Novel Stability Criteria for Linear Time-Delay Systems Using Lyapunov-Krasovskii Functionals With A Cubic Polynomial on Time-Varying Delay

Xian-Ming Zhang, *Senior Member, IEEE*, Qing-Long Han, *Fellow, IEEE*, and Xiaohua Ge, *Member, IEEE*

***Abstract*—One of challenging issues on stability analysis of time-delay systems is how to obtain a stability criterion from a matrix-valued polynomial on a time-varying delay. The first contribution of this paper is to establish a *necessary and sufficient* condition on a matrix-valued polynomial inequality over a certain closed interval. The degree of such a matrix-valued polynomial can be an arbitrary finite positive integer. The second contribution of this paper is to introduce a *novel* Lyapunov-Krasovskii functional, which includes a cubic polynomial on a time-varying delay, in stability analysis of time-delay systems. Based on the novel Lyapunov-Krasovskii functional and the necessary and sufficient condition on matrix-valued polynomial inequalities, two stability criteria are derived for two cases of the time-varying delay. A well-studied numerical example is given to show that the proposed stability criteria are of less conservativeness than some existing ones.**

***Index Terms*—Bessel-Legendre inequality, matrix-valued polynomial inequalities, stability, time-varying delay, time-delay systems.**

I. Introduction

TIME-DELAY systems have received considerable attention in the field of control during the past two decades. On the one hand, time-delay systems have found more and more applications in industrial control. For example, networked control systems [1] including active control systems for unmanned marine vehicles and offshore platforms in network environments [2], [3] can be modeled as time-delay systems. The problem of coordination and formation control of multi-agent systems can be solved by employing time-delay system theory [4]. On the other hand, although it is well known that time-delays usually play the negative effects on a control system, their potential positive effects are often disclosed. It is proven that for networked harmonic oscillators, synchronization cannot be reached using current position data, but can be achieved using delayed position data [5]. For offshore platforms, by intentionally introducing a small time-delay into the feedback channel, oscillation amplitudes and control forces can be reduced significantly [3]. Therefore, time-delay systems are still an important topic to research both in theory and in practice.

Manuscript received January 2, 2020; revised February 12, 2020; accepted March 2, 2020. This work was supported in part by the Australian Research Council Discovery Project (Grant No. DP160103567). Recommended by Associate Editor Dianwei Qian. *(Corresponding author: Qing-Long Han.)*

Citation: X.-M. Zhang, Q.-L. Han, and X. Ge, "Novel stability criteria for linear time-delay systems using Lyapunov-Krasovskii functionals with a cubic polynomial on time-varying delay," *IEEE/CAA J. Autom. Sinica*, vol. 8, no. 1, pp. 77–85, Jan. 2021.

The authors are with the School of Software and Electrical Engineering, Swinburne University of Technology, Melbourne, VIC 3122, Australia (e-mail: xianmingzhang@swin.edu.au; qhan@swin.edu.au; xge@swin.edu.au).

Digital Object Identifier 10.1109/JAS.2020.1003111

Delay-dependent stability of time-delay systems has been studied for a long time, see, e.g. [6]–[11]. Its objective is to derive a stability condition such that the allowable delay upper bound is as large as possible. To achieve this goal, a number of notable methods (techniques) have been proposed, such as a free-weighting matrix approach, an integral inequality approach, a quadratic convex approach, a reciprocally convex combination inequality and a Wirtinger-based inequality [12], see the survey paper [13]. Since 2013, boosted by the Wirtinger-based inequality, much progress has been made on delay-dependent stability analysis of time-delay systems. One can obtain some less conservative stability criteria using a Bessel-Legendre inequality, which is an extension of the Wirtinger-based inequality. However, when a Bessel-Legendre inequality is used, the time-derivative of some certain Lyapunov-Krasovskii functional $\mathcal{V}(t)$ may be estimated as a polynomial with respect to the time-varying delay $d(t)\in[0,\bar{h}]$, that is

$$\dot{\mathcal{V}}(t)\le \xi^T(t)f(d(t))\xi(t),\ \ f(d(t))\triangleq \Sigma_{i=0}^{m}d^i(t)\mathcal{M}_i,\tag{1}$$

where $\mathcal{M}_i\ (i=0,1,\cdots,m)$ with $m\ge 2$ are symmetric real matrices irrespective of $d(t)$; $\xi(t)$ is a state-related vector, and $\bar{h}$ is a positive constant. Then a hard nut to crack is how to derive a stability criterion from the matrix inequality $f(d(t))<0$ for $d(t)\in[0,\bar{h}]$. Although some sufficient conditions on $f(d(t))<0$ with $m=2$ for $d(t)\in[0,\bar{h}]$ are presented in [14], [15], a necessary and sufficient condition on such a matrix inequality has not been reported yet, which motivates the current study.

In this paper, we first establish a necessary and sufficient condition on $f(d(t))<0$ (or $f(d(t))>0$) for $d(t)\in[0,\bar{h}]$. Then, the obtained necessary and sufficient condition is applied to stability analysis of time-delay systems. If the time-varying delay $d(t)$ is differentiable, and its derivative function is bounded from below and above, a novel Lyapunov-Krasovskii functional with a cubic matrix-valued polynomial like $d^3(t)P_3+d^2(t)P_2+d(t)P_1+P_0$ is introduced. If the time-varying delay $d(t)$ is just continuous while not differentiable, a novel Lyapunov-Krasovskii functional is also introduced. A common feature of these novel Lyapunov-Krasovskii functionals is that their time-derivatives are estimated as $\xi^T(t)\bar{f}(d(t))\xi(t)$, where $\bar{f}(d(t))$ is a *quartic* matrix-valued polynomial as $\sum_{j=0}^{4}d^j(t)\Phi_j$ with $\Phi_j\ (j=0,1,\ldots,4)$ being symmetric real matrices irrespective of $d(t)$. The obtained necessary and sufficient condition $\bar{f}(d(t)) < 0$ for $d(t) \in [0, \bar{h}]$ is utilized to deliver some less conservative stability criteria, which is demonstrated through a well-studied numerical example.

Notations: The notations throughout this paper are standard. diag$\{\cdots\}$ and col$\{\cdots\}$ denote a block-diagonal matrix and a block-column matrix (vector), respectively. The set $\mathbb{S}^n(\mathbb{S}^n_+)$ represents the set of symmetric (positive definite) matrices of $\mathbb{R}^{n\times n}$. He$\{X\} = X + X^T$.

II. Necessary and Sufficient Condition on Matrix-valued Polynomial Inequalities

Consider the following matrix-valued polynomial described by

$$F_m(s) = s^{2m}\Phi_{2m} + s^{2m-1}\Phi_{2m-1} + \cdots + \Phi_0 \tag{2}$$

where $m \geq 1$ is an integer, and $s \in [0, \bar{h}]$ with $\bar{h}$ being a constant; and $\Phi_j \in \mathbb{S}^{q\times q}$ $(j = 0, 1, 2, \cdots, 2m)$. If $\Phi_{2m} = 0$,

$$F_m(s) = s^{2m-1}\Phi_{2m-1} + s^{2m-2}\Phi_{2m-2} + \cdots + \Phi_0$$

which is an odd matrix-valued polynomial on s if $\Phi_{2m-1} \neq 0$. Thus, (2) represents both even and odd matrix-valued polynomials.

For $m = 1$, $F_1(s)$ with $\Phi_2 = 0$ is convex on $s \in [0, \bar{h}]$. However, $F_1(s)$ with $\Phi_2 \neq 0$ is *not* necessarily convex on $s \in [0, \bar{h}]$. Thus, an emerging topic in recent years is to seek conditions such that $F_1(s)$ with $\Phi_2 \neq 0$ is strictly less than zero for $\forall s \in [0, \bar{h}]$, see, e.g. [14], [15]. In the following, we present a necessary and sufficient condition on $F_m(s) < 0$ for $\forall s \in [0, \bar{h}]$. To begin with, we introduce a key lemma as follows.

Lemma 1: For given matrices $\Omega \in \mathbb{S}^p$, $H_1, H_2 \in \mathbb{R}^{k\times p}$ with $p > k$, the following statements are equivalent:

1) The inequality $\zeta^T\Omega\zeta < 0$ holds for all nonzero vectors $\zeta \in \mathbb{R}^p$ that satisfy $(H_2 - \delta H_1)\zeta = 0$ for some real scalar δ such that $|\delta| \leq 1$.

2) There exist a matrix $D \in \mathbb{S}^k_+$ and a skew-symmetric matrix $G \in \mathbb{R}^{k\times k}$ such that

$$\begin{bmatrix} H_1 \\ H_2 \end{bmatrix}^T \begin{bmatrix} D & G \\ G^T & -D \end{bmatrix} \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} + \Omega < 0. \tag{3}$$

3) There exists $X \in \mathbb{R}^{p\times k}$ such that

$$X(H_2 + H_1) + (H_2 + H_1)^T X^T + \Omega < 0 \tag{4}$$

$$X(H_2 - H_1) + (H_2 - H_1)^T X^T + \Omega < 0 \tag{5}$$

4) There exist $\varepsilon_i \in \mathbb{R}$ $(i = 1, 2)$ such that

$$\Omega - \varepsilon_1(H_2 + H_1)^T(H_2 + H_1) < 0 \tag{6}$$

$$\Omega - \varepsilon_2(H_2 - H_1)^T(H_2 - H_1) < 0 \tag{7}$$

Proof: The equivalence between 1), 2) and 3) can be found in [16]. The equivalence between 3) and 4) is derived from the Finsler Lemma [17].

Let $\zeta = \text{col}\{I, sI, \cdots, s^m I\}\zeta_0$ with $\forall\zeta_0 \in \mathbb{R}^q$ and $\zeta_0 \neq 0$. Then $f_m(s) := \zeta_0^T F_m(s)\zeta_0$ can be rewritten as

$$f_m(s) = \zeta_0^T F_m(s)\zeta_0 = \zeta^T\Omega_m\zeta, \tag{8}$$

where

$$\Omega_m = \begin{bmatrix} \Phi_0 & \frac{1}{2}\Phi_1 & \cdots & 0 & 0 \\ \frac{1}{2}\Phi_1 & \Phi_2 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \Phi_{2m-2} & \frac{1}{2}\Phi_{2m-1} \\ 0 & 0 & \cdots & \frac{1}{2}\Phi_{2m-1} & \Phi_{2m} \end{bmatrix} \tag{9}$$

■

By applying Lemma 1, we have the following result.

