Stability of Nonlinear Differential-Algebraic Systems Via Additive Identity

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***Abstract*—The stability analysis for nonlinear differential-algebraic systems is addressed using tools from classical control theory. Sufficient stability conditions relying on matrix inequalities are established via Lyapunov Direct Method. In addition, a novel interpretation of differential-algebraic systems as feedback interconnection of a purely differential system and an algebraic system allows reducing the stability analysis to a small-gain-like condition. The study of stability properties for constrained mechanical systems, for a class of Lipschitz differential-algebraic systems and for an academic example is used to illustrate the theory.**

***Index Terms*—Differential-algebraic systems, Lyapunov method, small-gain theorem, stability analysis.**

I. Introduction

DIFFERENTIAL-algebraic systems (also known as DAE systems, descriptor systems or singular systems) provide a generalization of the classical state-space framework which allows a simpler characterization of many physical phenomena, such as conservation of mass and flow, topological and environmental constraints and/or thermodynamical relations. The range of engineering applications which can be naturally described by DAE systems includes mechanical systems [1], robot manipulators with constrained end-effector [2], chemical processes [3], electrical networks with nonlinear elements [4], as well as models arising in social and economic sciences [5]. Recently, modern simulation tools based on object-oriented languages [6] have considerably spread the use of DAE systems for the modelling of physical systems to such an extent that the interest in studying the problems of numerical integration and control of dynamical systems in the DAE formulation has grown rapidly (see e.g., [7]–[10]).

Classical approaches to the stability analysis of DAE systems are based on index[1] or coordinates reduction techniques which, by means of multiple time differentiations and algebraic manipulations, reveal the underlying differential representation of the system to which classical results can be applied. One of the first systematic contributions in this area has been provided by [12], in which state-space equivalent forms for linear time-invariant DAE systems have been presented. In [13] a state space realization for index-3 nonlinear DAE systems has been derived and the feedback stabilization problem has been solved by means of linearization techniques. A similar approach has been adopted in [14], i.e., a state space realization and an output feedback stabilization methodology have been developed for nonlinear index-2 DAE systems. However, the multiple differentiation of the algebraic equation and the need for further algebraic manipulations required by these methods poorly suit the scale of many engineering problems. This is the case, for instance, in power system models and switching networks [15]. In addition, nonlinearities in the model equations and model uncertainties may prevent the applicability of coordinates reduction methods [16]. Therefore, an approach to the problems of stability analysis and control directly in the DAE formulation is needed.

Another approach to the stability analysis of DAE systems consists in extending tools from classical and modern control theory to this class of systems. Lyapunov stability theory has been extended to nonlinear DAE systems in [17], in which the robust control problem for DAE systems with uncertainties has also been discussed. Other examples of application of Lyapunov stability theory can be found in [18], in which DAE systems with delays are considered, in [19], in which estimations of the domain of attraction of equilibria of DAE systems are provided, and in [10], in which the task of finding a Lyapunov function for a DAE system is transformed into an optimization problem subject to algebraic constraints, thus yielding sufficient stability conditions.

A further line of research focuses on the study of the $\mathcal{H}_\infty$ control problem for DAE systems. In [20], sufficient conditions for the existence of solutions to the $\mathcal{H}_\infty$ control problem have been derived for linear DAE systems, while [21] presents similar results using linear matrix inequalities. In [22], the authors have derived necessary and sufficient conditions for the existence of a controller solving the $\mathcal{H}_\infty$ control problem for a general class of nonlinear DAE systems, considering both state and output feedback controllers, in terms of Hamilton-Jacobi inequalities. Finally, $\mathcal{H}_\infty$ control and robust adaptive control for a class of nonlinear DAE systems with external disturbances and parametric uncertainties have been studied in [23].

We study stability properties for DAE systems using tools

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[1]Loosely speaking, the index indicates the number of time-differentiations required to reduce a DAE system to a system of ordinary differential equations, see [11] for a precise definition.

such as Lyapunov Direct Method and small-gain-like arguments. We consider primarily DAE systems with index one and in semi-explicit[2] form described by equations of the form

$$\begin{aligned}\dot{x} &= f(x,w)\\ 0 &= h(x,w)\end{aligned} \tag{1}$$

where $x(t)\in\mathbb{R}^n$ and $w(t)\in\mathbb{R}^m$ denote the differential and the algebraic variables of the system at time t, respectively, and $f:\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^n$ and[3] $h:\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^m$ are smooth mappings[4]. Note that the formulation in (1) is general and includes, for instance, constrained mechanical systems (see Section II-A), electrical networks [25] and pneumatic actuators [26]. The solution[5] of a DAE system can exhibit all the behaviour of ordinary differential equations plus additional behaviour such as bifurcation of solutions. In the remaining of the paper, we assume that the solution exists and is uniquely defined on the interval of interest. In addition, we also assume that the origin is an equilibrium point. Before undertaking the stability analysis a clarification on the nature of such a problem is required. As outlined in [28], any solution of a DAE system with index ν must stay on the solution manifold

$$\mathcal{M}=\left\{(x,w):\frac{d^kh(x,w)}{dt^k}=0,\ k=0,\ldots,\nu-1\right\} \tag{2}$$

hence, it must satisfy the algebraic equation in (1) for all time. Note that, in general, the solution manifold is invariant but not attractive. Hence, any perturbation of the state may cause the solution to move away from the manifold. In conclusion, in the definition of stability used in this paper only perturbations of the solutions corresponding to consistent initial conditions are considered, i.e., only perturbations such that the initial condition remains on the manifold $\mathcal{M}$. In particular, let $B_\delta\subseteq\mathbb{R}^n\times\mathbb{R}^m$ be the set of all the consistent initial conditions contained in a ball of radius δ around the origin, namely,

$$B_\delta:=\{(x,w)\in\mathcal{M}:\|(x,w)-(0,0)\|<\delta\}.$$

We make use of stability concepts specified by the definitions below.

Definition 1: The origin of the DAE system (1) is said to be stable if for every $\varepsilon>0$ and any $t_0\in\mathbb{R}_{\geq 0}$ there exists a $\delta=\delta(\varepsilon,t_0)>0$ such that if $(x(t_0),w(t_0))\in B_\delta$, then $\|(x(t),w(t))-(0,0)\|<\varepsilon$ for all $t\geq t_0$.

Definition 2: The origin of the DAE system (1) is said to be locally asymptotically stable if it is stable and, in addition, there exists a $\delta_0(t_0)>0$ with the property that if $(x(t_0),w(t_0))\in B_{\delta_0}$, then $\lim_{t\to\infty}\|(x(t),w(t))-(0,0)\|=0$.

The approach proposed is based on the simple observation that any linear combination (even by means of nonlinear weights) of the algebraic equations in (2) must be equal to zero, that is for any mapping

$$\Gamma(x,w)=[\Gamma_0(x,w)\ \ldots\ \Gamma_{\nu-1}(x,w)]$$

in which $\Gamma_i:\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^{p\times m}$, for $i=0,\ldots,\nu-1$ and $p>0$, the equation

$$\sum_{k=0}^{\nu-1}\Gamma_k(x,w)\frac{d^kh(x,w)}{dt^k}=0 \tag{3}$$

holds for all $(x,w)\in\mathcal{M}$. A direct consequence of this observation is the fact that the left hand side of (3) can be added to the first of equations (1) without affecting the solutions. On the other hand, a suitable selection of the mapping Γ may yield a formulation which better fits the study of the stability properties of an equilibrium point. Exploiting this simple idea families of methods to establish the stability of DAE systems have been derived as follows. We call these families *stabilization methods via additive identity* because the elements $\frac{d^kh(x,w)}{dt^k}=0$, for $k=0,\ldots,\nu-1$, are additive identities of (1).

Differently from existing approaches, the proposed method allows establishing stability properties for a class of DAE systems without the explicit calculation of the reduced unconstrained system. On the one hand, this approach avoids further time differentiation and algebraic manipulations required by classical approaches to reduce the index to zero. On the other hand, there is a broad class of processes in which a reduction to ordinary differential equations may be prevented by model uncertainties or by nonlinearities, see for instance [16]. Therefore, an approach to the problem of stability analysis directly in the differential-algebraic formulation is needed.

A. Contributions

The first contribution of the paper is the application of Lyapunov Direct Method to establish stability properties of DAE systems. In particular, sufficient conditions in the form of state-dependant matrix inequalities are given (Section II). The second contribution of the paper consists in reducing the problem of stability analysis for DAE systems to a stability analysis problem for a purely differential system (Section III). In particular, by interpreting the DAE system as the feedback interconnection of a differential system and an algebraic system, sufficient conditions for stability are derived by means of a small-gain-like condition. The application of the method to linear DAE systems is also discussed. The third contribution of the paper is the application of the aforementioned results to two widely studied classes of systems (Section IV): constrained mechanical systems, for which it is shown that the property of local asymptotic stability of the zero equilibrium can be inferred by a detectability condition; and a class of Lipschitz DAE systems, for which stability conditions reduce to the feasibility of a linear matrix inequality. Finally in Section V we report our conclusions.

