On Performance Gauge of Average Multi-Cue Multi-Choice Decision Making: A Converse Lyapunov Approach

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***Abstract*—Motivated by the converse Lyapunov technique for investigating converse results of semistable switched systems in control theory, this paper utilizes a constructive induction method to identify a cost function for performance gauge of an average, multi-cue multi-choice (MCMC), cognitive decision making model over a switching time interval. It shows that such a constructive cost function can be evaluated through an abstract energy called Lyapunov function at initial conditions. Hence, the performance gauge problem for the average MCMC model becomes the issue of finding such a Lyapunov function, leading to a possible way for designing corresponding computational algorithms via iterative methods such as adaptive dynamic programming. In order to reach this goal, a series of technical results are presented for the construction of such a Lyapunov function and its mathematical properties are discussed in details. Finally, a major result of guaranteeing the existence of such a Lyapunov function is rigorously proved.**

***Index Terms*—Cognitive modeling, decision making, Lyapunov function, multi-cue multi-choice tasks, performance gauge.**

I. Introduction

FOR many critical infrastructure systems, such as power transmission networks, water distribution systems, and gas pipeline networks, human operators are naturally part of the system decision making process in which they have absolute authority to hold off the decision made by automation or control systems. The question of how the human operators' decision will affect the performance of such systems is the main challenge in designing functional, human-intelligent control systems [1]. Also the idea of exploring the possibility of optimizing human-in-the-loop decision making by mainly controlling the parts that may affect human decision performance sounds plausible with the ever increasing presence of artificial intelligence in daily life. For this purpose, the first logical step is to develop appropriate models for human decision process in the control system by integrating cognitive perspectives on human decision making.

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The concept of temporal integration of evidence through sequential probability ratio test (SPRT) has been widely employed in decision-making modeling studies [2]. According to the drift diffusion model (DDM) to which SPRT converges in its continuum limit, decisions are made by accumulating noisy stimulus information until the decision variable reaches either positive or negative threshold in two-alternative forced choice (2AFC) tasks. The DDM has been proven to be successful in emulating the process of decision, and due to its connection to SPRT, it is optimal in a sense of maximizing any reward criterion that is monotonically decreasing with respect to decision time [3]. In other words, DDM renders the shortest possible decision time, given a specific accuracy. However, as pointed out by [4], this optimality description does not consider any cost associated with behavior, or a cost function for gathering information dynamically.

By increasing the number of decision choices and attributes in multi-choice multi-cue tasks (MCMC), race models with mutual inhibition were adopted to depict the decision process [3]. Although these models are intuitively plausible in describing the decision process, the notion of asymptotic optimality of SPRT [5] cannot be applied. Note that for the specific case of two-choice tasks, inhibition models can be reduced to DDM, and hence, renders the optimal solution under specific circumstances [6]. This leads to the question of how to address optimality for general MCMC tasks. Even before we talk about optimality, a cost associated with optimal performance needs to be defined and its evaluation needs to be tackled.

In this work we take a control-theoretic approach to tackle the performance evaluation of MCMC tasks modeled by mutual inhibition race pools, as discussed later in Section II. Our focus is to construct a performance gauge, which accounts for the performance of average MCMC and at the same time deals with the potential sources of deviation from optimality at the psychological level [7]. To do so, we first briefly review the mathematical abstract models in decision making in Section II and then formulate our proposed problem in Section III. In Section IV, some mathematical preliminaries of a novel method, called converse Lyapunov approach, are developed to prepare for construction of such a performance gauge later, which is based on converse Lyapunov results for switched and nonlinear semistable systems. As a major contribution of the paper, a performance gauge function is constructed in Section V. Finally, some conclusion about this work is drawn in Section VI.

II. Review of Decision Making Models

In cognitive and behavioral sciences, models of describing the process of making decision in human brain start with analyzing the simple 2AFC decision task. Using the optimal SPRT test, the process on its continuum limit converges to the DDM if symmetric threshold is assumed. In this case, the decision variable for a noisy evidence can be modeled by one-dimensional Wiener process bounded by positive and negative thresholds, θ_A and θ_B, in which an integrator accumulates the difference of evidences between two choices [3].

For 2AFC, only one cue is concerned. In real world situations, always several cues are involved. One method is to combine and integrate all cues in favor of each choice into single source of evidence and this source is being used throughout the decision process. More involved treatment includes separate processes for each cue. In this approach the order of considering the cues and the process time devoted to each cue are two important aspects. The time frame of the decision process is divided to subintervals with different lengths during which the attention focus is only one cue [8].

In order to model multi-choice tasks, a more general race model, which is comprised of separate leaky competing integrators, representing each choice, with mutual inhibition, was proposed in [9], where each integrator gathers information in favor of or against the associated choice based on the value of cue. Based on this idea, in our preliminary work [10], we have proposed the following leaky integrator race model to describe the dynamics of MCMC tasks. This model combines the race model and time and order scheduling concept as follows:

$$dx_i(t) = (-k_{m(t)}x_i(t) - \sum_{j\neq i} w_{m(t)}x_j(t) + S_{i,m(t)})dt + \sigma_i d\mathbf{W}_i(t), \quad m(t)\in\mathcal{M}, \quad t_{l-1}\le t < t_l \tag{1}$$

where $x_i(t)\in\mathbb{R}$ is the decision variable for choice i which is an abstract measure of amount of evidence information gathered at time t [3], $\mathbb{R}$ is the set of real numbers, $m:[0,\infty)\to\mathcal{M}$ is a piecewise constant cue switching signal, k_m denotes the leak, w_m is the mutual inhibition strength among pools, $S_{i,m}$ is the external input, σ_i is the diffusion rate, $\mathbf{W}_i(t)$ is the standard Wiener process, and $d\mathbf{W}_i(t)$ is the standard white noise. If finite decision time span is divided to L consecutive time intervals, $[t_{l-1}, t_l)$ for $l = 1,\ldots,L$, where t_l denotes the time instant switching from one cue to another, we assume that one cue is processed in each time interval based on the given order schedule.

However, the model (1) does not consider a different impact on the corresponding integrator at each time span $[t_{l-1}, t_l)$. This scenario is quite common when a decision maker faces different situations and prioritizes some tasks by weighing associated choices differently based on different choices. Hence, the MCMC model given by (1) can be further generalized as [11]:

$$dx_i(t) = (-k_{i,m(t)}x_i(t) - \sum_{j\neq i} w_{i,j,m(t)}x_j(t) + S_{i,m(t)})dt + \sigma_i d\mathbf{W}_i(t), \quad m(t)\in\mathcal{M}, \quad t_{l-1}\le t < t_l \tag{2}$$

where $k_{i,m}$ depends on both choice i and cue m, $w_{i,j,m}$ depends on choice i, choice j, and cue m, and $w_{i,j,m} = w_{j,i,m}$. The leak parameter $k_{i,m}$ is determined by the inhibitory and excitatory gains [10], [11] derived from the spiking neural network [12] for human brain. Depending on the decision making situation (normal versus emergency) and rationality of decision, these parameters may lead (2) to generate optimal decision if the situation is normal [10], [11], while they may lead (2) to generate heuristic decision if the situation is an emergency [10], [11], or polarized decision if antagonistic information is inevitable for group decision [13], [14].

III. Problem Formulation

In this paper, we consider the mean or average dynamics of (2) under the case where its initial condition $x_i(0)$ is random with a mean that may not be zero, meaning that the initial perception may be biased. Let $z_i(t) = E[x_i(t)]$, where E denotes the expectation operator. Then the mean or first moment equation of (2) is given by

$$\dot z_i(t) = -k_{i,m(t)}z_i(t) - \sum_{j\neq i} w_{i,j,m(t)}z_j(t) + S_{i,m(t)}$$

or in vector form

$$\dot Z(t) = G_{m(t)}Z(t) + S_{m(t)},\ m(t)\in\mathcal{M},\ t_{l-1}\le t < t_l \tag{3}$$

where $t\ge 0$, $\dot{()} = d()/dt$, $z(t) = [z_1(t),\ldots,z_n(t)]^T\in\mathbb{R}^n$, $(\cdot)^T$ denotes the transpose operation, $\mathbb{R}^n$ denotes the set of n-dimensional real column vectors, $G_{m(t)}\in\mathbb{R}^{n\times n}$ is a square matrix whose (i,i)th element is $-k_{i,m(t)}$ and (i,j)th element is $-\sum_{j\neq i} w_{i,j,m(t)}$, $i\neq j$, $i,j = 1,\ldots,n$, $\mathbb{R}^{p\times q}$ denotes the set of p-by-q real matrices, $S_{m(t)} = [S_{1,m(t)},\ldots,S_{n,m(t)}]^T\in\mathbb{R}^n$, $l = 1,\ldots,L$, and $\mathcal{M} = \{1,2,\ldots,M\}$ which is *finite*.

In this paper, we hypothesize that the incoming expected evidence vector $S_{m(t)}$ can be decomposed into the following way.

Hypothesis 1: For each $p\in\mathcal{M}$, there exist a symmetric matrix H_p and a vector b_p such that $S_p = H_pZ + b_p$.

Hypothesis 1 states that the incoming evidence can be decomposed into a mutual inhibition term H_pZ and a remained stimulus term b_p. From the control-theoretic perspective, this implies that S_p can be divided into a state feedback term H_pZ plus a residual disturbance term b_p. This postulate has been used to derive the well established mutual inhibition model for decision making, based on the race model (see Example 1).

Under Hypothesis 1, (3) can be written as

$$\dot Z(t) = A_{m(t)}Z(t) + b_{m(t)},\ m(t)\in\mathcal{M},\ t_{l-1}\le t < t_l \tag{4}$$

where $A_{m(t)} = G_{m(t)} + H_{m(t)}$, which is symmetric.

We call a function $Z^*(t)$ a *piecewise constant solution* of (4) if $Z^*(t)$ is a constant over $t\in[t_{l-1}, t_l)$ and $A_{m(t)}Z^*(t) + b_{m(t)} = 0$ for all $t\ge 0$. If (4) has a piecewise constant solution $Z^*(t)$, then define $Y(t) = Z(t) - Z^*(t)$, and hence, for $t\in[t_{l-1}, t_l)$, the error dynamics is given by $\dot Y(t) = \dot Z(t) = A_{m(t)}Z(t) + b_{m(t)} = A_{m(t)}Z(t) - A_{m(t)}Z^*(t) = A_{m(t)}(Z(t) - Z^*(t)) = A_{m(t)}Y(t)$, i.e.,

$$\dot{Y}(t) = A_{m(t)}Y(t), \quad m(t) \in \mathcal{M}, \quad t_{l-1} \le t < t_l \tag{5}$$

which is an autonomous, switched linear system. Hence, throughout the paper, we make the following hypothesis.

Hypothesis 2: (4) has a piecewise constant solution.

Hypothesis 2 assumes that if we consider a single-cue subsystem for (4), then it is possible to reach its steady state through some feedback stimulus. The following result gives a necessary and sufficient condition for (4) having a piecewise constant solution, which can be used to test Hypothesis 2.

Lemma 1: The system (4) has a piecewise constant solution if and only if $\text{rank}[A_{m(t)}] = \text{rank}\begin{bmatrix} A_{m(t)} & -b_{m(t)} \end{bmatrix}$ for all $t \ge 0$, where $\text{rank}[\cdot]$ denotes the rank operation. A sufficient condition to guarantee (4) having a piecewise constant solution is $\text{rank}[A_m] = \text{rank}\begin{bmatrix} A_m & -b_m \end{bmatrix}$ for all $m \in \mathcal{M}$.

