\boldsymbol{u}^0 = [1.5,2,1,4,3]^T$, so it is ϕ-stable. According to Theorem 3 of [25], all roots of $\delta_{pb}(s,\boldsymbol{u})$ lie in the area $S_1 = \{s \in \mathbb{C} | 1.7199\times10^{-5} \le |s| \le 5.8142\times10^{4}\}$. For using Theorems 1 and 2 consider $\max|\alpha_i(\boldsymbol{u})| = |\alpha_i(\max u_1,\ldots,\max u_5)|$ and $\min|\alpha_i(\boldsymbol{u})| = |\alpha_i(\min u_1,\ldots,\min u_5)|$ over $\boldsymbol{u} \in U$ for any $i \in W^{\le 3}$. By using Theorem 1, the roots lie in the area $S_2 = \{s \in \mathbb{C} | E_{\min} \le |s| \le E_{\max}\}$ where $E_{\min} = 8.0127\times10^{-5}$ and $E_{\max} = 1.2480\times10^{4}$. S_2 is smaller and has less conservatism than S_1. However, by using Theorem 2, where $f_c = f_s = 1$, the roots of $\delta_{pb}(s,\boldsymbol{u})$ on the half-line $\{s \in \mathbb{C} | \arg(s) = \pi/2\}$ lie in the area $S_3 = \{s \in \mathbb{C} | \arg(s) = \pi/2, 5.4200\times10^{-2} \le |s| \le 7.7760\times 10^{3}\}$. It can be found that S_3 has much less conservatism than S_1 and S_2, and accordingly it reduces the computational burden in order to check the zero exclusion condition. Regarding Remarks 4–6 it is sufficient to plot the value set $\delta_{vs}(z,U)$ for all $z \in S_D = \{s \in \mathbb{C} | \arg(s) = \pi/2, 5.4200\times10^{-2} \le |s| \le 7.7760\times10^{3}\}$. Moreover, according to Remark 8, one can plot $\delta_{vs}(z,U_E)$ instead of $\delta_{vs}(z,U)$ for any $z \in S_D$. The value set $\delta_{vs}(z,U_E)$ is plotted for 3000 vectors $\boldsymbol{u}$, chosen from U_E, per $z \in S_D$ with the step 0.01 and 1 over $|s|$ for $|s| \le 1$ and $|s| > 1$, respectively. The graph of $\delta_{vs}(z,U_E)$ is demonstrated in Fig. 2 after zooming in on the origin. According to Fig. 2 and Remark 8, it follows that $0 \notin \delta_{vs}(z,U)$ for all $z \in S_D$. Thus, regarding Theorem 3, $\delta(s,\boldsymbol{u})$, and consequently the closed loop control system is robust ϕ-stable, or in other words is robust stable. This result is the same result obtained in [25].

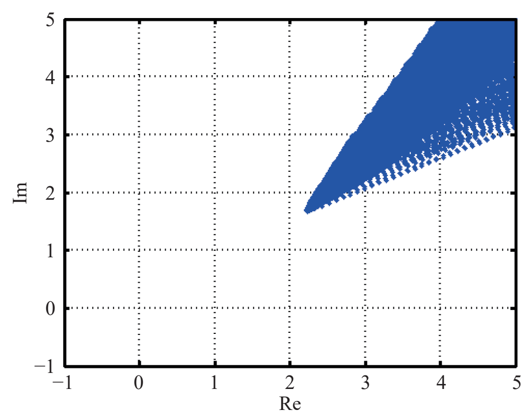

Fig. 2. Drawing $\delta_{vs}(z,U_E)$ for all $z \in S_D$ in Example 1.

Example 2: Consider the deployment of a tethered satellite from a space shuttle studied in [43]. Let λ is the normalized length of the tether such that $\lambda = 1$ when the tether deploys completely. Also, allow θ and T be the pitch angle and the tether tension as the input, respectively. The differential equations of this system have been introduced in [43]. Supposing 10 percent uncertainties in the system parameters and defining the state vector $x = [x_1,x_2,x_3,x_4]^T = [\lambda-1,\dot{\lambda},\theta,\dot{\theta}]^T$, the state-space equations of the system are formed as

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ u_1 & 0 & 0 & u_2 \\ 0 & 0 & 0 & 1 \\ 0 & u_3 & u_4 & 0 \end{bmatrix} x - T \tag{3}$$

where $\boldsymbol{u} \in U$ and

$$U = \left\{ \begin{array}{l} \boldsymbol{u} = [u_1,u_2,u_3,u_4]^T \big| u_1 \in [2.7,3.3], u_2 \in [1.8,2.2], \\ u_3 \in [-2.2,-1.8], u_4 \in [-3.3,-2.7] \end{array} \right\}.$$