Preliminary results have been published in [25], [29], and

[2]See [24] for detail on the transformation of fully-implicit DAE systems to the semi-explicit form and vice versa.

[3]The dependence of h on the algebraic variable w is explicit only when the index is one. However, with some abuse of notation, we use $h(x,w)$ for any index.

[4]Throughout the paper all mappings are assumed to be smooth.

[5]We consider the notion of "classical" solution as formulated in [27].

[30]. The additional contributions of the present paper are as follows: the results are presented in a more organized way with formal proofs; the result in [25] obtained via Lyapunov Direct Method has been generalized and a new numerical example has been provided; the application of the small-gain theorem for DAE systems, which has been initially presented in [25], has been reformulated in the more general framework in which the output mappings of the nonlinear and algebraic subsystems are design parameters; in addition, the concept of $\mathcal{L}_2$-gain is in the spirit of the notion introduced in [31]; the results on the class of constrained mechanical systems (presented in [29] for the linear case) have been extended to the nonlinear case; the application of the method to a class of Lipschitz DAE systems, which has been presented in [30] for the stabilization problem, has been revisited in the setting of stability analysis; moreover, the feedback decomposition (i.e., Proposition 1 in [30]) has been revisited to weaken some restrictions on the choice of the parameter ϵ_2.

B. Notations

We use standard notation. Given a matrix A, the symbols A^T and A^{-T} represent the transpose of A and A^{-1} (provided the inverse exists), respectively. Let $A \in \mathbb{R}^{n\times n}$, the symbol $\det(A)$ indicates the determinant of the matrix A. The symbols $\mathbb{R}_{>0}$ and $\mathbb{R}_{\geq 0}$ indicate, respectively, the set of strictly positive real numbers and the set of non-negative real numbers. Given a mapping $f : \mathbb{R}^n \to \mathbb{R}^n$, we use the symbols $\frac{\partial f}{\partial x}$ or f_x to denote the Jacobian of f with respect to the vector $x \in \mathbb{R}^n$. Given a manifold $\mathcal{M}$, the symbol $f|_{\mathcal{M}}$ indicates the restriction of f to $\mathcal{M}$. Given a matrix A the symbols $\underline{\sigma}(A)$ and $\overline{\sigma}(A)$ represent the smallest and the largest singular value, respectively, of the matrix A, while $\sigma(A)$ denotes the spectrum of A. The symbol $\|A\|_2$ represents the induced 2-norm of the matrix A, that is, $\|A\|_2 = \overline{\sigma}(A)$. Given the set $\mathcal{H}$ of Hamiltonian matrices of order $2n$, we define the set $\mathrm{dom(Ric)} := \{\mathrm{H} \in \mathcal{H} : \mathrm{Re}(\lambda_i) \neq 0, \forall \lambda_i \in \sigma(H)\}$, see [32]. Given a complex number s, we define $\mathbb{C}_{<0} = \{s \in \mathbb{C} : \mathrm{Re}(s) < 0\}$.

II. Stability by Lyapunov Direct Method

In this section we provide a family of stability conditions for nonlinear DAE systems using Lyapunov Direct Method. To begin with note that, under the stated smoothness assumptions, system (1) can always be rewritten as[6]

$$\begin{aligned} \dot{x} &= A_{11}(x,w)x + A_{12}(x,w)w \\ 0 &= A_{21}(x,w)x + A_{22}(x,w)w \end{aligned} \tag{4}$$

where[7]

$$\begin{aligned} &A_{11} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{n\times n}, \quad A_{12} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{n\times m} \\ &A_{21} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{m\times n}, \quad A_{22} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{m\times m}. \end{aligned}$$

For simplicity of notation the explicit dependence of the mappings A_{11}, A_{12}, A_{21}, and A_{22} from the state variables is often omitted.

The following result provides a family of stability conditions which is then exploited in the remainder of the section.

Theorem 1: Consider the DAE system (4). Let $O \subseteq \mathbb{R}^n \times \mathbb{R}^m$ be an open set which contains the origin and assume that the mapping A_{22} is invertible for all $(x,w) \in O$. Suppose there exist a constant matrix $P \in \mathbb{R}^{n\times n}$, such that $P = P^T > 0$, and mappings $M_1 : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{n\times m}$ and $M_2 : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{m\times m}$ such that

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{bmatrix} < 0 \tag{5}$$

for all $(x,w) \in O$, where

$$\begin{aligned} S_{11} &= A_{11}^T P + PA_{11} + M_1 A_{21} + A_{21}^T M_1^T \\ S_{12} &= PA_{12} + M_1 A_{22} + A_{21}^T M_2^T \\ S_{22} &= M_2 A_{22} + A_{22}^T M_2^T. \end{aligned}$$

Then the origin of the DAE system (4) is a locally asymptotically stable equilibrium point.

Proof: Consider the Lyapunov function candidate

$$V(x) = x^T P x. \tag{6}$$

Differentiating (6) with respect to time along the trajectories of the system yields

$$\dot{V} = x^T(A_{11}^T P + PA_{11})x + x^T PA_{12} w + w^T A_{12}^T P x. \tag{7}$$

The basic idea behind the next calculation is based on the use of (3), with $\nu = 1$ and $\Gamma = x^T M_1 + w^T M_2$, to rewrite (7) as

$$\begin{aligned} \dot{V} = {} & x^T(A_{11}^T P + PA_{11})x + x^T PA_{12} w + w^T A_{12}^T P x \\ & + (x^T M_1 + w^T M_2)(A_{21} x + A_{22} w) \\ & + (A_{21} x + A_{22} w)^T (x^T M_1 + w^T M_2)^T. \end{aligned} \tag{8}$$

Rearranging all the terms in (8) yields

$$\dot{V} = [x^T w^T] S \begin{bmatrix} x \\ w \end{bmatrix}.$$

Note, in addition, that since A_{22} is invertible,

$$\|A_{22}^{-1}(x,w)\| \leq M < +\infty.$$

Thus, $V > 0$ and $\dot{V} < 0$ for all $(x,w) \in O \setminus (0,0)$, implies that x is bounded and such that

$$\lim_{t\to\infty} \|x(t)\| = 0.$$

As a consequence

$$\lim_{t\to\infty} \|w(t)\| \leq \lim_{t\to\infty} M \|A_{21}(x(t), w(t))\| \|x(t)\| = 0.$$

■

Remark 1: The condition expressed by (5), implying that $\dot{V} < 0$ for all $(x,w) \in O \setminus (0,0)$ is, as a matter of fact, only valid on the solution manifold $\mathcal{M}$, since it relies on the property expressed by (3).

Note that different selections of M_1 and M_2 yield different sufficient conditions, as shown in the next propositions.

Proposition 1: Consider the DAE system (4). Let $O \subseteq \mathbb{R}^n \times \mathbb{R}^m$ be an open set which contains the origin. Assume that the mapping A_{22} is invertible for all $(x,w) \in O$ and that there exist a constant matrix $P \in \mathbb{R}^{n\times n}$, such that $P = P^T > 0$, and a mapping $R : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{m\times m}$, such that $R = R^T > 0$ and

$$X < 0 \tag{9}$$

[6]See Hadamard's Lemma [33].

[7]The matrices A_{ij}, for $i = 1, 2$ and $j = 1, 2$, are not uniquely defined.

where

$$X = \left(A_{11}^T - A_{21}^T A_{22}^{-T} A_{12}^T\right) P + P\left(A_{11} - A_{12} A_{22}^{-1} A_{21}\right) + A_{21}^T A_{22}^{-T} R A_{22}^{-1} A_{21}$$

for all $(x,w) \in O$. Then the origin of system (4) is a locally asymptotically stable equilibrium point.

Proof: The selection

$$M_1 = (-PA_{12} - A_{21}^T M_2^T) A_{22}^{-1}$$
$$M_2 = -\frac{1}{2} R A_{22}^{-1} \quad (10)$$

yields $S_{11} = X$, $S_{22} = -R$, and $S_{12} = 0$ in (5). By (9), $S < 0$ for all $(x,w) \in O$, hence the claim follows by Theorem 1. ■

Consider now the case in which one wishes to avoid computing the inverse of the mapping A_{22}.

Proposition 2: Consider the DAE system (4). Let $O \subseteq \mathbb{R}^n \times \mathbb{R}^m$ be an open set which contains the origin. Assume that there exist a constant matrix $P \in \mathbb{R}^{n\times n}$, such that $P = P^T > 0$, a mapping $R : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{m\times m}$, such that $R = R^T > 0$, and a constant $\gamma > 0$ such that

$$Y < 0 \quad (11)$$

where

$$Y = A_{11}^T P + PA_{11} + A_{21}^T R A_{21} + \frac{1}{\gamma^2} P A_{12} A_{12}^T P$$

and

$$\gamma^2 I - A_{22}^T R A_{22} < 0 \quad (12)$$

for all $(x,w) \in O$. Then the origin of system (4) is a locally asymptotically stable equilibrium point.

