# Stability of Delayed Switched Systems With State-Dependent Switching

Chao Liu, Zheng Yang, Xiaoyang Liu, and Xianying Huang

***Abstract*— This paper investigates the stability of switched systems with time-varying delay and all unstable subsystems. According to the stable convex combination, we design a state-dependent switching rule. By employing Wirtinger integral inequality and Leibniz-Newton formula, the stability results of nonlinear delayed switched systems whose nonlinear terms satisfy Lipschitz condition under the designed state-dependent switching rule are established for different assumptions on time delay. Moreover, some new stability results for linear delayed switched systems are also presented. The effectiveness of the proposed results is validated by three typical numerical examples.**

***Index Terms*—Stability, statedependent switching, switched system, time delay.**

## I. Introduction

A switched system is composed of a set of continuous time or discrete-time subsystems and a law which governs the switching among them [1]. As a class of typical hybrid systems, switched systems have widespread practical backgrounds and engineering value. On the one hand, many natural or engineering systems which generate different modes owing to sudden changes of their environment can be modeled as switched systems. For example, aircraft control systems [2], traffic management systems [3], robotic walking control systems [4], computer disk drives [5] and some chemical processes [6] can fall into switched systems. On the other hand, switching can be viewed as an effective way of solving certain control problems. Multi-controller switching scheme [7], Bang-Bang control [8], intermittent control [9] and supervisory control [10] are derived from the idea of switching. Until now, switched systems have been a popular research focus in the control field and many significant results have been deduced in recent years.

System switching can generate high nonlinearities, which indicates that stability analysis of switched systems is more complicated than those of ordinary dynamical systems. Under some typical examples, Decarlo *et al*. have pointed out that a switched system with all unstable subsystems (or with all asymptotically stable subsystems), may be asymptotically stable (or unstable) under some specified switching rules [11]. In [12], the authors show that the delayed random switched systems derived from exponentially stable subsystems may lose stability with decreases in dwell times. This is because switching may generate destabilizing effects which destroy the stability of switched systems. Therefore, we must concentrate on both subsystems and switching rules in the study of stability. Liberzon has summed up the stability investigation of switched systems as three basic problems [13]: 1) stability under arbitrary switching signals, 2) stability under constrained switching signals, and 3) the construction of switching signal to stabilize switched systems. In order to cope with the stability of switched systems, many effective analysis tools such as the common Lyapunov function [14], multiple Lyapunov functions [15], average dwell time [16], and mode-dependent average dwell time [17] have been employed to derive novel stability results.

Designing appropriate switching rules to stabilize switched systems with entirely unstable subsystems is a valuable and challenging problem. This problem can be solved with two strategies: time-dependent switching and state-dependent switching. For the first, dwell time of the admissible switching rules is required to have both upper and lower bounds. According to discretized Lyapunov function, the time-dependent switching rule to guarantee the asymptotic stability of linear switched systems was designed [18]. Based on the same method, the switching rule is constructed in [19] to stabilize switched neural networks with time-varying delays and unstable subsystems. The bound of dwell time for the asymptotic stability of linear delayed switched systems is presented in [20]. In these results, the stabilization property of switching behaviors is utilized to compensate the divergent effect of unstable subsystems. When a switched system is not stable under any time-dependent switching rule, the only alternative is to employ a state-dependent switching rule to stabilize the switched system. As early as 2003, Liberzon has proved that linear switched systems with two unstable subsystems is asymptotically stable under some state-

Manuscript received December 25, 2018; accepted February 8, 2019. This work was supported by the National Natural Science Foundation of China (61503052, 61503050, 61603065), the Scientific and Technological Research Program of Chongqing Municipal Education Commission (KJQN201801120, KJQN201801104, KJ1709206), the Opening Foundation of Hubei Key Laboratory of Applied Mathematics (Hubei University) (HBAM201805), and Chongqing Science and Technology Commission Technology Innovation and Application Demonstration (Social Livelihood General) Project (cstc2018jscx-msybX0049). Recommended by Associate Editor Choon Ki Ahn. *(Corresponding author: Chao Liu.)*

Citation: C. Liu, Z. Yang, X. Y. Liu, and X. Y. Huang, "Stability of delayed switched systems with state-dependent switching," *IEEE/CAA J. Autom. Sinica*, vol. 7, no. 3, pp. 872–881, May 2020.

C. Liu is with the School of Computer Science and Engineering, Chongqing University of Technology, Chongqing 400054, and also with Hubei Key Laboratory of Applied Mathematics (Hubei University), Wuhan 430062, China (e-mail: xiuwenzheng2000@163.com).

Z. Yang, X. Y. Liu, and X. Y. Huang are with the School of Computer Science and Engineering, Chongqing University of Technology, Chongqing 400054, China (e-mail: yazen@sina.com; lxy3103@163.com; hxy@cqut.edu.cn).

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Digital Object Identifier 10.1109/JAS.2019.1911624

dependent switching rule, if there exists a Hurwitz convex combination of coefficient matrices [13]. In [1], this result is extended to linear switched systems with multi-subsystems. At present, the switching rule design can be usually summarized with the following three steps: 1) based on Hurwitz linear convex combination, divide state space into several switching regions, 2) construct the state-dependent switching rule, and 3) establish the stability conditions that the switched system is asymptotically stable under the constructed state-dependent switching rule. According to the common Lyapunov function and mixed mode, Kim *et al*. designed state-dependent switching rules for linear switched systems with a constant time delay. However, the proposed results are only applicable for cases where the time delay is sufficiently small [21]. In [22], the new state-dependent switching rule is designed for linear switched systems with time delays to weaken the restriction on time delay. Under time delay approximation and small gain theorem, Li *et al*. present new stability results under the state-dependent switching rule for linear switched systems with time delay and show that time delay restriction is weaker [23]. Owing to common Lyapunov function and multiple Lyapunov functions, the state-dependent switching rule for delayed switched Hopfield neural networks is designed in [24]. The analogous switching rule for linear positive switched systems is designed in [25]. Furthermore, some design methods for state-dependent switching rules are presented in [26]–[28] by guaranteeing the convergence of Lyapunov function. However, these results are only valid for linear switched systems.

The discussed content indicates that the stability of switched systems with all unstable subsystems (under the designed state-dependent switching rule) has been studied extensively. However, some inadequacies still remain. First, until now, most stability results have been suitable for linear switched systems with or without time delays [1], [13], [20]–[23], [25]–[28]. Unfortunately, the researchers gave less attention on the stability for nonlinear switched systems with time delay under state-dependent switching rules. Although the stability results for delayed Hopfield neural networks are presented in [24], they may not be valid for more general nonlinear switched systems. Second, the rigorous restrictions on time delay in the existing results should be further weakened, especially for the case that the bound of the derivative of time delay is known.

In this paper, we also cope with the stability of delayed switched systems with all unstable subsystems. First, we focus on the nonlinear delayed switched systems whose nonlinear terms satisfy Lipschitz condition. Based on the Hurwitz linear convex combination, we design the state-dependent switching rule. Then, some stability results under different assumptions are derived by the Wirtinger integral inequality and Leibniz-Newton formula. Second, by viewing linear delayed switched systems as special nonlinear delayed switched systems, some new stability results for linear switched systems with time-varying delay are also proposed. Three numerical examples are employed to show the effectiveness of the proposed results. The main contributions of this paper are listed as follows.

1) Certain stability results for nonlinear switched systems with all unstable subsystems and time-varying delay under state-dependent switching rule are derived.

2) For linear switched systems with all unstable subsystems and time-varying delay, the proposed stability results weaken the restriction on time delay if the bound of the derivative of time delay is known.