The goal is to deploy the tether from the initial state $[\lambda,\dot{\lambda},\theta,\dot{\theta}]^T = [0,0,0,0]^T$ to the final state $[\lambda,\dot{\lambda},\theta,\dot{\theta}]^T = [1,0,0,0]^T$ in a fast and low overshoot manner. Let us achieve this goal by σ-stabilizing the system for $\sigma = -0.3$. Consider a fractional order control law as

$$T = d^{\sqrt{2}/10}x_2 - \dot{x}_2 + 4x_1 + 1.75x_2 \tag{4}$$

where $d^{\sqrt{2}/10}x_2$ is $\sqrt{2}/10$th order derivative of x_2. Substituting (4) in (3), and then taking Laplace transform, the characteristic equation of the closed loop control system can be obtained as

$$\delta(s,\boldsymbol{u}) = s^{3+\sqrt{2}/10} + 1.75s^3 + (4-u_1-u_2u_3)s^2 \\ - u_4 s^{1+\sqrt{2}/10} - 1.75u_4 s - 4u_4 + u_1u_4. \quad (5)$$

$\delta(s,\boldsymbol{u})$ is of non-commensurate order. In the following, it is demonstrated that the closed loop control system is robust σ-stable. For using Theorem 1, one can calculate $\max|\alpha_i(\boldsymbol{u})|$ and $\min|\alpha_i(\boldsymbol{u})|$ similar to the way described in Example 1. According to Theorem 1, all roots of $\delta_{pb}(s,\boldsymbol{u})$ lie in the area $\{s\in\mathbb{C}|E_{\min}\le|s|\le E_{\max}\}$ where $E_{\min}=0.1052$ and $E_{\max} = 1.0878\times10^6$. Let $\boldsymbol{u}^0=[3,2,-2,-3]^T$. According to Remark 7, consider Ω^+ and Γ^+ as displayed in Fig. 3. When the point s traverses Γ^+ once over in the direction indicated, the path due to $\delta_{pb}(s,\boldsymbol{u}^0)$ denoted by K^+ is obtained as drawn in Fig. 4. It is visible that K^+ does not pass the origin, and also the net number of times that K^+ winds about the origin in the clockwise direction is zero, i.e., $p^+=0$. Moreover, $\delta(s,\boldsymbol{u}^0)$ is analytical inside Γ^+ and continuous on it. Therefore, regarding Remark 7, $\delta(s,\boldsymbol{u}^0)$ is σ-stable. Regarding Theorem 3 and Remark 5, the value set should be plotted for the set $S_D = \{s\in\mathbb{C}|\mathrm{Im}(s)=0,-0.3\le\mathrm{Re}(s)\le-0.1052\}\cup\{s\in\mathbb{C}|\mathrm{Re}(s)=-0.3, 0\le\mathrm{Im}(s)\le1.0878\times10^6\}$. Note that Remark 8 can not be applied here because the coefficients of $\delta(s,\boldsymbol{u})$ have a nonlinear structure. The value set $\delta_{vs}(z,U)$ for 400 vectors $\boldsymbol{u}$, chosen from U and its edges, per element of S_D with the step 0.005 over $\mathrm{Re}(s)$ or $\mathrm{Im}(s)$, depending on which is varying, is plotted in Fig. 5. It is seen from Fig. 5 that the value set does not include the origin. Hence, according to Theorem 3, the closed loop control system is robust σ-stable.

Let us compare the results achieved in this paper with those were published in the literature. There is no condition by which the control system studied in this example can be analyzed for the robust D-stability. However, someone can use approximations to be able to employ the theorems introduced in [32] to test the robust σ-stability of the characteristic equation in (5). For this, imagine that $\sqrt{2}/10$ is approximated as 0.14. With this approximation, the characteristic equation $\delta(s,\boldsymbol{u})$ in (5) transforms to a fractional order function of commensurate order $\beta=0.02$. Consider

$$\delta_{\text{int}}(s,\boldsymbol{u}) = s^{3.14/\beta} + 1.75s^3 + (4-u_1-u_2u_3)s^2 \\ - u_4 s^{1.14/\beta} - 1.75u_4 s - 4u_4 + u_1u_4.$$

Regarding part 1 of Theorem 3.7 of [32], the closed loop control system is robust σ-stable if $\delta_{\text{int}}(s,\boldsymbol{u})$ is robust stable. Because $\delta_{\text{int}}(s,\boldsymbol{u})$ is an integer order polynomial with the order 157, investigating its stability is too insufferable, but using the condition presented here and a simple graphical approach it was illustrated that the control system is robust σ-stable, without approximating $\sqrt{2}/10$. Now, Assume that $\sqrt{2}/10$ can be approximated as 0.1. Using the theorems presented in this paper, it can be illustrated that the closed loop control system is not robust σ-stable. In this case, using Theorem 3.7 of [32] does not provide any result, because the control system is not robust σ-stable, and Theorem 3.7 presents just sufficient, not necessary and sufficient, conditions for the robust σ-stability. Note that the results presented in [33], [34] are associated with systems whose coefficients are equal to the uncertain parameters and accordingly are not applicable here where the coefficients are nonlinear functions of the uncertain parameters.