Proof: Similarly to the proof of Proposition 1, selecting

$$M_1 = \frac{1}{2} A_{21}^T R$$
$$M_2 = -\frac{1}{2} A_{22}^T R \quad (13)$$

yields

$$S = \begin{bmatrix} A_{11}^T P + PA_{11} + A_{21}^T R A_{21} & PA_{12} \\ A_{12}^T P & -A_{22}^T R A_{22} \end{bmatrix}.$$

Applying Young's inequality to the quadratic form

$$[x^T w^T] S \begin{bmatrix} x \\ w \end{bmatrix}$$

yields

$$[x^T\ w^T] \begin{bmatrix} A_{11}^T P + PA_{11} + A_{21}^T R A_{21} & PA_{12} \\ A_{12}^T P & -A_{22}^T R A_{22} \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} \le [x^T\ w^T] \begin{bmatrix} Y & 0 \\ 0 & \gamma^2 I - A_{22}^T R A_{22} \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix}.$$

Note also that (12) implies that A_{22} has full rank, i.e., A_{22} is invertible. By (11) and (12), $S < 0$ for all $(x,w) \in O$, hence the claim follows by Theorem 1. ■

Remark 2: For the selection of M_1 and M_2 given in (10), the condition (5) requires the inversion of the mapping A_{22}, which may be high-demanding in terms of number of computations for large-scale systems. On the other hand, such an inversion can be avoided by means of a suitable selection of the parameters M_1 and M_2 in (13). However, this may come at the cost of a reduced basin of attraction of the zero equilibrium, see the next example for a direct comparison of the two methods.

A. Application: Nonlinear Damp-Spring-Disc System

To illustrate the results developed in this section we study the stability properties of the nonlinear mechanism depicted in Fig. 1. The equations of motion of the system are given by

$$\dot z_1 = z_2$$
$$\dot z_2 = -\frac{k_1}{M} z_1 - \frac{k_2}{M} z_1^3 - \frac{b}{M} z_2 + \frac{\lambda}{M}$$
$$\dot z_3 = -\frac{r}{J}\lambda + \frac{1}{J} u$$
$$0 = z_2 - r z_3 \quad (14)$$

where z_1 is the distance from the rest position to the disc center, z_2 is the corresponding velocity, and z_3 is the disc angular velocity. The contact force between the disc and the surface is λ. The values of the parameters have been taken from [10]: the disc has radius $r = 2$ m, mass $M = 1$ kg and inertia $J = 4\ \mathrm{kg} \times \mathrm{m}^2$. It is assumed that the disc rolls without slipping, in absence of gravity, and it is connected to a wall by a linear damper with coefficient $b = 2\ \frac{N}{m/s}$, a linear spring with coefficient $k_1 = 1\ \frac{N}{m}$, and a nonlinear spring with coefficient $k_2 = 1\ \frac{N}{m^3}$. The dynamics of the system are subject to an external input torque u and a kinematic constraint. According to [34], the control law $u = -2brx_2$ renders the origin a locally asymptotically stable equilibrium. Replacing the stabilizing control law and applying the index reduction technique described in [35], (14) can be rewritten as the index-1 DAE system

$$\dot x_1 = x_2$$
$$\dot x_2 = -\frac{k_1}{M} x_1 - \frac{k_2}{M} x_1^3 - \frac{b}{M} x_2 + \frac{w_2}{M}$$
$$0 = x_2 - r w_1$$
$$0 = -\frac{k_2}{M} x_1^3 + \left(\frac{2r^2}{J} - \frac{1}{M}\right) b x_2 - \frac{k_1}{M} x_1 + \left(\frac{r^2}{J} + \frac{1}{M}\right) w_2 \quad (15)$$

in which $(x_1, x_2) = (z_1, z_2)$ and $(w_1, w_2) = (z_3, \lambda)$.

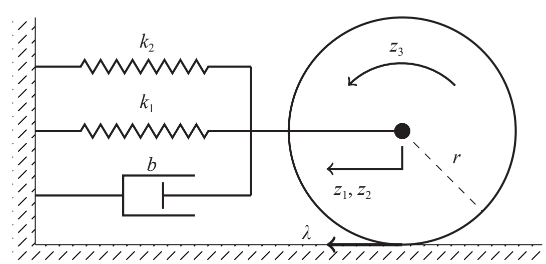

Fig. 1. A nonlinear damp-spring-disc system.

To estimate the basin of attraction of the zero equilibrium we apply the methods developed in this section. To begin with note that (15) is in the form (4) with $x = (x_1, x_2)$, $w = (w_1, w_2)$ and

$$A_{11}(x) = \begin{bmatrix} 0 & 1 \\ -\frac{k_1}{M} - \frac{k_2}{M}x_1^2 & -\frac{b}{M} \end{bmatrix}$$
$$A_{12} = \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{M} \end{bmatrix}$$
$$A_{21}(x) = \begin{bmatrix} 0 & 1 \\ -\frac{k_1}{M} - \frac{k_2}{M}x_1^2 & \left(\frac{2r^2}{J} - \frac{1}{M}\right)b \end{bmatrix}$$
$$A_{22} = \begin{bmatrix} -r & 0 \\ 0 & \frac{r^2}{J} + \frac{1}{M} \end{bmatrix}.$$

The mappings above contain only one non-constant term, i.e., x_1^2. Consider first the (9) with

$$P = \begin{bmatrix} 66.7341 & 10.9974 \\ 10.9974 & 14.6770 \end{bmatrix} \tag{16}$$

which is positive definite, and the parameter $R = 2 \times 10^{-6} I$. Fig. 2(a) shows the real part of the largest eigenvalue of X (black-solid line) for different values of x_1. Observe that the eigenvalues of X are negative in the compact set $C = \{x : |x_1| < 4.551\}$ and thus, by Proposition 1, the origin is a locally asymptotically stable equilibrium. Alternatively, consider the (11) with

$$P = \begin{bmatrix} 1.6993 & 0.2822 \\ 0.2822 & 0.7352 \end{bmatrix} \tag{17}$$

which is positive definite, and the parameters $R = 0.3I$ and $\gamma = 1.0786$. The real part of the largest eigenvalue of Y is shown in Fig. 2(a) (red-dashed line) for different values of x_1, while the eigenvalues of the left-hand side of (12) are

$$\begin{bmatrix} \bar{\lambda}_1 \\ \bar{\lambda}_2 \end{bmatrix} = \begin{bmatrix} -0.0366 \\ -0.0366 \end{bmatrix}.$$

In this case, the eigenvalues of Y are negative in the compact set $\mathcal{D} = \{x : |x_1| \leq 0.429\}$ and thus, by Proposition 2, the origin is a locally asymptotically stable equilibrium point.

Simulations, for different consistent initial conditions, have been carried out using the solver for DAE systems *ode15s* of MATLAB with default absolute and relative tolerances. Fig. 2(b) displays the phase portrait of the DAE system (15). In the same figure the estimated basins of attraction are indicated. One basin (solid-black line) has been computed as the largest level line of (6) which belongs to C, with P given in (16). The other basin (red-dashed line) has been computed as the largest level line of (6) which belongs to $\mathcal{D}$, with P given in (17). Observe that, although the method in Proposition 2 avoids the calculation of the inverse of the matrix A_{22}, the estimated basin of attraction of the zero equilibrium is smaller than the one estimated by means of Proposition 1.

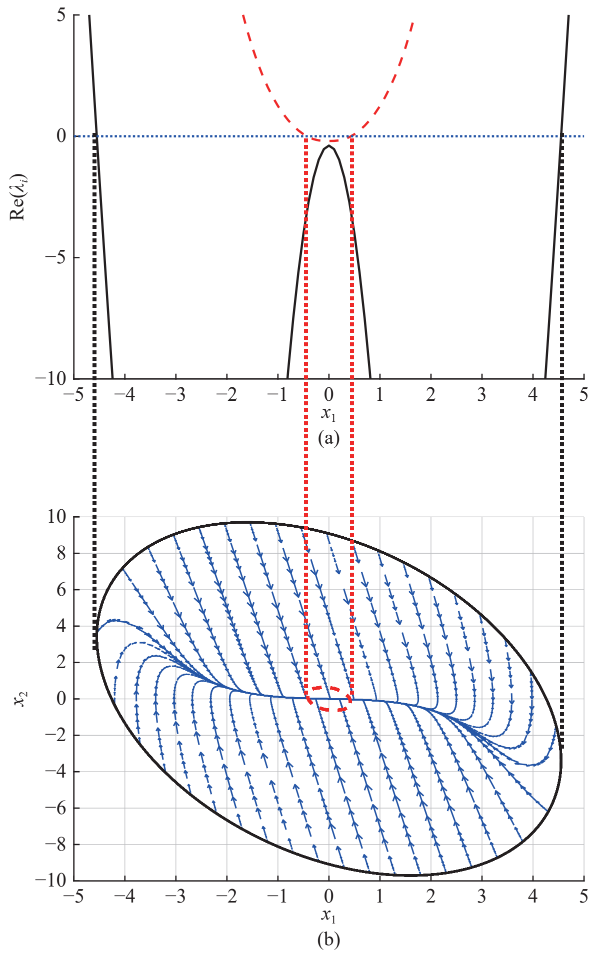

Fig. 2. Nonlinear damp-spring-disc system: (a) largest eigenvalue of X (black-solid line) and of Y (red-dashed line) for different values of x_1; (b) phase-portrait and estimated basins of attraction of the zero equilibrium: level lines of (6)–(16) (black-solid line) and (6)–(17) (red-dashed line).