Theorem 1: For the matrix-valued polynomial $F_m(s)$ in (2), then

i) $F_m(s) < 0$ for $\forall s \in [0, \bar{h}]$ if and only if there exist an $X \in \mathbb{S}^{mq}_+$ and a skew-symmetric matrix $S \in \mathbb{R}^{mq\times mq}$ such that

$$\Omega_m + \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{bmatrix}^T \begin{bmatrix} X & S \\ S^T & -X \end{bmatrix} \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{bmatrix} < 0; \tag{10}$$

ii) $F_m(s) > 0$ for $\forall s \in [0, \bar{h}]$ if and only if there exist an $X \in \mathbb{S}^{mq}_+$ and a skew-symmetric matrix $S \in \mathbb{R}^{mq\times mq}$ such that

$$\Omega_m - \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{bmatrix}^T \begin{bmatrix} X & S \\ S^T & -X \end{bmatrix} \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{bmatrix} > 0, \tag{11}$$

where Ω_m is given in (9); and

$$\mathcal{H}_1 = \begin{bmatrix} \bar{h}I & & & 0 \\ & \bar{h}I & & 0 \\ & & \ddots & \vdots \\ & & \bar{h}I & 0 \end{bmatrix}_{mq\times(m+1)q}$$

$$\mathcal{H}_2 = \begin{bmatrix} \bar{h}I & -2I & & \\ & \bar{h}I & -2I & \\ & & \ddots & \ddots \\ & & \bar{h}I & -2I \end{bmatrix}_{mq\times(m+1)q}$$

Proof: i) Note that

$$(\mathcal{H}_2 - \delta\mathcal{H}_1)\zeta = \begin{bmatrix} [\bar{h}(1-\delta) - 2s]I \\ s[\bar{h}(1-\delta) - 2s]I \\ \vdots \\ s^{m-1}[\bar{h}(1-\delta) - 2s]I \end{bmatrix}\zeta_0.$$

Then $(\mathcal{H}_2 - \delta\mathcal{H}_1)\zeta = 0$ for some real scalar δ such that $|\delta| \leq 1$ if and only if $s \in [0, \bar{h}]$. In fact

$$(\mathcal{H}_2 - \delta\mathcal{H}_1)\zeta = 0 \iff \bar{h}\delta = \bar{h} - 2s.$$

Then it is not difficult to verify that

$$|\bar{h}\delta| = |\bar{h} - 2s| \leq \bar{h} \iff 0 \leq s \leq \bar{h}.$$

Applying Lemma 1, $f_m(s) < 0$ for $\forall s \in [0, \bar{h}]$ if and only if there exist an $X \in \mathbb{S}^{mq}_+$ and a skew-symmetric real matrix $S \in \mathbb{R}^{mq\times mq}$ such that (10) is satisfied, which completes the proof of i).

ii) Set $\tilde{F}_m(s) = -F_m(s)$. Then the proof is straightforward from the proof of i). ■

Remark 1: Theorem 1 provides necessary and sufficient conditions on matrix-valued polynomial inequalities $F_m(s) < 0$ and $F_m(s) > 0$ for $\forall s \in [0, \bar{h}]$, respectively. Specifically, for $m = 1$, $F_1(s) = s^2\Phi_2 + s\Phi_1 + \Phi_0 < 0$ for $\forall s \in [0, \bar{h}]$ *if and only if* there exist an $X \in \mathbb{S}^q_+$ and a skew-symmetric real matrix $S \in \mathbb{R}^{q\times q}$ such that

$$\begin{bmatrix} \Phi_0 & \frac{1}{2}\Phi_1 \\ \frac{1}{2}\Phi_1 & \Phi_2 \end{bmatrix} + \begin{bmatrix} \bar{h}I & 0 \\ \bar{h}I & -2I \end{bmatrix}^T \begin{bmatrix} X & S \\ S^T & -X \end{bmatrix} \begin{bmatrix} \bar{h}I & 0 \\ \bar{h}I & -2I \end{bmatrix} < 0.$$

which is equivalent to that in [18].

In the next section, we are to establish some novel stability criteria for time-delay systems by using Theorem 1. To end this section, we introduce a canonical Bessel-Legendre inequality as follows [19], [20].

Lemma 2: For an integer $N \ge 0$, two scalars a and b with $b > a$, an $n \times n$ real matrix $R > 0$, and a differentiable function $x : [a,b] \to \mathbb{R}^n$ such that the integrations below are well defined, then

$$-(b-a)\int_a^b \dot{x}^T(s)R\dot{x}(s)ds \le -\varpi_N^T \Lambda_N^T \Theta_N^T \mathcal{R}_N \Theta_N \Lambda_N \varpi_N$$

where

$$\mathcal{R}_N = \mathrm{diag}\{R, 3R, \cdots, (2N+1)R\}$$

$$\Theta_N = \begin{bmatrix} I & 0 & \cdots & 0 \\ I & (-1)^1\binom{1}{1}\binom{2}{1}I & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ I & (-1)^1\binom{N}{1}\binom{N+1}{1}I & \cdots & (-1)^N\binom{N}{N}\binom{2N}{N}I \end{bmatrix} \quad (12)$$

$$\Lambda_N = \begin{bmatrix} I & -I & 0 & 0 & \cdots & 0 \\ 0 & -I & I & 0 & \cdots & 0 \\ 0 & -I & 0 & 2I & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & -I & 0 & 0 & \cdots & NI \end{bmatrix} \quad (13)$$

$$\varpi_N = \mathrm{col}\Big\{x(b), x(a), \gamma_1(a,b), \ldots, \gamma_N(a,b)\Big\}$$

$$\gamma_k(a,b) = \int_a^b \frac{(b-s)^{k-1}}{(b-a)^k} x(s)ds, \ \ (k = 1,2,\ldots,N).$$

III. Application to Stability Analysis of Time-delay Systems

Consider the following time-delay system described by

$$\begin{cases} \dot{x}(t) = Ax(t) + Bx(t-d(t)) \\ \phi(\theta) = \phi_0, \ \ \theta \in [-\bar{h}, 0] \end{cases} \quad (14)$$

where $x(t) \in \mathbb{R}^n$ is the system state and ϕ_0 is the initial condition; $A, B \in \mathbb{R}^{n\times n}$. Suppose that the time delay $d(t)$ satisfies one of two cases as

Case 1: $d(t)$ is a differentiable function satisfying

$$0 \le d(t) \le \bar{h}, \ \ \mu_1 \le \dot{d}(t) \le \mu_2 \quad (15)$$

Case 2: $d(t)$ is a continuous function satisfying

$$0 \le d(t) \le \bar{h} \quad (16)$$

where $\bar{h}$, μ_1 and μ_2 are constants with $\mu_1 < 0$ and $\mu_2 > 0$.

A. Stability Criteria for the System (14) in Case 1

To begin with, we denote

$$\begin{cases} \nu_1(t) = \mathrm{col}\{\nu_{11}(t), \nu_{12}(t), \nu_{13}(t), \nu_{14}(t)\} \\ \nu_2(t) = \mathrm{col}\{\nu_{21}(t), \nu_{22}(t), \nu_{23}(t), \nu_{24}(t)\} \end{cases} \quad (17)$$

where

$$\nu_{1i}(t) = \int_{t-d(t)}^t \frac{(t-s)^{i-1}x(s)}{d^i(t)}ds$$

$$\nu_{2i}(t) = \int_{t-\bar{h}}^{t-d(t)} \frac{(t-d(t)-s)^{i-1}x(s)}{(\bar{h}-d(t))^i}ds$$

Construct the following Lyapunov-Krasovskii functional candidate as

$$V(t,x_t) = V_1(t,x_t) + V_2(t,x_t) + V_3(t,x_t) \quad (18)$$

where

$$V_1(t,x_t) = \psi_1^T(t)P(d(t))\psi_1(t)$$

$$V_2(t,x_t) = \int_{t-d(t)}^t \psi_2^T(s,t)Q_1\psi_2(s,t)ds + \int_{t-\bar{h}}^{t-d(t)} \psi_3^T(s,t)Q_2\psi_3(s,t)ds$$

$$V_3(t,x_t) = \bar{h}\int_{t-d(t)}^t (\bar{h}-t+s)\dot{x}^T(s)R_1\dot{x}(s)ds + \bar{h}\int_{t-\bar{h}}^{t-d(t)} (\bar{h}-t+s)\dot{x}^T(s)R_2\dot{x}(s)ds$$

where $Q_1 > 0, Q_2 > 0, R_1 > 0, R_2 > 0$ and

$$P(d(t)) = d^3(t)P_3 + d^2(t)P_2 + d(t)P_1 + P_0$$

$$\psi_1(t) = \mathrm{col}\{\psi_0(t), d(t)\nu_1(t), (\bar{h}-d(t))\nu_2(t)\}$$

$$\psi_0(t) = \mathrm{col}\{x(t), x(t-d(t)), x(t-\bar{h})\}$$

$$\psi_2(s,t) = \mathrm{col}\Big\{\dot{x}(s), x(s), x(t), \int_{t-d(t)}^s x(\theta)d\theta, \int_s^t x(\theta)d\theta, \int_{t-d(t)}^s (s-\theta)x(\theta)d\theta, \int_s^t (\theta-s)x(\theta)d\theta\Big\}$$

$$\psi_3(s,t) = \mathrm{col}\Big\{\dot{x}(s), x(s), x(t), \int_{t-\bar{h}}^s x(\theta)d\theta, \int_s^{t-d(t)} x(\theta)d\theta, \int_{t-\bar{h}}^s (s-\theta)x(\theta)d\theta, \int_s^{t-d(t)} (\theta-s)x(\theta)d\theta\Big\} \quad (19)$$

with $P_i \in \mathbb{S}^{11n}$.

Remark 2: The Lyapnuv-Krasovskii functional $V(t,x_t)$ is different from some existing ones published in the literature. On the one hand, a *cubic matrix-valued polynomial* $P(d(t))$ in (19) is introduced in $V_1(t,x_t)$, leading to the fact that $V_1(t,x_t)$ is a cubic polynomial on $d(t)$. The positive finiteness of $P(d(t))$ for $\forall d(t) \in [0,\bar{h}]$ does not need all P_i $(i = 0,1,2,3)$ to be positive definite. By Theorem 1 (ii), $P(d(t)) > 0$ for $\forall d(t) \in [0,\bar{h}]$ if and only if there exist an $X > 0$ and a skew-symmetric real matrix S such that

$$\begin{bmatrix} P_0 & \frac{1}{2}P_1 & 0 \\ \frac{1}{2}P_1 & P_2 & \frac{1}{2}P_3 \\ 0 & \frac{1}{2}P_3 & 0 \end{bmatrix} - \begin{bmatrix} \tilde{\mathcal{H}}_1 \\ \tilde{\mathcal{H}}_2 \end{bmatrix}^T \begin{bmatrix} X & S \\ S^T & -X \end{bmatrix} \begin{bmatrix} \tilde{\mathcal{H}}_1 \\ \tilde{\mathcal{H}}_2 \end{bmatrix} > 0$$

which, by Lemma 1, is equivalent to that there exist $\epsilon_i \in \mathbb{R}$ $(i = 1,2)$ such that

$$\begin{bmatrix} P_0 & \frac{1}{2}P_1 & 0 \\ \frac{1}{2}P_1 & P_2 & \frac{1}{2}P_3 \\ 0 & \frac{1}{2}P_3 & 0 \end{bmatrix} + \epsilon_1(\tilde{\mathcal{H}}_2 + \tilde{\mathcal{H}}_1)^T(\tilde{\mathcal{H}}_2 + \tilde{\mathcal{H}}_1) > 0 \quad (20)$$

$$\begin{bmatrix} P_0 & \frac{1}{2}P_1 & 0 \\ \frac{1}{2}P_1 & P_2 & \frac{1}{2}P_3 \\ 0 & \frac{1}{2}P_3 & 0 \end{bmatrix} + \epsilon_2(\tilde{\mathcal{H}}_2 - \tilde{\mathcal{H}}_1)^T(\tilde{\mathcal{H}}_2 - \tilde{\mathcal{H}}_1) > 0 \quad (21)$$

where

$$\tilde{\mathcal{H}}_1 = \begin{bmatrix} \bar{h}I & 0 & 0 \\ 0 & \bar{h}I & 0 \end{bmatrix}, \tilde{\mathcal{H}}_2 = \begin{bmatrix} \bar{h}I & -2I & 0 \\ 0 & \bar{h}I & -2I \end{bmatrix}.$$

On the other hand, four vectors $\int_{t-d(t)}^{s}(s-\theta)x(\theta)d\theta$, $\int_{s}^{t}(\theta-s)x(\theta)d\theta$, $\int_{t-\bar{h}}^{s}(s-\theta)x(\theta)d\theta$ and $\int_{s}^{t-d(t)}(\theta-s)x(\theta)d\theta$ are included in $V_2(t,x_t)$, which brings more information on past system states into the derivative of the Lyapnuv-Krasovskii functional.