For the numerical simulation of the closed loop control system, consider a 220 km orbit altitude, 1.1804×10^{-3} rad/s orbital rate, and 100 km tether length [44]. The tether length obtained from numerical simulations is plotted in Fig. 6 with red solid line for 10 vectors $\boldsymbol{u}\in U$. In [44], an integer order control law was suggested by which the results of the simulations are also plotted in Fig. 6 with blue dashed line. From Fig. 6, it follows that the responses of the closed loop control system by the fractional order control law are faster and have lower overshoot than the integer order one.

Example 3: The characteristic equation of motion of a satellite, obtained from nonlinear equations by linearizing as stated in [38], is

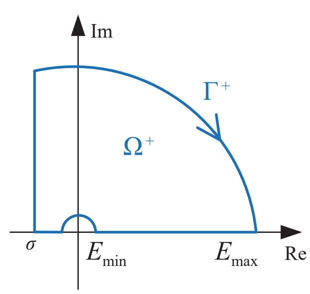

Fig. 3. Curve Γ^+ for checking the σ-stability of $\delta(s,\boldsymbol{u}^0)$ in Example 2.

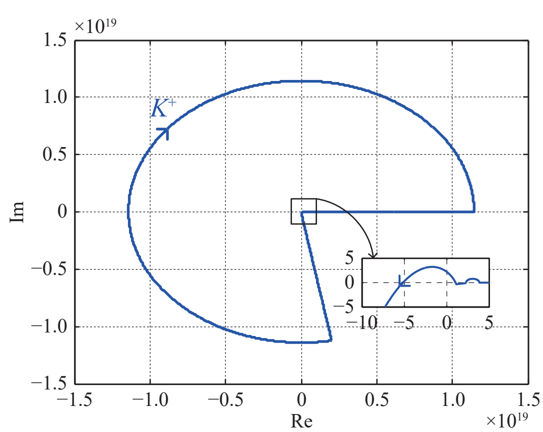

Fig. 4. The path obtained by $\delta_{pb}(s,\boldsymbol{u}^0)$ when s traverses Γ^+ in Fig. 3.

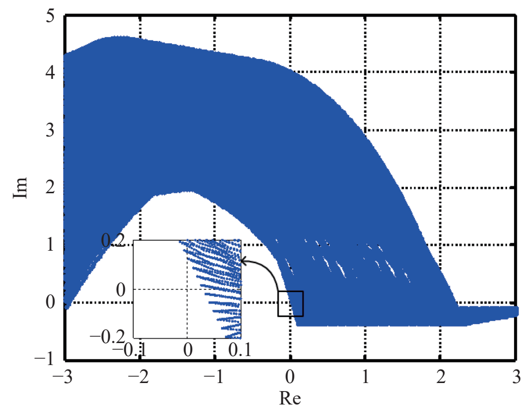

Fig. 5. Plotting $\delta_{vs}(z,U)$ for all $z\in S_D$ in Example 2.

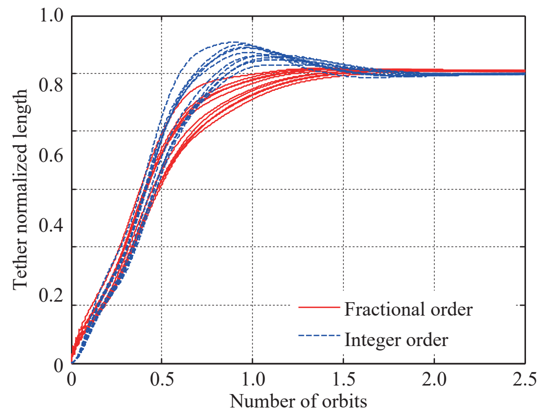

Fig. 6. The responses of system (3) by the fractional order control law (4) and the integer order one suggested in [44].

$$\delta(s) = s^3 + (0.0359 + jq_3)s^2 - (q_2 - j0.0134)s + jq_1$$

where $q_1 = 0.1646$, $q_2 = 0.1583$, and $q_3 = 2.3292$. Consider the state feedback control input $T_{0\lambda} = -x_c^{-1} d^{q_4} x_c / dt^{q_4}$, where $T_{0\lambda}$, x_c, and d^{q_4}/dt^{q_4} respectively are a parameter corresponding to the thrust force of motor, displacement, and fractional order derivative operator with the order $q_4 \in [0.3, 0.5]$. By applying $T_{0\lambda}$, and embedding some uncertainties in the parameters q_1, q_2, and q_3, the characteristic equation of the closed loop control system is obtained as

$$\delta(s, \boldsymbol{u}) = s^3 + (0.0359 + ju_3)s^2 - (u_2 - j0.0134)s - s^{u_4} + ju_1$$

where $\boldsymbol{u} \in U$ and

$$U = \left\{ \begin{array}{l} [u_1, u_2, u_3, u_4]^T \big| u_1 \in [-0.2, 0.2], \\ u_2 \in [0.1, 0.5], u_3 \in [2, 3], u_4 \in [0.3, 0.5] \end{array} \right\}.$$