*Notations*: $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote the $n$-dimensional Euclidean space and the set of all $n\times m$ real matrices, respectively, $\Sigma^T$ denotes the transpose of square matrix $\Sigma$, $\Sigma>0\,(\Sigma<0)$ denotes that $\Sigma$ is a symmetric positive definite (negative definite) matrix, $I$ denotes the identity matrix with appropriate dimension, $\mathrm{diag}\{\cdot\}$ denotes the block-diagonal matrix, $\begin{pmatrix} X & Y \\ * & Z \end{pmatrix}$ denotes $\begin{pmatrix} X & Y \\ Y^T & Z \end{pmatrix}$.

## II. Preliminaries

This paper concerns delayed switched systems with the form

$$\begin{cases} \dot{x}(t) = A_\sigma x(t) + B_\sigma f_\sigma(x(t-\tau(t))), & t>t_0 \\ x(t_0+s) = \phi(s), & s\in[-\tau,0] \end{cases} \tag{1}$$

where $x(t)\in\mathbb{R}^n$ is the state vector, $\sigma(t)\in M=\{1,2,\dots,m\}$ is a piecewise continuous function called switching signal, $A_p$, $B_p\in\mathbb{R}^{n\times n}$, $\tau(t)$ is the time-varying delay, $\phi(s)$ is a piecewise continuous function, $f_p(z)=(f_{p1}(z_1),\dots,f_{pn}(z_n))$ is a nonlinear function. If $\sigma(t)=p$, we say that the $p$th subsystem $\dot{x}(t)=A_p(t)x(t)+B_pf_p(x(t-\tau(t)))$ is activated. Time $t$ is called as a switching instant if $\sigma(t^+)\neq\sigma(t^-)$.

In this paper, we always assume that for any $p\in M$ and $i=1,2,\dots,n$, $f_{pi}(0)=0$. Moreover, we assume that the function $f_{pi}$ satisfies Lipschitz condition. Namely, there exists positive constant $l_{pi}$ such that

$$0\le \frac{f_{pi}(z_1)-f_{pi}(z_2)}{z_1-z_2}\le l_{pi},\quad z_1,z_2\in\mathbb{R}^n, z_1\neq z_2. \tag{2}$$

For convenience, we denote $L_p=\mathrm{diag}\{l_{p1},\dots,l_{pn}\}$. Then, system (1) can be rewritten as

$$\begin{cases} \dot{x}(t) = (A_\sigma + B_\sigma L_\sigma)x(t) - B_\sigma L_\sigma \int_{t-\tau(t)}^{t}\dot{x}(s)ds \\ \qquad + B_\sigma(f_\sigma(x(t-\tau(t))) - L_\sigma x(t-\tau(t))), & t>t_0 \\ x(t_0+s)=\phi(s), & s\in[-\tau,0]. \end{cases} \tag{3}$$

We always assume that there exists a Hurwitz linear convex combination $F$ of $A_p+B_pL_p$. Namely

$$F=\sum_{p\in M}\alpha_p\left(A_p+B_pL_p\right) \tag{4}$$

where $\alpha_p\in[0,1]$ and $\sum_{p\in M}\alpha_p=1$. For a given symmetric positive definite matrix $\Xi$, there exists some symmetric positive definite matrix $P$ such that

$$F^TP+PF=-\Xi. \tag{5}$$

Define region $\Theta_p$ as

$$\Theta_p=\left\{x \middle| x^T\left(\left(A_p+B_pL_p\right)^TP + P\left(A_p+B_pL_p\right)\right)x\le -x^T\Xi x\right\}. \tag{6}$$

Under the proof of [21, Proposition 2], it is easy to see

$$\bigcup_{p\in M}\Theta_p=\mathbb{R}^n. \tag{7}$$

The state space can be divided into the following switching areas

$$\begin{aligned}\bar{\Theta}_1&=\Theta_1\\ \bar{\Theta}_2&=\frac{\Theta_2}{\Theta_2\bigcap\bar{\Theta}_1}\\ &\vdots\\ \bar{\Theta}_i&=\frac{\Theta_i}{\Theta_i\bigcap\bigcup_{j=1}^{i-1}\bar{\Theta}_j}\\ &\vdots\\ \bar{\Theta}_m&=\frac{\Theta_m}{\Theta_m\bigcap\bigcup_{j=1}^{m-1}\bar{\Theta}_j}.\end{aligned}$$

Then, the switching rule for system (1) can be designed as

$$\sigma(t)=p,\ \text{ if } x(t)\in\bar{\Theta}_p. \tag{8}$$

The above switching rule indicates that, for any instant $t$, the index of the activated subsystem is $p$, if $x(t)$ is located in the area $\bar{\Theta}_p$. Noting that there is no restriction on $t$, the switching rule (8) is thus only dependent on the state.

The main purpose of this paper is to deduce the sufficient conditions for the global asymptotic stability of system (1) under the state-dependent switching rule (8). The following assumptions are essential for our results.

*Assumption 1:* There exist positive constants $\tau$, $\bar{\tau}$ and constant $\tilde{\tau}$ such that

$$0\le\tau(t)\le\tau,\ \ \tilde{\tau}\le\dot{\tau}(t)\le\bar{\tau}. \tag{9}$$

*Assumption 2:* There exists positive constant $\tau$ such that

$$0\le\tau(t)\le\tau. \tag{10}$$

Now we give a lemma which is the core of this investigation.

*Lemma 1 (Wirtinger-based integral inequality, [29]):* For a given matrix $R>0$, positive constants $t_1$ and $t_2$ where $t_1<t_2$, and vector function $y$ which is continuously differentiable in $[t_1,t_2]$, the following inequality is true

$$\int_{t_1}^{t_2}\dot{y}^T(t)R\dot{y}(t)dt\ge\frac{1}{t_2-t_1}\left(\beta_1^TR\beta_1+3\beta_2^TR\beta_2\right) \tag{11}$$

where $\beta_1=y(t_2)-y(t_1)$, $\beta_2=y(t_2)+y(t_1)-\frac{1}{t_2-t_1}\int_{t_1}^{t_2}y(t)dt$.

## III. Main Results

This section proposes some stability results for the delayed switched systems under state-dependent switching rule (8) via the Lyapunov functionals. In Section III-A, we give the stability results for system (1) with Assumptions 1 and 2. Some stability results for linear delayed switched systems are derived in Section III-B.

### *A. Stability for Nonlinear Delayed Switched Systems*

*Theorem 1:* Under Assumption 1, if there exist matrices $P>0$, $Z>0$, $R_l>0\ (l=1,2)$, diagonal matrices $N_{ph}=\text{diag}\{n_{p1}^h,\ldots,n_{pn}^h\}>0\ (p\in M,h=1,2)$, $n\times n$ matrices $Y_{pj}$, $T_{pj}$, $S_{pj}$, $U_{pj}$, $H_{pj}\ (j=1,2)$, and $5n\times 5n$ symmetric matrices $X_p^1$, $X_p^2$, such that

$$X_p^h=\begin{pmatrix}X_{p11}^h & X_{p12}^h & X_{p13}^h & X_{p14}^h & X_{p15}^h\\ * & X_{p22}^h & X_{p23}^h & X_{p24}^h & X_{p25}^h\\ * & * & X_{p33}^h & X_{p34}^h & X_{p35}^h\\ * & * & * & X_{p44}^h & X_{p45}^h\\ * & * & * & * & X_{p55}^h\end{pmatrix}>0,\ h=1,2 \tag{12}$$

$$\Psi_{pj}^h=\begin{pmatrix}\Psi_{pj11}^h & \Psi_{pj12}^h & \Psi_{pj13}^h & \Psi_{pj14}^h & \Psi_{pj15}^h\\ * & \Psi_{pj22}^h & \Psi_{pj23}^h & \Psi_{pj24}^h & \Psi_{pj25}^h\\ * & * & \Psi_{pj33}^h & \Psi_{pj34}^h & \Psi_{pj35}^h\\ * & * & * & \Psi_{pj44}^h & \Psi_{pj45}^h\\ * & * & * & * & \Psi_{pj55}^h\end{pmatrix}<0,\ j=1,2,h=1,2 \tag{13}$$