Based on the Lyapnuv-Krasovskii functional $V(t,x_t)$, we now state and establish the following result.

Proposition 1: For given constants μ_1,μ_2, and $\bar{h}$, the system described by (14) and (15) is asymptotically stable if there exist $Q_i>0$, $R_i>0$, symmetric real matrices P_0,P_1,P_2,P_3,Z_i, real matrices Y_i with appropriate dimensions and scalars $\epsilon_i,\varepsilon_{i1},\varepsilon_{i2}$ $(i=1,2)$ such that (20), (21) and

$$\begin{bmatrix} \mathcal{R}_1 - Z_1 & Y_1 \\ Y_1^T & \mathcal{R}_2 \end{bmatrix} \geq 0, \begin{bmatrix} \mathcal{R}_1 & Y_2 \\ Y_2^T & \mathcal{R}_2 - Z_2 \end{bmatrix} \geq 0 \tag{22}$$

$$\Phi(\mu_i) - \varepsilon_{i1}(\mathcal{H}_2+\mathcal{H}_1)^T(\mathcal{H}_2+\mathcal{H}_1) < 0 \tag{23}$$

$$\Phi(\mu_i) - \varepsilon_{i2}(\mathcal{H}_2-\mathcal{H}_1)^T(\mathcal{H}_2-\mathcal{H}_1) < 0,\ i=1,2 \tag{24}$$

where $\mathcal{R}_i = \mathrm{diag}\{R_i,3R_i,5R_i,7R_i,9R_i\}$ $(i=1,2)$, and

$$\mathcal{H}_1 = \begin{bmatrix} \bar{h}I & 0 & 0 \\ 0 & \bar{h}I & 0 \end{bmatrix}, \mathcal{H}_2 = \begin{bmatrix} \bar{h}I & -2I & 0 \\ 0 & \bar{h}I & -2I \end{bmatrix},$$

$$\Phi(\dot{d}(t)) = \begin{bmatrix} \Phi_0 & \frac{1}{2}\Phi_1 & 0 \\ \frac{1}{2}\Phi_1 & \Phi_2 & \frac{1}{2}\Phi_3 \\ 0 & \frac{1}{2}\Phi_3 & \Phi_4 \end{bmatrix},$$

where

$$\begin{aligned}
\Phi_0 = {} & \mathrm{He}\{C_{11}^T P_0 C_2\} + \dot{d}(t)C_{11}^T P_1 C_{11} + C_{31}^T Q_1 C_{31} \\
& - C_{41}^T Q_2 C_{41} + (1-\dot{d}(t))(C_{61}^T Q_2 C_{61} - C_{51}^T Q_1 C_{51}) \\
& + \mathrm{He}\Big\{\aleph_{11}^T Q_1 D_{10} + \sum_{j=1}^{3} \aleph_{2j}^T Q_2 D_{(3+j)0}\Big\} \\
& + \bar{h}^2 C_0^T R_1 C_0 + \bar{h}^2(1-\dot{d}(t))e_4^T(R_2-R_1)e_4 \\
& - C_7^T(\mathcal{R}_1+Z_1)C_7 - C_8^T\mathcal{R}_2 C_8 - \mathrm{He}\big\{C_7^T Y_2 C_8\big\} \\
\Phi_1 = {} & \mathrm{He}\{(C_{11}^T P_1 + C_{12}^T P_0)C_2 + \dot{d}(t)C_{11}^T P_1 C_{12}\} \\
& + \mathrm{He}\{C_{31}^T Q_1 C_{32} - C_{41}^T Q_2 C_{42}\} + 2\dot{d}(t)C_{11}^T P_2 C_{11} \\
& + (1-\dot{d}(t))\mathrm{He}\{C_{61}^T Q_2 C_{62} - C_{51}^T Q_1 C_{52}\} \\
& + \mathrm{He}\Big\{\sum_{j=1}^{3}\big[\aleph_{1j}^T Q_1 D_{j1} + \aleph_{2j}^T Q_2 D_{(3+j)1}\big]\Big\} \\
& + \tfrac{1}{\bar{h}}\Big[C_7^T Z_1 C_7 - C_8^T Z_2 C_8 - \mathrm{He}\big\{C_7^T(Y_1-Y_2)C_8\big\}\Big] \\
& - \bar{h}(1-\dot{d}(t))e_4^T(R_2-R_1)e_4 \\
\Phi_2 = {} & \mathrm{He}\{(C_{11}^T P_2 + C_{12}^T P_1)C_2 + 2\dot{d}(t)C_{11}^T P_2 C_{12}\} \\
& + \mathrm{He}\{C_{31}^T Q_1 C_{33} - C_{41}^T Q_2 C_{43}\} + C_{32}^T Q_1 C_{32} \\
& - C_{42}^T Q_2 C_{42} + \dot{d}(t)(3C_{11}^T P_3 C_{11} + C_{12}^T P_1 C_{12}) \\
& + (1-\dot{d}(t))\mathrm{He}\{C_{61}^T Q_2 C_{63} - C_{51}^T Q_1 C_{53}\} \\
& + (1-\dot{d}(t))(C_{62}^T Q_2 C_{62} - C_{52}^T Q_1 C_{52}) \\
& + \mathrm{He}\Big\{\sum_{j=1}^{3}\big[\aleph_{1j}^T Q_1 D_{j2} + \aleph_{2j}^T Q_2 D_{(3+j)2}\big]\Big\} \\
\Phi_3 = {} & \mathrm{He}\{(C_{11}^T P_3 + C_{12}^T P_2)C_2 + 3\dot{d}(t)C_{11}^T P_3 C_{12}\} \\
& + \mathrm{He}\{C_{32}^T Q_1 C_{33} - C_{42}^T Q_2 C_{43}\} + 2\dot{d}(t)C_{12}^T P_2 C_{12} \\
& + (1-\dot{d}(t))\mathrm{He}\{C_{62}^T Q_2 C_{63} - C_{52}^T Q_1 C_{53}\} \\
& + \mathrm{He}\Big\{\sum_{j=1}^{3}\big[\aleph_{1j}^T Q_1 D_{j3} + \aleph_{2j}^T Q_2 D_{(3+j)3}\big]\Big\} \\
\Phi_4 = {} & \mathrm{He}\{C_{12}^T P_3 C_2\} + 3\dot{d}(t)C_{12}^T P_3 C_{12} + C_{33}^T Q_1 C_{33} \\
& - C_{43}^T Q_2 C_{43} + (1-\dot{d}(t))(C_{63}^T Q_2 C_{63} - C_{53}^T Q_1 C_{53}) \\
& + \mathrm{He}\Big\{\sum_{j=2}^{3}\big[\aleph_{1j}^T Q_1 D_{j4} + \aleph_{2j}^T Q_2 D_{(3+j)4}\big]\Big\}
\end{aligned}$$

with $C_0 = Ae_1 + Be_2$, and $\mathrm{col}\{e_1,e_2,e_3,e_4,e_5,e_6,e_7\}$ being a 13×13 identity matrix with $e_6 = \mathrm{col}\{e_{61},e_{62},e_{63},e_{64}\}$ and $e_7 = \mathrm{col}\{e_{71},e_{72},e_{73},e_{74}\}$; and

$$\begin{aligned}
& C_{11} = \mathrm{col}\{e_1,e_2,e_3,0,\bar{h}e_7\},\ C_{12} = \mathrm{col}\{0,0,0,e_6,-e_7\} \\
& C_2 = \mathrm{col}\{C_0,(1-\dot{d}(t))e_4,e_5,\Gamma_1(\dot{d}(t)),\Gamma_2(\dot{d}(t))\} \\
& \Gamma_1(\dot{d}(t)) = \mathrm{col}\{e_1-(1-\dot{d}(t))e_2,\ell_{11},\ell_{12},\ell_{13}\} \\
& \Gamma_2(\dot{d}(t)) = \mathrm{col}\{(1-\dot{d}(t))e_2-e_3,\ell_{21},\ell_{22},\ell_{23}\} \\
& \ell_{1i} = -(1-\dot{d}(t))e_2 + i[e_{6i} - \dot{d}(t)e_{6(i+1)}] \\
& \ell_{2i} = -e_3 + i[(1-\dot{d}(t))e_{7i} + \dot{d}(t)e_{7(i+1)}] \\
& C_{31} = \mathrm{col}\{C_0,e_1,e_1,e_{40}\},\ C_{32} = \mathrm{col}\{e_{30},e_{61},e_{30}\}, \\
& C_{33} = \mathrm{col}\{e_{50},e_{62},0\},\ C_{42} = \mathrm{col}\{e_{40},-e_{71},0,-2\bar{h}\rho_1\}, \\
& C_{41} = \mathrm{col}\{e_5,e_3,e_1,0,\bar{h}e_{71},0,\bar{h}^2\rho_1\},\ C_{43} = \mathrm{col}\{e_{60},\rho_1\}, \\
& C_{51} = \mathrm{col}\{e_4,e_2,e_1,e_{40}\}, C_{52} = \mathrm{col}\{e_{40},e_{61},e_{20}\}, \\
& C_{53} = \mathrm{col}\{e_{60},\sigma_1\},\ C_{61} = \mathrm{col}\{e_4,e_2,e_1,\bar{h}e_{71},0,\bar{h}^2e_{72},0\}, \\
& C_{62} = \mathrm{col}\{e_{30},-e_{71},0,-2\bar{h}e_{72},0\},\ C_{63} = \mathrm{col}\{e_{50},e_{72},0\} \\
& C_7 = \Theta_4\Lambda_4\mathrm{col}\{e_1,e_2,e_6\},\ C_8 = \Theta_4\Lambda_4\mathrm{col}\{e_2,e_3,e_7\} \\
& \aleph_{11} = \mathrm{col}\{e_{20},C_0,-(1-\dot{d}(t))e_2,e_1,e_{20}\}, \\
& \aleph_{12} = \mathrm{col}\{e_{50},(1-\dot{d}(t))e_2,0\},\ \aleph_{13} = \mathrm{col}\{e_{60},e_1\} \\
& \aleph_{21} = \mathrm{col}\{e_{20},C_0,-e_3,(1-\dot{d}(t))e_2,e_{20}\}, \\
& \aleph_{22} = \mathrm{col}\{e_{60},(1-\dot{d}(t))e_2\},\ \aleph_{23} = \mathrm{col}\{e_{50},e_3,0\}
\end{aligned}$$

where Θ_4 and Λ_4 are defined in (12) and (13) with $N=4$, respectively; and $e_{j0} = 0_{jn\times13n}$ $(j\geq2)$; and