Due to the changes of the order u_4, $\delta(s, \boldsymbol{u})$ is a fractional order function of both commensurate and non-commensurate orders. Let the aim be to check the robust ϕ-stability of the closed loop control system for $\phi_1 = \phi_2 = 0.05\pi$. Using Theorem 2, all roots of $\delta_{pb}(s, \boldsymbol{u})$ lie in the area $\{s \in \mathbb{C} | E_{\min} \le |s| \le E_{\max}\}$ where $E_{\min} = 0$ and $E_{\max} = 4.7004$. Suppose $\boldsymbol{u}^0 = [0.1, 0.3, 2.5, 0.4]^T$. Regarding Remark 7, consider Ω^+, Ω^-, Γ^+, and Γ^- as shown in Fig. 7. When s traverses Γ^+ and Γ^- once over in the directions indicated, the paths due to $\delta_{pb}(s, \boldsymbol{u}^0)$ respectively denoted by K^+ and K^- are obtained as plotted in Fig. 8. As it is specified in Fig. 8, a window zooming in around the origin of Fig. 8 is also displayed in Fig. 9. From Figs. 8 and 9, it is visible that K^+ and K^- do not pass the origin, but the net number of times that K^+ and K^- wind about the origin in the clockwise direction is 0 and 2, respectively. Therefore, $p^+ = p^- = 0$ does not hold. It follows that $\delta(s, \boldsymbol{u}^0)$ is not ϕ-stable, and accordingly, the closed loop control system is not robust ϕ-stable.

Although it was determined that $\delta(s, \boldsymbol{u})$ is not robust ϕ-stable, let us plot the value set of $\delta(s, \boldsymbol{u})$. One can use Remark 4, but let us ignore that here. Regarding Theorem 3, the value set should be plotted for $S_D = \{s \in \mathbb{C} | \arg(s) = 0.55\pi,\ 0 < |s| \le 4.7004\} \cup \{0\} \cup \{s \in \mathbb{C} | \arg(s) = -0.55\pi, 0 < |s| \le 4.7004\}$. Note that Remark 8 can not be used here. The value set $\delta_{vs}(z, U)$ for 800 vectors $\boldsymbol{u}$, chosen from U and its edges, per element of S_D with the step 0.05 over $|s|$ is depicted in Fig. 10. It can be seen

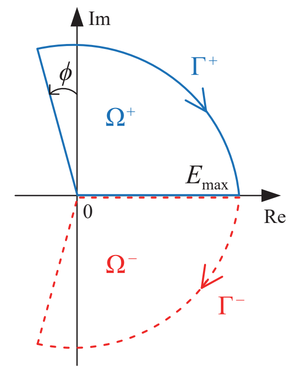

Fig. 7. Curves Γ^+ and Γ^- for checking the ϕ-stability of $\delta(s, \boldsymbol{u}^0)$ in Example 3.

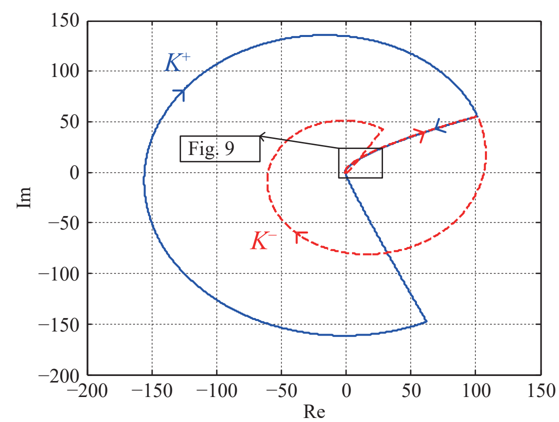

Fig. 8. The path obtained by $\delta_{pb}(s, \boldsymbol{u}^0)$ when s traverses Γ^+ and Γ^- in Fig. 7.

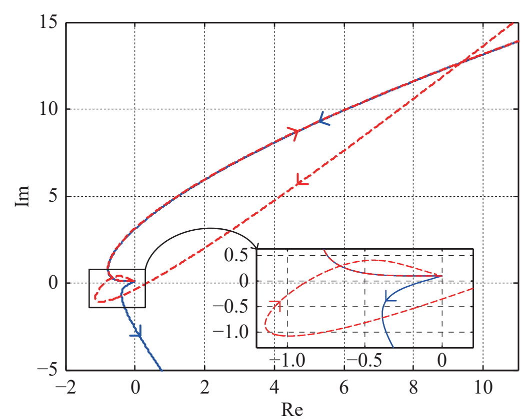

Fig. 9. Zoomed in around the origin of Fig. 8.

that the value set includes the origin. Therefore, in addition to that $\delta(s, \boldsymbol{u}^0)$ is not ϕ-stable, the zero exclusion condition is not also held.