$$\Phi_{p1}=\left(\begin{array}{c|c} X_p^1 & \begin{matrix}\Phi_{p1}^{11}\\ \Phi_{p1}^{12}\\ U_{p1}\\ S_{p1}\\ \Phi_{p1}^{15}\end{matrix}\\ \hline * & \Phi_{p1}^{16}\end{array}\right)>0 \tag{14}$$

$$\Phi_{p2}=\left(\begin{array}{c|c} X_p^2 & \begin{matrix}Y_{p2}\\ T_{p2}\\ U_{p2}\\ S_{p2}\\ H_{p2}\end{matrix}\\ \hline * & Z\end{array}\right)>0 \tag{15}$$

where

$$\begin{aligned}\Psi_{p111}^h=\Psi_{p211}^h=&\,F^TP+PF+R_1\\ &+\tau\left(A_p+B_pL_p\right)^TZ\left(A_p+B_pL_p\right)\\ &-3\tau L_p^TB_p^TZB_pL_p+Y_{p1}+Y_{p1}^T+\tau X_{p11}^h\end{aligned}$$

$$\begin{aligned}\Psi_{p122}^h=&\,(1-\bar{\tau})(R_2-R_1)-2\tau L_p^TB_p^TZB_pL_p\\ &-T_{p1}-T_{p1}^T+T_{p2}+T_{p2}^T+\tau X_{p22}^h\end{aligned}$$

$$\begin{aligned}\Psi^h_{p222} = {}& (1-\tilde{\tau})(R_2-R_1) - 2\tau L_p^T B_p^T Z B_p L_p \\ & - T_{p1} - T_{p1}^T + T_{p2} + T_{p2}^T + \tau X^h_{p22}\end{aligned}$$

$$\begin{aligned}\Psi^h_{p133} &= \Psi^h_{p233} = -R_2 - U_{p2} - U_{p2}^T + \tau X^h_{p33} \\ \Psi^h_{p144} &= \Psi^h_{p244} = -3\tau L_p^T B_p^T Z B_p L_p + \tau X^h_{p44} \\ \Psi^h_{p155} &= \Psi^h_{p255} = -N_{ph} + \tau B_p^T Z B_p + \tau X^h_{p55} \\ \Psi^h_{p112} &= \Psi^h_{p212} = -PB_pL_p - \tau\left(A_p + B_pL_p\right)^T Z \\ & \quad \times B_pL_p - 3\tau L_p^T B_p^T Z B_p L_p - Y_{p1} + T_{p1}^T \\ & \quad + Y_{p2} + \tau X^h_{p12}\end{aligned}$$

$$\begin{aligned}\Psi^h_{p113} &= \Psi^h_{p213} = -Y_{p2} + U_{p1}^T + \tau X^h_{p13} \\ \Psi^h_{p114} &= \Psi^h_{p214} = 3\tau L_p^T B_p^T Z B_p L_p + S_{p1}^T + \tau X^h_{p14} \\ \Psi^h_{p115} &= \Psi^h_{p215} = PB_p + \tau\left(A_p + B_pL_p\right)^T Z B_p \\ & \quad + H_{p1}^T + \tau X^h_{p15}\end{aligned}$$

$$\begin{aligned}\Psi^h_{p123} &= \Psi^h_{p223} = -U_{p1}^T - T_{p2} + U_{p2}^T + \tau X^h_{p23} \\ \Psi^h_{p124} &= \Psi^h_{p224} = 3\tau L_p^T B_p^T Z B_p L_p - S_{p1}^T + S_{p2}^T \\ & \quad + \tau X^h_{p24} \\ \Psi^h_{p125} &= \Psi^h_{p225} = -\tau L_p^T B_p^T Z B_p - H_{p1}^T + H_{p2}^T \\ & \quad + 0.5 N_{ph} L_p + \tau X^h_{p25}\end{aligned}$$

$$\begin{aligned}\Psi^h_{p134} &= \Psi^h_{p234} = -S_{p2}^T + \tau X^h_{p34} \\ \Psi^h_{p135} &= \Psi^h_{p235} = -H_{p2}^T + \tau X^h_{p35} \\ \Psi^h_{p145} &= \Psi^h_{p245} = \tau X^h_{p45} \\ \Phi^{11}_{p1} &= PB_pL_p + \tau\left(A_p + B_pL_p\right)^T Z B_p L_p + Y_{p1} \\ \Phi^{12}_{p1} &= T_{p1} - \tau L_p^T B_p^T Z B_p L_p \\ \Phi^{15}_{p1} &= H_{p1} + \tau B_p^T Z B_p L_p \\ \Phi^{16}_{p1} &= -\tau^2 L_P^T B_p^T Z B_p L_p + Z\end{aligned}$$

then, system (1) is globally asymptotically stable under the state-dependent switching rule (8).

*Proof:* We choose the Lyapunov functional as follows

$$V(t) = V_1(t) + V_2(t) + V_3(t) \tag{16}$$

where

$$\begin{aligned}V_1(t) &= x^T(t)Px(t) \\ V_2(t) &= \int_{t-\tau(t)}^{t} x^T(s)R_1x(s)ds \\ & \quad + \int_{t-\tau}^{t-\tau(t)} x^T(s)R_2x(s)ds \\ V_3(t) &= \int_{-\tau}^{0}\int_{t+\theta}^{t} \dot{x}(s)Z\dot{x}(s)dsd\theta.\end{aligned}$$

When $x(t) \in \bar{\Theta}_p$, according to the switching rule (8), we know that the $p$th subsystem is activated. It follows from (5) and (6) that

$$\begin{aligned}& x^T(t)\left(\left(A_p + B_pL_p\right)^T P + P\left(A_p + B_pL_p\right)\right)x(t) \\ & \quad \le x^T(t)\left(F^TP + PF\right)x(t).\end{aligned} \tag{17}$$

Differentiating $V_1$, $V_2$ and $V_3$, we have

$$\begin{aligned}\dot{V}_1(t) = {}& x^T(t)\left(\left(A_p + B_pL_p\right)^T P + P\left(A_p + B_pL_p\right)\right)x(t) \\ & - x^T(t)PB_pL_p\int_{t-\tau(t)}^{t}\dot{x}(s)ds \\ & - \int_{t-\tau(t)}^{t}\dot{x}^T(s)dsL_p^TB_p^TPx(t) \\ & - x^T(t)PB_pL_px(t-\tau(t)) \\ & - x^T(t-\tau(t))L_p^TB_p^TPx(t) \\ & + x^T(t)PB_pf_p(x(t-\tau(t))) \\ & - f_p^T(x(t-\tau(t)))B_p^TPx(t)\end{aligned} \tag{18}$$

$$\begin{aligned}\dot{V}_2(t) = {}& x^T(t)R_1x(t) + (1-\dot{\tau}(t))x^T(t-\tau(t)) \\ & \times (R_2 - R_1)x(t-\tau(t)) \\ & - x^T(t-\tau)R_2x(t-\tau)\end{aligned} \tag{19}$$

$$\begin{aligned}\dot{V}_3(t) = {}& \tau\dot{x}^T(t)Z\dot{x}(t) - \int_{t-\tau(t)}^{t}\dot{x}^T(s)Z\dot{x}(s)ds \\ & - \int_{t-\tau}^{t-\tau(t)}\dot{x}^T(s)Z\dot{x}(s)ds.\end{aligned} \tag{20}$$