Proof: The conclusions follow from the fact that $Ax = b$ has a solution x if and only if $\text{rank}[A] = \text{rank}\begin{bmatrix} A & b \end{bmatrix}$, where $A \in \mathbb{R}^{n\times n}$, $x \in \mathbb{R}^n$, and $b \in \mathbb{R}^n$ (e.g., see [15]). ■

Example 1: Consider the mutual inhibition model [9] given by

$$dx_1(t) = (-kx_1(t) - wx_2(t) + b_1)dt + \sigma d\mathbf{W}_1(t) \tag{6}$$

$$dx_2(t) = (-kx_2(t) - wx_1(t) + b_2)dt + \sigma d\mathbf{W}_2(t) \tag{7}$$

where $k, w \ne 0$. This model can be viewed as a derivation from the following race model [16] based on Hypothesis 1:

$$dx_1(t) = S_1 dt + \sigma d\mathbf{W}_1(t) \tag{8}$$

$$dx_2(t) = S_2 dt + \sigma d\mathbf{W}_2(t) \tag{9}$$

where $\begin{bmatrix} S_1 \\ S_2 \end{bmatrix} = \begin{bmatrix} -k & -w \\ -w & -k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$. In this case, the mean model of (6) and (7) can be written as $\dot{Z}(t) = AZ(t) + b$, where $A = \begin{bmatrix} -k & -w \\ -w & -k \end{bmatrix}$ and $b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$. Clearly A is symmetric.

If $k \ne \pm w$, then $\text{rank}[A] = 2 = \text{rank}\begin{bmatrix} A & -b \end{bmatrix}$ for any $b \in \mathbb{R}^2$. In this case, the system (6) and (7) has a constant solution and $\ker(A) = \{0\}$, where $\ker(A)$ denotes the null space of A.

Alternatively, if $k = w$ and $b_1 = b_2$, or $k = -w$ and $b_1 = -b_2$, then $\text{rank}[A] = 1 = \text{rank}\begin{bmatrix} A & -b \end{bmatrix}$. In this case, the system (6) and (7) still has a constant solution and $\ker(A) = \{\alpha[1,1]^T : \alpha \in \mathbb{R}\}$ or $\ker(A) = \{\alpha[1,-1]^T : \alpha \in \mathbb{R}\}$.

It has been shown in [3] that the DDM solves 2AFC problems optimally: It will on average return a decision in the shortest possible time for a specified level of accuracy. However, as pointed out by [4], this optimality description does not consider any cost associated with behavior, or a cost function for gathering information dynamically. Thus, in this paper, we will focus on the performance cost function that involves dynamic behavior of information gathering for the average MCMC model (4). The core question that this paper attempts to address is

Question 1: Does there exist a cost function associated with (4) to gauge its dynamic performance over finite or infinite horizon?

The answer to this question is imperative since it can serve as the first step toward finding optimal decision making strategies for MCMC tasks by evaluating their performance gauge. We will give a positive answer to this question by constructing such a performance cost function for average MCMC models. Before we show this main result, some mathematical preliminaries are needed in the next section.

IV. Mathematical Preliminaries

In this section, we will present some mathematical preliminaries for a general switched linear system motivated by (5). Specifically, consider the following switched linear system:

$$\dot{x}(t) = A_{\sigma(t)}x(t): \quad t \ge 0, \quad x(0) = x_0 \tag{10}$$

where $x(t) \in \mathbb{R}^n$, $A_\sigma \in \mathbb{R}^{n\times n}$, and $\sigma : [0,\infty) \to \mathcal{M} = \{1,\ldots,M\}$ is a piecewise constant switching signal. Define the *equilibrium set* for (10) as $\ker(A_p)$ for every $p \in \mathcal{M}$. Recall from Example 1 that $\ker(A_p)$ may not always be trivial. In order to construct a performance cost function for (10), we need to find a metric to gauge the long-term dynamic behavior of (10). To this end, we will make several assumptions for (10) to narrow down our discussion and to have a meaningful result. The first assumption is motivated by the MCMC model (2) or symmetric matrix A in Example 1.

Assumption 1: For every $p \in \mathcal{M}$, A_p is symmetric.

The second one is for the equilibrium set of (10) inspired by the discussion for $\ker(A)$ in Example 1.

Assumption 2: For every $p, q \in \mathcal{M}$, $\ker(A_p) = \ker(A_q)$.

Assumption 2 ensures that the possible results for steady-state decision making under different cues will be consistent. Let $\mathcal{E}_s = \ker(A_p)$ under Assumption 2. The next result gives a necessary and sufficient condition for (10) having an identical equilibrium set for every $p \in \mathcal{M}$, which can be used to verify Assumption 2.

Lemma 2: For every $p, q \in \mathcal{M}$, $\ker(A_p) = \ker(A_q)$ if and only if $\text{rank}[A_p] = \text{rank}[A_q] = \text{rank}\begin{bmatrix} A_p \\ A_q \end{bmatrix}$ for every $p, q \in \mathcal{M}$.

Proof: The result follows from the fact that $\ker(A) = \ker(B)$ if and only if $\text{rank}(A) = \text{rank}(B) = \text{rank}\begin{bmatrix} A \\ B \end{bmatrix}$ for $A \in \mathbb{R}^{n\times m}$ and $B \in \mathbb{R}^{n\times l}$ (e.g., see [15]). ■

Let $[\cdot,\cdot]$ denote the Lie bracket [17]. The next assumption is motivated by the mutual inhibition model in Example 1 and the fact that $\begin{bmatrix} -k & -w \\ -w & -k \end{bmatrix}\begin{bmatrix} -k' & -w' \\ -w' & -k' \end{bmatrix} = \begin{bmatrix} -k' & -w' \\ -w' & -k' \end{bmatrix}\begin{bmatrix} -k & -w \\ -w & -k \end{bmatrix}$, where $k, k', w, w' \in \mathbb{R}$.

Assumption 3: For every $p, q \in \mathcal{M}$ and any $x \in \mathbb{R}^n$, $[A_p x, A_q x] = 0$.

Assumption 3 indicates that the flows of (10) following the directions of $A_p x$ and $A_q x$ define a surface, with $A_p x$ and $A_q x$ as coordinate vector fields. It guarantees the commutativity property of pairwise local flows of (10) [18], [19] and two matrices A_p and A_q [20], i.e., $A_p A_q = A_q A_p$ for every $p, q \in \mathcal{M}$. This assumption reflects the fact that decision making performance of (2) occurs on the plane generated by the excitatory and inhibitory gains [10], [11].

Recall from Definition 2.11 in [21, p. 33] that $A \in \mathbb{R}^{n\times n}$ is called *Lyapunov stable* if all the eigenvalues λ of A satisfy either $\text{Re}\lambda < 0$ or $\text{Re}\lambda = 0$, and if $\text{Re}\lambda = 0$, then λ is semisimple (e.g., see [15]), where $\text{Re}\lambda$ denotes the real part of

λ. A is called *semistable* if all the eigenvalues λ of A satisfy either $\mathrm{Re}\lambda < 0$ or $\lambda = 0$, and if $\lambda = 0$, then λ is semisimple. Next, let $y(t)$ denote the solution to $\dot{y}(t) = A_p y(t)$ for a fixed $p \in \mathcal{M}$, $t \geq 0$. Recall from Proposition 2.6 in [21, p. 33] that there exists a class $\mathcal{K}$ function $\alpha : [0,\infty) \to [0,\infty)$ (strictly increasing, continuous, and $\alpha(0) = 0$ [22]) such that $\|y(t)\| \leq \alpha(\|y(0)\|)$ if and only if A_p is Lyapunov stable, $\lim_{t\to\infty} y(t)$ exists if and only if A_p is semistable, where $\|\cdot\|$ denotes the 2-norm. And if A_p is semistable, then $\lim_{t\to\infty} y(t) = (I_n - A_p A_p^{\#})y(0)$, where I_n denotes the n-by-n identity matrix and $A^{\#}$ denotes the group inverse of A [15].

It can be seen from these definitions that in general, semistability of a matrix is a stronger notion than Lyapunov stability of a matrix. However, for the special case where A_p is symmetric, the following result holds.

Lemma 3: Assume that Assumption 1 holds. Then for every $p \in \mathcal{M}$, A_p is Lyapunov stable if and only if A_p is semistable.

Proof: Since A_p is symmetric for every $p \in \mathcal{M}$, all the eigenvalues of A_p are real. Now the result follows from the definitions of Lyapunov stability and semistability of A_p. ■

Motivated by Lemma 3 and Simon's notion of bounded rationality [23], we make the following assumption.

Assumption 4: For every $p \in \mathcal{M}$, A_p is Lyapunov stable.

Assumption 4 restricts the dynamic behavior of all the subsystems of (10) to be bounded. If we specialize (10) into (5), this assumption implies that the gathered evidence information always falls into some finite range which indicates the limiting, extreme cases that a decision maker can take into account. This situation gives the decision maker a certain level of confidence to make rational decision by knowing the best case and worse case scenarios, which happens in many decision making problems [23].

Theorem 1: Consider (10). Assume that Assumptions 1–4 hold. Then there exists a smooth function $V : \mathbb{R}^n \to [0,\infty)$ such that $V(x) = 0$ for every $x \in \mathcal{E}_s$, $V(x) > 0$ for all $x \in \mathbb{R}^n \backslash \mathcal{E}_s$, and $\nabla V(x) A_p x < 0$ for all $x \in \mathbb{R}^n \backslash \mathcal{E}_s$ and $p \in \mathcal{M}$, where $A \backslash B = \{x : x \in A, x \notin B\}$ and ∇ denotes the nabla operator.

Proof: See Appendix. ■

Theorem 1 states the existence of an abstract energy function $V(\cdot)$, called *Lyapunov function for sets*, to have a minimum energy at the equilibrium set and always decrease its value everywhere else. Its proof is based on a novel *nested* construction of an approximation of such a function via multiple intermediate results in Appendix. The technique used for such a construction is motivated by the constructive proof of converse Lyapunov theorems for semistable switched systems [19] and semistable nonlinear systems [24].

It is important to note that Theorem 1 is *different* from the traditional converse Lyapunov theorems for (asymptotic or exponential) stability of (10) given in the literature, which state the existence of a Lyapunov function by assuming that (10) is stable. In contrast, Theorem 1 does *not* assume that (10) is semistable but all its subsystems are semistable. Moreover, it takes Assumptions 1–4 to arrive at the existence of a (common) Lyapunov function. It is known that if all the subsystems of (10) are asymptotically stable, one can construct a counterexample to show that (10) may *not* be asymptotically stable [25]. This observation may lead us to wonder if a similar situation would occur to semistability of (10), and if not, what would the relationship between Assumptions 1–4 and semistability of (10) be?

Indeed, a stunning application of Theorem 1 is that it can be used to establish a semistability property of (10). To clarify this point, recall that an equilibrium point x_e of (10) is Lyapunov stable under arbitrary switchings if there exists a class $\mathcal{K}$ function $\alpha : [0,\infty) \to [0,\infty)$ such that $\|x(t) - x_e\| \leq \alpha(\|x(0) - x_e\|)$, where $x(t)$ denotes a trajectory of (10). We define an equilibrium point of (10) to be semistable under arbitrary switchings if it is Lyapunov stable and every trajectory starting in a neighborhood of this point is convergent under arbitrary switchings. Equation (10) is called semistable under arbitrary switchings if every equilibrium point of (10) is semistable under arbitrary switchings.

First, we present a necessary and sufficient condition for (10) being Lyapunov stable under arbitrary switchings, provided that Assumptions 1–3 hold.

Lemma 4: Consider (10). Assume that Assumptions 1–3 hold. Then (10) is Lyapunov stable under arbitrary switchings if and only if all its subsystems are Lyapunov stable.

Proof: If (10) is Lyapunov stable under arbitrary switchings, then we pick $\sigma(t) = p$ for an arbitrary $p \in \mathcal{M}$. In this case, (10) becomes a linear time-invariant system $\dot{y}(t) = A_p y(t)$ with the constant system matrix A_p. Since (10) is Lyapunov stable under $\sigma(t) = p$, it follows that $\dot{y}(t) = A_p y(t)$ is Lyapunov stable. Finally, due to the arbitrary pick of $p \in \mathcal{M}$, it follows that all the subsystems of (10) are Lyapunov stable.