It is notable that the general fractional order control system studied in this example is of both commensurate and incommensurate orders with complex coefficients, and uncertainties in both the coefficients and the orders. There is no condition to analyze the D-stability of such systems in the literature while the condition presented in this paper is applicable to these systems.

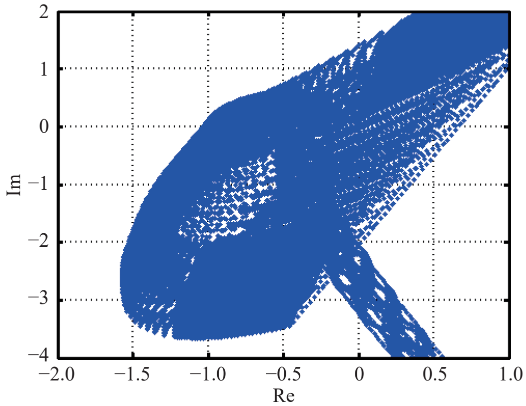

Fig. 10. Plotting $\delta_{vs}(z, U)$ for all $z \in S_D$ in Example 3.

V. Conclusion

This paper has investigated the robust D-stability test of LTI general fractional order control systems. The characteristic equation of these systems may be of both commensurate and non-commensurate orders, may have complex coefficients, and may have uncertainties in both its coefficients and its orders. Moreover, the uncertainties can have a nonlinear structure. For the roots of the characteristic equation, some new specific areas have been found. These areas reduce the computational burden of testing the robust D-stability in some important cases, specially in the robust stability case. The zero exclusion condition has been extended for the robust D-stability of these systems, and a necessary and sufficient condition has been derived. Furthermore, in the case that the coefficients have a linear structure and the uncertainties do not exist in the orders, the condition has been adjusted for further computational burden reduction. Three numerical examples have been studied to verify the merits of the presented results. For future works, extending the results achieved here to systems with time delays and also deriving conditions for designing a robustly D-stabilizing fractional order controller may be considered. It is notable that since the D-stability deals with the location of roots of the characteristic equation, it is not extendable to linear parameter-varying systems and nonlinear systems.

Acknowledgment

The authors would like to thank Dr. Mohsen Fathi Jegarkandi, with the Department of Aerospace Engineering, Sharif University of Technology, Tehran, Iran, for his support.

Appendix

Proof of Theorem 1: Because $U \subset \mathbb{R}^M$ is a bounded and closed set, according to Theorem 17.T of [45], it is a compact set. Moreover, since α_i and β_j are continuous for any $i \in W^{\le I}$ and $j \in \mathbb{N}^{\le I}$, regarding Theorem 17.T of [45], there are finite values for $\min|\alpha_i|$, $\max|\alpha_i|$, $\min\beta_j$, and $\max\beta_j$ over $\boldsymbol{u} \in U$ for any $i \in W^{\le I}$ and $j \in \mathbb{N}^{\le I}$. Now, for $\hat{E}_{1\min}$ and $\hat{E}_{2\min}$ stated in Lemma 1, regarding that the orders β_j are uncertain, one can write

$$\min_{\boldsymbol{u}\in U}\left[\min\left\{1,\left(|\alpha_0|/\sum_{i=1}^{I}|\alpha_i|\right)^{\frac{1}{\min_{i\in\mathbb{N}^{\le I}}\beta_i}}\right\}\right] \ge \min\left\{1,\left(\frac{\min_{\boldsymbol{u}\in U}|\alpha_0|}{\sum_{i=1}^{I}\max_{\boldsymbol{u}\in U}|\alpha_i|}\right)^{\frac{1}{\min_{\boldsymbol{u}\in U,\, i\in\mathbb{N}^{\le I}}\beta_i}}\right\} = E_{1\min}$$

$$\min_{\boldsymbol{u}\in U}\left[\left(\frac{|\alpha_0|}{|\alpha_0|+\max\left\{|\alpha_i|\,\middle|\, i\in\mathbb{N}^{\le I}\right\}}\right)^{\max\left\{\frac{i}{\beta_i}\middle| i\in\mathbb{N}^{\le I}\right\}}\right] \ge \left(\frac{\min_{\boldsymbol{u}\in U}|\alpha_0|}{\min_{\boldsymbol{u}\in U}|\alpha_0|+\max_{\boldsymbol{u}\in U,\, i\in\mathbb{N}^{\le I}}|\alpha_i|}\right)^{P_{2\min}} = E_{2\min}.$$

It is notable that $\hat{E}_{\min} \le 1$, so minimizing $\hat{E}_{1\min}$ and $\hat{E}_{2\min}$ involves maximizing their powers. Using Lemma 1 on $\delta(s, \boldsymbol{u})$ for any $\boldsymbol{u} \in U$ and regarding the relations provided for $E_{1\min}$, $E_{2\min}$, and $E_{\min}$, it follows that for any $\boldsymbol{u} \in U$ the function $\delta_{pb}(s, \boldsymbol{u})$ does not have any roots in the area $\{s \in \mathbb{C} \| s| < E_{\min}\}$. One can similarly prove that for any $\boldsymbol{u} \in U$ the function $\delta_{pb}(s, \boldsymbol{u})$ does not have any roots in the area $\{s \in \mathbb{C} \| s| > E_{\max}\}$.