III. A Small-Gain-Like Condition for the Stability of DAE Systems

In this section it is shown that the DAE system (1) can be decomposed as the feedback interconnection of a differential system and an algebraic system. In this framework, the algebraic variable assumes the role of an external disturbance and the stability analysis of the DAE system reduces to a small-gain-like condition with internal stability.

Consider the differential system (see Fig. 3)

$$\Sigma_D : \begin{cases} \dot{x} = f(x, \alpha(v)) + \Gamma(x, \alpha(v))h(x, \alpha(v)) \\ z = \beta^{-1}(x) \end{cases} \tag{18}$$

in which $x(t) \in \mathbb{R}^n$ is the state variable, $v(t) \in \mathbb{R}^m$ is the input, $z(t) \in \mathbb{R}^n$ is the output, $\beta : \mathbb{R}^n \to \mathbb{R}^n$ and $\alpha : \mathbb{R}^m \to \mathbb{R}^m$ are diffeomorphisms, and $\Gamma : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{n \times m}$ is a smooth mapping. Consider now the algebraic system

$$\Sigma_A : \begin{cases} 0 = h(\beta(u), w) \\ \tau = \alpha^{-1}(w) \end{cases} \tag{19}$$

in which $w(t) \in \mathbb{R}^m$ is the algebraic variable, $u(t) \in \mathbb{R}^n$ is the input, $\tau(t) \in \mathbb{R}^m$ is the output.

Lemma 1: The DAE system (1) and the system obtained from the feedback interconnection of Σ_D and Σ_A through the interconnection equations $u = z$ and $v = \tau$ (see Fig. 3) have the same solutions for all consistent initial conditions.

Proof: Replacing the interconnection equations $u = z$ and $v = \tau$ in systems (18)-(19) yields the DAE system

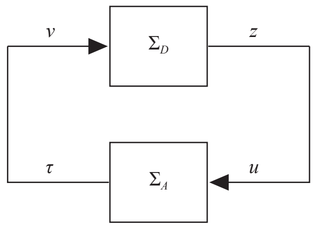

Fig. 3. Feedback decomposition of the DAE system (1).

$$\begin{aligned}\dot{x} &= f(x,w)+\Gamma(x,w)h(x,w)\\ 0 &= h(x,w).\end{aligned} \tag{20}$$

Since (3) with $\nu = 1$ holds, then the equations describing system (20) are equal to the equations describing system (1) for any mapping $\Gamma : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{n\times m}$. ■

On the basis of the feedback decomposition in Fig. 3, the stability analysis of the DAE system (1) is now addressed. Observe that, under the stated smoothness assumptions, the DAE system (1) can always be rewritten as[8]

$$\begin{aligned}\dot{x} &= f(x,w)\\ 0 &= h_0(x)+h_1(x,w)w\end{aligned} \tag{21}$$

in which $h_0(x) = h(x,0)$ and $h_1 : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{m\times m}$. Assume that

$$\operatorname{rank}\left(\frac{\partial h(x,w)}{\partial w}\right) = m \tag{22}$$

for all $(x,w) \in O \subseteq \mathbb{R}^n \times \mathbb{R}^m$.

Remark 3: Assumption (22) implies that all the components of the vector w appear explicitly in the mapping h, i.e., the DAE system (1) has index equal to one for all $(x,w) \in O$.

Under assumption (22), the following preliminary result holds.

Lemma 2: Consider the DAE system (21) and assume that (22) holds. Then there exist an open set $O_1 \times O_2 \subseteq O$ which contains the origin and a scalar $\overline{M} > 0$ such that

$$\|h_1(x,w)^{-1}\| \le \overline{M} < +\infty \tag{23}$$

for all $(x,w) \in O_1 \times O_2$.

Proof: Inequality (23) is equivalent to

$$\det(h_1(x,w)) \neq 0 \tag{24}$$

for all $(x,w) \in O_1 \times O_2$. To prove the claim, note that

$$\det\left(\left.\frac{\partial h(x,w)}{\partial w}\right|_{\substack{x=0\\w=0}}\right) = \det(h_1(0,0)) \neq 0.$$

Hence, by the standing smoothness assumptions, there exists a neighborhood of the origin, say $O_1 \times O_2$, such that (24) holds. ■

In the next result we show that the $\mathcal{L}_2$-gain of the algebraic system Σ_A is bounded by a known function of the input, for some selection of the output mapping.

Lemma 3: Consider the algebraic system (19) and the open set $O_1 \times O_2$ introduced in Lemma 2. Assume that there exists a scalar $M > 0$ such that

$$\|\alpha^{-1}(w)\| \le M\|w\| \tag{25}$$

for all $w \in O_2$. Then,

$$\frac{\|\tau\|}{\|u\|} \le M\overline{M}\frac{\|h_0(\beta(u))\|}{\|u\|} \tag{26}$$

with $\overline{M} > 0$ satisfying (23), holds.

Proof: Manipulating the first equation of system (19) written with the notation introduced in (21), one obtains

$$w = -h_1(\beta(u),w)^{-1}h_0(\beta(u))$$

which yields

$$\frac{\|\tau\|}{\|u\|} = \frac{\|\alpha^{-1}(w)\|}{\|u\|} = \frac{\|\alpha^{-1}(-h_1(\beta(u),w)^{-1}h_0(\beta(u)))\|}{\|u\|}.$$

Since (25) holds

$$\begin{aligned}\frac{\|\tau\|}{\|u\|} &\le M\frac{\|h_1(\beta(u),w)^{-1}h_0(\beta(u))\|}{\|u\|}\\ &\le M\frac{\|h_1(\beta(u),w)^{-1}\|\|h_0(\beta(u))\|}{\|u\|}\\ &\le M\overline{M}\frac{\|h_0(\beta(u))\|}{\|u\|}.\end{aligned}$$

■

Remark 4: The choice of the mapping β in equations (18) plays an important role on the bound derived in Lemma 3. For instance, let

$$\beta(u) = h_0^{-1}(u) \tag{27}$$

provided it exists. Then, replacing (27) in (26) yields

$$\frac{\|\tau\|}{\|u\|} \le M\overline{M}\frac{\|u\|}{\|u\|} = M\overline{M}$$

therefore, the system Σ_A is finite-gain stable. Similar conclusions can be obtained in the special case in which the mapping h_0 is Lipschitz. In fact, assume that

$$\|h_0(\beta(u))\| \le N\|\beta(u)\|$$

for some $N > 0$, and that

$$\|\beta(u)\| \le \overline{N}\|u\|$$

for some $\overline{N} > 0$. This yields

$$\frac{\|\tau\|}{\|u\|} \le M\overline{M}N\frac{\|\beta(u)\|}{\|u\|} \le M\overline{M}N\overline{N}$$

proving the claim.

For a smooth positive definite function $V : \mathbb{R}^n \to \mathbb{R}_{>0}$, locally bounded on O_1 and such that $V(0) = 0$ and $V_x(0) = 0$, let $\mathcal{H}(x, V_x^T, v, \mu)$ be the Hamiltonian function associated with the system Σ_D and defined as

$$\begin{aligned}\mathcal{H}(x,V_x^T,v,\mu) = V_x\,(f(x,\alpha(v)) + \Gamma(x,\alpha(v))h(x,\alpha(v)))\\ +\|\beta^{-1}(x)\|^2 - \mu(x)^2\|v\|^2\end{aligned}$$

where[9] $\mu : \mathbb{R}^n \to \mathbb{R}_{>0}$. We are now ready to provide a small-gain-like theorem for the stability analysis of the DAE system (1).

Theorem 2: Consider the DAE system (1), the differential system (18) and the algebraic system (19). Assume that condition (22) holds in an open set O which contains the

[8]Again, see Hadamard's Lemma [33].

[9]See [31] for the concept of *generalized* $\mathcal{L}_2$-gain.

origin and consider the open set $\mathcal{O}_1 \times \mathcal{O}_2 \subseteq \mathcal{O}$ introduced in Lemma 2. Suppose that there exist α^{-1}, β^{-1}, Γ, and a Lyapunov function candidate V such that:

(A_1)(25) holds for some $M > 0$ and for all $w \in \mathcal{O}_2$;

(A_2) the system Σ_D, described by (18), is detectable in $\mathcal{O}_1$;

(A_3)

$$\mathcal{H}(x, V_x^T, v, \mu) \leq 0 \tag{28}$$

holds for some μ such that

$$\mu(x) < \frac{1}{M\overline{M}} \frac{\|\beta^{-1}(x)\|}{\|h_0(x)\|} \tag{29}$$

for all $x \in \mathcal{O}_1$.

Then the origin of the DAE system (1) is a locally asymptotically stable equilibrium point.