Conversely, assume that all the subsystems of (10) are Lyapunov stable. In this case, Assumptions 1–4 all hold. Let $x_e \in \mathcal{E}_s$ be arbitrary. We claim that $\|x(t) - x_e\| \leq \|x_0 - x_e\|$ for all $t \geq 0$. To see this, let t_k denote the switching time instant for $\sigma(t)$. Note that $x_e \in \ker(A_p)$ for any $p \in \mathcal{M}$. Then for every $t \in [t_{k-1}, t_k)$ and any $x_e \in \mathcal{E}_s$, we have

$$x(t) - x_e = e^{A_{p_k}(t - t_{k-1})} e^{A_{p_{k-1}}(t_{k-1} - t_{k-2})} \cdots e^{A_{p_0}(t_0 - t_{-1})}(x_0 - x_e)$$

where $p_k \in \mathcal{M}$ and $t_{-1} = 0$. By Lemma 3 and symmetry of A_{p_k}, $\|e^{A_{p_k}(t - t_{k-1})}\| \leq 1$ for all $k = 0, 1, \ldots$, and $t \geq 0$, which leads to

$$\|x(t) - x_e\| \leq \|e^{A_{p_k}(t - t_{k-1})}\| \times \|e^{A_{p_{k-1}}(t_{k-1} - t_{k-2})}\| \cdots \|e^{A_{p_0}(t_0 - t_{-1})}\| \times \|x_0 - x_e\| \leq \|x_0 - x_e\|$$

for all $t \geq 0$, $x_0 \in \mathbb{R}^n$, and $k = 0, 1, \ldots$. With $\alpha(s) = s$, it follows that (10) is Lyapunov stable under arbitrary switchings. ■

The next result gives a necessary and sufficient condition for (10) being semistable under arbitrary switchings, provided that Assumptions 1–3 hold.

Theorem 2: Consider (10). Assume that Assumptions 1–3 hold. Then (10) is semistable under arbitrary switchings if and only if all its subsystems are semistable.

Proof: If (10) is semistable under arbitrary switchings, then we pick $\sigma(t) = p$ for an arbitrary $p \in \mathcal{M}$. In this case, (10) becomes a linear time-invariant system $\dot{y}(t) = A_p y(t)$ with the constant system matrix A_p. Since (10) is semistable under $\sigma(t) = p$, it follows that $\dot{y}(t) = A_p y(t)$ is semistable. Finally, due to the arbitrary pick of $p \in \mathcal{M}$, it follows that all the subsystems of (10) are semistable.

Conversely, assume that all the subsystems of (10) are semistable. In this case, Assumptions 1–4 all hold. Then it follows from Theorem 1 that there exists a smooth function $V:\mathbb{R}^n \to [0,\infty)$ such that all the conditions in Theorem 1 hold. Next, by using $V(\cdot)$ for the set $\mathcal{E}_s$, it follows from the Lyapunov stability theorem for (noncompact) sets (Theorem 1 of [26]) that $\mathcal{E}_s$ is asymptotically stable, i.e., $x(t) \to \mathcal{E}_s$ as $t \to \infty$ for $x(0)$ in a neighborhood of $\mathcal{E}_s$. Since, by Lemma 4, every point in $\mathcal{E}_s$ is Lyapunov stable under arbitrary switchings, it follows from Proposition 2.2 of [27] that $x(t) \to z$ as $t \to \infty$ for $x(0)$ in a neighborhood of $\mathcal{E}_s$, where $z \in \mathcal{E}_s$. Now by definition, every point in $\mathcal{E}_s$ is semistable under arbitrary switchings. Hence, (10) is semistable under arbitrary switchings. ■

It follows from Theorem 2 that if Assumptions 1–4 hold, then (10) is indeed semistable under arbitrary switchings. This answers the previous question about the relationship between Assumptions 1–4 and semistability of (10), i.e., Assumptions 1–4 do imply semistability of (10).

V. Performance Gauge: Main Result

In this section, we will state our main result on the performance gauge of (5) by means of Theorem 1, which answers Question 1 raised at the end of Section II. To this end, for the switched system (5), we call the piecewise constant switching signal $m:[0,\infty) \to \mathcal{M}$ a *switching path*. Let $\mathcal{M}$ denote the set of all possible switching paths. Next, let $\psi(m,t,x)$ denote the solution to (5) along the switching path $m \in \mathcal{M}$ at time instant $t \in [0,\infty)$ with initial condition $Y(0) = x \in \mathbb{R}^n$.

The following theorem states the main result of the paper.

Theorem 3: Consider (5). Assume that Hypotheses 1, 2, and Assumptions 2–4 hold, where $A_p = G_p + H_p$ is symmetric. Define a performance cost functional

$$J(m,x_0) = \int_0^\infty -\nabla V(\psi(m,t,x_0))A_{m(t)}\psi(m,t,x_0)dt \tag{11}$$

where $m \in \mathcal{M}$, $x_0 \in \mathbb{R}^n$, and the existence of such $V(\cdot)$ is guaranteed by Theorem 1. Then $J(m,x_0) = V(x_0)$ for all $x_0 \in \mathbb{R}^n$ and $m \in \mathcal{M}$.

Proof: First, we claim that $\|\psi(m,t,x_0)\| \le \|x_0\|$ for all $t \ge 0$. To see this, for a particular $m \in \mathcal{M}$, let the switching time sequence be given by $\{t_0,t_1,\ldots,t_l,\ldots\}$, which may be finite $(L<\infty)$ or infinite $(L=\infty)$. Then for any $t \in [t_{l-1},t_l)$, $l = 0,1,\ldots,$

$$\psi(m,t,x_0) = e^{A_{p_l}(t-t_{l-1})}e^{A_{p_{l-1}}(t_{l-1}-t_{l-2})}\cdots e^{A_{p_0}(t_0-t_{-1})}x_0$$

where $p_l \in \mathcal{M}$ and $t_{-1} = 0$. Hence, by Lemma 3 and symmetry of A_{p_l}, $\|e^{A_{p_l}(t-t_{l-1})}\| \le 1$ for all $l = 0,1,\ldots,$ and $t \ge 0$, which leads to

$$\begin{aligned}\|\psi(m,t,x_0)\| \le \|e^{A_{p_l}(t-t_{l-1})}\| \times \|e^{A_{p_{l-1}}(t_{l-1}-t_{l-2})}\| \\ \cdots \|e^{A_{p_0}(t_0-t_{-1})}\| \times \|x_0\| \le \|x_0\|\end{aligned} \tag{12}$$

for all $t \ge 0$, $x_0 \in \mathbb{R}^n$, and $m \in \mathcal{M}$.

Since (12) holds for all $t \ge 0$, it follows from the Bolzano-Weierstrass theorem [22] that there exists an increasing unbounded sequence $\{\tau_k\}_{k=0}^\infty$, with $\tau_0 = 0$, such that $\lim_{k\to\infty}\psi(m,\tau_k,x_0) = p_m$ for every $x_0 \in \mathbb{R}^n$ and $m \in \mathcal{M}$. For a fixed switching path $m \in \mathcal{M}$, let $\omega_m(x_0)$ be the set of all such limit points p_m for $x_0 \in \mathbb{R}^n$.

Next, it follows from (12) that $\|p_m\| \le \|x_0\|$, and hence, $\omega_m(x_0)$ is bounded. To show that $\omega_m(x_0)$ is closed, let $\{p_{m,i}\}_{i=0}^\infty$ be a sequence in $\omega_m(x_0)$ such that $\lim_{i\to\infty} p_{m,i} = p_m$. Then it follows that for every $\varepsilon > 0$, there exists an i such that $\|p_m - p_{m,i}\| < \varepsilon/2$. Next, since $p_{m,i} \in \omega_m(x_0)$, it follows that there exists $t > T$, where $0 < T < \infty$, such that $\|p_{m,i} - \psi(m,t,x_0)\| < \varepsilon/2$. Now it follows that $\|p_m - \psi(m,t,x_0)\| \le \|p_{m,i} - \psi(m,t,x_0)\| + \|p_m - p_{m,i}\| < \varepsilon$ for sufficiently large t and i. Hence, $p_m \in \omega_m(x_0)$, which indicates that $\omega_m(x_0)$ is closed. Finally, since $\omega_m(x_0)$ is bounded and closed, it is compact.

We claim that for all $\psi(m,0,x_0) \in \omega_m(x_0)$, $\psi(m,t,x_0) \in \omega_m(x_0), t \ge 0$. Let $p_m \in \omega_m(x_0)$ be such that $\lim_{k\to\infty}\psi(m,\tau_k,x_0) = p_m$ for an increasing bounded sequence $\{\tau_k\}_{k=0}^\infty$. Consider $\psi(m,\tau_k,x_0)$. Then for $t+\tau_k \ge 0$, it follows from the semigroup property and continuity of $\psi(m,t,x_0)$ that $\lim_{k\to\infty}\psi(m,t+\tau_k,x_0) = \lim_{k\to\infty}\psi(m,t,\psi(m,\tau_k,x_0)) = \psi(m,t,p_m)$, which implies that $\psi(m,t,p_m) \in \omega_m(x_0)$ for all $t \ge 0$.

To show $\psi(m,t,x_0) \to \omega_m(x_0)$ as $t \to \infty$, suppose that this is not true. In this case, there exists a sequence $\{t_k\}_{k=0}^\infty$, with $t_k \to \infty$ as $k \to \infty$, such that $\inf_{p_m \in \omega_m(x_0)}\|\psi(m,t_k,x_0) - p_m\| > 0$ for all $k \ge 0$. However, since $\psi(m,t,x_0)$ is bounded for all $t \ge 0$, the bounded sequence $\{\psi(m,t_k,x_0)\}_{k=0}^\infty$ has a convergent subsequence $\{\psi(m,t_k^*,x_0)\}_{k=0}^\infty$ such that $\psi(m,t_k^*,x_0) \to p_m^* \in \omega_m(x_0)$ as $k \to \infty$, which contradicts the assumption. Hence, $\psi(m,t,x_0) \to \omega_m(x_0)$ as $t \to \infty$.

Next, it follows from Theorem 1 that there exists a smooth $V(\cdot)$ such that all the conditions in Theorem 1 are satisfied. Since by Theorem 1, $V(\psi(m,\tau,x_0)) - V(\psi(m,s,x_0)) = \int_s^\tau \dot{V}(\psi(m,t,x_0))dt = \int_s^\tau \nabla V(\psi(m,t,x_0))A_{m(t)}\psi(m,t,x_0)dt \le 0$ for all $\tau > s \ge 0$ and $x_0 \in \mathbb{R}^n$, it follows that $V(\psi(m,t,x_0))$ is a nonincreasing function of t. Note that $V(x) \ge 0$ for all $x \in \mathbb{R}^n$, it follows that $\gamma_{m,x_0} = \lim_{t\to\infty} V(\psi(m,t,x_0))$ exists and $\gamma_{m,x_0} \ge 0$.

Now, for any $p_m \in \omega_m(x_0)$, there exists an increasing unbounded sequence $\{t_k\}_{k=0}^\infty$, with $t_0 = 0$, such that $\lim_{k\to\infty}\psi(m,t_k,x_0) = p_m$. Hence, $V(p_m) = V(\lim_{k\to\infty}\psi(m,t_k,x_0)) = \lim_{k\to\infty} V(\psi(m,t_k,x_0)) = \gamma_{m,x_0}$, which implies that $V(x) = \gamma_{m,x_0}$ for all $x \in \omega_m(x_0)$.

If $\gamma_{m,x_0} = 0$, then $V(x) = 0$ for all $x \in \omega_m(x_0)$. It follows from Theorem 1 that $x \in \mathcal{E}_s$, and hence, $\omega_m(x_0) \subseteq \mathcal{E}_s$. In this case, $\psi(m,t,x_0) \to \mathcal{E}_s$ as $t \to \infty$.

Otherwise, assume that $\gamma_{m,x_0} > 0$. In this case, $V(x) > 0$ for all $x \in \omega_m(x_0)$. It follows from Theorem 1 that $x \notin \mathcal{E}_s$. Since for all $\psi(m,0,x_0) \in \omega_m(x_0)$, $\psi(m,t,x_0) \in \omega_m(x_0)$, $t \ge 0$, it follows that $V(\psi(m,t,x_0)) = \gamma_{m,x_0} > 0$ for all $t \ge 0$ and $\psi(m,0,x_0) \in \omega_m(x_0)$. Hence, $\psi(m,t,x_0) \notin \mathcal{E}_s$ for all $t \ge 0$ and $x_0 \in \omega_m(x_0)$. Now for any $\tau > s \ge 0$ and $x_0 \in \omega_m(x_0)$, it follows from Theorem 1 that $V(\psi(m,\tau,x_0)) - V(\psi(m,s,x_0)) = \int_s^\tau \dot{V}(\psi(m,t,x_0))dt = \int_s^\tau \nabla V(\psi(m,t,x_0))A_{m(t)}\psi(m,t,x_0)dt < 0$, and hence, $\gamma_{m,x_0} = V(\psi(m,\tau,x_0)) < V(\psi(m,s,x_0)) = \gamma_{m,x_0}$, which is a contradiction. Thus, the assumption $\gamma_{m,x_0} > 0$ is invalid.