Proof of Theorem 2: It is obvious that if $\exists \boldsymbol{u} \in U : \alpha_0 = 0$, then $\delta(0, \boldsymbol{u}) = 0$, and consequently $R_{\min} = 0$. For proving the rest of the theorem, it is sufficient to prove that if $|s| > R_{1\max}, R_{2\max}, R_{3\max}, R_{4\max}, R_{5\max}, R_{6\max}$ holds and also if $|s| < R_{1\min}, R_{2\min}$ when $\alpha_0 \ne 0$ for any $\boldsymbol{u} \in U$ holds, then one will have $|\delta(s, \boldsymbol{u})| > 0$. For the sake of brevity in the proof, let us prove only two cases $|s| > R_{1\max}$ and $|s| < R_{2\min}$ when $\alpha_0 \ne 0$ for any $\boldsymbol{u} \in U$. Similarly, the other cases can be proved. From (1) we have

$$\delta_{pb}\left(|s|e^{j\Lambda}, \boldsymbol{u}\right) = \sum_{i=0}^{I}\alpha_i\cos(\beta_i\Lambda)|s|^{\beta_i} + j\sum_{i=1}^{I}\alpha_i\sin(\beta_i\Lambda)|s|^{\beta_i}.$$

Hence, for we have $|\delta_{pb}(s, \boldsymbol{u})| > 0$, we must have for any $\boldsymbol{u} \in U$

$$\mu = \left|\sum_{i=0}^{I}\alpha_i\cos(\beta_i\Lambda)|s|^{\beta_i}\right| + \underbrace{\left|\sum_{i=1}^{I}\alpha_i\sin(\beta_i\Lambda)|s|^{\beta_i}\right|}_{Z_1 \ge 0} > 0. \tag{6}$$

Furthermore, according to the triangle inequality, we can write

$$\forall \xi, \rho, \tau \in \mathbb{R} : |\xi + \rho + \tau| \ge |\xi| - |\rho + \tau| \ge |\xi| - |\rho| - |\tau|. \tag{7}$$

Case $|s| > R_{1\max}$: If $f_c = 1$, for any $\boldsymbol{u} \in U$ one can write

$$(6),(7) \Rightarrow \mu \ge |\alpha_I\cos(\beta_I\Lambda)||s|^{\beta_I} - \sum_{i=0}^{I-1}|\alpha_i\cos(\beta_i\Lambda)||s|^{\beta_i} + Z_1 = \mu_1. \tag{8}$$

It is notable that if $f_c = 0$, one cannot drive (8), and accordingly it cannot be obtained some result for $R_{1\max}$. Therefore, $R_{1\max} = \infty$. Supposing $f_c = 1$ the proof can be pursued for any $\boldsymbol{u} \in U$ as follows:

$$|s| > R_{1\max} \Rightarrow |s| > 1 \tag{9}$$

$$\max_{\boldsymbol{u}\in U, i\in W^{\leq I-1}} \beta_i = \gamma_2 \tag{10}$$

$$(9),(10) \Rightarrow \forall i \in W^{\leq I-1} : |s|^{\gamma_2} \geq |s|^{\beta_i} \tag{11}$$

$$|s| > R_{1\max} \overset{(9)}{\Rightarrow}, H_1 > 0 |s|^{\frac{1}{H_1}} > \frac{\sum_{i=0}^{I-1} |\alpha_i \cos(\beta_i \Lambda)|}{|\alpha_I \cos(\beta_I \Lambda)|} \tag{12}$$

$$\begin{aligned}(8),(11) \Rightarrow \mu_1 &\geq |\alpha_I \cos(\beta_I \Lambda)| |s|^{\min\limits_{\boldsymbol{u}\in U}\beta_I} \\ &\quad - \sum_{i=0}^{I-1} |\alpha_i \cos(\beta_i \Lambda)| |s|^{\gamma_2} + Z_1 \\ &= |\alpha_I \cos(\beta_I \Lambda)| |s|^{\gamma_2} \left[|s|^{\frac{1}{H_1}} - \frac{\sum_{i=0}^{I-1} |\alpha_i \cos(\beta_i \Lambda)|}{|\alpha_I \cos(\beta_I \Lambda)|} \right] \\ &\quad + Z_1 \overset{(12)}{\Rightarrow} \mu_1 > 0 \overset{(8)}{\Rightarrow} \mu > 0.\end{aligned}$$

Case $|s| < R_{2\min}$ when $\alpha_0 \neq 0$ for any $\boldsymbol{u} \in U$: we can write for any $\boldsymbol{u} \in U$

$$(6),(7) \Rightarrow \mu \geq |\alpha_0| - \sum_{i=1}^{I} |\alpha_i \cos(\beta_i \Lambda)| |s|^{\beta_i} + Z_1. \tag{13}$$