Under (3), we conclude that

$$\begin{aligned}\tau\dot{x}^T(t)Z\dot{x}(t) = {}& \tau x^T(t)\left(\left(A_p + B_pL_p\right)^T Z\left(A_p + B_pL_p\right)\right)x(t) \\ & + \tau\int_{t-\tau(t)}^{t}\dot{x}^T(s)dsL_p^TB_p^TZB_pL_p\int_{t-\tau(t)}^{t}\dot{x}(s)ds \\ & + \tau f_p^T(x(t-\tau(t)))B_p^TZB_pf_p(x(t-\tau(t))) \\ & + \tau x^T(t-\tau(t))L_p^TB_p^TZB_pL_px(t-\tau(t)) \\ & - 2\tau x^T(t)\left(A_p + B_pL_p\right)^T ZB_pL_p\int_{t-\tau(t)}^{t}\dot{x}(s)ds \\ & + 2\tau x^T(t)\left(A_p + B_pL_p\right)^T ZB_pf_p(x(t-\tau(t))) \\ & - 2\tau x^T(t)\left(A_p + B_pL_p\right)^T ZB_pL_px(t-\tau(t)) \\ & - 2\tau x^T(t-\tau(t))L_p^TB_p^TZB_pf_p(x(t-\tau(t))) \\ & - 2\tau f_p^T(x(t-\tau(t)))B_p^TZB_pL_p\int_{t-\tau(t)}^{t}\dot{x}(s)ds \\ & + 2\tau x^T(t-\tau(t))L_p^TB_p^TZB_pL_p\int_{t-\tau(t)}^{t}\dot{x}(s)ds.\end{aligned} \tag{21}$$

According to Lemma 1, we gain

$$\begin{aligned}& \tau\int_{t-\tau(t)}^{t}\dot{x}^T(s)dsL_p^TB_p^TZB_pL_p\int_{t-\tau(t)}^{t}\dot{x}(s)ds \\ & \quad \le \tau^2\int_{t-\tau(t)}^{t}\dot{x}^T(s)L_p^TB_p^TZB_pL_p\dot{x}(s)ds \\ & \qquad - 3\tau(x(t) + x(t-\tau(t)) - \vartheta)^T L_p^TB_p^TZ \\ & \qquad \times B_pL_p(x(t) + x(t-\tau(t)) - \vartheta)\end{aligned} \tag{22}$$

where $\vartheta = \dfrac{2}{\tau(t)}\int_{t-\tau(t)}^{t}\dot{x}(s)ds$.

Based on (2), we have

$$f_{pi}(x(t-\tau(t)))\left(f_{pi}(x(t-\tau(t))) - l_{pi}x(t-\tau(t))\right) \le 0 \tag{23}$$

which yields that

$$0 \le f_p^T(x(t-\tau(t)))N_{ph}L_p x(t-\tau(t)) - f_p^T(x(t-\tau(t)))N_{ph}f_p(x(t-\tau(t))), \quad h=1,2. \tag{24}$$

For any matrix $Y_{pj}$, $T_{pj}$, $U_{pj}$, $S_{pj}$ and $H_{pj}$, $j=1,2$, owing to the Leibniz-Newton formula, it is known that

$$2\Big(x^T(t)Y_{p1} + x^T(t-\tau(t))T_{p1} + x^T(t-\tau)U_{p1} + \vartheta^T S_{p1} + f_p^T(x(t-\tau(t)))H_{p1}\Big) \times \Big(x(t) - x(t-\tau(t)) - \int_{t-\tau(t)}^{t} \dot{x}(s)ds\Big) = 0 \tag{25}$$

and

$$2\Big(x^T(t)Y_{p2} + x^T(t-\tau(t))T_{p2} + x^T(t-\tau)U_{p2} + \vartheta^T S_{p2} + f_p^T(x(t-\tau(t)))H_{p2}\Big) \times \Big(x(t-\tau(t)) - x(t-\tau) - \int_{t-\tau}^{t-\tau(t)} \dot{x}(s)ds\Big) = 0. \tag{26}$$

For any matrix $X_p^h$ $(h=1,2)$ satisfying (12), we have

$$\tau(t)\eta^T(t)X_p^1\eta(t) - \int_{t-\tau(t)}^{t} \eta^T(t)X_p^1\eta(t)ds = 0 \tag{27}$$

and

$$(\tau-\tau(t))\eta^T(t)X_p^2\eta(t) - \int_{t-\tau}^{t-\tau(t)} \eta^T(t)X_p^2\eta(t)ds = 0 \tag{28}$$

where

$$\eta(t) = \Big(x^T(t), x^T(t-\tau(t)), x^T(t-\tau), \vartheta^T, f_p^T(x(t-\tau(t)))\Big)^T.$$

Then, it follows from (17)–(22) and (24)–(28) that

$$\dot{V}(t) \le \frac{\tau(t)}{\tau}\eta^T(t)\Psi_p^1(t)\eta(t) + \frac{\tau-\tau(t)}{\tau}\eta^T(t)\Psi_p^2(t)\eta(t) - \int_{t-\tau(t)}^{t} \varsigma^T(t,s)\Phi_{p1}\varsigma(t,s)ds - \int_{t-\tau}^{t-\tau(t)} \varsigma^T(t,s)\Phi_{p2}\varsigma(t,s)ds \tag{29}$$

where $\varsigma(t,s) = \big(\eta^T(t), \dot{x}^T(s)\big)^T$ and

$$\Psi_p^1(t) = \frac{\dot{\tau}(t)-\tilde{\tau}}{\bar{\tau}-\tilde{\tau}}\Psi_{p1}^1 + \frac{\bar{\tau}-\dot{\tau}(t)}{\bar{\tau}-\tilde{\tau}}\Psi_{p2}^1$$
$$\Psi_{p2}(t) = \frac{\dot{\tau}(t)-\tilde{\tau}}{\bar{\tau}-\tilde{\tau}}\Psi_{p1}^2 + \frac{\bar{\tau}-\dot{\tau}(t)}{\bar{\tau}-\tilde{\tau}}\Psi_{p2}^2.$$

On the grounds of (13) and (14), we know that $\dot{V}(t) < 0$ for any $x(t) \in \bar{\Theta}_p$ and $\eta \ne 0$. Accordingly, due to (16), system (1) is globally asymptotically stable under the state-dependent switching rule (8). ■

If we restrict $R_1 = R_2 = R$, the stability results under Assumption 2 can be derived.

*Theorem 2:* Under Assumption 2, if there exists matrices $P>0$, $Z>0$, $R>0$, diagonal matrices $N_{ph} = \mathrm{diag}\{n_{p1}^h,\ldots,n_{pn}^h\} > 0$ $(p\in M, h=1,2)$, $n\times n$ matrices $Y_{pj}$, $T_{pj}$, $S_{pj}$, $U_{pj}$, $H_{pj}$ $(j=1,2)$, and $5n\times 5n$ symmetric matrices $X_p^1$, $X_p^2$, such that (12), (14), (15) and

$$\Psi_p^h < 0, \quad h=1,2 \tag{30}$$

where $\Psi_p^h = \Psi_{p1}^h$ with $R_1 = R_2 = R$, then system (1) is globally asymptotically stable under the state-dependent switching rule (8).

*Remark 1:* In [24], the researchers have investigated the stability of delayed switched neural networks

$$\dot{x}(t) = \bar{A}_\sigma x(t) + \bar{B}_\sigma g(x(t-\tau(t))) \tag{31}$$

under state-dependent switching, where $\bar{A}_p = \mathrm{diag}\{-a_{p1},\ldots,-a_{pn}\}$ with $a_{pi}>0$, $i=1,\ldots,n$. The switching region $\Pi_p$ is constructed by

$$\Pi_p = \Big\{x|x^T\big(\bar{A}_pP + P\bar{A}_p\big)x \le -x^T\Lambda x\Big\}$$

where $\Lambda = -\bar{F}^TP - P\bar{F}$, $\bar{F}$ is the Hurwitz linear convex combination of $\bar{A}_p$. Thus, the switching region $\Pi_p$ always exists because of the negative definiteness of $\bar{A}_p$. The proposed results in [24] can also be applied to nonlinear delayed switched systems. In system (1), the matrix $A_p$ may not be negative definite and the nonlinear terms of subsystems are not identical. Therefore, the stability results in [24] are not suitable for system (1) with unstable matrices $A_p$ and non-identical nonlinear terms. Therefore, our results are more effective than those presented in [24].