Next, it follows from Lemma 4 that (5) is Lyapunov stable. Now by Proposition 2.2 in [27], it follows that $\lim_{t\to\infty}\psi(m,t,x_0) = \psi_e$ exists and $\psi_e \in \mathcal{E}_s$ for all $x_0 \in \mathbb{R}^n$ and $m \in \mathcal{M}$. Finally, it follows that for all $x_0 \in \mathbb{R}^n$ and $m \in \mathcal{M}$:

$$\begin{aligned} J(m, x_0) &= \int_0^\infty -\nabla V(\psi(m,t,x_0)) A_{m(t)} \psi(m,t,x_0) dt \\ &= V(\psi(m,0,x_0)) - \lim_{t\to\infty} V(\psi(m,t,x_0)) \\ &= V(x_0) - V(\lim_{t\to\infty} \psi(m,t,x_0)) \\ &= V(x_0) - V(\psi_e) = V(x_0) \end{aligned}$$

which proves the result. ■

The constructed cost functional (11) can be interpreted as the accumulated performance of the state change rate for (5) projected on the negative gradient direction of an abstract energy function, i.e., $\int_0^\infty \langle -\nabla^T V(Z(t)), \dot{Z}(t)\rangle dt$, where $Z(\cdot)$ denotes the solution to (5). It measures how fast the current state of (5) can reach the steady-state one along the dynamic evolution of (5). Theorem 3 indicates that such an accumulated performance measure can be gauged by the initial bias, which in some cases greatly reduces the complexity of evaluating such a cost functional. Note that the cost functional (11) represents a large spectrum of widely used performance metrics in dynamics and control [22], [28]. For example, if $V(\cdot)$ is quadratic, then (11) becomes the quadratic cost functional used in optimal control and H_2/H_∞ theory [29]. Hence, the average MCMC model (3) implies some type of optimal performance under Hypotheses 1, 2, and Assumptions 2–4, which has not been discovered before. This point floats out a way to explain optimality performance of many existing decision making models, such as mutual inhibition models, which cannot be done through the DDM framework. For instance, let $\mathcal{D} \subset \mathbb{R}^n$ be compact. According to Theorem 3, the stimulus input given by Hypothesis 1, satisfying Hypothesis 2 and Assumptions 2–4, is optimal if the initial condition $Z(0)$ for (5) is chosen in the *optimal reachable set* $\mathcal{R}_m = \{Z^* \in \mathcal{D} : Z^* = \arg\min_{Z(0)\in\mathcal{D}} J(m, Z(0))\}$ for $m \in \mathcal{M}$.

Example 2: Consider an average mutual inhibition model given by

$$\dot{x}_1(t) = -k_{\sigma(t)} x_1(t) - w_{\sigma(t)} x_2(t) + b_{1,\sigma(t)} \quad (13)$$

$$\dot{x}_1(t) = -k_{\sigma(t)} x_2(t) - w_{\sigma(t)} x_1(t) + b_{2,\sigma(t)}. \quad (14)$$

Let $\mathcal{M} = \{1, 2\}$ and pick $k_p = -w_p = p^2$ and $b_{1,p} = b_{2,p} = 0$ (no external stimulus), $p = 1, 2$. Then the model (13) and (14) has the form (10) with $A_p = -p^2 H$ and $H = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$, $p =, 1, 2$. Note that A_p is symmetric, $A_p A_q = A_q A_p$, $\ker(A_p) = \{\alpha[1,1]^T, \alpha \in \mathbb{R}\}$, and the eigenvalues of A_p are given by 0 and –2, $p, q = 1, 2$. Hence, Assumptions 1–4 hold. Then it follows from Theorem 1 that there exists a smooth Lyapunov function for sets $V(\cdot)$ for (13) and (14).

To explicitly construct such a Lyapunov function for sets, we borrow the construction concept from (15)–(17). Specifically, consider $V(x) = (Hx)^T(Hx)/2 = x^T Hx = (x_1 - x_2)^2$, where $x = [x_1, x_2]^T$. Clearly, $V(x) = 0$ if $x \in \ker(A_p)$ and $V(x) > 0$ if $x \notin \ker(A_p)$, $p = 1, 2$. Next, $\nabla V(x) A_p x = 2x^T H A_p x = -2p^2 x^T H^2 x = -4p^2 x^T Hx = -4p^2(x_1 - x_2)^2 < 0$ if $x \notin \ker(A_p)$, $p = 1, 2$. Hence, $V(x) = x^T Hx = (x_1 - x_2)^2$ satisfies all the conditions in Theorem 1.

According to Theorem 3, one can construct the following cost functional for performance gauge of autonomous (13) and (14):

$$\begin{aligned} J(\sigma, x(0)) &= \int_0^\infty 4(\sigma(t))^2 (x_1(t) - x_2(t))^2 dt \\ &= \int_0^\infty (-\nabla V(x(t)) A_{\sigma(t)} x(t)) dt \\ &= \int_0^\infty \Big(-\frac{d}{dt}(V(x(t)))\Big) dt \\ &= V(x(0)) = x^T(0) H x(0) = (x_1(0) - x_2(0))^2 \end{aligned}$$

which implies that the performance gauge of autonomous (13) and (14) solely depends on the initial bias $x(0)$, regardless of $\sigma(t)$. Finally, let $\mathcal{D} \subset \mathbb{R}^2$ be a compact set denoting the possible range of initial bias $x(0)$. Then $\min_{x(0)\in\mathcal{D}} J(\sigma, x(0)) = \min_{x_0\in\mathcal{D}} x_0^T H x_0$. For the simulation of autonomous (13) and (14), we take $\sigma(t) = 1$ for $t \in \bigcup_{k=0}^\infty [2k, 2k+1)$ and $\sigma(t) = 2$ for $t \in \bigcup_{k=0}^\infty [2k+1, 2k+2)$. Fig. 1 shows its state trajectories versus time for the initial bias $x(0) = [4, -2]^T$.

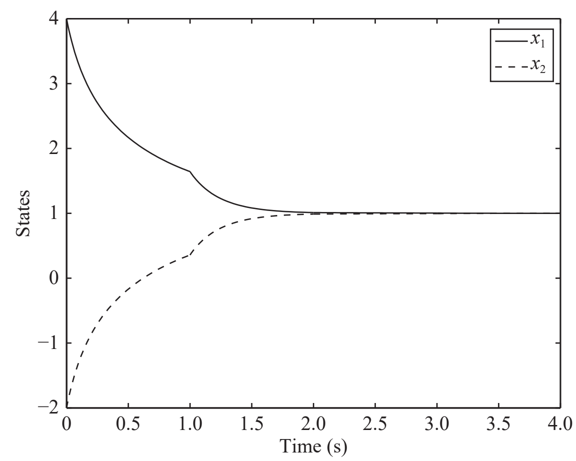

Fig. 1. State trajectories versus time.

VI. Conclusion

This paper utilized a converse Lyapunov approach to solve a longstanding problem regarding the performance gauge of average MCMC models for decision making. The developed result can be used to evaluate the cost for performance comparison of decision making under various inputs and initial conditions, and hence, lead to possible algorithmic ways of finding optimal performance for average MCMC decision making via a computational scheme. This idea seems tangible due to the extensive development of semidefinite programming methods [30] for searching appropriate Lyapunov functions within the past two decades, and hence, the proposed approach lays a theoretical foundation and suggests a possible path toward seeking optimal decision making in MCMC situations.

Appendix

This appendix contains multiple intermediate results used for proving Theorem 1. First, it follows from Lemma 3 that this bounded behavior is actually a steady-state convergence one. Using Assumptions 1–4, we can start our construction for the cost function toward (10) by defining a series of the compositional functions $U_p : \mathbb{R}^n \to \mathbb{R}$ and $U : \mathbb{R}^n \to \mathbb{R}$ as follows, which is motivated by the iterative process in [19]. To this end, note that $\mathcal{E}_s = \ker(A_1)$ and $\mathcal{E}_s$ is a subspace.

$$U_1(x) = \sup_{t\geq 0}\{w_1(t)\|e^{A_1 t}x - p_1(t,x)\|\} \tag{15}$$

$$U_p(x) = \sup_{t\geq 0}\{w_p(t)U_{p-1}(e^{A_p t}x)\}, \quad p = 2,\ldots,M \tag{16}$$

$$U(x) = U_M(x) \tag{17}$$

where $x \in \mathbb{R}^n$, $p_1(t,x) = \mathrm{proj}_{\mathcal{E}_s}(e^{A_1 t}x)$ is the orthogonal projection of $e^{A_1 t}x$ onto $\mathcal{E}_s$, $w_i : [0,\infty) \to \mathbb{R}$ is a strictly increasing, smooth function satisfying the following properties for every $i = 1,2,\ldots,M$: i) $c_1 \leq w_i(t) \leq c_2$ for all $t \geq 0$ and some $0 < c_1 < c_2 < 2c_1$; and ii) there exists a decreasing continuous function $v : [0,\infty) \to (0,\infty)$ such that $dw_i(t)/dt \geq v(t)$, $t \geq 0$.

The following result provides an explicit expression for $\|e^{A_1 t}x - p_1(t,x)\|$ in (15), which can be used to show the wellposedness of $U_1(\cdot)$.

Lemma 5: Assume that A_1 is symmetric. Then for any $x \in \mathbb{R}^n$ and every $t \geq 0$,

$$\begin{aligned}\min_{y\in\mathcal{E}_s}\|e^{A_1 t}x - y\| &= \|e^{A_1 t}x - p_1(t,x)\| \\ &= \|\sum_{l=1}^{r_1}(u_{1,l}^T x)e^{\lambda_{1,l}t}u_{1,l}\| \\ &= [\sum_{l=1}^{r_1}(u_{1,l}^T x)^2 e^{2\lambda_{1,l}t}]^{\frac{1}{2}}\end{aligned} \tag{18}$$

where $r_1 = \mathrm{rank}[A_1] \leq n$, $\{u_{1,1},\ldots,u_{1,n}\}$ is an orthonormal basis for $\mathbb{R}^n$ such that $\{u_{1,r_1+1},\ldots,u_{1,n}\}$ is an orthonormal basis for $\ker(A_1)$, and $\lambda_{1,i}$ are the nonzero eigenvalues of A_1, $i = 1,\ldots,r_1$.

Proof: First, since $\mathcal{E}_s = \ker(A_1)$ is a subspace in $\mathbb{R}^n$, it follows that:

$$\min_{y\in\mathcal{E}_s}\|e^{A_1 t}x - y\| = \|e^{A_1 t}x - p_1(t,x)\|. \tag{19}$$

Next, since A_1 is symmetric, it follows from diagonalization that there exists an orthogonal matrix P_1 such that $P_1^{-1}A_1P_1 = \mathrm{diag}(\lambda_{1,1},\ldots,\lambda_{1,r_1},0,\ldots,0)$, where $\lambda_{1,i} \in \mathbb{R}$, $i = 1,\ldots,r_1$, denote the nonzero eigenvalues of A_1 and $\mathrm{diag}(X)$ denotes a diagonal matrix whose ith diagonal entry is the ith component of X. Hence, $P_1^{-1}A_1^TA_1P_1 = \mathrm{diag}(\lambda_{1,1}^2,\ldots,\lambda_{1,r_1}^2, 0,\ldots,0)$. Let $P_1 = [u_{1,1},\ldots,u_{1,n}] \in \mathbb{R}^{n\times n}$ and $P_1^T = [v_{1,1},\ldots,v_{1,n}] \in \mathbb{R}^{n\times n}$, where $u_{1,i} \in \mathbb{R}^n$ and $v_{1,i} \in \mathbb{R}^n$. Then $A_1u_{1,l} = \lambda_{1,l}u_{1,l}$ for $l = 1,\ldots,r_1$, and $A_1u_{1,j} = 0$ for $j = r_1+1,\ldots,n$. Furthermore, $e^{A_1 t}u_{1,l} = e^{\lambda_{1,l}t}u_{1,l}$ for $l = 1,\ldots,r_1$ and $e^{A_1 t}u_{1,j} = u_{1,j}$ for $j = r_1+1,\ldots,n$.