Proof: We exploit the feedback decomposition of system (1) illustrated in Fig. 3. Observe that (A_1) and condition (23) imply (26). Replacing the interconnection equation $u = z$ in (26) yields

$$\frac{\|\tau\|}{\|u\|} \leq M\overline{M} \frac{\|h_0(x)\|}{\|\beta^{-1}(x)\|} = \eta(x)$$

from which it follows that:

$$\eta(x)^2 \|u\|^2 - \|\tau\|^2 \geq 0. \tag{30}$$

The next steps are inspired by the results in Proposition 3 of [36] and in Corollary 10.8.2 of [37]. Since (A_3) holds, then

$$\dot{V} \leq \mu(x)^2 \|v\|^2 - \|z\|^2. \tag{31}$$

Let[10] $a : \mathbb{R}^n \to \mathbb{R}_{>0}$, note that (31) can be rewritten as

$$\begin{aligned} \dot{V} \leq\ & \mu(x)^2\|v\|^2 - \|z\|^2 + a(x)\Big(\eta(x)^2\|u\|^2 - \|\tau\|^2\Big) \\ & - a(x)\Big(\eta(x)^2\|u\|^2 - \|\tau\|^2\Big). \end{aligned} \tag{32}$$

Replacing the interconnection equation $v = \tau$ and $u = z$ in the first and third term, respectively, of (32) yields

$$\begin{aligned} \dot{V} \leq\ & -a(x)\Big(\eta(x)^2\|u\|^2 - \|\tau\|^2\Big) + \Big(\mu(x)^2 - a(x)\Big)\|\tau\|^2 \\ & + \Big(a(x)\eta^2(x) - 1\Big)\|z\|^2. \end{aligned} \tag{33}$$

Since $\mu(x)\eta(x) < 1$, then there exists a mapping $a(x) > 0$ such that

$$\mu^2(x) < a(x) < \frac{1}{\eta^2(x)}$$

which satisfies

$$\mu^2(x) - a(x) < 0, \quad a(x)\eta^2(x) - 1 < 0 \tag{34}$$

for all $x \in \mathcal{O}_1$. Using (30) and (34) in (33) yields $\dot{V} \leq 0$. Hence, the state x of system Σ_D is bounded and the origin is stable. Asymptotic stability of the origin of Σ_D follows from (A_2) and from La Salle's invariance principle [38]. Finally, note that by (A_1) $w(t)$ is such that

$$\|w\| \leq \eta(x)\|u\| = M\overline{M}\|h_0(x)\|$$

for all $x \in \mathcal{O}_1$ and thus, since $h_0(0) = 0$, we obtain $\lim_{t \to +\infty} w(t) = 0$, proving the last part of the claim. ■

Note that, while the solutions of the DAE system (1) do not change by adding the algebraic equation multiplied by a mapping Γ to the differential equation, the solutions of the differential system Σ_D defined in (18) are affected by the addition of such a term. The choice of Γ plays, indeed, an important role in the calculation of the $\mathcal{L}_2$-gain of the system Σ_D as shown in the following result.

Corollary 1: Consider system (1). Assume condition (22) holds in an open set $\mathcal{O}$ which contains the origin and consider the open set $\mathcal{O}_1 \times \mathcal{O}_2 \subseteq \mathcal{O}$ introduced in Lemma 2. Suppose the following conditions hold.

(C_1) There exists Γ such that

$$\frac{\partial}{\partial w}\left[(f(x,w) + \Gamma(x,w)h(x,w)\right] = 0 \tag{35}$$

for all $(x, w) \in \mathcal{O}_1 \times \mathcal{O}_2$.

(C_2) $x = 0$ is a locally asymptotically stable equilibrium of the system $\dot{x} = f(x,0) + \Gamma(x,0)h_0(x)$.

Then the origin of the DAE system (1) is a locally asymptotically stable equilibrium point.

Proof: Consider the feedback decomposition of (1) as illustrated in Fig. 3. Condition (C_1) implies that the system Σ_D is not affected by the input v. Provided that the origin is a locally asymptotically stable equilibrium, which is the case by (C_2), then Σ_D has $\mathcal{L}_2$-gain equal to zero. In addition, (C_2) implies that the conditions (A_1) and (A_2) of Theorem 2 are satisfied for the trivial choices $\alpha^{-1}(w) = w$ and $\beta^{-1}(x) = x$, therefore the claim follows by Theorem 2. ■

The use of the corollary is illustrated in the next section.

A. The Linear Case

The application of the proposed method to linear DAE systems yields classical results [12] which can be reinterpreted in this framework. Consider the linear DAE system

$$\begin{aligned} \dot{x} &= A_{11}x + A_{12}w \\ 0 &= A_{21}x + A_{22}w \end{aligned} \tag{36}$$

where $A_{11} \in \mathbb{R}^{n\times n}$, $A_{12} \in \mathbb{R}^{n\times m}$, $A_{21} \in \mathbb{R}^{m\times n}$, and $A_{22} \in \mathbb{R}^{m\times m}$ are constant matrices. The following result holds.

Corollary 2: Consider the DAE system (36). Suppose that $\det(A_{22}) \neq 0$. The system (36) is asymptotically stable if and only if

$$\sigma(A_{11} - A_{12}A_{22}^{-1}A_{21}) \subset \mathbb{C}_{<0}. \tag{37}$$

Proof: The sufficiency can be proved by means of Corollary 1. Note that condition (C_1) of Corollary 1 holds for $\Gamma = -A_{12}A_{22}^{-1}$. In addition, condition (37) is equivalent to condition (C_2) of Corollary 1, proving the claim. To prove the necessity we use standard coordinates reduction techniques [12]. Manipulating the algebraic equation in (36) yields $w = -A_{22}^{-1}A_{21}x$. By replacing the previous expression in the differential equation of (36) yields

$$\dot{x} = (A_{11} - A_{12}A_{22}^{-1}A_{21})x$$

which is asymptotically stable only if (37) holds. ■

Remark 5: In the special case of linear DAE systems there exists a particular choice of the matrix Γ such that the $\mathcal{L}_2$-gain of system (18) from the input d to the output z is equal to

[10]In [37] a is constant.

zero, namely $\Gamma = -A_{12}A_{22}^{-1}$. Based on this observation, it is clear that, for the nonlinear case, the choice of the constant matrix

$$\Gamma = -\left.\frac{\partial f(x,w)}{\partial w}\right|_{\substack{x=0\\w=0}} \left.\left(\frac{\partial h_1(x,w)}{\partial w}\right)^{-1}\right|_{\substack{x=0\\w=0}}$$

renders the $\mathcal{L}_2$-gain arbitrarily small in a neighborhood of the origin, provided this is a locally asymptotically stable equilibrium for the resulting system and (22) holds. However, for global results this choice might not be the best.

IV. Further Results and Applications

In this section we illustrate how the methods presented in Sections III can be exploited. In particular, we show that:

1) for a class of constrained mechanical systems asymptotic stability of the zero equilibrium can be inferred by a detectability condition;

2) for a class of Lipschitz DAE systems asymptotic stability of the origin can be deduced from the feasibility of a linear matrix inequality.

A. Constrained Mechanical Systems

The equations of motion of mechanical systems subject to *holonomic* and *scleronomic* constraints can be described by DAE equations of index three, see [7]. High-index DAE systems may be reduced to index-1 DAE systems by means of index reduction techniques, see [24], [39], [41].

Consider a constrained mechanical system described by the equations

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + f(q) = G^T(q)w, \;\; g(q) = 0 \tag{38}$$

where $q(t) \in \mathbb{R}^n$ represents the displacement vector, $M : \mathbb{R}^n \to \mathbb{R}^{n\times n}$, $M(q) = M(q)^T > 0$ is the mass or inertia matrix, $C : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$, $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^m$, and $G(q) = \dfrac{\partial g(q)}{\partial q} \in \mathbb{R}^{m\times m}$ is the constraint matrix. The equation $g(q) = 0$ represents a set of geometric constraints, while $w \in \mathbb{R}^m$ characterizes m Lagrange multipliers (constraint forces). We assume that $f(0) = 0$ and

$$\text{rank}\, G(q) = m \tag{39}$$

for all $q(t) \in \mathbb{R}^n$. Systems (38) can be rewritten in state space form as

$$\begin{aligned}\dot{x} &= \begin{bmatrix} x_2 \\ -M^{-1}(x_1)(C(x_1,x_2)x_2 + f(x_1)) \end{bmatrix} \\ &\quad + \begin{bmatrix} 0 \\ M^{-1}(x_1)G^T(x_1) \end{bmatrix} w \\ 0 &= g(x_1)\end{aligned} \tag{40}$$

where $x = [x_1^T \; x_2^T]^T = [q^T \; \dot{q}^T]^T$. With similar arguments as those introduced in Corollary 1 for index-1 DAE systems, the stability analysis of the DAE system (40) reduces to the stability analysis of a differential system. To show this, consider the differential system

$$\begin{aligned}\dot{x} &= \begin{bmatrix} 0 \\ k(x_1) \end{bmatrix} + \begin{bmatrix} I \\ D(x_1,x_2) \end{bmatrix} x_2 \\ &\quad + \begin{bmatrix} \Gamma_{01}(x)g(x_1) + \Gamma_{11}(x)G(x_1)x_2 \\ \Gamma_{02}(x)g(x_1) + \Gamma_{12}(x)G(x_1)x_2 \end{bmatrix}\end{aligned} \tag{41}$$

where $\Gamma_{ij} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$, for $i = 0,1,\; j = 1,2$,

$$\begin{aligned}&D(x_1,x_2) \\ &= -M^{-1}(x_1)\Big(C(x_1,x_2) - G^T(x_1)\big(G(x_1)M^{-1}(x_1)G^T(x_1)\big)^{-1} \\ &\qquad \times \big(z(x_1,x_2) - G(x_1)M^{-1}(x_1)C(x_1,x_2)\big)\Big)\end{aligned}$$

with $z(x_1,x_2) = \dfrac{\partial (G(x_1)x_2)}{\partial x_1}$ and

$$\begin{aligned}k(x_1) = \Big(&M^{-1}(x_1)G^T(x_1)\big(G(x_1)M^{-1}(x_1)G^T(x_1)\big)^{-1}G(x_1) \\ &- I\Big)M^{-1}(x_1)f(x_1).\end{aligned}$$

The following preliminary result holds.