By defining

$$G = \frac{\min\limits_{\boldsymbol{u}\in U} |\alpha_0|}{\min\limits_{\boldsymbol{u}\in U} |\alpha_0| + \max\limits_{\boldsymbol{u}\in U, i\in\mathbb{N}^{\leq I}} |\alpha_i \cos(\beta_i \Lambda)|} \leq 1 \tag{14}$$

$$H = \max\left\{ \frac{i}{\min\limits_{\boldsymbol{u}\in U} \beta_i} \middle| i \in \mathbb{N}^{\leq I} \right\} \tag{15}$$

one has

$$(15) \Rightarrow H = 1/\min\left\{ \frac{\min\limits_{\boldsymbol{u}\in U} \beta_i}{i} \middle| i \in \mathbb{N}^{\leq I} \right\} \tag{16}$$

$$|s| < R_{2\min} \Rightarrow |s| < G^H \tag{17}$$

$$(17) \Rightarrow |s| < 1 \tag{18}$$

$$(13) \Rightarrow \mu \geq |\alpha_0| \left(1 - \sum_{i=1}^{I} \frac{|\alpha_i \cos(\beta_i \Lambda)|}{|\alpha_0|} |s|^{\min\limits_{\boldsymbol{u}\in U}\beta_i} \right) + Z_1 = \mu_2 \tag{19}$$

$$(14),(19) \Rightarrow \mu_2 \geq |\alpha_0| \left(1 - \left(\frac{1}{G} - 1 \right) \sum_{i=1}^{I} |s|^{i \frac{\min_{\boldsymbol{u}\in U} \beta_i}{i}} \right) + Z_1 = \mu_3 \tag{20}$$

$$\begin{aligned}&(16),(18),(20) \Rightarrow \mu_3 \\ &\geq |\alpha_0| \left(1 - \left(\frac{1}{G} - 1 \right) \sum_{i=1}^{I} \left(|s|^{1/H} \right)^i \right) + Z_1 = \mu_4 \end{aligned} \tag{21}$$

$$\begin{aligned}(17),(21) &\Rightarrow \mu_4 > |\alpha_0| \left(1 - \left(\frac{1}{G} - 1 \right) \sum_{i=1}^{\infty} G^i \right) + Z_1 \\ &= |\alpha_0| \left(1 - \left(\frac{1}{G} - 1 \right) \frac{G}{1-G} \right) + Z_1 \\ &= Z_1 \Rightarrow \mu_4 > 0 \Rightarrow \mu > 0.\end{aligned}$$

Proof of Theorem 3: For proving this theorem, the following lemmas are needed. Here, for $S_1, S_2 \subset \mathbb{C}$ define $\bar{S}_1 = S_1 \cup \partial S_1$ and $S_1 \backslash S_2 = \{x \in S_1 | x \notin S_2\}$.

Lemma 3 (Theorem 4 of [46]): Assume Q is a bounded and open subset of $\mathbb{C}^N$ where $N \in \mathbb{N}$. Suppose that $h : \bar{Q} \times [0,1] \to \mathbb{C}^N$ is a continuous mapping in its domain and analytical for any $s \in Q$. Further, suppose that $h(s,t) \neq 0$ holds for any $s \in \partial Q$ and $t \in [0,1]$. Then, $h(s,0)$ and $h(s,1)$ have the same number of zeros in Q.

Lemma 4: Consider the characteristic equation of a general fractional order control system, $\delta(s,\boldsymbol{u})$, as described in Definition 1. Suppose that U is pathwise connected. Let $D \subset \mathbb{C}$ be an open set. If $0 \notin \delta_{vs}(z,U)$ for all $z \in S_D = \{s \in \partial D^+ \cup \partial D^- | E_{\min} \leq |s| \leq E_{\max}\}$, where $E_{\min}$ and $E_{\max}$ are define as in Theorem 1, and D^+ and D^- as in Theorem 3, then $\delta_{pb}(s,\boldsymbol{u}^1)$ and $\delta_{pb}(s,\boldsymbol{u}^2)$ have the same number of roots in D^C for any $\boldsymbol{u}^1, \boldsymbol{u}^2 \in U$.