Note that $z \in \ker(A_1)$ if and only if $z \in \ker(A_1^TA_1)$. For every $z \in \ker(A_1^TA_1)$, let $z = \sum_{i=1}^n \alpha_i u_{1,i}$, where $\alpha_i \in \mathbb{R}$. It follows from the diagonalization of A_1 that $A_1 = \sum_{i=1}^{r_1}\lambda_{1,i}v_{1,i}u_{1,i}^T$. Hence, $A_1^TA_1 = \sum_{i=1}^{r_1}\lambda_{1,i}^2 u_{1,i}u_{1,i}^T$ and $z^TA_1^TA_1z = \sum_{i=1}^{r_1}\lambda_{1,i}^2\alpha_i^2$. Hence, $z \in \ker(A_1)$ if and only if $\alpha_1 = \cdots = \alpha_{r_1} = 0$, which is equivalent to $z = \sum_{i=r_1+1}^n \alpha_i u_{1,i}$, i.e., $z \in \mathrm{span}\{u_{1,r_1+1},\ldots,u_{1,n}\}$. Hence, $\{u_{1,r_1+1},\ldots,u_{1,n}\}$ is an orthonormal basis for $\ker(A_1)$. Now it follows that $p_1(t,x) = \mathrm{proj}_{\mathcal{E}_s}(e^{A_1 t}x) = \sum_{i=r_1+1}^n \langle e^{A_1 t}x, u_{1,i}\rangle u_{1,i}$, where $\langle\cdot,\cdot\rangle$ denotes the inner product [17].

Finally, note that $\{u_{1,1},\ldots,u_{1,n}\}$ is an orthonormal basis for $\mathbb{R}^n$. Let $x = \sum_{k=1}^n \beta_k u_{1,k}$, where $\beta_k = \langle x, u_{1,k}\rangle = u_{1,k}^T x$, $k = 1,\ldots,n$. Then for every $i = r_1+1,\ldots,n$,

$$\begin{aligned}\langle e^{A_1 t}x, u_{1,i}\rangle u_{1,i} &= (u_{1,i}^T e^{A_1 t}x)u_{1,i} \\ &= (u_{1,i}^T e^{A_1 t}(\sum_{k=1}^n \beta_k u_{1,k}))u_{1,i} \\ &= (\sum_{k=1}^n \beta_k u_{1,i}^T e^{A_1 t}u_{1,k})u_{1,i} \\ &= (\sum_{l=1}^{r_1}\beta_l u_{1,i}^T e^{A_1 t}u_{1,l} \\ &\quad + \sum_{j=r_1+1}^n \beta_j u_{1,i}^T e^{A_1 t}u_{1,j})u_{1,i} \\ &= (\sum_{l=1}^{r_1}\beta_l e^{\lambda_{1,l}t}u_{1,i}^T u_{1,l} \\ &\quad + \sum_{j=r_1+1}^n \beta_j u_{1,i}^T u_{1,j})u_{1,i} \\ &= \beta_i u_{1,i} = (u_{1,i}^T x)u_{1,i}.\end{aligned}$$

Therefore, $p_1(t,x) = \mathrm{proj}_{\mathcal{E}_s}(e^{A_1 t}x) = \sum_{i=r_1+1}^n (u_{1,i}^T x)u_{1,i}$ and

$$\begin{aligned}e^{A_1 t}x - p_1(t,x) &= \sum_{k=1}^n \beta_k e^{A_1 t}u_{1,k} - \sum_{i=r_1+1}^n (u_{1,i}^T x)u_{1,i} \\ &= \sum_{l=1}^{r_1}\beta_l e^{\lambda_{1,l}t}u_{1,l} + \sum_{j=r_1+1}^n \beta_j u_{1,j} \\ &\quad - \sum_{i=r_1+1}^n (u_{1,i}^T x)u_{1,i} \\ &= \sum_{l=1}^{r_1}\beta_l e^{\lambda_{1,l}t}u_{1,l} = \sum_{l=1}^{r_1}(u_{1,l}^T x)e^{\lambda_{1,l}t}u_{1,l}.\end{aligned}$$

Consequently, $\|e^{A_1 t}x - p_1(t,x)\|^2 = \langle e^{A_1 t}x - p_1(t,x), e^{A_1 t}x - p_1(t,x)\rangle = \sum_{l=1}^{r_1}(u_{1,l}^T x)^2 e^{2\lambda_{1,l}t}$, which proves the result.

Lemma 6: Assume that A_1 is symmetric and Lyapunov stable. Then $U_1(\cdot)$ is well defined.

Proof: It follows from Lemma 5 that:

$$U_1(x) = \sup_{t\geq 0}\{w_1(t)[\sum_{l=1}^{r_1}(u_{1,l}^T x)^2 e^{2\lambda_{1,l}t}]^{1/2}\}. \tag{20}$$

Since A_1 is symmetric and Lyapunov stable, it follows that $\lambda_{1,l} < 0$. By the Cauchy-Schwarz inequality, we have

$$\begin{aligned}U_1(x) &\leq \sup_{t\geq 0}\{w_1(t)[\sum_{l=1}^{r_1}(u_{1,l}^T x)^2]^{1/2}\} \\ &\leq \sup_{t\geq 0}\{w_1(t)[\sum_{l=1}^{r_1}\|u_{1,l}\|^2\|x\|^2]^{1/2}\} \\ &= \sup_{t\geq 0}\{w_1(t)\sqrt{r_1}\|x\|\} \\ &\leq c_2\sqrt{r_1}\|x\| < \infty.\end{aligned}$$

Hence, $U_1(\cdot)$ is well defined. ■

Next, we present a convexity property of $U_1(\cdot)$.

Lemma 7: Assume that A_1 is symmetric and Lyapunov stable. Then $U_1(\cdot)$ is convex on $\mathbb{R}^n$.

Proof: Let $z=\mu x+(1-\mu)y$, where $\mu\in[0,1]$ and $x,y\in\mathbb{R}^n$. Then it follows from Lemma 5 and the definition of norm that:

$$\begin{aligned}||e^{A_1t}z-p_1(t,z)|| &= ||\sum_{l=1}^{r_1}(u_{1,l}^Tz)e^{\lambda_{1,l}t}u_{1,l}||\\ &= ||\sum_{l=1}^{r_1}\mu(u_{1,l}^Tx)e^{\lambda_{1,l}t}u_{1,l}\\ &\quad+\sum_{l=1}^{r_1}(1-\mu)(u_{1,l}^Ty)e^{\lambda_{1,l}t}u_{1,l}||\\ &\le \mu||\sum_{l=1}^{r_1}\mu(u_{1,l}^Tx)e^{\lambda_{1,l}t}u_{1,l}||\\ &\quad+(1-\mu)||\sum_{l=1}^{r_1}(u_{1,l}^Ty)e^{\lambda_{1,l}t}u_{1,l}||\\ &= \mu||e^{A_1t}x-p_1(t,x)||\\ &\quad+(1-\mu)||e^{A_1t}y-p_1(t,y)||.\end{aligned}$$

That is, $||e^{A_1t}x-p_1(t,x)||$ is convex in terms of x. Now since

$$\begin{aligned}&\sup_{t\ge0}(w_1(t)\,||e^{A_1t}z-p_1(t,z)||)\\ &\le \sup_{t\ge0}(w_1(t)\mu||e^{A_1t}x-p_1(t,x)||\\ &\quad+w_1(t)(1-\mu)||e^{A_1t}y-p_1(t,y)||)\\ &\le \mu\sup_{t\ge0}(w_1(t)||e^{A_1t}x-p_1(t,x)||)\\ &\quad+(1-\mu)\sup_{t\ge0}(w_1(t)(1-\mu)||e^{A_1t}y-p_1(t,y)||)\end{aligned}$$

it follows that $U_1(z)\le\mu U_1(x)+(1-\mu)U_1(y)$, which proves the convexity property of $U_1(\cdot)$. ■

Next, we present a continuity property of $U_1(\cdot)$.

Lemma 8: Assume that A_1 is symmetric and Lyapunov stable. Then $U_1(\cdot)$ is continuous on $\mathbb{R}^n$.

Proof: Define $T_1:\mathbb{R}^n\backslash\mathcal{E}_s\to[0,\infty)$ by $T_1(z)=\inf\{h:[\sum_{l=1}^{r_1}(u_{1,l}^Tz)^2e^{2\lambda_{1,l}t}]^{1/2}<[\sum_{l=1}^{r_1}(u_{1,l}^Tz)^2]^{1/2}/2,\forall t\ge h>0\}$, where $z\in\mathbb{R}^n\backslash\mathcal{E}_s$. Note that since A_1 is symmetric and Lyapunov stable, it follows that $\lambda_{1,l}<0$ for all $l=1,\ldots,r_1$. Next, the proof of Lemma 5 gives the fact that $\mathcal{E}_s=\operatorname{span}\{u_{1,r_1+1},\ldots,u_{1,n}\}$, where $\operatorname{span}S$ denotes the span of S. Since $u_{1,i}^Tu_{1,j}=0$ for $l=1,\ldots,r_1$ and $j=r_1+1,\ldots,n$, it follows that $u_{1,l}^Tz\ne0$, $l=1,\ldots,r_1$. Therefore, for $t>\epsilon=\max_{1\le l\le r_1}(1/\lambda_{1,l})\ln(1/2)>0$, it follows that $e^{2\lambda_{1,l}t}<1/4$ for all $l=1,\ldots,r_1$. Consequently, $[\sum_{l=1}^{r_1}(u_{1,l}^Tz)^2e^{2\lambda_{1,l}t}]^{1/2}<[\sum_{l=1}^{r_1}(u_{1,l}^Tz)^2]^{1/2}/2$ for all $t>\epsilon$, and hence, $T_1(z)$ is well defined.

It follows from Lemma 5 that $\min_{y\in\mathcal{E}_s}||e^{A_1t}z-y||=[\sum_{l=1}^{r_1}(u_{1,l}^Tz)^2e^{2\lambda_{1,l}t}]^{1/2}$ for all $t\ge0$. In particular, for $t=0$, $\min_{y\in\mathcal{E}_s}||z-y||=[\sum_{l=1}^{r_1}(u_{1,l}^Tz)^2]^{1/2}$. Consider $z\in\mathbb{R}^n\backslash\mathcal{E}_s$ and define $\lambda=[\sum_{l=1}^{r_1}(u_{1,l}^Tz)^2]^{1/2}>0$. Next, denote $\mathcal{B}_\mu(z)=\{x\in\mathbb{R}^n:||x-z||<\mu\}$ and $\overline{\mathcal{B}_\mu(z)}=\{x\in\mathbb{R}^n:||x-z||\le\mu\}$, where $\mu>0$ and $\overline{S}$ denotes the closure of set $S\subset\mathbb{R}^n$. For any $\mu<\lambda$ and any $x\in\overline{\mathcal{B}_\mu(z)}$, it follows that $||z-z||\le\mu<\lambda=[\sum_{l=1}^{r_1}(u_{1,l}^Tz)^2]^{1/2}=\min_{y\in\mathcal{E}_s}||z-y||$. Hence, $\overline{\mathcal{B}_\mu(z)}\cap\mathcal{E}_s=\emptyset$.