*Remark 2:* In many stability results for switched systems with time-varying delay, the derivative of the time delay must satisfy $\dot{\tau}(t) < 1$ [22], [24], which implies that $t-\tau(t)$ is monotonically increasing. Generally speaking, this restriction is rigorous. In order to remove this restriction, in Assumption 1 we required that the derivative of the delay has both upper and lower bounds, which is consistent with that proposed in [30].

### *B. Stability for Linear Delayed Switched Systems*

By restricting $f_p = I$ for any $p \in M$, system (1) can be written as the following linear switched systems with time-varying delay

$$\begin{cases} \dot{x}(t) = A_\sigma x(t) + B_\sigma x(t-\tau(t)), & t > t_0 \\ x(t_0+s) = \phi(s), & s\in[-\tau,0]. \end{cases} \tag{32}$$

Similarly, the above system can be rewritten as

$$\begin{cases} \dot{x}(t) = (A_\sigma + B_\sigma)x(t) + B_\sigma \int_{t-\tau(t)}^{t} \dot{x}(s)ds, & t > t_0 \\ x(t_0+s) = \phi(s), & s\in[-\tau,0]. \end{cases} \tag{33}$$

Analogous to Theorems 1 and 2, we can obtain the stability results for system (32) under Assumptions 1 and 2.

*Theorem 3:* Under Assumption 1, if there exist $n\times n$ matrices $P>0$, $Z>0$, $R_i>0$ $(i=1,2)$, $n\times n$ matrices $Y_{pj}$, $T_{pj}$, $S_{pj}$, $U_{pj}$ $(p\in M, j=1,2)$, and $4n\times 4n$ symmetric matrices $\bar{X}_p^h$ $(h=1,2)$, such that

$$\bar{X}_p^h = \begin{pmatrix} \bar{X}_{p11}^h & \bar{X}_{p12}^h & \bar{X}_{p13}^h & \bar{X}_{p14}^h \\ * & \bar{X}_{p22}^h & \bar{X}_{p23}^h & \bar{X}_{p24}^h \\ * & * & \bar{X}_{p33}^h & \bar{X}_{p34}^h \\ * & * & * & \bar{X}_{p44}^h \end{pmatrix} > 0, \quad h=1,2 \tag{34}$$

$$\bar{\Psi}^h_{pj} = \begin{pmatrix} \bar{\Psi}^h_{pj11} & \bar{\Psi}^h_{pj12} & \bar{\Psi}^h_{pj13} & \bar{\Psi}^h_{pj14} \\ * & \bar{\Psi}^h_{pj22} & \bar{\Psi}^h_{pj23} & \bar{\Psi}^h_{pj24} \\ * & * & \bar{\Psi}^h_{pj33} & \bar{\Psi}^h_{pj34} \\ * & * & * & \bar{\Psi}^h_{pj44} \end{pmatrix} < 0, \quad j = 1,2, h = 1,2 \tag{35}$$

$$\bar{\Phi}_{p1} = \left( \begin{array}{c|c} \bar{X}^1_p & \begin{matrix} \bar{\Phi}^{11}_{p1} \\ T_{p1} \\ U_{p1} \\ S_{p1} \end{matrix} \\ \hline * & \bar{\Phi}^{15}_{p1} \end{array} \right) > 0 \tag{36}$$

$$\bar{\Phi}_{p2} = \left( \begin{array}{c|c} \bar{X}^2_p & \begin{matrix} Y_{p2} \\ T_{p2} \\ U_{p2} \\ S_{p2} \end{matrix} \\ \hline * & Z \end{array} \right) > 0 \tag{37}$$

where

$$\begin{aligned} \bar{\Psi}^h_{p111} = \bar{\Psi}^h_{p211} &= F^T P + PF + R_1 \\ &\quad + \tau\left(A_p + B_p\right)^T \left(A_p + B_p\right) - 3\tau B_p^T Z B_p \\ &\quad + Y_{p1} + Y_{p1}^T + \tau \bar{X}^h_{p11} \\ \bar{\Psi}^h_{p122} &= (1-\bar{\tau})(R_2 - R_1) - 3\tau B_p^T Z B_p - T_{p1} \\ &\quad - T_{p1}^T + T_{p2} + T_{p2}^T + \tau \bar{X}^h_{p22} \\ \bar{\Psi}^h_{p222} &= (1-\tilde{\tau})(R_2 - R_1) - 3\tau B_p^T Z B_p - T_{p1} \\ &\quad - T_{p1}^T + T_{p2} + T_{p2}^T + \tau \bar{X}^h_{p22} \end{aligned}$$

$$\begin{aligned} \bar{\Psi}^h_{p133} &= \bar{\Psi}^h_{p233} = -R_2 - U_{p2} - U_{p2}^T + \tau \bar{X}^h_{p33} \\ \bar{\Psi}^h_{p144} &= \bar{\Psi}^h_{p244} = -3\tau B_p^T Z B_p + \tau \bar{X}^h_{p44} \\ \bar{\Psi}^h_{p112} &= \bar{\Psi}^h_{p212} = -3\tau B_p^T Z B_p - Y_{p1} + T_{p1}^T + Y_{p2} \\ &\quad + \tau \bar{X}^h_{p12} \\ \bar{\Psi}^h_{p113} &= \bar{\Psi}^h_{p213} = -Y_{p2} + U_{p1}^T + \tau \bar{X}^h_{p13} \\ \bar{\Psi}^h_{p114} &= \bar{\Psi}^h_{p214} = 3\tau B_p^T Z B_p + S_{p1}^T + \tau \bar{X}^h_{p14} \\ \bar{\Psi}^h_{p123} &= \bar{\Psi}^h_{p223} = -U_{p1}^T - T_{p2} + U_{p2}^T + \tau \bar{X}^h_{p23} \\ \bar{\Psi}^h_{p124} &= \bar{\Psi}^h_{p224} = 3\tau B_p^T Z B_p - S_{p1}^T + S_{p2}^T + \tau \bar{X}^h_{p24} \\ \bar{\Psi}^h_{p134} &= \bar{\Psi}^h_{p234} = -S_{p2}^T + \tau \bar{X}^h_{p34} \\ \bar{\Phi}^{11}_{p1} &= PB_p + \tau\left(A_p + B_p\right)^T Z B_p + Y_{p1} \\ \bar{\Phi}^{15}_{p1} &= -\tau^2 B_p^T Z B_p + Z \end{aligned}$$

then, system (32) must be globally asymptotically stable under the state-dependent switching rule (8).

*Theorem 4:* Under Assumption 2, if there exist $n \times n$ matrices $P > 0$, $Z > 0$, $R > 0$, $n \times n$ matrices $Y_{pj}$, $T_{pj}$, $S_{pj}$, $U_{pj}$ $(p \in M, j = 1,2)$, and $4n \times 4n$ symmetric matrices $\bar{X}^h_p$ $(h = 1,2)$, such that (34), (36), (37) and

$$\bar{\Psi}'^h_p < 0, \quad h = 1,2 \tag{38}$$

where $\bar{\Psi}'^h_p = \bar{\Psi}^h_{p1}$ with $R_1 = R_2 = R$, then, system (32) is globally asymptotically stable under the state-dependent switching rule (8).

As we know, the stability of system (32) under state-dependent switching rules have been discussed in [22]. In order to directly show the comparison between our results and the proposed ones, we give some stability results under the assumption presented in [22].