$$\begin{aligned}
& D_{10} = \mathrm{col}\{e_1-e_2,e_{60}\},\ D_{11} = \mathrm{col}\{0,e_{61},e_1,e_{40}\} \\
& D_{12} = \mathrm{col}\{e_{30},e_{62},\sigma_1,e_{20}\},\ \ D_{13} = \mathrm{col}\{e_{50},\tfrac{1}{2}e_{63},\sigma_2\} \\
& D_{21} = \mathrm{col}\{e_{61}-e_1,e_{60}\},\ D_{22} = \mathrm{col}\{0,-\sigma_1,-\tfrac{1}{2}e_1,e_{40}\} \\
& D_{23} = \mathrm{col}\{e_{30},\sigma_3,-\sigma_2,e_{20}\},\ D_{24} = \mathrm{col}\{e_{50},\sigma_5,\sigma_4\} \\
& D_{31} = \mathrm{col}\{\sigma_7,e_{60}\},\ D_{32} = \mathrm{col}\{0,e_{62},\tfrac{1}{2}e_1,e_{40}\} \\
& D_{33} = \mathrm{col}\{e_{30},\tfrac{1}{2}e_{63},\sigma_6,e_{20}\},\ D_{34} = \mathrm{col}\{e_{50},\tfrac{1}{6}e_{64},\sigma_8\} \\
& D_{40} = \mathrm{col}\{e_2-e_3,\bar{h}e_{71},\bar{h}e_1,\bar{h}^2e_{72},\bar{h}^2\rho_1,\tfrac{1}{2}\bar{h}^3e_{73},\bar{h}^3\rho_2\} \\
& D_{41} = -\mathrm{col}\{0,e_{71},e_1,2\bar{h}e_{72},2\bar{h}\rho_1,\tfrac{3}{2}\bar{h}^2e_{73},3\bar{h}^2\rho_2\} \\
& D_{42} = \mathrm{col}\{e_{30},e_{72},\rho_1,\tfrac{3}{2}\bar{h}e_{73},3\bar{h}\rho_2\} \\
& D_{43} = \mathrm{col}\{e_{50},-\tfrac{1}{2}e_{73},-\rho_2\} \\
& D_{50} = \mathrm{col}\{\bar{h}\rho_0,\bar{h}^2e_{72},\tfrac{1}{2}\bar{h}^2e_1,\tfrac{1}{2}\bar{h}^3e_{73},\bar{h}^3\rho_6,\tfrac{1}{6}\bar{h}^4e_{74},\bar{h}^4\rho_8\} \\
& D_{51} = -\mathrm{col}\{\rho_0,2\bar{h}e_{72},\bar{h}e_1,\tfrac{3}{2}\bar{h}^2e_{73},3\bar{h}^2\rho_6,\tfrac{2}{3}\bar{h}^3e_{74},4\bar{h}^3\rho_8\} \\
& D_{52} = \mathrm{col}\{0,e_{72},\tfrac{1}{2}e_1,\tfrac{3}{2}\bar{h}e_{73},3\bar{h}\rho_6,\bar{h}^2e_{74},6\bar{h}^2\rho_8\} \\
& D_{53} = -\mathrm{col}\{e_{30},\tfrac{1}{2}e_{73},\rho_6,\tfrac{2}{3}\bar{h}e_{74},4\bar{h}\rho_8\} \\
& D_{54} = \mathrm{col}\{e_{50},\tfrac{1}{6}e_{74},\rho_8\},\ D_{64} = \mathrm{col}\{e_{50},\rho_5,\rho_4\}
\end{aligned}$$

$$D_{60} = \mathrm{col}\{\bar{h}\rho_7, -\bar{h}^2\rho_1, -\tfrac{1}{2}\bar{h}^2 e_1, \bar{h}^3\rho_3, -\bar{h}^3\rho_2, \bar{h}^4\rho_5, \bar{h}^4\rho_4\}$$
$$D_{61} = \mathrm{col}\{-\rho_7, 2\bar{h}\rho_1, \bar{h}e_1, -3\bar{h}^2\rho_3, 3\bar{h}^2\rho_2, -4\bar{h}^3\rho_5, -4\bar{h}^3\rho_4\}$$
$$D_{62} = \mathrm{col}\{0, -\rho_1, -\tfrac{1}{2}e_1, 3\bar{h}\rho_3, -3\bar{h}\rho_2, 6\bar{h}^2\rho_5, 6\bar{h}^2\rho_4\}$$
$$D_{63} = \mathrm{col}\{e_{30}, -\rho_3, \rho_2, -4\bar{h}\rho_5, -4\bar{h}\rho_4\}$$

with $\rho_0 = e_{71} - e_3$ and

$$\sigma_1 = e_{61} - e_{62},\ \ \sigma_2 = \tfrac{1}{2}(e_{63} - 2e_{62} + e_{61})$$
$$\sigma_3 = \tfrac{1}{2}(e_{63} - 2e_{62}),\ \ \sigma_4 = \tfrac{1}{6}(e_{64} - 3e_{63} + 3e_{62} - e_{61})$$
$$\sigma_5 = \tfrac{1}{6}(e_{64} - 3e_{63}),\ \ \sigma_6 = \tfrac{1}{2}(e_{61} - e_{63})$$
$$\sigma_7 = e_{61} - e_2,\ \ \sigma_8 = \tfrac{1}{6}(e_{64} - 3e_{62} + 2e_{61})$$
$$\rho_1 = e_{71} - e_{72},\ \ \rho_2 = \tfrac{1}{2}(e_{73} - 2e_{72} + e_{71})$$
$$\rho_3 = \tfrac{1}{2}(e_{73} - 2e_{72}),\ \ \rho_4 = \tfrac{1}{6}(e_{74} - 3e_{73} + 3e_{72} - e_{71})$$
$$\rho_5 = \tfrac{1}{6}(e_{74} - 3e_{73}),\ \ \rho_6 = \tfrac{1}{2}(e_{71} - e_{73})$$
$$\rho_7 = e_{71} - e_2,\ \ \rho_8 = \tfrac{1}{6}(e_{74} - 3e_{72} + 2e_{71}).$$

Proof: First, the conditions (20) and (21) ensure that the real matrix $P(d(t))$ is positive definite for $\forall d(t) \in [0, \bar{h}]$. Thus, $V(t, x_t)$ constructed in (18) is a Lyapunov-Krasovskii functional candidate. Then, we take the derivative of $V(t, x_t)$ along with the trajectory of the system (14) to obtain

$$\dot{V}(t, x_t) = \dot{V}_1(t, x_t) + \dot{V}_2(t, x_t) + \dot{V}_3(t, x_t) \tag{25}$$

where

$$\begin{aligned}
\dot{V}_1(t, x_t) &= 2\psi_1^T(t)P(d(t))\dot{\psi}_1(t) + \psi_1^T(t)\dot{P}(d(t))\psi_1(t)\\
\dot{V}_2(t, x_t) &= \psi_2^T(t,t)Q_1\psi_2(t,t) - \psi_3^T(t-\bar{h},t)Q_2\psi_3(t-\bar{h},t)\\
&\quad - (1-\dot{d}(t))\psi_2^T(t-d(t),t)Q_1\psi_2(t-d(t),t)\\
&\quad + (1-\dot{d}(t))\psi_3^T(t-d(t),t)Q_2\psi_3(t-d(t),t)\\
&\quad + \int_{t-d(t)}^{t} 2\psi_2^T(s,t)Q_1\frac{\partial}{\partial t}\psi_2(s,t)ds\\
&\quad + \int_{t-\bar{h}}^{t-d(t)} 2\psi_3^T(s,t)Q_2\frac{\partial}{\partial t}\psi_3(s,t)ds\\
\dot{V}_3(t, x_t) &= \bar{h}^2\dot{x}^T(t)R_1\dot{x}(t) + \bar{h}(1-\dot{d}(t))(\bar{h}-d(t))\\
&\quad \times \dot{x}^T(t-d(t))(R_2-R_1)\dot{x}(t-d(t))\\
&\quad + \mathcal{I}_1(t) + \mathcal{I}_2(t)
\end{aligned}$$

where $\mathcal{I}_1(t) = -\bar{h}\int_{t-d(t)}^{t} \dot{x}^T(s)R_1\dot{x}(s)ds$ and $\mathcal{I}_2(t) = -\bar{h}\int_{t-\bar{h}}^{t-d(t)} \dot{x}^T(s)R_2\dot{x}(s)ds$. Denote

$$\xi(t) = \mathrm{col}\{x(t), x(t-d(t)), x(t-\bar{h}), \dot{x}(t-d(t)), \dot{x}(t-\bar{h}), \nu_1(t), \nu_2(t)\}$$

where $\nu_1(t)$ and $\nu_2(t)$ are defined in (17). Note that

$$\frac{d}{dt}[d(t)\nu_1(t)] = \Gamma_1(\dot{d}(t))\xi(t)$$
$$\frac{d}{dt}[(\bar{h}-d(t))\nu_2(t)] = \Gamma_2(\dot{d}(t))\xi(t)$$

where $\Gamma_1(\cdot)$ and $\Gamma_2(\cdot)$ are defined in Proposition 1. Hereafter, the notations used are also defined in Proposition 1 without declaration. Then

$$\psi_1(t) = C_1\xi(t),\ \ \dot{\psi}_1(t) = C_2\xi(t)$$

where $C_1 = C_{11} + d(t)C_{12}$. Hence, we have that

$$\dot{V}_1(t, x_t) = \xi^T(t)\Psi_1(d(t), \dot{d}(t))\xi(t) \tag{26}$$

where

$$\begin{aligned}
\Psi_1(d(t), \dot{d}(t)) &= \dot{d}(t)C_1^T\left[3d^2(t)P_3 + 2d(t)P_2 + P_1\right]C_1\\
&\quad + \mathrm{He}\{C_1^T P(d(t))C_2\}
\end{aligned}$$

Some algebraic manipulations follow that

$$\psi_2(t,t) = C_3\xi(t), \psi_3(t-\bar{h},t) = C_4\xi(t)$$
$$\psi_2(t-d(t),t) = C_5\xi(t), \psi_3(t-d(t),t) = C_6\xi(t)$$

where

$$C_j = C_{j1} + d(t)C_{j2} + d^2(t)C_{j3},\ \ j = 3,4,5,6$$

Since

$$\frac{\partial}{\partial t}\psi_2(s,t) = [\aleph_{11} + g_1(s)\aleph_{12} + (t-s)\aleph_{13}]\xi(t)$$
$$\frac{\partial}{\partial t}\psi_3(s,t) = [\aleph_{21} + g_1(s)\aleph_{22} + g_2(s)\aleph_{23}]\xi(t)$$

where $g_1(s) = t - d(t) - s$ and $g_2(s) = t - \bar{h} - s$. Then

$$\begin{aligned}
&\int_{t-d(t)}^{t} 2\psi_2^T(s,t)Q_1\frac{\partial}{\partial t}\psi_2(s,t)ds\\
&\quad = 2\xi^T(t)\aleph_{11}^T Q_1\int_{t-d(t)}^{t}\psi_2(s,t)ds\\
&\qquad + 2\xi^T(t)\aleph_{12}^T Q_1\int_{t-d(t)}^{t} g_1(s)\psi_2(s,t)ds\\
&\qquad + 2\xi^T(t)\aleph_{13}^T Q_1\int_{t-d(t)}^{t}(t-s)\psi_2(s,t)ds,
\end{aligned}$$

and

$$\begin{aligned}
&\int_{t-\bar{h}}^{t-d(t)} 2\psi_3^T(s,t)Q_2\frac{\partial}{\partial t}\psi_3(s,t)ds\\
&\quad = 2\xi^T(t)\aleph_{21}^T Q_2\int_{t-\bar{h}}^{t-d(t)}\psi_3(s,t)ds\\
&\qquad + 2\xi^T(t)\aleph_{22}^T Q_2\int_{t-\bar{h}}^{t-d(t)} g_1(s)\psi_3(s,t)ds\\
&\qquad + 2\xi^T(t)\aleph_{23}^T Q_2\int_{t-\bar{h}}^{t-d(t)} g_2(s)\psi_3(s,t)ds
\end{aligned}$$