Lemma 4: Consider the DAE system (40) and the differential system (41). Assume there exist mappings Γ_{01}, Γ_{02}, Γ_{11}, and Γ_{12} such that the zero equilibrium of the system (41) is locally asymptotically stable. Then the origin of the DAE system (40) is a locally asymptotically stable equilibrium point.

Proof: Differentiating the constraint equation in (40) twice with respect to time we obtain two more algebraic equations that the solutions of (40) must satisfy, namely

$$\begin{aligned}0 &= G(x_1)x_2 \\ 0 &= G(x_1)\dot{x}_2 + z(x_1,x_2)x_2 \\ &= -G(x_1)M^{-1}(x_1)(C(x_1,x_2)x_2 + f(x_1)) + z(x_1,x_2)x_2 \\ &\quad + G(x_1)M^{-1}(x_1)G^T(x_1)w.\end{aligned} \tag{42}$$

Note that the solutions of the DAE system (40)–(42) are not modified if we add the algebraic equations premultiplied by some functions of the state to the differential equations, that is if we consider the system

$$\begin{aligned}\dot{x} &= \begin{bmatrix} x_2 \\ -M^{-1}(x_1)(C(x_1,x_2)x_2 + f(x_1)) \end{bmatrix} + \begin{bmatrix} 0 \\ M^{-1}(x_1)G^T(x_1) \end{bmatrix} w \\ &\quad + \begin{bmatrix} \Gamma_{01}(x) \\ \Gamma_{02}(x) \end{bmatrix} g(x_1) + \begin{bmatrix} \Gamma_{11}(x) \\ \Gamma_{12}(x) \end{bmatrix} G(x_1)x_2 \\ &\quad + \begin{bmatrix} \Gamma_{21}(x) \\ \Gamma_{22}(x) \end{bmatrix}\Big(-G(x_1)M^{-1}(x_1)(C(x_1,x_2)x_2 + f(x_1)) \\ &\quad + z(x_1,x_2)x_2 + G(x_1)M^{-1}(x_1)G^T(x_1)w\Big) \\ 0 &= g(x_1) \\ 0 &= G(x_1)x_2 \\ 0 &= -G(x_1)M^{-1}(x_1)(C(x_1,x_2)x_2 + f(x_1)) + z(x_1,x_2)x_2 \\ &\quad + G(x_1)M^{-1}(x_1)G^T(x_1)w\end{aligned}$$

where $\Gamma_{2j} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$, for $j = 1,2$. After some manipulations one obtains

$$\begin{aligned}\dot{x} = &\left(\begin{bmatrix} I \\ -M^{-1}(x_1)C(x_1,x_2) \end{bmatrix} + \begin{bmatrix} \Gamma_{11}(x)G(x_1) \\ \Gamma_{12}(x)G(x_1) \end{bmatrix}\right. \\ &\left.+\begin{bmatrix} \Gamma_{21}(x)\left(z(x_1,x_2)-G(x_1)M^{-1}(x_1)C(x_1,x_2)\right) \\ \Gamma_{22}(x)\left(z(x_1,x_2)-G(x_1)M^{-1}(x_1)C(x_1,x_2)\right) \end{bmatrix}\right)x_2 \\ &+\begin{bmatrix} 0 \\ -M^{-1}(x_1)f(x_1) \end{bmatrix} + \begin{bmatrix} -\Gamma_{21}(x)G(x_1)M^{-1}(x_1)f(x_1) \\ -\Gamma_{22}(x)G(x_1)M^{-1}(x_1)f(x_1) \end{bmatrix} \\ &+\begin{bmatrix} \Gamma_{01}(x)g(x_1) \\ \Gamma_{02}(x)g(x_1) \end{bmatrix} + \left(\begin{bmatrix} 0 \\ M^{-1}(x_1)G^T(x_1) \end{bmatrix}\right. \\ &\left.+\begin{bmatrix} \Gamma_{21}(x)G(x_1)M^{-1}(x_1)G^T(x_1) \\ \Gamma_{22}(x)G(x_1)M^{-1}(x_1)G^T(x_1) \end{bmatrix}\right)w \\ 0 = &\, g(x_1) \\ 0 = &\, G(x_1)x_2 \\ 0 = &\, -G(x_1)M^{-1}(x_1)\left(C(x_1,x_2)x_2+f(x_1)\right)+z(x_1,x_2)x_2 \\ &+G(x_1)M^{-1}(x_1)G^T(x_1)w. \end{aligned} \tag{43}$$

Observe now that the differential subsystem in (43), together with the last algebraic constraint, forms an index-1 DAE system. In fact, since (39) holds and $M(x_1) > 0$, the condition (22) is satisfied. According to Corollary 1, if there exist Γ_{21} and Γ_{22} such that the algebraic variable w is removed from the differential equation of (43), then asymptotic stability of the zero equilibrium of the DAE system (40) can be inferred by asymptotic stability of the zero equilibrium of a differential system. Selecting $\Gamma_{21} = 0$ and $\Gamma_{22}(x) = -M^{-1}(x_1)G^T(x_1)\times \left(G(x_1)M^{-1}(x_1)G^T(x_1)\right)^{-1}$ we obtain the system (41). ■

We now study the stability properties of the zero equilibrium by means of the linearized equations. We demonstrate that, if a certain detectability condition holds, the zero equilibrium of the DAE system (40) is locally asymptotically stable. To this end, let

$$A = \begin{bmatrix} 0 & I \\ A_{21} & A_{22} \end{bmatrix} \tag{44}$$

with

$$\begin{aligned} A_{21} &= \left.\frac{\partial(k(x_1)+D(x_1,x_2)x_2)}{\partial x_1}\right|_{\substack{x_1=0\\x_2=0}} = \left.\frac{\partial(k(x_1))}{\partial x_1}\right|_{x_1=0} \\ A_{22} &= \left.\frac{\partial(D(x_1,x_2)x_2)}{\partial x_2}\right|_{\substack{x_1=0\\x_2=0}} = D(0,0) \end{aligned}$$

and

$$C = \begin{bmatrix} G(0) & 0 \end{bmatrix}. \tag{45}$$

The following result holds.

Proposition 3: Consider the DAE system (40). Assume that the pair (A,C) is detectable, with A and C as given in (44) and (45), respectively. Then the origin of the DAE system (40) is a locally asymptotically stable equilibrium point.

Proof: Consider the differential system (41) and select Γ_{01}, Γ_{02}, Γ_{11}, and Γ_{12} as constant matrices of appropriate dimensions. Performing the linearization around the zero equilibrium yields the linear system

$$\dot{x} = \left(\underbrace{\begin{bmatrix} 0 & I \\ A_{21} & A_{22} \end{bmatrix}}_{A} + \underbrace{\begin{bmatrix} \Gamma_{01} & \Gamma_{11} \\ \Gamma_{02} & \Gamma_{12} \end{bmatrix}}_{L}\underbrace{\begin{bmatrix} G(0) & 0 \\ 0 & G(0) \end{bmatrix}}_{\hat{C}}\right)x \tag{46}$$

which is asymptotically stabilizable if and only if the pair $(A,\widehat{C})$ is detectable, or equivalently, if and only if

$$\operatorname{rank}\left(\begin{bmatrix} \lambda_i I - A \\ \hat{C} \end{bmatrix}\right) = 2n$$

for all $\lambda_i \in \sigma(A)$, such that $\operatorname{Re}(\lambda_i) \geq 0$. Simple operations of Gaussian elimination yields

$$\operatorname{rank}\left(\begin{bmatrix} \lambda_i I - A \\ \hat{C} \end{bmatrix}\right) = \operatorname{rank}\left(\begin{bmatrix} \lambda_i I - A \\ C \end{bmatrix}\right)$$

proving that the linear system (46) is stabilizable if and only if the pair (A,C) is detectable. Thus, the claim follows by Lemma 4. ■

B. Lipschitz DAE Systems

Lipschitz DAE systems represent a subclass of DAE systems in which the nonlinearities are slope-restricted. Although they describe a special class of systems, these are widely studied, see [9], [41]–[44] and references therein. A further extension of the method developed in Section III, according to which the DAE system is interpreted as the feedback interconnection of a differential system and an algebraic system, is to consider the nonlinear terms as additional algebraic variables. In this setting, the stability analysis of a class of DAE systems with Lipschitz nonlinearities reduces to the stability analysis of a linear system.