*Assumption 3:* There exist positive constants $\tau$ and $\bar{\tau}$ such that

$$0 \le \tau(t) \le \tau, \quad \dot{\tau}(t) \le \bar{\tau} < 1. \tag{39}$$

*Corollary 1:* Under Assumption 3, if there exist $n \times n$ matrices $P > 0$, $Z > 0$, $R_i > 0$ $(i = 1,2)$, $n \times n$ matrices $Y_{pj}$, $T_{pj}$, $S_{pj}$, $U_{pj}$ $(p \in M, j = 1,2)$, and $4n \times 4n$ symmetric matrices $\bar{X}^h_p$ $(h = 1,2)$, such that (34), (36), (37) and

$$\tilde{\Psi}^h_p < 0, \quad h = 1,2 \tag{40}$$

$$R_2 - R_1 < 0 \tag{41}$$

where $\tilde{\Psi}^h_{pij} = \bar{\Psi}^h_{p1ij}$, then, system (32) is globally asymptotically stable under the state-dependent switching rule (8).

*Remark 3:* In [22], researchers coped with the stability of system (32) with state-dependent switching under Assumption 3. According to the proof of [22, Theorem 1], one can derive that the following two inequalities are employed

$$\begin{aligned} &\int_{t-\tau(t)}^t \dot{x}^T(s) ds B_p^T Z B_p \int_{t-\tau(t)}^t \dot{x}(s) ds \\ &\le \tau^2 \int_{t-\tau(t)}^t \dot{x}^T(s) B_p^T Z B_p \dot{x}(s) ds \end{aligned} \tag{42}$$

$$\tau \xi^T(t) X_p \xi(t) - \int_{t-\tau(t)}^t \xi^T(t) X_p \xi(t) ds \ge 0 \tag{43}$$

where $\xi(t) = \left(x^T(t), x^T(t-\tau(t))\right)^T$. Based on the proof of Theorem 1, it is easy to see that the inequality

$$\begin{aligned} &\int_{t-\tau(t)}^t \dot{x}^T(s) ds B_p^T Z B_p \int_{t-\tau(t)}^t \dot{x}(s) ds \\ &\le \tau^2 \int_{t-\tau(t)}^t \dot{x}^T(s) B_p^T Z B_p \dot{x}(s) ds \\ &\quad - 3\tau(x(t) + x(t-\tau(t)) - \vartheta)^T B_p^T Z B_p \\ &\quad \times (x(t) + x(t-\tau(t)) - \vartheta) \end{aligned} \tag{44}$$

and the equations

$$\tau(t) \tilde{\eta}^T(t) \bar{X}^1_p \tilde{\eta}(t) - \int_{t-\tau(t)}^t \tilde{\eta}^T(t) \bar{X}^1_p \tilde{\eta}(t) ds = 0 \tag{45}$$

$$(\tau - \tau(t)) \tilde{\eta}^T(t) \bar{X}^2_p \tilde{\eta}(t) - \int_{t-\tau}^{t-\tau(t)} \tilde{\eta}^T(t) \bar{X}^2_p \tilde{\eta}(t) ds = 0 \tag{46}$$

are used, where $\tilde{\eta}(t) = (x^T(t), x^T(t-\tau(t)), x^T(t-\tau), \vartheta^T)^T$. Clearly, (44), (45) and (46) are more precise than (42) and (43), respectively. This yields that Corollary 1 is less conservative than Theorem 1 of [22].

## IV. EXAMPLES

*Example 1:* Consider the switched system (1) with $m = 2$,

$$A_1 = \begin{pmatrix} -2 & \frac{1}{3} \\ \frac{1}{2} & 1 \end{pmatrix}, \quad B_1 = \begin{pmatrix} -1 & -\frac{1}{2} \\ \frac{1}{3} & \frac{1}{4} \end{pmatrix}$$

$$A_2 = \begin{pmatrix} 1 & \frac{1}{3} \\ \frac{1}{2} & -1 \end{pmatrix}, \quad B_2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & -3 \end{pmatrix}$$

$$f_1(x) = f_2(x) = (\sin x_1, \sin x_2)^T.$$

It is easy to obtain that $L_1 = L_2 = I$ and $A_p + B_p L_p$, $p \in M$, is unstable. Letting $\alpha_1 = \alpha_2 = 0.5$, we obtain the Hurwitz linear convex combination

$$F = \begin{pmatrix} -\frac{3}{4} & \frac{1}{4} \\ \frac{11}{12} & -\frac{11}{8} \end{pmatrix}.$$

Assume that $\bar{\tau} = -\tilde{\tau} = \mu$. Then, owing to Theorem 1, we can obtain the upper bound $\tau$ of $\tau(t)$ for different $\mu$ such that this system is globally asymptotically stable under the state-dependent switching law (8) (see Table I). In order to show its computational complexity, we also give the number of decision variables (Novs) in Table I, where $N_1 = 35\,mn^2 + 7\,mn + 2\,n^2 + 2\,n$.

For numerical simulation, we choose $\tau(t) = 0.06 - 0.06\sin\frac{25}{3}t$. Then, according to Theorem 1, we obtain (47). Fig. 1 shows the stable time response curves for this system for different initial conditions and the state-dependent switching rules, respectively. Obviously, one can see that, corresponding with different initial conditions, the switching instants of switching rule are not identical.

TABLE I
THE UPPER BOUND $\tau$ OF TIME DELAY AND NOVS FOR DIFFERENT $\bar{\tau} = -\tilde{\tau} = \mu$ IN EXAMPLE 1

| $\mu = \bar{\tau} = -\tilde{\tau}$ | | | | Novs |
|---|---|---|---|---|
| 0.1 | 0.2 | 0.5 | 1 | |
| 0.2449 | 0.2332 | 0.2180 | 0.1264 | $N_1$ |

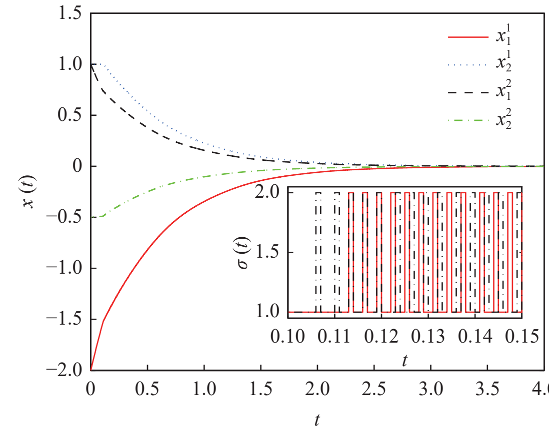


Fig. 1. The stable response curves and the corresponding switching rules for system in Example 1 with $\tau(t) = 0.06 - 0.06\sin\left(\frac{25t}{3}\right)$.

Now we validate that the results presented in [24] are not valid for this example. Let $\alpha \in [0,1]$. The linear convex combination of $A_1$ and $A_2$ can be expressed as

$$\bar{F} = \alpha A_1 + (1-\alpha)A_2 = \begin{pmatrix} 1-3\alpha & \frac{1}{3} \\ \frac{1}{2} & 2\alpha - 1 \end{pmatrix}.$$