Note that

$$\int_{t-d(t)}^{t}\psi_2(s,t)ds = \mathcal{D}_1(d(t))\xi(t) = \sum_{i=0}^{3} d^i(t)D_{1i}\xi(t)$$
$$\int_{t-d(t)}^{t} g_1(s)\psi_2(s,t)ds = \mathcal{D}_2(d(t))\xi(t) = \sum_{i=1}^{4} d^i(t)D_{2i}\xi(t)$$
$$\int_{t-d(t)}^{t}(t-s)\psi_2(s,t)ds = \mathcal{D}_3(d(t))\xi(t) = \sum_{i=1}^{4} d^i(t)D_{3i}\xi(t)$$
$$\int_{t-\bar{h}}^{t-d(t)}\psi_3(s,t)ds = \mathcal{D}_4(d(t))\xi(t) = \sum_{i=0}^{3} d^i(t)D_{4i}\xi(t)$$
$$\int_{t-\bar{h}}^{t-d(t)} g_1(s)\psi_3(s,t)ds = \mathcal{D}_5(d(t))\xi(t) = \sum_{i=0}^{4} d^i(t)D_{5i}\xi(t)$$
$$\int_{t-\bar{h}}^{t-d(t)} g_2(s)\psi_3(s,t)ds = \mathcal{D}_6(d(t))\xi(t) = \sum_{i=0}^{4} d^i(t)D_{6i}\xi(t),$$

which lead to

$$\dot{V}_2(t,x_t) + \dot{V}_3(t,x_t) = \xi^T(t)\Psi_2(d(t),\dot{d}(t))\xi(t) + \mathcal{I}_1(t)+\mathcal{I}_2(t) \quad (27)$$

where

$$\begin{aligned}\Psi_2(d(t),\dot{d}(t)) &= (1-\dot{d}(t))(C_6^TQ_2C_6 - C_5^TQ_1C_5)\\ &+\mathrm{He}\Big\{\sum_{j=1}^{3}\big[\aleph_{1j}^TQ_1\mathcal{D}_j(d(t))+\aleph_{2j}^TQ_2\mathcal{D}_{3+j}(d(t))\big]\Big\}\\ &+C_3^TQ_1C_3 - C_4^TQ_2C_4+\bar{h}^2C_0^TR_1C_0\\ &+\bar{h}(1-\dot{d}(t))(\bar{h}-d(t))e_4^T(R_2-R_1)e_4\end{aligned}$$

For the integral term in (27), applying Lemma 2, one has

$$\mathcal{I}_1(t) \le -\frac{1}{\alpha}\xi^T(t)C_7^T\mathcal{R}_1C_7\xi(t)$$
$$\mathcal{I}_2(t) \le -\frac{1}{1-\alpha}\xi^T(t)C_8^T\mathcal{R}_2C_8\xi(t)$$

where $\alpha = d(t)/\bar{h}$; and

$$C_7 = \Theta_4\Lambda_4\mathrm{col}\{e_1,e_2,e_6\},\ C_8 = \Theta_4\Lambda_4\mathrm{col}\{e_2,e_3,e_7\}.$$

Use the improved reciprocally convex inequality [21] to get

$$\mathcal{I}_1(t)+\mathcal{I}_2(t) \le \xi^T(t)\Psi_3(d(t))\xi(t) \quad (28)$$

where $\Psi_3(d(t)) = \Psi_{31}+d(t)\Psi_{32}$ with

$$\Psi_{31} = -C_7^T(\mathcal{R}_1+Z_1)C_7 - C_8^T\mathcal{R}_2C_8-\mathrm{He}\big\{C_7^TY_2C_8\big\}$$
$$\Psi_{32} = \tfrac{1}{\bar{h}}\Big[C_7^TZ_1C_7-C_8^TZ_2C_8-\mathrm{He}\big\{C_7^T(Y_1-Y_2)C_8\big\}\Big]$$

Substituting (28) into (27) and into (25) yields

$$\dot{V}(t,x_t) \le \xi^T(t)\Psi(d(t),\dot{d}(t))\xi(t) \quad (29)$$

where

$$\begin{aligned}\Psi(d(t),\dot{d}(t)) &= \Psi_1(d(t),\dot{d}(t))+\Psi_2(d(t),\dot{d}(t))+\Psi_3(d(t))\\ &=\sum_{i=0}^{4}d^i(t)\Phi_i\end{aligned} \quad (30)$$

Since $\Psi(d(t),\dot{d}(t))$ is linear on $\dot{d}(t)\in[\mu_1,\mu_2]$, $\Psi(d(t),\dot{d}(t)) < 0$ for $\dot{d}(t)\in[\mu_1,\mu_2]$ if and only if $\Psi(d(t),\mu_1)<0$ and $\Psi(d(t),\mu_2)<0$. By Theorem 1, $\Psi(d(t),\mu_1)<0$ and $\Psi(d(t),\mu_2)<0$ are equivalent to that there exist $L_1>0, L_2>0$ and skew-symmetric real matrices T_1 and T_2 such that

$$\Phi(\mu_1)+\mathcal{H}^T\begin{bmatrix}L_1 & T_1\\ T_1^T & -L_1\end{bmatrix}\mathcal{H}<0 \quad (31)$$

$$\Phi(\mu_2)+\mathcal{H}^T\begin{bmatrix}L_2 & T_2\\ T_2^T & -L_2\end{bmatrix}\mathcal{H}<0 \quad (32)$$

which, respectively, are equivalent to (23) and (24) for $i=1$, and (23) and (24) for $i=2$. Thus, if the conditions in (20)–(24) are satisfied, there exists a scalar $\varepsilon_0>0$ such that $\dot{V}(t,x_t)\le -\varepsilon_0\xi^T(t)\xi(t)\le -\varepsilon_0x^T(t)x(t)$, which means that the system (14) subject to (15) is asymptotically stable. ■

Remark 3: The proof of Proposition 1 presents an approach to stability analysis of time-delay systems. The defining feature of it lies in that: i) A *cubic* matrix-valued polynomial, i.e. $\sum_{i=0}^{3}d^i(t)P_i$ (see (19), is introduced in the Lyapunov-Krasovskii functional; ii) A *quartic* matrix-valued polynomial, i.e. $\sum_{i=0}^{4}d^i(t)\Phi_i$ (see (30)), is produced in the derivative of the Lyapunov-Krasovskii functional; and iii) Theorem 1 is employed to obtain two *necessary and sufficient conditions* such that $\sum_{i=0}^{3}d^i(t)P_i>0$ and $\sum_{i=0}^{4}d^i(t)\Phi_i<0$ for $d(t)\in[0,\bar{h}]$, respectively. If we do not employ Theorem 1, some other methods should be used to estimate them, which definitely yields conservative stability criteria. Moreover, it should be mentioned that, to the best of the authors' knowledge, there is no effective method available to estimate such cubic and quartic polynomials on the time-varying delay. A well-studied numerical example in Section IV shows that Proposition 1 can deliver some larger delay upper bounds than some existing ones.

B. Stability Criteria for the System (14) in Case 2

In Case 2, since information on delay-derivative is unknown, the Lyapunov-Krasovskii functional candidate is chosen as

$$\tilde{V}(t,x_t) = \tilde{\psi}_1^T(t)P\tilde{\psi}_1(t)+\tilde{V}_1(t,x_t)+\tilde{V}_2(t,x_t) \quad (33)$$

where

$$\tilde{V}_1(t,x_t) = \int_{t-\bar{h}}^{t}\tilde{\psi}_2^T(s,t)Q\tilde{\psi}_2(s,t)ds$$
$$\tilde{V}_2(t,x_t) = \bar{h}\int_{t-\bar{h}}^{t}(\bar{h}-t+s)\dot{x}^T(s)R\dot{x}(s)ds.$$

with $P>0, Q>0, R>0$; and

$$\tilde{\psi}_1(t) = \mathrm{col}\{x(t),\int_{t-\bar{h}}^{t}x(s)ds,\int_{t-\bar{h}}^{t}(t-s)x(s)ds\}$$
$$\begin{aligned}\tilde{\psi}_2(s,t) = \mathrm{col}\Big\{&x(s),x(t),\int_{t-\bar{h}}^{s}x(\theta)d\theta,\int_{s}^{t}x(\theta)d\theta,\\ &\int_{t-\bar{h}}^{s}(s-\theta)x(\theta)d\theta,\int_{s}^{t}(\theta-s)x(\theta)d\theta\Big\}.\end{aligned}$$

Proposition 2: For a given $\bar{h}>0$, the system (14) with (16) is asymptotically stable if there exist real matrices $P>0$, $Q>0$, $R>0$, symmetric real matrices Z_1,Z_2, real matrices Y_1,Y_2 with appropriate dimensions and two scalars ε_1 and ε_2 such that

$$\begin{bmatrix}\mathcal{R}-Z_1 & Y_1\\ Y_1^T & \mathcal{R}\end{bmatrix}\ge 0,\ \begin{bmatrix}\mathcal{R} & Y_2\\ Y_2^T & \mathcal{R}-Z_2\end{bmatrix}\ge 0 \quad (34)$$

$$\tilde{\Phi}-\varepsilon_1(\bar{\mathcal{H}}_2+\bar{\mathcal{H}}_1)^T(\bar{\mathcal{H}}_2+\bar{\mathcal{H}}_1)<0 \quad (35)$$

$$\tilde{\Phi}-\varepsilon_2(\bar{\mathcal{H}}_2-\bar{\mathcal{H}}_1)^T(\bar{\mathcal{H}}_2-\bar{\mathcal{H}}_1)<0 \quad (36)$$

where $\mathcal{R} = \mathrm{diag}\{R,3R,5R,7R,9R\}$, and

$$\bar{\mathcal{H}}_1 = \begin{bmatrix}\bar{h}I & 0 & 0\\ 0 & \bar{h}I & 0\end{bmatrix},\ \bar{\mathcal{H}}_2 = \begin{bmatrix}\bar{h}I & -2I & 0\\ 0 & \bar{h}I & -2I\end{bmatrix},$$
$$\tilde{\Phi} = \begin{bmatrix}\tilde{\Phi}_0 & \frac{1}{2}\tilde{\Phi}_1 & 0\\ \frac{1}{2}\tilde{\Phi}_1 & \tilde{\Phi}_2 & \frac{1}{2}\tilde{\Phi}_3\\ 0 & \frac{1}{2}\tilde{\Phi}_3 & \tilde{\Phi}_4\end{bmatrix},$$