Proof: Since U is a pathwise connected set, there exists the continuous function $\boldsymbol{u} : [0,1] \to U$ such that $\boldsymbol{u}(0) = \boldsymbol{u}^1$ and $\boldsymbol{u}(1) = \boldsymbol{u}^2$ for any $\boldsymbol{u}^1, \boldsymbol{u}^2 \in U$. Let $U_s = \{\boldsymbol{u}(t) | t \in [0,1]\}$. U_s is a bounded and closed set. From Theorem 1 it is deduced that $\delta_{pb}(s,\boldsymbol{u})$ does not have any roots in $E = \{s \in D^C || s| > E_{\max}\}$ for any $\boldsymbol{u} \in U_s$, and also if $0 \notin \delta_{vs}(z,U)$ holds for all $z \in S_D = \{s \in \partial D^+ \cup \partial D^- | E_{\min} \leq |s| \leq E_{\max}\}$ then it holds for all $z \in \partial D$, too. Consider $Q = \{s \in D^C \backslash \partial D || s| < E_{\max} + \kappa, \kappa \in \mathbb{R}^+\}$ and $h : \bar{Q} \times [0,1] \to \mathbb{C}$ as $h(s,t) = \delta_{pb}(s,\boldsymbol{u}(t))$. Considering that $\delta_{pb}(s,\boldsymbol{u})$ for any $\boldsymbol{u} \in U_s$ and accordingly the function $h(s,t)$ for any $t \in [0,1]$ does not have any roots in E, and also regarding that $0 \notin \delta_{vs}(z,U)$ for all $z \in \partial D^+ \cup \partial D^-$, it is concluded that $h(s,t) \neq 0$ for any $s \in \partial Q$ and $t \in [0,1]$. Hence, based on Lemma 3, the functions $h(s,0)$ and $h(s,1)$, which are equal to $\delta_{pb}(s,\boldsymbol{u}^1)$ and $\delta_{pb}(s,\boldsymbol{u}^2)$, respectively, have the same number of roots in Q. Therefore, $\delta_{pb}(s,\boldsymbol{u}^1)$ and $\delta_{pb}(s,\boldsymbol{u}^2)$ for any $\boldsymbol{u}^1, \boldsymbol{u}^2 \in U$ do not have any roots in E, and have the same number of roots in Q. Further, considering that $0 \notin \delta_{vs}(z,U)$ for all $z \in \partial D^+ \cup \partial D^-$, the functions $\delta_{pb}(s,\boldsymbol{u}^1)$ and $\delta_{pb}(s,\boldsymbol{u}^2)$ do not have any roots on ∂D. Therefore, these functions have the same number of roots in D^C.

Now, the proof of Theorem 3 can be presented as follows: Necessary condition: regarding Lemma 4, $\delta_{pb}(s,\boldsymbol{u})$ for any $\boldsymbol{u} \in U$ has no roots in D^C. Therefore, $\delta(s,\boldsymbol{u})$ is robust D-stable. Sufficient condition: according to Definition 4, if $\delta(s,\boldsymbol{u})$ is robust D-stable, then $\delta_{pb}(s,\boldsymbol{u}) \neq 0$ for any $s \in \{\partial D^+ \cup \partial D^-\} \subset D^C$ and $\boldsymbol{u} \in U$.

Proof of Theorem 4: By defining $\boldsymbol{u} = [u_1, u_2, \ldots, u_M]^T$, the functions $\alpha_i(\boldsymbol{u})$ for any $i \in W^{\leq I}$ can be rewritten as

$$\alpha_i(\boldsymbol{u}) = \alpha_{i0} + \alpha_{i1} u_1 + \alpha_{i2} u_2 + \cdots + \alpha_{iM} u_M \tag{22}$$

where $\alpha_{im} \in \mathbb{C}$ for any $i \in W^{\leq I}$ and $m \in W^{\leq M}$. For a $s \in \mathbb{C}$ and $\hat{\beta} \in \mathbb{R}$ we have

$$s^{\hat{\beta}} = |s|^{\hat{\beta}}\left[\cos\left(\arg(s)\hat{\beta}\right) + j\sin\left(\arg(s)\hat{\beta}\right)\right]. \tag{23}$$

Substituting (22) in (2) and employing (23, one gets

$$\begin{aligned}\hat{\delta}(z,\boldsymbol{u}) = &\sum_{m=1}^{M} u_m\left[\sum_{i=1}^{I}\alpha_{im}|z|^{\beta_i}\cos(\arg(z)\beta_i) + \alpha_{0m}\right]\\ &+ \sum_{i=1}^{I}\alpha_{i0}|z|^{\beta_i}\cos(\arg(z)\beta_i) + \alpha_{00}\\ &+ j\left\{\sum_{m=1}^{M} u_m\left[\sum_{i=1}^{I}\alpha_{im}|z|^{\beta_i}\sin(\arg(z)\beta_i) + \alpha_{0m}\right]\right.\\ &\left.+ \sum_{i=1}^{I}\alpha_{i0}|z|^{\beta_i}\sin(\arg(z)\beta_i) + \alpha_{00}\right\}.\end{aligned} \tag{24}$$

From (24) it is seen that the real part is related with the imaginary part linearly, and viceversa. It follows that (24), which is a linear mapping from U to $\mathbb{C}$, is a polygon whose exposed edges and vertices are the mapping of the exposed edges and the vertices of U. Therefore, $\partial\hat{\delta}_{vs}(z,U)$ can be obtained from U_E. Hence, $\partial\hat{\delta}_{vs}(z,U) \subseteq \hat{\delta}_{vs}(z,U_E)$ for any $z \in \mathbb{C}$.

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