Let $z_e=p_1(0,z)$. Then $A_1z_e=0$. Define $q(t)=x(t)-z_e$, where $\dot{x}(t)=A_1x(t)$. Then it follows that $\dot{q}(t)=A_1q(t)$ for all $t\ge0$, which implies that $q(t)=e^{A_1t}q(0)$ for all $t\ge0$. Hence, $||q(t)||=||e^{A_1t}q(0)||\le||e^{A_1t}||||q(0)||=\sigma_{\max}(e^{A_1t})||q(0)||$, where $\sigma_{\max}(A)$ denotes the maximum singular value of A. Since A_1 is symmetric and Lyapunov stable, it follows that $\sigma_{\max}(e^{A_1t})=\max_{1\le i\le r_1}\{e^{2\lambda_{1,i}t},1\}=1$ for all $t\ge0$. Thus, $||q(t)||\le||q(0)||$ for all $t\ge0$, i.e., $||e^{A_1t}x(0)-z_e||\le||x(0)-z_e||$ for all $x(0)\in\mathbb{R}^n$ and $t\ge0$. Consider $\mathcal{B}_\varepsilon(z_e)=\{x\in\mathbb{R}^n:||x-z_e||<\varepsilon\}$. Clearly for any $\varepsilon>0$ and $t\ge0$, $x(t)=e^{A_1t}x\in\mathcal{B}_\varepsilon(z_e)$ for $x\in\mathcal{B}_\varepsilon(z_e)$. In particular, choose $\varepsilon=\lambda/2$. It then follows that $e^{A_1t}x\in\mathcal{B}_{\lambda/2}(z_e)$ for all $t\ge0$ and $x\in\mathcal{B}_{\lambda/2}(z_e)$. Note that by Lemma 5, $[\sum_{l=1}^{r_1}(u_{1,l}^Te^{A_1t}x)^2]^{1/2}=\min_{y\in\mathcal{E}_s}||e^{A_1t}x-y||\le||e^{A_1t}x-z_e||$ for all $t\ge0$. Denote $\mathcal{W}_\varepsilon=\{x\in\mathbb{R}^n:[\sum_{l=1}^{r_1}(u_{1,l}^Tx)^2]^{1/2}<\varepsilon\}$ and note that $\mathcal{W}_\varepsilon$ open. Since $||e^{A_1t}x-z_e||\le||x-z_e||$ for all $t\ge0$, it follows that $e^{A_1t}x\in\mathcal{W}_{\lambda/2}$ for all $t\ge0$ and $x\in\mathcal{B}_{\lambda/2}(z_e)$.

Note that $||z-z_e||=\min_{y\in\mathcal{E}_s}||z-y||=\lambda$. If $\lambda/2<\mu<\lambda$, then it follows that $\mathcal{B}_\mu(z)\cap\mathcal{B}_{\lambda/2}(z_e)\ne\emptyset$. Hence, for all $y\in\mathcal{B}_\mu(z)\cap\mathcal{B}_{\lambda/2}(z_e)$, $||e^{A_1T_1(z)}y-z_e||\le||y-z_e||<\lambda/2$, i.e., $e^{A_1T_1(z)}y\in\mathcal{B}_{\lambda/2}(z_e)$. Now, it follows that $e^{A_1(T_1+t)(z)}y\in\mathcal{W}_{\lambda/2}$ for all $t\ge0$ and $y\in\mathcal{B}_\mu(z)\cap\mathcal{B}_{\lambda/2}(z_e)$. Then, for every $t>T_1(z)$ and $y\in\mathcal{B}_\mu(z)\cap\mathcal{B}_{\lambda/2}(z_e)$, $w_1(t)||e^{A_1t}y-p_1(t,y)||\le c_2||e^{A_1t}y-p_1(t,y)||<c_2\lambda/2$.

Alternatively, since A_1 is symmetric and Lyapunov stable, it follows from Lemma 3 that $\lim_{t\to\infty}e^{A_1t}x=(I_n-A_1A_1^\#)x\in\ker(A_1)$ for any $x\in\mathbb{R}^n$. Hence, $0\le\lim_{t\to\infty}||e^{A_1t}x-p_1(t,x)||\le\lim_{t\to\infty}||e^{A_1t}x-(I_n-A_1A_1^\#)x||=0$ for any $x\in\mathbb{R}^n$, it follows that $\lim_{t\to\infty}||e^{A_1t}x-p_1(t,x)||=0$ for any $x\in\mathbb{R}^n$. Hence, for $y\in\mathcal{B}_\mu(z)\backslash\mathcal{B}_{\lambda/2}(z_e)$, it follows from $\lim_{t\to\infty}||e^{A_1t}y-p_1(t,y)||=0$ that there exists $h_1=h_1(\lambda,y)>0$ such that $||e^{A_1t}y-p_1(t,y)||<\lambda/2$. Note that $\lambda=[\sum_{l=1}^{r_1}(u_{1,l}^Tz)^2]^{1/2}$. Hence, h_1 depends on z and y, i.e., $h_1=h_1'(z,y)$. Define $T_1'(z,y)=\inf\{h_1'(z,y):||e^{A_1t}y-p_1(t,y)||<\lambda/2,\forall t\ge h_1'(z,y)\}$. In this case, for every $t>T_1'(z,y)$ and $y\in\mathcal{B}_\mu(z)\backslash\mathcal{B}_{\lambda/2}(z_e)$, $w_1(t)||e^{A_1t}y-p_1(t,y)||<c_2\lambda/2$.

Next, define $T_{1,\max}(z)=\sup\{T_1'(z,y):y\in\overline{\mathcal{B}_\mu(z)\backslash\mathcal{B}_{\lambda/2}(z_e)}\}$. We claim that $T_{1,\max}(\cdot)$ is well defined. Since by Lemma 5, $||e^{A_1t}y-p_1(t,y)||=[\sum_{l=1}^{r_1}(u_{1,l}^Ty)^2e^{2\lambda_{1,l}t}]^{1/2}<\lambda/2$ for every $t>T_1'(z,y)$, it follows from the continuity of the map $y\mapsto[\sum_{l=1}^{r_1}(u_{1,l}^Ty)^2e^{2\lambda_{1,l}t}]^{1/2}$ and the fact that $e^{2\lambda_{1,l}t}\in(0,1)$ for all $t>T_1'(z,y)$ that, there exists an open set $\mathcal{B}_\eta(y)\subset\overline{\mathcal{B}_\mu(z)\backslash\mathcal{B}_{\lambda/2}(z_e)}$ such that $[\sum_{l=1}^{r_1}(u_{1,l}^Tw)^2e^{2\lambda_{1,l}t}]^{1/2}<\lambda/2$ for every $w\in\mathcal{B}_\eta(y)$ and $t>T_1'(z,y)$. Hence, $\limsup_{w\to y}T_1'(z,w)\le T_1'(z,y)$ implying that the function $y\mapsto T_1'(z,y)$ is upper semicontinuous at the arbitrarily chosen point y, and hence on $\overline{\mathcal{B}_\mu(z)\backslash\mathcal{B}_{\lambda/2}(z_e)}$. Since an upper semicontinuous function defined on a compact set achieves its supremum, it follows that $T_{1,\max}(z)$ is well defined. Therefore, for every $t>T_{1,\max}(z)$ and $y\in\mathcal{B}_\mu(z)\backslash\mathcal{B}_{\lambda/2}(z_e)$, $w_1(t)||e^{A_1t}y-p_1(t,y)||\le c_2||e^{A_1t}y-p_1(t,y)||<c_2\lambda/2$.

Alternatively, if $\mu\le\lambda/2$, it follows that $\mathcal{B}_\mu(z)\cap\mathcal{B}_{\lambda/2}(z_e)=\emptyset$. In this case, for $y\in\mathcal{B}_\mu(z)$, it follows from $\lim_{t\to\infty}||e^{A_1t}y-p_1(t,y)||=0$ that there exists $\hbar_1=\hbar_1(\lambda,y)>0$ such that $||e^{A_1t}y-p_1(t,y)||<\lambda/2$. Using the similar arguments as above, one can define $T'_{1,\max}(z)$, which is similar to $T_{1,\max}(z)$, such that for every $t > T'_{1,\max}(z)$ and $y \in \mathcal{B}_\mu(z)$, $w_1(t)\|e^{A_1 t}y - p_1(t,y)\| < c_2\lambda/2$.

In summary, as long as $0 < \mu < \lambda$, for every $t > T_m(z) = \max\{T_1(z), T_{1,\max}(z), T'_{1,\max}(z)\}$ and $y \in \mathcal{B}_\mu(z)$, $w_1(t)\|e^{A_1 t}y - p_1(t,y)\| < c_2\lambda/2$. Now, note that $w_1(0)\|y - p_1(0,y)\| \geq c_1 \inf_{y\in\mathcal{B}_\mu(z), x\in\mathcal{E}_s}\|y - x\| \geq c_1(\lambda - \mu)$ for every $y \in \mathcal{B}_\mu(z)$ and $w_1(0)\|z - p_1(0,z)\| = w_1(0)\|z - z_e\| \geq c_1\lambda$. Choose $\mu \leq (1 - c_2/2c_1)\lambda$. Clearly $\mu < \lambda$. Moreover, it follows from $c_1 > 2c_2$ that $c_1(\lambda - \mu) \geq c_2\lambda/2$ and $c_1\lambda > c_2\lambda/2$. Hence,

$$\begin{aligned}U_1(z) - U_1(y) = &\sup_{t\geq 0}\{w_1(t)\|e^{A_1 t}z - p_1(t,z)\|\}\\ &- \sup_{t\geq 0}\{w_1(t)\|e^{A_1 t}y - p_1(t,y)\|\}\\ = &\sup_{0\leq t\leq T_m(z)}\{w_1(t)\|e^{A_1 t}z - p_1(t,z)\|\}\\ &- \sup_{0\leq t\leq T_m(z)}\{w_1(t)\|e^{A_1 t}y - p_1(t,y)\|\}.\end{aligned}$$

Thus, by the triangle inequality and Lemmas 5 and 3,

$$\begin{aligned}&|U_1(z) - U_1(y)|\\ &\leq \sup_{0\leq t\leq T_m(z)}\Big|w_1(t)(\|e^{A_1 t}z - p_1(t,z)\|\\ &\quad - \|e^{A_1 t}y - p_1(t,y)\|)\Big|\\ &\leq c_2 \sup_{0\leq t\leq T_m(z)}\Big|\|e^{A_1 t}z - p_1(t,z)\| - \|e^{A_1 t}y - p_1(t,y)\|\Big|\\ &= c_2 \sup_{0\leq t\leq T_m(z)}\Big|\|\sum_{l=1}^{r_1}(u_{1,l}^T z)e^{\lambda_{1,l}t}u_{1,l}\|\\ &\quad - \|\sum_{l=1}^{r_1}(u_{1,l}^T y)e^{\lambda_{1,l}t}u_{1,l}\|\Big|\\ &\leq c_2 \sup_{0\leq t\leq T_m(z)}\Big|\|\sum_{l=1}^{r_1}(u_{1,l}^T z)e^{\lambda_{1,l}t}u_{1,l}\\ &\quad - \sum_{l=1}^{r_1}(u_{1,l}^{\mathrm{T}} y)e^{\lambda_{1,l}t}u_{1,l}\|\Big|\\ &= c_2 \sup_{0\leq t\leq T_m(z)}\|\sum_{l=1}^{r_1}(u_{1,l}^T(z-y))e^{\lambda_{1,l}t}u_{1,l}\|\\ &= c_2 \sup_{0\leq t\leq T_m(z)}\big[\sum_{l=1}^{r_1}(u_{1,l}^T(z-y))^2 e^{2\lambda_{1,l}t}\big]^{1/2}\\ &\leq c_2\big[\sum_{l=1}^{r_1}(u_{1,l}^T(z-y))^2\big]^{1/2}\\ &\leq c_2\big[\sum_{l=1}^{r_1}\|u_{1,l}\|^2\|z-y\|^2\big]^{1/2}\\ &= c_2\sqrt{r_1}\|z-y\|, \quad z\in\mathbb{R}^n\backslash\mathcal{E}_s, \quad y\in\mathcal{B}_\mu(z)\end{aligned} \tag{21}$$

where the last inequality holds due to the Cauchy-Schwarz inequality. Now, it follows from (21) that $U_1(z)$ is continuous at z. Since $z \in \mathbb{R}^n\backslash\mathcal{E}_s$ was chosen arbitrarily, it follows that $U_1(\cdot)$ is continuous on $\mathbb{R}^n\backslash\mathcal{E}_s$. In fact, (21) shows that $U_1(\cdot)$ is locally Lipschitz continuous on $\mathbb{R}^n\backslash\mathcal{E}_s$.