$$\begin{aligned}
&P = \begin{pmatrix} 0.2016 & -0.2131 \\ -0.2131 & 0.2400 \end{pmatrix}, \ \Xi = \begin{pmatrix} 0.6931 & -0.7232 \\ -0.7232 & 0.7665 \end{pmatrix}, \ Z = \begin{pmatrix} 0.6811 & 0.5196 \\ 0.5196 & 1.0569 \end{pmatrix}\\
&R_1 = \begin{pmatrix} 1.0595 & 0.2836 \\ 0.2836 & 2.7501 \end{pmatrix}, \ R_2 = \begin{pmatrix} 0.0026 & -0.0005 \\ -0.0005 & 0.0035 \end{pmatrix}, \ Y_{11} = \begin{pmatrix} -0.9977 & -0.1495 \\ -0.4663 & -2.0434 \end{pmatrix}\\
&Y_{12} = \begin{pmatrix} -0.3513 & -0.0637 \\ -0.1953 & -0.6222 \end{pmatrix}, \ Y_{21} = \begin{pmatrix} -1.1707 & -0.1614 \\ 0.1558 & -2.5904 \end{pmatrix}, \ Y_{22} = \begin{pmatrix} -0.2154 & -0.0327 \\ -0.0120 & -0.1604 \end{pmatrix}\\
&T_{11} = \begin{pmatrix} -0.0839 & -0.0544 \\ -0.0782 & -1.0210 \end{pmatrix}, \ T_{12} = \begin{pmatrix} 0.0632 & 0.0816 \\ 0.1567 & 0.2915 \end{pmatrix}, \ T_{21} = \begin{pmatrix} 0.3426 & -0.1570 \\ -0.6227 & 0.0169 \end{pmatrix}\\
&T_{22} = \begin{pmatrix} 0.1017 & -0.0483 \\ 0.2691 & 0.9222 \end{pmatrix}, \ U_{11} = \begin{pmatrix} 0.0106 & -0.0546 \\ 0.0690 & -0.2953 \end{pmatrix}, \ U_{12} = \begin{pmatrix} 0.5506 & 0.1247 \\ 0.0940 & 0.5850 \end{pmatrix}\\
&U_{21} = \begin{pmatrix} 0.2222 & 0.1528 \\ 0.0835 & -0.1347 \end{pmatrix}, \ U_{22} = \begin{pmatrix} 0.6222 & 0.1930 \\ 0.0482 & 0.7227 \end{pmatrix}, \ S_{11} = \begin{pmatrix} -0.0570 & -0.0437 \\ -0.0267 & -0.0218 \end{pmatrix}\\
&S_{12} = \begin{pmatrix} -0.0747 & -0.0550 \\ -0.0350 & -0.0275 \end{pmatrix}, \ S_{21} = \begin{pmatrix} -0.0144 & -0.0108 \\ -0.0100 & -0.0075 \end{pmatrix}, \ S_{22} = \begin{pmatrix} -0.0107 & 0.1440 \\ -0.0922 & -0.5975 \end{pmatrix}\\
&H_{11} = \begin{pmatrix} -0.0861 & 0.1157 \\ -0.1010 & 0.1525 \end{pmatrix}, \ H_{12} = \begin{pmatrix} -0.0659 & -0.0915 \\ 0.1477 & 0.0795 \end{pmatrix}, \ H_{21} = \begin{pmatrix} -0.3829 & 0.2242 \\ 0.1076 & -0.7291 \end{pmatrix}\\
&H_{22} = \begin{pmatrix} -0.2494 & -0.3178 \\ 0.0525 & -0.1362 \end{pmatrix}, \ N_{11} = \mathrm{diag}\{0.2146, 0.3116\}, N_{12} = \mathrm{diag}\{0.8172, 0.4280\}\\
&N_{21} = N_{22} = \mathrm{diag}\{1.0536, 1.3519\}.
\end{aligned} \tag{47}$$

In order to ensure that $\bar{F}$ is Hurwitz, the following inequality must be satisfied

$$(1-3\alpha)(2\alpha-1)-\frac{25}{144}>0.$$

Namely,

$$6\alpha^2-5\alpha+\frac{169}{144}<0. \quad (48)$$

Let $y(\alpha)=6\alpha^2-5\alpha+169/144$. According to the extreme value theorem, it is the case that $y(\alpha)\geq 19/144$ for any $\alpha\in[0,1]$, which yields that (48) is not satisfied. This shows that all the linear convex combinations of $A_1$ and $A_2$ are not Hurwitz. Therefore, one cannot design the state-dependent switching rule to stabilize this example under the results proposed in [24].

*Example 2:* Consider the switched system (32) with $m=2$ and

$$A_1=\begin{pmatrix}-2 & 2\\ -20 & -2\end{pmatrix},\ B_1=\begin{pmatrix}-1 & -7\\ 23 & 6\end{pmatrix}$$

$$A_2=\begin{pmatrix}-2 & 10\\ -4 & -2\end{pmatrix},\quad B_2=\begin{pmatrix}4 & -5\\ 1 & -8\end{pmatrix}.$$

Similar to [22], [23], by choosing $\alpha_1=0.6$, $\alpha_2=0.4$, we have

$$F=\begin{pmatrix}-1 & 1\\ -0.6 & -1.6\end{pmatrix}.$$

By letting $\bar{\tau}=-\tilde{\tau}=\mu$, according to Theorem 3, we can obtain the upper bound $\tau$ of time delay for different $\mu$, which is shown in Table II. Moreover, the Novs of Theorem 3 is also given in Table II, where $N_2=24mn^2+4mn+2n^2+2n$. For numerical simulation, we choose $\tau(t)=0.009-0.009\sin\left(\frac{1000t}{9}\right)$. It is known that $\tau=0.018$ and $\bar{\tau}=-\tilde{\tau}=1$. Based on Theorem 3, one can conclude that

TABLE II
THE UPPER BOUND $\tau$ OF TIME DELAY AND NOVS FOR DIFFERENT $\bar{\tau}=-\tilde{\tau}=\mu$ IN EXAMPLE 2

| $\mu=\bar{\tau}=-\tilde{\tau}$ | | | | Novs |
|---|---|---|---|---|
| 0.1 | 0.2 | 0.5 | 1 | |
| 0.0457 | 0.0430 | 0.0321 | 0.0183 | $N_2$ |

$$P=\begin{pmatrix}0.0644 & -0.0097\\ -0.0097 & 0.0339\end{pmatrix},\ \Xi=\begin{pmatrix}0.1404 & 0.0187\\ 0.0187 & 0.0889\end{pmatrix}.$$

The stable time response curves for initial conditions $\phi(s)=(-3,2)^T$ and $\phi(s)=(2,-1)^T$ are shown in Fig. 2 under the corresponding state-dependent switching rules (8), which are shown in the subfigure of Fig. 2. This indicates the effectiveness of the proposed control strategy. It is worth noting that $\dot{\tau}(t)<1$ is not satisfied; therefore, the results presented in [22] are not available for this case.

In order to show the superiority of our results, we can draw some comparisons with existing results. Table III gives the upper bound $\tau$ of time delay for different $\bar{\tau}$ and the Novs under the identical assumptions. It can be seen that the upper bound $\tau$ derived by our results is more precise than those obtained by the method proposed in [22] at the cost of greater computation. For example, for $\tau(t)=0.022-0.022\sin\left(\frac{50t}{11}\right)$, [22, Theorem 1] fails because $\tau=0.044$. However, owing to $\bar{\tau}=0.1$ and $\tau=0.044$, by solving the LMI presented in Corollary 1 we obtain

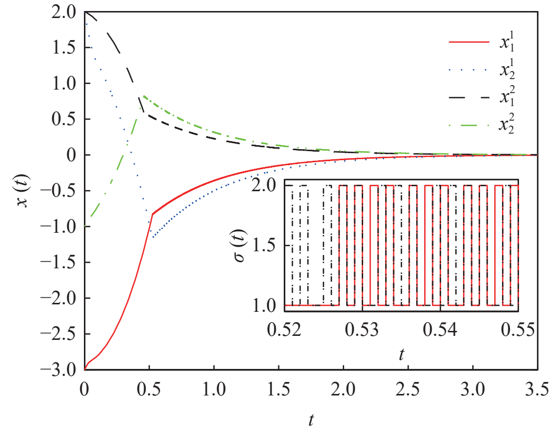


Fig. 2. The stable response curves and the corresponding switching rules for system in Example 2 with $\tau(t)=0.009-0.009\sin\left(\frac{1000t}{9}\right)$.

$$P=\begin{pmatrix}1.0455 & 0.0001\\ 0.0001 & 0.2275\end{pmatrix},\ \Xi=\begin{pmatrix}2.0910 & 0.9092\\ 0.9092 & 0.7280\end{pmatrix}$$

which yields that this switched system is globally asymptotically stable under the state-dependent switching rule (8). The stable time response curves and the corresponding switching rules are plotted in Fig. 3 for different initial conditions.