Consider a DAE system in semi-explicit form described by equations of the form

$$\begin{aligned} \dot{x} &= Ax + Bw + \widetilde{f}(x) + \widetilde{g}(w) \\ 0 &= Cx + Dw + \widetilde{h}(x) + \widetilde{r}(w) \end{aligned} \tag{47}$$

where $x(t) \in \mathbb{R}^n$ is the differential variable, $w(t) \in \mathbb{R}^m$ is the algebraic variable, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$, $D \in \mathbb{R}^{m\times m}$, $\widetilde{f}: \mathbb{R}^n \to \mathbb{R}^n$, $\widetilde{g}: \mathbb{R}^m \to \mathbb{R}^n$, $\widetilde{h}: \mathbb{R}^n \to \mathbb{R}^m$ and $\widetilde{r}: \mathbb{R}^m \to \mathbb{R}^m$. We also assume that (22) holds, i.e.

$$\operatorname{rank}\left(D + \frac{\partial \widetilde{r}(w)}{\partial w}\right) = m \tag{48}$$

for all $w \in \mathbb{R}^m$. In addition, the mappings $\widetilde{f}$, $\widetilde{g}$, $\widetilde{h}$, and $\widetilde{r}$ are such that the Lipschitz conditions

$$\begin{aligned} \|\widetilde{f}(x_2) - \widetilde{f}(x_1)\| &\leq k_f\|(x_2 - x_1)\| \\ \|\widetilde{h}(x_2) - \widetilde{h}(x_1)\| &\leq k_h\|(x_2 - x_1)\| \\ \|\widetilde{g}(w_2) - \widetilde{g}(w_1)\| &\leq k_g\|(w_2 - w_1)\| \\ \|\widetilde{r}(w_2) - \widetilde{r}(w_1)\| &\leq k_r\|(w_2 - w_1)\| \end{aligned} \tag{49}$$

are satisfied for all $x_i \in \mathbb{R}^n$, $w_i \in \mathbb{R}^m$, $i = 1,2$, and some $k_f > 0$, $k_h > 0$, $k_g > 0$, and $k_r > 0$. Note that, by the standing smoothness assumptions, the conditions (49) are always satisfied in any neighborhood of the origin. However, to achieve global results we assume that the conditions (49) hold globally.

In what follows we reformulate the equations describing the DAE system (47) by interpreting the nonlinear terms as additional algebraic variables. This allows exploiting the results derived in Section III in a simple way. To this end, let

$$\begin{aligned} d_1 &= \frac{1}{\epsilon_1}\left(\widetilde{f}(x)+\widetilde{g}(w)+\epsilon_2\widehat{\Gamma}d_2\right)\\ d_2 &= \frac{1}{\epsilon_2}\left(\widetilde{h}(x)+\widetilde{r}(w)\right)\end{aligned}$$

where $\epsilon_1 \in \mathbb{R}_{>0}$, $\epsilon_2 \in \mathbb{R}_{>0}$, and $\widehat{\Gamma} \in \mathbb{R}^{n\times m}$. Using these definitions, the DAE system (47) can be rewritten as

$$\begin{aligned} \dot{x} &= Ax+Bw+\epsilon_1 d_1-\epsilon_2\widehat{\Gamma}d_2\\ 0 &= Cx+Dw+\epsilon_2 d_2\\ 0 &= \frac{1}{\epsilon_1}\left(\widetilde{f}(x)+\widetilde{g}(w)+\epsilon_2\widehat{\Gamma}d_2\right)-d_1\\ 0 &= \frac{1}{\epsilon_2}\left(\widetilde{h}(x)+\widetilde{r}(w)\right)-d_2. \end{aligned}\tag{50}$$

Note now that the mappings $\widetilde{f}$, $\widetilde{g}$, $\widetilde{h}$, and $\widetilde{r}$ can be rewritten as

$$\widetilde{f}(x)=\widetilde{f}(x)\frac{x^T}{\|x\|^2}x,\qquad \widetilde{h}(x)=\widetilde{h}(x)\frac{x^T}{\|x\|^2}x$$
$$\widetilde{g}(w)=\widetilde{g}(w)\frac{w^T}{\|w\|^2}w,\qquad \widetilde{r}(w)=\widetilde{r}(w)\frac{w^T}{\|w\|^2}w.$$

Note also that, although the mappings

$$\widetilde{f}(x)\frac{x^T}{\|x\|^2},\ \ \widetilde{h}(x)\frac{x^T}{\|x\|^2},\ \ \widetilde{g}(w)\frac{w^T}{\|w\|^2},\ \ \widetilde{r}(w)\frac{w^T}{\|w\|^2}$$

may not be smooth, these are however bounded. To see this, observe that

$$\left\|\widetilde{f}(x)\frac{x^T}{\|x\|^2}\right\|_2 \le \|\widetilde{f}(x)\|\frac{\|x^T\|}{\|x\|^2} \le k_f\frac{\|x\|^2}{\|x\|^2}=k_f \tag{51}$$

in which the second inequality holds by (49). Similarly,

$$\left\|\widetilde{h}(x)\frac{x^T}{\|x\|^2}\right\|_2 \le k_h, \left\|\widetilde{g}(w)\frac{w^T}{\|w\|^2}\right\|_2 \le k_g, \left\|\widetilde{r}(w)\frac{w^T}{\|w\|^2}\right\|_2 \le k_r. \tag{52}$$

Let now $\widehat{w}=\left[w^T\ d_1^T\ d_2^T\right]^T$, then the DAE system (50) can be rewritten in compact form as

$$\begin{aligned} \dot{x} &= Ax+\left[B\ \ \epsilon_1 I\ \ -\epsilon_2\widehat{\Gamma}\right]\widehat{w}\\ 0 &= \widetilde{C}(x)x+\widetilde{D}(\widehat{w})\widehat{w} \end{aligned}\tag{53}$$

where

$$\widetilde{C}(x)=\begin{bmatrix} C\\ \frac{1}{\epsilon_1}\widetilde{f}(x)\frac{x^T}{\|x\|^2}\\ \frac{1}{\epsilon_2}\widetilde{h}(x)\frac{x^T}{\|x\|^2}\end{bmatrix}$$

$$\widetilde{D}(\hat{w})=\begin{bmatrix} D & 0 & \epsilon_2 I\\ \frac{1}{\epsilon_1}\widetilde{g}(w)\frac{w^T}{\|w\|^2} & -I & \frac{\epsilon_2}{\epsilon_1}\hat{\Gamma}\\ \frac{1}{\epsilon_2}\widetilde{r}(w)\frac{w^T}{\|w\|^2} & 0 & -I\end{bmatrix}.$$

Observe that, as a consequence of the reformulation in (53), the feedback decomposition described in Section III leads to the interconnection of a linear differential system and an algebraic system. To see this, note that the equations in (53) reduce to the equations in (1) with

$$\begin{aligned} f(x,\widehat{w}) &= Ax+\left[B\ \ \epsilon_1 I\ \ -\epsilon_2\widehat{\Gamma}\right]\widehat{w}\\ h(x,\widehat{w}) &= \widetilde{C}(x)x+\widetilde{D}(\widehat{w})\widehat{w}. \end{aligned}\tag{54}$$

Selecting the mappings

$$\alpha^{-1}(\hat{w})=\hat{w},\quad \beta^{-1}(x)=x,\quad \Gamma=\left[\begin{array}{ccc}\hat{\Gamma} & 0 & 0\end{array}\right] \tag{55}$$

and replacing (54) and (55) in (18) and (19) yields

$$\Sigma_D:\begin{cases}\dot{x}=\widetilde{A}x+\widetilde{B}v\\ z=x\end{cases} \tag{56}$$

and

$$\Sigma_A:\begin{cases}0=\widetilde{C}(u)u+\widetilde{D}(\widehat{w})\widehat{w}\\ \tau=\widehat{w}\end{cases}$$

in which

$$\begin{aligned}\widetilde{A} &= A+\widehat{\Gamma}C\\ \widetilde{B} &= \left[B+\widehat{\Gamma}D\ \ \epsilon_1 I\ \ 0\right].\end{aligned}\tag{57}$$

The application of Theorem 2 to the DAE system (47) yields the following stability result.

Proposition 4: Consider the DAE system (47). Assume that the conditions (48) and (49) hold. Let

$$\begin{aligned} \delta &= (D)\\ \rho &= \bar{\sigma}(C)\\ \gamma &= \frac{\min(\delta,1)}{\sqrt{\rho^2+\frac{k_f^2}{\epsilon_1^2}+\frac{k_h^2}{\epsilon_2^2}\left(\frac{\max(1,\delta)}{\delta-k_r}\sqrt{\frac{k_g^2}{\epsilon_1^2}+\frac{k_r^2}{\epsilon_2^2}}+1\right)}}\\ H &= \begin{bmatrix}\widetilde{A} & \frac{1}{\gamma^2}\widetilde{B}\widetilde{B}^T\\ -I & -\widetilde{A}^T\end{bmatrix}\end{aligned}\tag{58}$$

in which $\widetilde{A}$ and $\widetilde{B}$ are given in (57). Assume that there exist a matrix $\widehat{\Gamma}$ and constants $\epsilon_1>0$, $\epsilon_2>0$ such that

(a_1) $\delta > k_r$;

(a_2) $\sigma(\widetilde{A})\subset\mathbb{C}_{<0}$;

(a_3) $H\in \mathrm{dom(Ric)}$.