where

$$\begin{aligned}\tilde{\Phi}_0 &= \bar{h}^2\tilde{C}_0^TR\tilde{C}_0+\tilde{C}_{30}^TQ\tilde{C}_{30}-\tilde{C}_{40}^TQ\tilde{C}_{40}-\tilde{C}_6^T\mathcal{R}\tilde{C}_6\\ &-\tilde{C}_5^T(\mathcal{R}+Z_1)\tilde{C}_5+\mathrm{He}\{\Sigma_{j=1}^{3}\aleph_j^TQ\tilde{\mathcal{D}}_{j0}\}\\ &+\mathrm{He}\{\tilde{C}_{10}^TP\tilde{C}_{20}-\tilde{C}_5^TY_2\tilde{C}_6\}\end{aligned}$$

$$\begin{aligned}\tilde{\Phi}_1 &= \mathrm{He}\{\tilde{C}_{10}^T P\tilde{C}_{21} + \tilde{C}_{11}^T P\tilde{C}_{20} + \Sigma_{j=1}^3 \aleph_j^T Q\tilde{\mathcal{D}}_{j1}\}\\ &\quad + \tfrac{1}{\bar{h}}[\tilde{C}_5^T Z_1\tilde{C}_5 - \tilde{C}_6^T Z_2\tilde{C}_6 - \mathrm{He}\{\tilde{C}_5^T(Y_1-Y_2)\tilde{C}_6\}]\\ &\quad + \mathrm{He}\{\tilde{C}_{30}^T Q\tilde{C}_{31} - \tilde{C}_{40}^T Q\tilde{C}_{41}\}\\ \tilde{\Phi}_2 &= \mathrm{He}\{\tilde{C}_{11}^T P\tilde{C}_{21} + \tilde{C}_{12}^T P\tilde{C}_{20} + \tilde{C}_{30}^T Q\tilde{C}_{32} - \tilde{C}_{40}^T Q\tilde{C}_{42}\}\\ &\quad + \tilde{C}_{31}^T Q\tilde{C}_{31} - \tilde{C}_{41}^T Q\tilde{C}_{41} + \mathrm{He}\{\Sigma_{j=1}^3 \aleph_j^T Q\tilde{\mathcal{D}}_{j2}\}\\ \tilde{\Phi}_3 &= \mathrm{He}\{\tilde{C}_{12}^T P\tilde{C}_{21} + \tilde{C}_{31}^T Q\tilde{C}_{32} - \tilde{C}_{41}^T Q\tilde{C}_{42} + \Sigma_{j=1}^3 \aleph_j^T Q\tilde{\mathcal{D}}_{j3}\}\\ \tilde{\Phi}_4 &= \mathrm{He}\{\aleph_2^T Q\tilde{\mathcal{D}}_{24} + \aleph_3^T Q\tilde{\mathcal{D}}_{34}\} + \tilde{C}_{32}^T Q\tilde{C}_{32} - \tilde{C}_{42}^T Q\tilde{C}_{42}\end{aligned}$$

where $\tilde{C}_0 = A\tilde{e}_1 + B\tilde{e}_2$; and

$$\begin{aligned}&\tilde{C}_{10} = \mathrm{col}\{\tilde{e}_1, \bar{h}\tilde{e}_{51}, \bar{h}^2\tilde{e}_{52}\},\ \tilde{C}_{11} = \mathrm{col}\{0, \tilde{\rho}_1, \bar{h}\tilde{\rho}_3\}\\ &\tilde{C}_{12} = \mathrm{col}\{\tilde{e}_{20}, \tilde{\rho}_2\},\ \tilde{C}_{20} = \mathrm{col}\{\tilde{C}_0, \tilde{e}_1 - \tilde{e}_3, \bar{h}(\tilde{e}_{51} - \tilde{e}_3)\}\\ &\tilde{C}_{21} = \mathrm{col}\{\tilde{e}_{20}, \tilde{\rho}_1\},\ \tilde{C}_{30} = \mathrm{col}\{\tilde{e}_1, \tilde{e}_1, \bar{h}\tilde{e}_{51}, 0, \bar{h}^2\tilde{e}_{52}, 0\}\\ &\tilde{C}_{31} = \mathrm{col}\{\tilde{e}_{20}, \tilde{\rho}_1, 0, \bar{h}\tilde{\rho}_3, 0\},\ \tilde{C}_{32} = \mathrm{col}\{\tilde{e}_{40}, \tilde{\rho}_2, 0\}\\ &\tilde{C}_{40} = \mathrm{col}\{\tilde{e}_3, \tilde{e}_1, 0, \bar{h}\tilde{e}_{51}, 0, \bar{h}^2\tilde{\rho}_4\}\\ &\tilde{C}_{41} = \mathrm{col}\{\tilde{e}_{30}, \tilde{\rho}_1, 0, \bar{h}(\tilde{e}_{41} - 2\tilde{\rho}_4)\},\ \tilde{C}_{42} = \mathrm{col}\{\tilde{e}_{50}, -\tilde{\rho}_2\}\\ &\tilde{C}_5 = \Theta_4\Lambda_4\mathrm{col}\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_4\},\ \tilde{C}_6 = \Theta_4\Lambda_4\mathrm{col}\{\tilde{e}_2, \tilde{e}_3, \tilde{e}_5\}\\ &\tilde{\mathcal{D}}_{10} = \mathrm{col}\{\bar{h}\tilde{e}_{51}, \bar{h}\tilde{e}_1, \bar{h}^2\tilde{e}_{52}, \bar{h}^2\tilde{\rho}_4, \tfrac{1}{2}\bar{h}^3\tilde{e}_{53}, \tfrac{1}{2}\bar{h}^3\tilde{\rho}_8\}\\ &\tilde{\mathcal{D}}_{11} = \mathrm{col}\{\tilde{\rho}_1, 0, \bar{h}\tilde{\rho}_3, \bar{h}(\tilde{\rho}_1 - \tilde{\rho}_3), \tfrac{1}{2}\bar{h}^2\tilde{\rho}_5, \tfrac{1}{2}\bar{h}^2\tilde{\rho}_9\}\\ &\tilde{\mathcal{D}}_{12} = \mathrm{col}\{\tilde{e}_{20}, \tilde{\rho}_2, -\tilde{\rho}_2, \tfrac{1}{2}\bar{h}\tilde{\rho}_6, \tfrac{1}{2}\bar{h}\tilde{\rho}_{10}\}\\ &\tilde{\mathcal{D}}_{13} = \mathrm{col}\{\tilde{e}_{40}, \tfrac{1}{2}\tilde{\rho}_7, \tfrac{1}{2}\tilde{\rho}_7\},\ \tilde{\mathcal{D}}_{20} = \mathrm{col}\{-\bar{h}^2\tilde{\rho}_4, -\tfrac{1}{2}\bar{h}^2\tilde{e}_1,\\ &\quad \tfrac{1}{2}\bar{h}^3(\tilde{\rho}_8 - \tilde{e}_{51}), -\tfrac{1}{2}\bar{h}^3\tilde{\rho}_8, \tfrac{1}{6}\bar{h}^4(\tilde{e}_{54} - 3\tilde{e}_{53}), \tfrac{1}{6}\bar{h}^4\tilde{\rho}_{14}\}\\ &\tilde{\mathcal{D}}_{21} = \mathrm{col}\{\bar{h}(\tilde{\rho}_3 - \tilde{\rho}_1), 0, \tfrac{1}{2}\bar{h}^2(\tilde{\rho}_9 - \tilde{\rho}_1), -\tfrac{1}{2}\bar{h}^2\tilde{\rho}_9,\\ &\quad \tfrac{1}{6}\bar{h}^3(\tilde{\rho}_{11} - 3\tilde{\rho}_5), -\tfrac{1}{6}\bar{h}^3(\tilde{e}_{41} + 4\tilde{\rho}_{14})\}\\ &\tilde{\mathcal{D}}_{22} = \mathrm{col}\{\tilde{\rho}_2, 0, \tfrac{1}{2}\bar{h}\tilde{\rho}_{10}, -\tfrac{1}{2}\bar{h}\tilde{\rho}_{10}, \tfrac{1}{6}\bar{h}^2(\tilde{\rho}_{12} - 3\tilde{\rho}_6),\\ &\quad \tfrac{1}{2}\bar{h}^2(\tilde{e}_{42} + 2\tilde{\rho}_{14})\}\\ &\tilde{\mathcal{D}}_{23} = \mathrm{col}\{\tilde{e}_{20}, \tfrac{1}{2}\tilde{\rho}_7, -\tfrac{1}{2}\tilde{\rho}_7, \tfrac{1}{6}\bar{h}(\tilde{\rho}_{13} - 3\tilde{\rho}_7), \tfrac{1}{6}\bar{h}(\tilde{\rho}_{13} - 3\tilde{\rho}_7)\}\\ &\tilde{\mathcal{D}}_{24} = \mathrm{col}\{\tilde{e}_{40}, \tfrac{1}{6}(\tilde{e}_{44} + \tilde{\rho}_{14}), \tfrac{1}{6}(\tilde{e}_{44} + \tilde{\rho}_{14})\}\\ &\tilde{\mathcal{D}}_{30} = \mathrm{col}\{\bar{h}^2\tilde{e}_{52}, \tfrac{1}{2}\bar{h}^2\tilde{e}_1, \tfrac{1}{2}\bar{h}^3\tilde{e}_{53}, \tfrac{1}{2}\bar{h}^3(\tilde{e}_{51} - \tilde{e}_{53}),\\ &\quad \tfrac{1}{6}\bar{h}^4\tilde{e}_{54}, \tfrac{1}{6}\bar{h}^4(\tilde{\rho}_{14} + 3\tilde{\rho}_8)\}\\ &\tilde{\mathcal{D}}_{31} = \mathrm{col}\{\bar{h}\tilde{\rho}_3, 0, \tfrac{1}{2}\bar{h}^2\tilde{\rho}_5, \tfrac{1}{2}\bar{h}^2(\tilde{\rho}_1 - \tilde{\rho}_5), \tfrac{1}{6}\bar{h}^3\tilde{\rho}_{11}, \tfrac{1}{6}\bar{h}^3\tilde{\rho}_{16}\}\\ &\tilde{\mathcal{D}}_{32} = \mathrm{col}\{\tilde{\rho}_2, 0, \tfrac{1}{2}\bar{h}\tilde{\rho}_6, -\tfrac{1}{2}\bar{h}\tilde{\rho}_6, \tfrac{1}{6}\bar{h}^2\tilde{\rho}_{12}, \tfrac{1}{2}\bar{h}^2\tilde{\rho}_{15}\}\\ &\tilde{\mathcal{D}}_{33} = \mathrm{col}\{\tilde{e}_{20}, \tfrac{1}{2}\tilde{\rho}_7, -\tfrac{1}{2}\tilde{\rho}_7, \tfrac{1}{6}\bar{h}\tilde{\rho}_{13}, \tfrac{1}{6}\bar{h}\tilde{\rho}_{13}\}\\ &\tilde{\mathcal{D}}_{34} = \mathrm{col}\{\tilde{e}_{40}, \tfrac{1}{6}(\tilde{\rho}_{14} + \tilde{e}_{44}), \tfrac{1}{6}(\tilde{\rho}_{14} + \tilde{e}_{44})\}\end{aligned}$$