To show that $U_1(\cdot)$ is continuous on $\mathcal{E}_s$, consider $x_e \in \ker(A_1)$. Let $\{x_n\}_{n=1}^\infty$ be a sequence in $\mathbb{R}^n\backslash\mathcal{E}_s$ such that $\lim_{n\to\infty} x_n = x_e$. Note that $x_e \in \mathcal{E}_s$ and $\mathcal{E}_s = \mathrm{span}\{u_{1,r_1+1},\ldots,u_{1,n}\}$. It then follows that $u_{1,l}^T x_e = 0$ for all $l = 1,\ldots,r_1$. Hence, by (20), $U_1(x_e) = \sup_{t\geq 0}\{w_1(t)[\sum_{l=1}^{r_1}(u_{1,l}^T x_e)^2 e^{2\lambda_{1,l}t}]^{1/2}\} = 0$. Next, it follows from (20) and semistability of A_1 that:

$$\begin{aligned}U_1(x_n) &\leq c_2 \sup_{t\geq 0}\big[\sum_{l=1}^{r_1}(u_{1,l}^T x_n)^2 e^{2\lambda_{1,l}t}\big]^{1/2}\\ &\leq c_2\big[\sum_{l=1}^{r_1}(u_{1,l}^T x_n)^2\big]^{1/2}.\end{aligned}$$

Thus, $\lim_{n\to\infty}[\sum_{l=1}^{r_1}(u_{1,l}^T x_n)^2]^{1/2} = [\sum_{l=1}^{r_1}(u_{1,l}^T \lim_{n\to\infty} x_n)^2]^{1/2} = [\sum_{l=1}^{r_1}(u_{1,l}^T x_e)^2]^{1/2} = 0$. Therefore, $0 \leq \lim_{n\to\infty} U_1(x_n) \leq c_2 \lim_{n\to\infty}[\sum_{l=1}^{r_1}(u_{1,l}^T x_n)^2]^{1/2} = 0$, which implies that $\lim_{n\to\infty} U_1(x_n) = 0 = U_1(x_e)$. ■

Based on Lemmas 7 and 8, we have the following Lipschitz continuity property for $U_1(\cdot)$.

Lemma 9: Assume that A_1 is symmetric and Lyapunov stable. Then $U_1(\cdot)$ is locally Lipschitz continuous on $\mathbb{R}^n$.

Proof: The result follows from Lemmas 7 and 8, and the fact that continuous convex functions on $\mathbb{R}^n$ are locally Lipschitz continuous (e.g., see [31]). ■

The following result states that the values of $U_p(\cdot)$ will vanish on the equilibrium set.

Lemma 10: Assume that Assumptions 1–4 hold. Then for every $p = 1,\ldots,M$, $U_p(x_e) = 0$ if and only if $x_e \in \mathcal{E}_s$.

Proof: We prove this result by induction. For $p = 1$, the last part of the proof for Lemma 8 has showed that if $x_e \in \mathcal{E}_s$, $U_1(x_e) = 0$. On the other hand, if $U_1(x) = 0$, then it follows from (20) that for all $t \geq 0$, $0 = U_1(x) \geq c_1[\sum_{l=1}^{r_1}(u_{1,l}^T x)^2 e^{2\lambda_{1,l}t}]^{1/2} \geq 0$, which implies that $\sum_{l=1}^{r_1}(u_{1,l}^T x)^2 e^{2\lambda_{1,l}t} = 0$ for all $t \geq 0$. Since $e^{2\lambda_{1,l}t} > 0$ for all $t \geq 0$, it follows that $\sum_{l=1}^{r_1}(u_{1,l}^T x)^2 e^{2\lambda_{1,l}t} = 0$ if and only of $u_{1,l}^T x = 0$ for every $l = 1,\ldots,r_1$, i.e., $x \in \mathrm{span}\{u_{1,r_1+1},\ldots,u_{1,n}\} = \ker(A_1) = \mathcal{E}_s$.

Suppose that for $p = k$, $U_k(x_e) = 0$ if and only if $x_e \in \mathcal{E}_s$. Now consider $p = k+1$. Note that by assumption, $\ker(A_{k+1}) = \mathcal{E}_s$. Hence, for every $x_e \in \mathcal{E}_s$, $A_{k+1}x_e = 0$ and hence, $e^{A_{k+1}t}x_e = \sum_{m=0}^\infty \frac{A_{k+1}^m t^m}{m!}x_e = x_e + \sum_{m=1}^\infty \frac{t^m}{m!}A_{k+1}^m x_e = x_e$. By the construction of $U_{k+1}(\cdot)$, $U_{k+1}(x_e) = \sup_{t\geq 0}\{w_{k+1}(t)U_k(e^{A_{k+1}t}x_e)\} = \sup_{t\geq 0}\{w_{k+1}(t)U_k(x_e)\} = 0$. Alternatively, if $U_{k+1}(x) = 0$, then for all $t \geq 0$, $0 = U_{k+1}(x) = \sup_{t\geq 0}\{w_{k+1}(t)U_k(e^{A_{k+1}t}x)\} \geq c_1 U_k(e^{A_{k+1}t}x) \geq 0$, which implies that $U_k(e^{A_{k+1}t}x) = 0$ for all $t \geq 0$. By induction assumption, $e^{A_{k+1}t}x \in \mathcal{E}_s$ for all $t \geq 0$. In particular, letting $t = 0$ yields $x \in \mathcal{E}_s$. Thus, by mathematical induction, the conclusion holds. ■

Next, we extend Lemma 9 to all $U_p(\cdot)$.

Lemma 11: Assume that Assumptions 1–4 hold. For each $p = 2,\ldots,M$, $U_p(\cdot)$ is locally Lipschitz continuous on $\mathbb{R}^n$.

Proof: We prove this result by induction. For $p = 1$, it follows from Lemma 9 that $U_1(\cdot)$ is locally Lipschitz continuous on $\mathbb{R}^n$. Suppose that for $p = k$, $U_k(\cdot)$ is locally Lipschitz continuous on $\mathbb{R}^n$. Then consider the case where $p = k+1$. By definition, $U_{k+1}(x) = \sup_{t\geq 0}\{w_{k+1}(t)U_k(e^{A_{k+1}t}x)\} \leq c_2\sup_{t\geq 0}\{U_k(e^{A_{k+1}t}x)\}$ for any $x \in \mathbb{R}^n$. Since A_{k+1} is symmetric and Lyapunov stable, it follows from Lemma 3 that $\lim_{t\to\infty} e^{A_{k+1}t}x = x_e^{k+1} \in \ker(A_{k+1}) = \mathcal{E}_s$ for any $x \in \mathbb{R}^n$. Hence, by the induction assumption on continuity of $U_k(\cdot)$ and Lemma 10, we have $\lim_{t\to\infty} U_k(e^{A_{k+1}t}x) = U_k(\lim_{t\to\infty} e^{A_{k+1}t}x) = U_k(x_e^{k+1}) = 0$. Hence, $\sup_{t\geq 0}\{U_k(e^{A_{k+1}t}x)\} < \infty$, which implies that $U_{k+1}(\cdot)$ is well defined.

Next, we show that $U_p(\cdot)$ is convex on $\mathbb{R}^n$ for every $p\in\mathcal{M}$. Again, we use mathematical induction. For $p=1$, Lemma 7 shows that $U_1(\cdot)$ is convex on $\mathbb{R}^n$. Assume that for $p=k$, $U_k(\cdot)$ is convex on $\mathbb{R}^n$. Now consider $p=k+1$. Let $u=\mu x+(1-\mu)y$, where $\mu\in[0,1]$ and $x,y\in\mathbb{R}^n$. Then by induction assumption,

$$\begin{aligned}U_{k+1}(u) &= \sup_{t\geq 0}\{w_{k+1}(t)U_k(\mu e^{A_{k+1}t}x+(1-\mu)e^{A_{k+1}t}y)\}\\ &\leq \sup_{t\geq 0}\{w_{k+1}(t)\mu U_k(e^{A_{k+1}t}x)\\ &\quad + w_{k+1}(t)(1-\mu)U_k(e^{A_{k+1}t}y)\}\\ &\leq \mu\sup_{t\geq 0}\{w_{k+1}(t)U_k(e^{A_{k+1}t}x)\}\\ &\quad +(1-\mu)\sup_{t\geq 0}\{w_{k+1}(t)U_k(e^{A_{k+1}t}y)\}\\ &= \mu U_{k+1}(x)+(1-\mu)U_{k+1}(y)\end{aligned}$$

which implies that $U_{k+1}(\cdot)$ is convex on $\mathbb{R}^n$. Thus, by mathematical induction, $U_p(\cdot)$ is convex on $\mathbb{R}^n$ for every $p\in\mathcal{M}$.

Define $T_{k+1}:\mathbb{R}^n\backslash\mathcal{E}_s\to[0,\infty)$ by $T_{k+1}(z)=\inf\{h: U_k(e^{A_{k+1}t}z)< U_k(z)/2, \forall t\geq h>0\}$, where $z\in\mathbb{R}^n\backslash\mathcal{E}_s$, and denote $\mathcal{U}_\varepsilon=\{x\in\mathbb{R}^n: U_k(x)<\varepsilon\}$. Note that it follows from Lemma 10 that $\mathcal{U}_\varepsilon\supset\mathcal{E}_s$. Next, we claim that $\mathcal{U}_\varepsilon$ is open. To see this, consider $\mathbb{R}^n\backslash\mathcal{U}_\varepsilon=\{x\in\mathbb{R}^n: U_k(x)\geq\varepsilon\}$. Let $\{x_n\}_{n=1}^\infty$ be a sequence in $\mathbb{R}^n\backslash\mathcal{U}_\varepsilon$ that converges. Then $U_k(x_n)\geq\varepsilon$ for all $n=1,2,\ldots$. By induction assumption, $\varepsilon\leq\lim_{n\to\infty}U_k(x_n)=U_k(\lim_{n\to\infty}x_n)$, which implies that $\lim_{n\to\infty}x_n\in\mathbb{R}^n\backslash\mathcal{U}_\varepsilon$. Hence, $\mathbb{R}^n\backslash\mathcal{U}_\varepsilon$ is closed, which is equivalent to say that $\mathcal{U}_\varepsilon$ is open.

Now consider $z\in\mathbb{R}^n\backslash\mathcal{E}_s$. Define $\lambda=U_k(z)>0$ and let $z_e=\lim_{t\to\infty}e^{A_{k+1}t}z$. Next, define $q(t)=x(t)-z_e$, where $\dot{x}(t)=A_{k+1}x(t)$. Then $\dot{q}(t)=A_{k+1}q(t)$ for all $t\geq 0$, which implies that $q(t)=e^{A_{k+1}t}q(0)$ for all $t\geq 0$. Using the similar arguments as in the proof of Lemma 8, it follows that $\|q(t)\|\leq\|q(0)\|$ for all $t\geq 0$, i.e., $\|e^{A_{k+1}t}x(0)-z_e\|\leq\|x(0)-z_e\|$ for all $t\geq 0$. Hence, for any $\varepsilon>0$, if $x(0)\in\mathcal{B}_\varepsilon(z_e)$, then $e^{A_{k+1}t}x(0)\in\mathcal{B}_\varepsilon(z_e)$ for all $t\geq 0$.

Since $z_e\in\mathcal{U}_{\lambda/2}$ and $\mathcal{U}_{\lambda/2}$ is open, there exists $\eta>0$ such that $\mathcal{B}_\eta(z_e)\subset\mathcal{U}_{\lambda/2}$. Hence, for all $x(0)\in\mathcal{B}_\eta(z_e)$, it follows that $e^{A_{k+1}t}x(0)\in\mathcal{B}_\eta(z_e)\subset\mathcal{U}_{\lambda/2}$ for all $t\geq 0$.

Since A_{k+1} is symmetric and Lyapunov stable, it follows from Lemma 3 that A_{k+1} is semistable, and hence, $\lim_{t\to\infty}e^{A_{k+1}t}z=z_e\in\ker(A_{k+1})=\mathcal{E}_s$. Then it follows that there exists $h>0$ such that $e^{A_{k+1}h}z\in\mathcal{B}_\eta(z_e)$. Consequently, $e^{A_{k+1}(h+t)}z\in\mathcal{B}_\eta(z_e)\subset\mathcal{U}_{\lambda/2}$ for all $t\geq 0$, and hence, it follows that $T_{k+1}(z)$ is well defined.