Then the zero equilibrium of the DAE system (47) is globally asymptotically stable.

Proof: To prove the claim we use Theorem 2. Note first that, for the selection of $\alpha^{-1}(w)$ given in (55), (25) holds with $M=1$. Moreover, the linear system Σ_D described by (56) is detectable. The rest of the proof is performed in two steps: in the first step we compute the constant $\overline{M}$ defined in (23), where $h_1(x,\widehat{w})=\widetilde{D}(\widehat{w})$; in the second step we show that the condition (A_3) of Theorem 2 holds.

Step 1: Note that the mapping $\widetilde{D}$ can be rewritten as $\widetilde{D}(\widehat{w})=\widehat{D}+U(\widehat{w})V$, where $V=\left[\begin{array}{ccc}I & 0 & 0\end{array}\right]$,

$$U(\hat{w}) = \begin{bmatrix} 0 \\ \frac{1}{\epsilon_1}\widetilde{g}(w)\frac{w^T}{\|w\|^2} \\ \frac{1}{\epsilon_2}\widetilde{r}(w)\frac{w^T}{\|w\|^2} \end{bmatrix} \quad \hat{D} = \begin{bmatrix} D & 0 & \epsilon_2 I \\ 0 & -I & \frac{\epsilon_2}{\epsilon_1}\hat{\Gamma} \\ 0 & 0 & -I \end{bmatrix}$$

the matrix $\widehat{D}$ is nonsingular by (a_1) and

$$\begin{aligned} \|V\|_2 &= 1 \\ \|U(\widehat{w})\|_2 &\leq \sqrt{\overline{\sigma}\left(\frac{1}{\epsilon_1}g(w)\frac{w^T}{\|w\|^2}\right)^2 + \overline{\sigma}\left(\frac{1}{\epsilon_2}r(w)\frac{w^T}{\|w\|^2}\right)^2} \\ &\leq \sqrt{\frac{k_g^2}{\epsilon_1^2} + \frac{k_r^2}{\epsilon_2^2}} \end{aligned} \tag{59}$$

in which the last inequality holds by (52). Note now that

$$\begin{aligned} \widetilde{D}^{-1}(\widehat{w}) &= (\widehat{D} + U(\widehat{w})V)^{-1} \\ &= \widehat{D}^{-1} - \widehat{D}^{-1}U(\widehat{w})(I + V\widehat{D}^{-1}U(\widehat{w}))^{-1}V\widehat{D}^{-1} \end{aligned} \tag{60}$$

provided that the matrix $(I + V\widehat{D}^{-1}U(\widehat{w}))$ is nonsingular, see [45]. To show that this is the case, note that

$$\hat{D}^{-1} = \begin{bmatrix} D^{-1} & 0 & \epsilon_2 D^{-1} \\ 0 & -I & -\frac{\epsilon_2}{\epsilon_1}\hat{\Gamma} \\ 0 & 0 & -I \end{bmatrix}$$

and

$$\|\widehat{D}^{-1}\|_2 \leq \max\left(1, \frac{1}{\delta}\right). \tag{61}$$

Moreover,

$$\begin{aligned} \underline{\sigma}(I + V\widehat{D}^{-1}U(\widehat{w})) &= \underline{\sigma}\left(I + D^{-1}\widetilde{r}(w)\frac{w^T}{\|w\|^2}\right) \\ &\geq \underline{\sigma}(I) - \overline{\sigma}\left(D^{-1}\widetilde{r}(w)\frac{w^T}{\|w\|^2}\right) \\ &\geq \underline{\sigma}(I) - \overline{\sigma}\left(D^{-1}\right)\overline{\sigma}\left(\widetilde{r}(w)\frac{w^T}{\|w\|^2}\right) \\ &= 1 - \frac{\overline{\sigma}\left(\widetilde{r}(w)\frac{w^T}{\|w\|^2}\right)}{\underline{\sigma}(D)} \geq 1 - \frac{k_r}{\delta} > 0 \end{aligned} \tag{62}$$

where the last inequality holds by assumption (a_1). From (62) it follows that:

$$\|(I + V\widehat{D}^{-1}U(\widehat{w}))^{-1}\|_2 = \frac{1}{\underline{\sigma}(I + V\widehat{D}^{-1}U(\widehat{w}))} \leq \frac{\delta}{\delta - k_r}. \tag{63}$$

Using properties of the norm and replacing equations (59), (61), and (63) in (60) yields

$$\begin{aligned} \|\widetilde{D}^{-1}(w)\|_2 &\leq \|\widehat{D}^{-1}\|_2 + \|\widehat{D}^{-1}\|_2\|U(\widehat{w})\|_2 \\ &\quad \times \|(I + V\widehat{D}^{-1}U(\widehat{w}))^{-1}\|_2\|V\|_2\|\widehat{D}^{-1}\|_2 \\ &\leq \max\left(\frac{1}{\delta}, 1\right)\left(\frac{\max(1,\delta)}{\delta - k_r}\sqrt{\frac{k_g^2}{\epsilon_1^2} + \frac{k_r^2}{\epsilon_2^2}} + 1\right) \\ &= \overline{M} \end{aligned}$$

concluding the first part of the proof.

Step 2: We prove that the condition (A_3) of Theorem 2 is satisfied. Observe first that

$$\begin{aligned} \|\widetilde{C}(x)\|_2 &\leq \sqrt{\rho^2 + \overline{\sigma}\left(\frac{1}{\epsilon_1}\widetilde{f}(x)\frac{x^T}{\|x\|^2}\right)^2 + \overline{\sigma}\left(\frac{1}{\epsilon_2}\widetilde{h}(x)\frac{x^T}{\|x\|^2}\right)^2} \\ &\leq \sqrt{\rho^2 + \frac{k_f^2}{\epsilon_1^2} + \frac{k_h^2}{\epsilon_2^2}}. \end{aligned}$$

Moreover, replacing (55) and $h_0(x) = h(x,0) = \widetilde{C}(x)x$ in the right hand side of (29) yields

$$\frac{1}{\overline{M}}\frac{\|x\|}{\|\widetilde{C}(x)x\|} \geq \frac{1}{\overline{M}\|\widetilde{C}(x)\|_2} \geq \gamma.$$

Consider now the linear system (56). Let

$$G(s) = (sI - \widetilde{A})^{-1}\widetilde{B}$$

in which $s \in \mathbb{C}$, be the transfer matrix of (56). Since $\widetilde{A}$ is Hurwitz by assumption (a_2), then assumption (a_3) is equivalent to $\|G\|_\infty < \gamma$, see [32]. By Theorem 5.4 in [46], the $\mathcal{L}_2$-gain μ of the linear system (56) is such that $\mu = \|G\|_\infty < \gamma$, thus (28) and (29) are satisfied. ■

V. CONCLUSIONS

In this paper, exploiting the properties of the solution manifold, we have given sufficient stability conditions for classes of DAE systems. Using Lyapunov Direct Method we have shown that local asymptotic stability of the zero equilibrium can be inferred by the feasibility of a state-dependant matrix inequality (Theorem 1), which assumes different forms depending on the selection of the design parameters. On the one hand, the proposed method allows recovering classical results based on the inversion of the algebraic equation (Proposition 1). On the other hand, such an inversion can be avoided by a suitable selection of the design parameters (Proposition 2). A numerical example motivated by the analysis of a nonlinear mechanical system has been used to validate the technique (Section II-A). We have also proposed a novel interpretation of DAE systems as feedback interconnection of a differential system and an algebraic system (Section III). In this framework, the algebraic variable assumes the role of an external disturbance and the stability analysis reduces to a small-gain-like condition (Theorem 2). The application of this method to the linear case yields classical results which can be reinterpreted in this framework as a particular case (Corollary 2). For a class of constrained mechanical systems we have shown that the problem of stability analysis can be formulated as a stabilization problem of a differential system (Lemma 4). By means of linearization techniques, we have also shown that such a problem can be solved when a detectability condition is satisfied (Proposition 3). Finally, using the feedback interpretation introduced in Section III, we have shown that the stability analysis of a class of Lipschitz DAE systems reduces to finding the feasibility of a linear matrix inequality (Proposition 4).

Motivated by the results of this paper [26] has shown that the stabilization problem for a general class of DAE systems can be addressed by means of the feedback decomposition introduced in Section III of the present study. In addition, the stabilization problem for a class of Lipschitz DAE systems has been studied in [30]. Numerical examples which exploit the results derived in Sections III and IV can be found in [25, Section 3.4], in [29, Section III.A], in [26, Section IV] and in [30, Section V]. Following the same line of research, future works will focus on the application to the observer design and to the output feedback stabilization problems for classes of DAE systems.

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