with $\mathrm{col}\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4, \tilde{e}_5\}$ being an 11×11 identity matrix with $\tilde{e}_4 = \mathrm{col}\{\tilde{e}_{41}, \tilde{e}_{42}, \tilde{e}_{43}, \tilde{e}_{44}\}$ and $\tilde{e}_5 = \mathrm{col}\{\tilde{e}_{51}, \tilde{e}_{52}, \tilde{e}_{53}, \tilde{e}_{54}\}$, and $\tilde{e}_{j0} = 0_{jn\times 11n}$ $(j \geq 2)$; and

$$\begin{aligned}&\tilde{\rho}_1 = \tilde{e}_{41} - \tilde{e}_{51},\ \tilde{\rho}_2 = \tilde{e}_{42} + \tilde{e}_{52} - \tilde{e}_{51},\ \tilde{\rho}_3 = \tilde{e}_{51} - 2\tilde{e}_{52}\\ &\tilde{\rho}_4 = \tilde{e}_{51} - \tilde{e}_{52},\ \tilde{\rho}_5 = 2\tilde{e}_{52} - 3\tilde{e}_{53},\ \tilde{\rho}_6 = \tilde{\rho}_3 - \tilde{\rho}_5\\ &\tilde{\rho}_7 = \tilde{e}_{43} - \tilde{e}_{51} + 2\tilde{e}_{52} - \tilde{e}_{53},\ \tilde{\rho}_8 = \tilde{e}_{53} + \tilde{\rho}_3\\ &\tilde{\rho}_9 = \tilde{e}_{41} - 3\tilde{\rho}_8,\ \tilde{\rho}_{10} = 3\tilde{\rho}_8 - 2\tilde{e}_{42},\ \tilde{\rho}_{11} = 3\tilde{e}_{53} - 4\tilde{e}_{54}\\ &\tilde{\rho}_{12} = 3\tilde{e}_{52} - 9\tilde{e}_{53} + 6\tilde{e}_{54},\ \tilde{\rho}_{13} = \tilde{e}_{51} - 3\tilde{\rho}_5 - 4\tilde{e}_{54}\\ &\tilde{\rho}_{14} = \tilde{e}_{54} - \tilde{\rho}_4 + \tilde{\rho}_5,\ \tilde{\rho}_{15} = \tilde{e}_{51} - 3\tilde{e}_{53} + 2\tilde{e}_{54} - \tilde{e}_{42}\\ &\tilde{\rho}_{16} = 3\tilde{\rho}_9 - 4\tilde{\rho}_{14} - \tilde{e}_{41},\ \aleph_1 = \mathrm{col}\{0, \tilde{C}_0, -\tilde{e}_3, \tilde{e}_1, \tilde{e}_{20}\}\\ &\aleph_2 = \mathrm{col}\{\tilde{e}_{40}, \tilde{e}_3, 0\},\ \aleph_3 = \mathrm{col}\{\tilde{e}_{50}, \tilde{e}_1\}.\end{aligned}$$

Proof: First, taking the time-derivative of $\tilde{V}(t, x_t)$ yields

$$\begin{aligned}\dot{\tilde{V}}(t,x_t) &= 2\tilde{\psi}_1^T(t)P\dot{\tilde{\psi}}_1(t) - \tilde{\psi}_2^T(t-\bar{h},t)Q\tilde{\psi}_2(t-\bar{h},t)\\ &\quad + \tilde{\psi}_2^T(t,t)Q\tilde{\psi}_2(t,t) + \int_{t-\bar{h}}^t 2\tilde{\psi}_2^T(s,t)Q\frac{\partial}{\partial t}\tilde{\psi}_2(s,t)ds\\ &\quad + \bar{h}^2\dot{x}^T(t)R\dot{x}(t) - \bar{h}\int_{t-\bar{h}}^t \dot{x}^T(s)R\dot{x}(s)ds.\end{aligned} \tag{37}$$

Let $\tilde{\xi}(t) = \mathrm{col}\{x(t), x(t-d(t)), x(t-\bar{h}), v_1(t), v_2(t)\}$. Then

$$\begin{aligned}\tilde{\psi}_1(t) &= \tilde{C}_{10} + d(t)\tilde{C}_{11} + d^2(t)\tilde{C}_{12}\\ \dot{\tilde{\psi}}_1(t) &= \tilde{C}_{20} + d(t)\tilde{C}_{21}\\ \tilde{\psi}_2(t,t) &= \tilde{C}_{30} + d(t)\tilde{C}_{31} + d^2(t)\tilde{C}_{32}\\ \tilde{\psi}_2(t-\bar{h},t) &= \tilde{C}_{40} + d(t)\tilde{C}_{41} + d^2(t)\tilde{C}_{42}.\end{aligned}$$

Note that

$$\frac{\partial}{\partial t}\tilde{\psi}_2(s,t) = \aleph_1 + (t-\bar{h}-s)\aleph_2 + (t-s)\aleph_3$$

which leads to

$$\begin{aligned}\int_{t-\bar{h}}^t 2\tilde{\psi}_2^T(s,t)Q\frac{\partial}{\partial t}\tilde{\psi}_2(s,t)ds &= 2\aleph_1^T Q\int_{t-\bar{h}}^t \tilde{\psi}_2(s,t)ds\\ &\quad + 2\aleph_2^T Q\int_{t-\bar{h}}^t (t-\bar{h}-s)\tilde{\psi}_2(s,t)ds\\ &\quad + 2\aleph_3^T Q\int_{t-\bar{h}}^t (t-s)\tilde{\psi}_2(s,t)ds\end{aligned}$$

After some algebraic manipulations, one has

$$\begin{aligned}\int_{t-\bar{h}}^t \tilde{\psi}_2(s,t)ds &= \sum_{j=0}^3 d^j(t)\tilde{\mathcal{D}}_{1j}\\ \int_{t-\bar{h}}^t (t-\bar{h}-s)\tilde{\psi}_2(s,t)ds &= \sum_{j=0}^4 d^j(t)\tilde{\mathcal{D}}_{2j}\\ \int_{t-\bar{h}}^t (t-s)\tilde{\psi}_2(s,t)ds &= \sum_{j=0}^4 d^j(t)\tilde{\mathcal{D}}_{3j}.\end{aligned}$$

On the other hand

$$-\bar{h}\int_{t-\bar{h}}^t \dot{x}^T(s)R\dot{x}(s)ds \leq \tilde{\xi}^T(t)[\tilde{\Psi}_{11} + d(t)\tilde{\Psi}_{12}]\tilde{\xi}(t)$$

where

$$\begin{aligned}\tilde{\Psi}_{11} &= -\tilde{C}_5^T(\mathcal{R} + Z_1)\tilde{C}_5 - \tilde{C}_6^T\mathcal{R}\tilde{C}_6 - \mathrm{He}\{\tilde{C}_5^T Y_2\tilde{C}_6\}\\ \tilde{\Psi}_{12} &= \tfrac{1}{\bar{h}}[\tilde{C}_5^T Z_1\tilde{C}_5 - \tilde{C}_6^T Z_2\tilde{C}_6 - \mathrm{He}\{\tilde{C}_5^T(Y_1 - Y_2)\tilde{C}_6\}]\end{aligned}$$

To sum up, one has that

$$\dot{\tilde{V}}(t,x_t) \leq \tilde{\xi}^T(t)\sum_{j=0}^4 d^j(t)\tilde{\Phi}_j\tilde{\xi}(t) \tag{38}$$

where $\tilde{\Phi}_j$ $(j = 0, 1, \ldots, 4)$ are defined in Proposition 2. If the conditions in (34)–(36) are satisfied, there exists a scalar $\varepsilon > 0$ such that $\dot{\tilde{V}}(t,x_t) \leq -\varepsilon\tilde{\xi}^T(t)\tilde{\xi}(t) \leq -\varepsilon x^T(t)x(t)$, leading to the asymptotic stability of the system (14) subject to (15). ■

Remark 4: From the proof of Proposition 2, one can see that the introduction of $\int_{t-\bar{h}}^s (s-\theta)x(\theta)d\theta$ and $\int_s^t (\theta - s)x(\theta)d\theta$ into $\tilde{V}_1(t,x_t)$ yields a quartic polynomial on $d(t)$ in $\dot{\tilde{V}}(t,x_t)$, see, (38). If we introduce more vectors such as $\int_{t-\bar{h}}^s (s-\theta)^2 x(\theta)d\theta$ and $\int_s^t (\theta - s)^2 x(\theta)d\theta$ into $\tilde{V}_1(t,x_t)$, then a sixth-degree polynomial will appear in $\dot{\tilde{V}}(t,x_t)$. However, it is not an easy task to express explicitly such a sixth-degree polynomial.

IV. Numerical Example

In this section, we take a well-studied example to compare Propositions 1 and 2 with some existing stability criteria recently reported.

Example 1: Consider the system (14) with

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix},\ B = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} \tag{39}$$

Case 1: Suppose that the time-varying delay satisfies (15).

In this case, we calculate the maximum admissible upper bound of $\bar{h}$ for $\mu = -\mu_1 = \mu_2 \in \{0.1, 0.5, 0.8\}$. Table I lists the obtained results using some existing methods [23, Theorem 1], [24, Theorem 1], [19, Theorem 8], [13, Proposition 2] and [18, Corollary 2]. However, applying Proposition 1 in this paper gives much larger upper bounds of $\bar{h}$, which can be seen in Table I.

On the other hand, Table I also lists the number of decision variables (DVs) required in those methods. It is clear to see that Proposition 1 requires a larger number of DVs, which means that solving the matrix inequalities in Proposition 1 is much time-consuming. However, with the rapid development of computer technology, such a number of DVs is not a problem for high performance computers to solve the matrix inequalities in Proposition 1.

Case 2: Suppose that the time-varying delay satisfies (16).

In this case, we use Proposition 2 to compare with some existing methods [13], [25]–[27]. Table II lists both the obtained delay upper bounds and the required number of DVs by [26, Corollary 1], [27, Proposition 6], [25, Theorem 2], [13, Proposition 6] and Proposition 2 in this paper. From Table II, one can see that Proposition 2 delivers a larger upper bound $\bar{h} = 3.04$ than those in [13], [25]–[27]. Moreover, the number of DVs required in Proposition 2 is less than those of [25, Theorem 2] and [13, Proposition 6].

TABLE I
The Maximum Admissible Upper Bound $\bar{h}$ for Different Values of $\mu = -\mu_1 = \mu_2$

Method \ μ	0.1	0.5	0.8	Number of DVs
[22, Proposition 1]	4.910	3.233	2.789	$54.5n^2 + 6.5n$
[23, Theorem 1]	4.942	3.309	2.882	$108n^2 + 12n$
[24, Theorem 1]	4.996	3.251	2.867	$38n^2 + 9n$
[19, Theorem 8]	5.01	3.19	2.70	$146.5n^2 + 9.5n$
[13, Proposition 2]	4.929	3.252	2.823	$216n^2 + 11n$
[18, Corollary 2]	5.044	3.443	2.983	$235n^2 + 34n$
Proposition 1	5.147	3.673	3.258	$367n^2 + 35n + 6$

TABLE II
The Maximum Admissible Upper Bound $\bar{h}$ for Case 2

Method	$\bar{h}$	Number of DVs
[26, Corollary 1]	2.18	$54.5n^2 + 9.5n$
[27, Proposition 1]	2.18	$54n^2 + 9n$
[25, Theorem 2]	2.39	$154.5n^2 + 4.5n$
[13, Proposition 6]	2.53	$119.5n^2 + 3.5n$
Proposition 2	3.04	$98n^2 + 10n + 2$

V. Conclusion

Stability of linear systems with a time-varying delay has been studied. First, a necessary and sufficient condition on matrix-valued polynomial inequalities has been established. Then, this condition has been employed to formulate two stability criteria for two cases of the time-varying delay, respectively, where the time-varying delay is differentiable or only continuous. Simulation has shown that the obtained stability criteria can provide larger delay upper bounds than some existing ones.