One should note that when $\bar{\tau}$ is unknown, [23, Corollary 1] leads to $\tau=0.0259$, which is more effective than those calculated by our result. This is because the approximation of time-varying delay is employed. That is to say, $x(t-0.5\tau)$ is used to approximate $x(t-\tau(t))$. However, under the theoretical analysis presented in [23], $\tau(t)$ is not used, which implies that the stability results depending on $\bar{\tau}$ cannot be established. Therefore, when $\bar{\tau}$ is known, the results presented in [23] may be more conservative. Obviously, our results are more effective when $\bar{\tau}$ is known.

*Example 3:* In order to show the application of the proposed results, we consider the water quality system, which is expressed as [31], [32]

$$\begin{pmatrix}\dot{\rho}(t)\\ \dot{\varsigma}(t)\end{pmatrix}=A_\sigma\begin{pmatrix}\rho(t)\\ \varsigma(t)\end{pmatrix}+B_\sigma\begin{pmatrix}\rho(t-\tau(t))\\ \varsigma(t-\tau(t))\end{pmatrix} \quad (49)$$

where $\rho(t)$ and $\varsigma(t)$ are the concentrations per unit volume of biological oxygen demand and dissolved oxygen, respectively, at time $t$. When $m=2$, $\tau(t)=0.05+0.05\sin 2t$,

$$A_1=\begin{pmatrix}-1 & 3\\ 0.3 & 1\end{pmatrix},\ B_1=\begin{pmatrix}-1.1 & 2.9\\ 0.3 & 0.6\end{pmatrix}$$

$$A_2=\begin{pmatrix}0.8 & -2\\ -0.5 & -1\end{pmatrix},\ B_2=\begin{pmatrix}0.2 & -0.8\\ -0.5 & -1.1\end{pmatrix}$$

the two subsystems of (49) are unstable, which is shown in

TABLE III
THE UPPER BOUND $\tau$ OF TIME DELAY AND NOVS FOR DIFFERENT $\bar{\tau}$

| Criterion | $\bar{\tau}$ | | | | Novs |
|---|---|---|---|---|---|
| | 0 | 0.1 | 0.5 | $\bar{\tau}$ is unknown | |
| Sun [22] | 0.0202 | 0.0179 | 0.0176 | 0.0176 | $4mn^2+mn+1.5n^2+1.5n$ |
| Corollary 1 | 0.0478 | 0.0457 | 0.0321 | 0.0236 | $24mn^2+4mn+2n^2+2n$ |

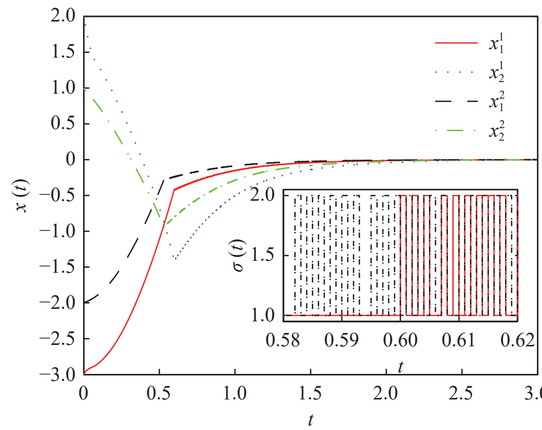


Fig. 3. The stable response curves for the system in Example 2 with $\tau(t)=0.022-0.022\sin\left(\frac{50t}{11}\right)$.

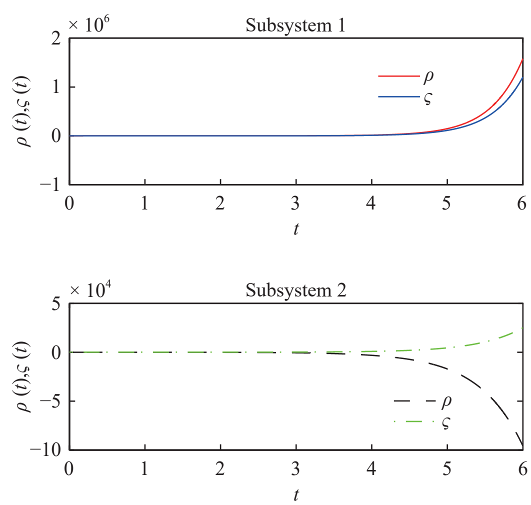


Fig. 4. The unstable response curves for the subsystems of (49).

Fig. 4. However, according to Theorem 3, one can obtain the feasible solution

$$P=\begin{pmatrix}10.0202 & -9.0651\\ -9.0651 & 38.3337\end{pmatrix},\ \Xi=\begin{pmatrix}7.3962 & -15.1168\\ -15.1168 & 47.2688\end{pmatrix}.$$

This indicates that the water quality system is asymptotically stable under the state-dependent switching rule (8). Fig. 5 and its subfigure show the stable response curves of the water quality system (49) and the corresponding state-dependent switching rules (8), respectively. This example indicates that the state-dependent switching strategy can effectively control the water quality.

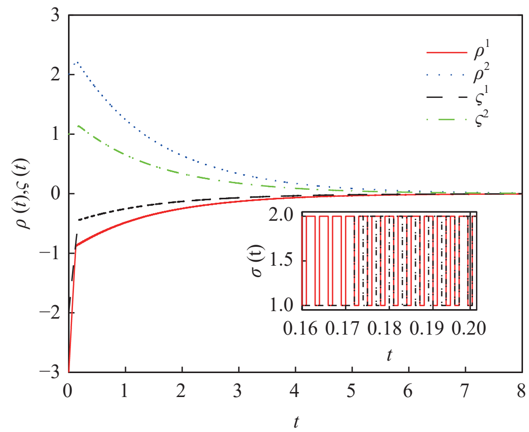


Fig. 5. The stable response curves of the water quality system (49).

## V. CONCLUSIONS

This paper has investigated the stability of switched systems with time-varying delay and all unstable subsystems under the state-dependent switching rule. Due to the Hurwitz linear convex combination, a state-dependent switching rule is designed. Based on Wirtinger integral inequality and Leibniz-Newton formula, the stability results for nonlinear delayed switched systems whose nonlinear terms satisfy Lipschitz condition and linear delayed switched systems have been derived, respectively. Through the numerical simulation, it has shown that the proposed results are more flexible than those presented in [24], and the restriction on time delay of the proposed results is weaker than those of the stability results in [22].

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**Chao Liu** received the B.S degree and the Ph.D. degree in computer science and technology from Chongqing University in 2006 and 2013, respectively. He has been an Associate Professor at the School of Computer Science and Engineering, Chongqing University of Technology. His research interests include impulsive systems, switched systems, and neural networks.

**Zheng Yang** received the M.A. degree and the Ph.D. degree in computer science and technology from Chongqing University in 2009 and in Ruhr-Universität Bochum in 2013. Now he is an Associate Professor at the School of Computer Science and Engineering, Chongqing University of Technology. His research interests include nonlinear systems and network security.

**Xiaoyang Liu** received the B.S. degree in electronic information science and technology form Huainan Normal University, China, in 2006, and the M.S. degree in communication and information systems from Kunming University of Science and Technology, in 2010, and Ph.D. degree in information and communication engineering form Northwestern Polytechnical University in 2013. Now he is an Associate Professor at the School of Computer Science and Engineering, Chongqing University of Technology. His research interests include signal processing, control systems, and wireless communication.

**Xianying Huang** received the M.A. degree in computer science and technology from Chongqing University in 2002. Now she is a Professor at the School of Computer Science and Engineering, Chongqing University of Technology. Her research interests include delayed systems and machine learning.