References

[1] X.-M. Zhang, Q.-L. Han, X. Ge, D. Ding, L. Ding, D. Yue, and C. Peng, "Networked control systems: a survey of trends and techniques," *IEEE/CAA J. Autom. Sinica*, vol. 7, no. 1, pp. 1–17, Jan. 2020.

[2] Y.-L. Wang and Q.-L. Han, "Network-based modelling and dynamic output feedback control for unmanned marine vehicles in network environments," *Automatica*, vol. 91, pp. 43–53, 2018.

[3] B.-L. Zhang, Q.-L. Han, X.-M. Zhang, and X. Yu, "Sliding mode control with mixed current and delayed states for offshore steel jacket platforms," *IEEE Trans. Control Syst. Technol*, vol. 22, no. 5, pp. 1769–1783, May 2014.

[4] W. Ren and Y. Cao, *Distributed Coordination of Multi-agent Networks*: *Emergent Problems*, *Models*, *and Issues*. Springer Science & Business Media, 2010.

[5] Q. Song, W. Yu, J. Cao, and F. Liu, "Reaching synchronization in networked harmonic oscillators with outdated position data," *IEEE Trans. Cybern.*, vol. 46, no. 7, pp. 1566–1578, Jul. 2016.

[6] V. B. Kolmanovskii and V. R. Nosov, *Stability of functional differential equations*, Academic Press, London, 1986.

[7] K. Gu, L. Kharitonov, and J. Chen, *Stability of Time-delay Systems*, Birkhaüser, Boston, 2003.

[8] J. Richard, "Time-delay systems: An overview of some recent advances and open problems," *Automatica*, vol. 39, pp. 1667–1694, 2003.

[9] X.-M. Zhang, Q.-L. Han, and J. Wang, "Admissible delay upper bounds for global asymptotic stability of neural networks with time-varying delays," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 29, no. 11, pp. 5319–5329, Nov. 2018.

[10] Y. Chen and G. Chen, "Stability analysis of systems with timevarying delay via a novel Lyapunov functional," *IEEE/CAA J. Autom. Sinica*, vol. 6, no. 4, pp. 1068–1073, Jul. 2019.

[11] Y. Zhang, K. Lou, and Y. Ge, "New result on delay-dependent stability for Markovian jump time-delay systems with partial information on transition probabilities," *IEEE/CAA J. Autom. Sinica*, vol. 6, no. 6, pp. 1499–1505, Nov. 2019.

[12] A. Seuret and F. Gouaisbaut, "Wirtinger-based integral inequality: Application to time-delay systems," *Automatica*, vol. 49, no. 9, pp. 2860–2866, Sep. 2013.

[13] X.-M. Zhang, Q.-L. Han, A. Seuret, F. Gouaisbaut, and Y. He, "Overview of recent advances in stability of linear systems with timevarying delays," *IET Control Theory Appl.*, vol. 13, no. 1, pp. 1–16, Jan. 2019.

[14] J.-H. Kim, "Further improvement of Jensen inequality and application to stability of time-delayed systems," *Automatica*, vol. 64, pp. 121–125, Feb. 2016.

[15] C.-K. Zhang, Y. He, L. Jiang, W.-J. Lin and M. Wu, "Delay-dependent stability analysis of neural networks with time-varying delay: A generalized free-weightingmatrix approach," *Appl. Math. Comput.*, vol. 294, pp. 102–120, 2017.

[16] M. R. Graham and M. C. de Oliveira, "Robust analysis with respect to real parameter uncertainty," in *Proc. 47th IEEE Conf. Decision Control*, Cancun, Mexico, pp. 2894–2899, Dec. 2008.

[17] M. C. de Oliveira and R. E. Skelton, *Stability Tests for Constrained*

Linear Systems. Berlin, Germany: Springer-Verlag, 2001.

[18] F. de Oliveira and F. Souza, "Further refinements in stability conditions for time-varying delay systems," *Appl. Math. Comput*.. vol. 369, Article 124866, Mar. 2020.

[19] A. Seuret and F. Gouaisbaut, "Stability of linear systems with timevarying delays using Bessel-Legendre inequalities," *IEEE Trans. Autom. Control*, vol. 63, no. 1, pp. 225–232, Jan. 2018.

[20] X.-M. Zhang, Q.-L. Han, and Z. Zeng, "Hierarchical type stability criteria for delayed neural networks via canonical Bessel-Legendre inequalities," *IEEE Trans. Cybern*, vol. 48, no. 5, pp. 1660–1671, May 2018.

[21] A. Seuret and F. Gouaisbaut, "Delay-dependent reciprocally convex combination lemma," *Rapport LAAS n*16006, 2016.

[22] X.-M. Zhang, Q.-L. Han, A. Seuret, and F. Gouaisbaut, "An improved reciprocally convex inequality and an augmented Lyapunov-Krasovskii functional for stability of linear systems with time-varying delay," *Automatica*, vol. 84, pp. 221–226, Oct. 2017.

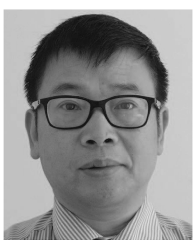
[23] J. Chen, J. H. Park, and S. Xu, "Stability analysis of continuoustime systems with time-varying delay using new Lyapunov-Krasovskii functionals," *J. Franklin Inst.*, vol. 355, no. 13, pp. 5957–5967, Sep. 2018.

[24] Z. Li, H. Yan, H. Zhang, X. Zhan, and C. Huang, "Improved inequalitybased functions approach for stability analysis of time delay system," *Automatica*, vol. 108, Article 108416, Oct. 2019.

[25] K. Liu, A. Seuret, and Y. Xia, "Stability analysis of systems with time-varying delays via the second-order Bessel-Legendre inequality," *Automatica*, vol. 79, pp. 138–142, 2017.

[26] H. Zeng, Y. He, M. Wu, and J. She, "Free-matrix-based integral inequality for stability analysis of systems with time-varying delay," *IEEE Trans. Autom. Control*, vol. 60, no. 10, pp. 2768–2772, Oct. 2015.

[27] J. Chen, S. Xu, B. Zhang, and G. Liu, "A note on relationship between two classes of integral inequalities," *IEEE Trans. Autom. Control*, vol. 62, no. 8, pp. 4044–4049, Aug. 2017.



Xian-Ming Zhang (M'16–SM'18) received the M.Sc. degree in applied mathematics and the Ph.D. degree in control theory and engineering from Central South University, Changsha, China, in 1992 and 2006, respectively.

In 1992, he joined Central South University, where he was an Associate Professor with the School of Mathematics and Statistics. From 2007 to 2014, he was a Post-Doctoral Research Fellow and a Lecturer with the School of Engineering and Technology, Central Queensland University, Rockhampton, QLD, Australia. From 2014 to 2016, he was a Lecturer with the Griffith School of Engineering, Griffith University, Gold Coast, QLD, Australia. In 2016, he joined the Swinburne University of Technology, Melbourne, VIC, Australia, where he is currently an Associate Professor with the School of Software and Electrical Engineering. His current research interests include H-infinity filtering, event-triggered control systems, networked control systems, neural networks, distributed systems, and time-delay systems.

Dr. Zhang was a recipient of second National Natural Science Award in China in 2013, and first Hunan Provincial Natural Science Award in Hunan Province in China in 2011, both jointly with Prof. M. Wu and Prof. Y. He, and the IET Premium Award in 2016, jointly with Prof. Q.-L. Han. He is an Associate Editor of the *IEEE Transactions on Cybernetics*, *Neural Processing Letters*, *Journal of the Franklin Institute*, *International Journal of Control*, *Automation*, *and Systems* and the *Neurocomputing*, and he is an Editorial Board Member of *Neural Computing and Applications*.

Qing-Long Han (M'09–SM'13–F'19) received the B.Sc. degree in Mathematics from Shandong Normal University, Jinan, China, in 1983, and the M.Sc. and Ph.D. degrees in Control Engineering and Electrical Engineering from East China University of Science and Technology, Shanghai, China, in 1992 and 1997, respectively.

From September 1997 to December 1998, he was a Post-doctoral Researcher Fellow with the Laboratoire d'Automatique et d'Informatique Industielle (currently, Laboratoire d'Informatique et d'Automatique pour les Systémes), École Supérieure d'Ing'enieurs de Poitiers (currently, École Nationale Supérieure d'Ingénieurs de Poitiers), Université de Poitiers, France. From January 1999 to August 2001, he was a Research Assistant Professor with the Department of Mechanical and Industrial Engineering at Southern Illinois University at Edwardsville, USA. From September 2001 to December 2014, he was Laureate Professor, an Associate Dean (Research and Innovation) with the Higher Education Division, and the Founding Director of the Centre for Intelligent and Networked Systems at Central Queensland University, Australia. From December 2014 to May 2016, he was Deputy Dean (Research), with the Griffith Sciences, and a Professor with the Griffith School of Engineering, Griffith University, Australia. In May 2016, he joined Swinburne University of Technology, Australia, where he is currently Pro Vice-Chancellor (Research Quality) and a Distinguished Professor. His research interests include networked control systems, multi-agent systems, time-delay systems, complex dynamical systems and neural networks.

Professor Han is a Highly Cited Researcher according to Clarivate Analytics (formerly Thomson Reuters). He is a Fellow of The Institution of Engineers Australia. He is an Associate Editor of several international journals, including the *IEEE Transactions on Cybernetics*, the *IEEE Transactions on Industrial Electronics*, the *IEEE Transactions on Industrial Informatics*, *IEEE Industrial Electronics Magazine*, the *IEEE/CAA Journal of Automatica Sinica*, *Control Engineering Practice*, and *Information Sciences*.

Xiaohua Ge (M'18) received the B.Eng. degree in electronics and information engineering from Nanchang Hangkong University, Nanchang, China, in 2008, the M.Eng. degree in control theory and control engineering from Hangzhou Dianzi University, Hangzhou, China, in 2011, and the Ph.D. degree in computer engineering from Central Queensland University, Rockhampton, QLD, Australia, in 2014.

From 2011 to 2013, he was a Research Assistant with the Centre for Intelligent and Networked Systems, Central Queensland University, Rockhampton, QLD, Australia, where he was a Research Fellow in 2014. From 2015 to 2017, he was a Research Fellow with the Griffith School of Engineering, Griffith University, Gold Coast, QLD, Australia. He is currently a Senior Lecturer with the School of Software and Electrical Engineering, Swinburne University of Technology, Melbourne, VIC, Australia.

His research interests include distributed estimation over sensor networks, distributed coordination in multi-agent systems, security and privacy preserving in cyber-physical systems.