Next, by continuity of the map $y\mapsto e^{A_{k+1}t}y$ over the compact time interval $t\in[0,T_{k+1}(z)]$, it follows that there exists $\rho>0$ such that $\overline{\mathcal{B}_\rho(z)}\cap\mathcal{E}_s=\emptyset$ and $e^{A_{k+1}T_{k+1}(z)}y\in\mathcal{B}_\eta(z_e)$ for all $y\in\mathcal{B}_\rho(z)$, and hence, $e^{A_{k+1}(T_{k+1}(z)+t)}y\in\mathcal{U}_{\lambda/2}$ for all $t\geq 0$ and $y\in\mathcal{B}_\rho(z)$. Then, for every $t>T_{k+1}(z)$ and $y\in\mathcal{B}_\rho(z)$, $w_{k+1}(t)U_k(e^{A_{k+1}t}y)\leq c_2U_k(e^{A_{k+1}t}y)<c_2\lambda/2$, and hence, $\sup_{t>T_{k+1}(z)}\{w_{k+1}(t)U_k(e^{A_{k+1}t}y)\}\leq c_2\lambda/2$.

On the other hand, note that $\sup_{t\geq 0}\{w_{k+1}(t)U_k(e^{A_{k+1}t}y)\}\geq w_{k+1}(0)U_k(y)\geq c_1U_k(y)$ for $y\in\mathcal{B}_\rho(z)$. Since by induction assumption, $U_k(\cdot)$ is locally Lipschitz continuous, it follows that for sufficiently small $\varrho\in(0,\rho)$, one can find $L_k>0$ such that $|U_k(z)-U_k(y)|\leq L_k\|z-y\|$ for all $y\in\mathcal{B}_\varrho(z)\subset\mathcal{B}_\rho(z)$. In this case, $U_k(y)\geq U_k(z)-L_k\|z-y\|\geq\lambda-L_k\sup_{y\in\mathcal{B}_\varrho(z)}\|z-y\|=\lambda-L_k\varrho$. Hence, $\sup_{t\geq 0}\{w_{k+1}(t)U_k(e^{A_{k+1}t}y)\}\geq c_1(\lambda-L_k\varrho)$ for $y\in\mathcal{B}_\varrho(z)$. Choose ϱ to be sufficiently small so that $\varrho\leq(1-c_2/2c_1)\lambda/L_k$. Then it follows that $c_1(\lambda-L_k\varrho)\geq c_2\lambda/2$. Therefore, for each $y\in\mathcal{B}_\varrho(z)$,

$$\begin{aligned}U_{k+1}(z)-U_{k+1}(y) &= \sup_{t\geq 0}\left\{w_{k+1}(t)U_k(e^{A_{k+1}t}z)\right\}\\ &\quad -\sup_{t\geq 0}\left\{w_{k+1}(t)U_k(e^{A_{k+1}t}y)\right\}\\ &= \sup_{0\leq t\leq T_{k+1}(z)}\left\{w_{k+1}(t)U_k(e^{A_{k+1}t}z)\right\}\\ &\quad -\sup_{0\leq t\leq T_{k+1}(z)}\left\{w_{k+1}(t)U_k(e^{A_{k+1}t}y)\right\}\end{aligned}\tag{22}$$

which leads to

$$\begin{aligned}&|U_{k+1}(z)-U_{k+1}(y)|\\ &\quad\leq\sup_{0\leq t\leq T_{k+1}(z)}|w_{k+1}(t)(U_k(e^{A_{k+1}t}z)-U_k(e^{A_{k+1}t}y))|\\ &\quad\leq c_2\sup_{0\leq t\leq T_{k+1}(z)}|U_k(e^{A_{k+1}t}z)-U_k(e^{A_{k+1}t}y)|\\ &\quad z\in\mathbb{R}^n\backslash\mathcal{E}_s,\quad y\in\mathcal{B}_\varrho(z).\end{aligned}\tag{23}$$

Now, it follows from (23) and the induction assumption on Lipschitz continuity of $U_k(\cdot)$ that $U_{k+1}(\cdot)$ is continuous at z. Since $z\in\mathbb{R}^n\backslash\mathcal{E}_s$ was chosen arbitrarily, it follows that $U_{k+1}(\cdot)$ is continuous on $\mathbb{R}^n\backslash\mathcal{E}_s$.

To show that $U_{k+1}(\cdot)$ is continuous on $\mathcal{E}_s$, consider $x_e\in\mathcal{E}_s$. Let $\{x_n\}_{n=1}^\infty$ be a sequence in $\mathbb{R}^n\backslash\mathcal{E}_s$ that converges to x_e. Let $\varepsilon>0$ and define $p(t)=x(t)-x_e$, where $\dot{x}(t)=A_{k+1}x(t)$. Then $\dot{p}(t)=A_{k+1}p(t)$ for all $t\geq 0$, which implies that $p(t)=e^{A_{k+1}t}p(0)$ for all $t\geq 0$. Using the similar arguments as in the proof of Lemma 8, it follows that $\|p(t)\|\leq\|p(0)\|$ for all $t\geq 0$, i.e., $\|e^{A_{k+1}t}x(0)-x_e\|\leq\|x(0)-x_e\|$ for all $t\geq 0$. Hence, for any $\varepsilon>0$, if $x(0)\in\mathcal{B}_\varepsilon(x_e)$, then $e^{A_{k+1}t}x(0)\in\mathcal{B}_\varepsilon(x_e)$ for all $t\geq 0$. Since $x_e\in\mathcal{U}_\varepsilon$ and $\mathcal{U}_\varepsilon$ is open, it follows that there exists $\delta>0$ such that $\mathcal{B}_\delta(x_e)\subset\mathcal{U}_\varepsilon$. Hence, for all $x(0)\in\mathcal{B}_\delta(x_e)$, it follows that $e^{A_{k+1}t}x(0)\in\mathcal{B}_\delta(x_e)\subset\mathcal{U}_\varepsilon$ for all $t\geq 0$.

Next, note that there exists a positive integer N_1 such that $x_n\in\mathcal{B}_\delta(x_e)$ for all $n\geq N_1$. Now, it follows from (16) that for $n\geq N_1$, $U_{k+1}(x_n)\leq c_2\sup_{t\geq 0}U_k(e^{A_{k+1}t}x_n)\leq c_2\varepsilon$, which implies that $\lim_{n\to\infty}U_{k+1}(x_n)=0=U_{k+1}(x_e)$.

Finally, using the fact that continuous convex functions on $\mathbb{R}^n$ are locally Lipschitz continuous, it follows that $U_{k+1}(\cdot)$ is locally Lipschitz continuous on $\mathbb{R}^n$. By mathematical induction, for each $p=1,\ldots,M$, $U_p(\cdot)$ is locally Lipschitz continuous on $\mathbb{R}^n$. ■

The next result evaluates the upper Dini derivative of $U(\cdot)$ given by (17).

Lemma 12: Assume that Assumptions 1–4 hold. For each $p\in\mathcal{M}$,

$$\limsup_{h\to 0^+}\frac{U(e^{A_p h}x)-U(x)}{h}\le -\varepsilon(x)U(x)<0,\quad x\in\mathbb{R}^n\backslash\mathcal{E}_s \tag{24}$$

where $\varepsilon(x)$ is some function satisfying $0<\varepsilon(x)\le\varpi$ for some $\varpi>0$ and for all $x\in\mathbb{R}^n\backslash\mathcal{E}_s$.

Proof: It follows from Assumption 3 that for any $p,q\in\mathcal{M}$ and any $x\in\mathbb{R}^n$, $0=[A_px,A_qx]=\frac{\partial}{\partial x}(A_qx)A_px-\frac{\partial}{\partial x}(A_px)A_qx=A_qA_px-A_pA_qx$, which implies that $A_pA_q=A_qA_p$. Hence, $A_p^kA_q^l=A_q^lA_p^k$ for any nonnegative integers k,l. Therefore, for any $p,q\in\mathcal{M}$ and any $t,s\ge 0$,

$$\begin{aligned}e^{A_pt}e^{A_qs}&=\sum_{k=0}^{\infty}\frac{A_p^kt^k}{k!}e^{A_qs}=\sum_{k=0}^{\infty}\frac{A_p^kt^k}{k!}(\sum_{l=0}^{\infty}\frac{A_q^ls^l}{l!})\\&=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{A_p^kA_q^lt^ks^l}{k!l!}=\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\frac{A_q^lA_p^ks^lt^k}{l!k!}\\&=\sum_{l=0}^{\infty}\frac{A_q^ls^l}{l!}(\sum_{k=0}^{\infty}\frac{A_p^kt^k}{k!})=\sum_{l=0}^{\infty}\frac{A_q^ls^l}{l!}e^{A_pt}\\&=e^{A_qs}e^{A_pt}\end{aligned}$$

which establishes the commutativity of local flows for (10). Consequently, for each $p\in\mathcal{M}$, we have

$$\begin{aligned}U(x)&=\sup_{t_M\ge 0}\Big\{\{\cdots\sup_{t_1\ge 0}\{\min_{y\in\mathcal{E}_s}\|\prod_{i=1}^{M}e^{A_it_i}x-y\|\\&\quad\times w_1(t_1)\}\cdots\}w_M(t_M)\Big\}\\&=\sup_{t_p\ge 0,p\in\mathcal{M}}\Big\{\min_{y\in\mathcal{E}_s}\|\prod_{i=1}^{M}e^{A_it_i}x-y\|\\&\quad\times w_1(t_1)\cdots w_M(t_M)\Big\}.\end{aligned}$$

To show that $U(\cdot)$ is strictly decreasing along the solution of (10) on $\mathbb{R}^n\backslash\mathcal{E}_s$, note that for every $x\in\mathbb{R}^n\backslash\mathcal{E}_s$ and $h>0$ such that $e^{A_ph}x\in\mathbb{R}^n\backslash\mathcal{E}_s$, it follows from the arguments preceding (22) that, for sufficiently small h, the supremum in the definition of $\min_{y\in\mathcal{E}_s}\|\prod_{i=1}^{M}e^{A_it_i}e^{A_ph}x-y\|\prod_{i=1}^{M}w_i(t_i)$ is reached at some time $\hat{t}_i$, $i\in\mathcal{M}$ such that $0\le\hat{t}_i\le T(x)$, where $T(x)=\max_{p\in\mathcal{M}}T_p(x)$. Then

$$\begin{aligned}U(e^{A_ph}x)&=\min_{y\in\mathcal{E}_s}\|\prod_{i=1}^{M}e^{A_i\hat{t}_i}e^{A_ph}x-y\|\prod_{i=1}^{M}w_i(\hat{t}_i)\\&=\min_{y\in\mathcal{E}_s}\|\prod_{i=1,i\ne p}^{M}e^{A_i\hat{t}_i}e^{A_p(\hat{t}_p+h)}x-y\|\prod_{i=1}^{M}w_i(\hat{t}_i)\\&=\min_{y\in\mathcal{E}_s}\|\prod_{i=1,i\ne p}^{M}e^{A_i\hat{t}_i}e^{A_p(\hat{t}_p+h)}x-y\|\\&\quad\times\prod_{i=1,i\ne p}^{M}w_i(\hat{t}_i)w_p(\hat{t}_p+h)\Big[1-\frac{w_p'(\hat{t}_p+\theta_ph)h}{w_p(\hat{t}_p+h)}\Big]\\&\le U(x)\Big[1-\frac{v(h+T(x))h}{w_p(h+T(x))}\Big]\end{aligned}$$

for some $\theta_p\in(0,1)$, where we used the mean-value theorem for $w_p(t)$ on $[\hat{t}_p,\hat{t}_p+h]$: $w_p(\hat{t}_p+h)-w_p(\hat{t}_p)=w_p'(\hat{t}_p+\theta_ph)h$ for some $\theta_p\in(0,1)$. Thus,

$$\limsup_{h\to 0^+}\frac{U(e^{A_ph}x)-U(x)}{h}\le-\frac{v(h+T(x))}{w_p(h+T(x))}U(x)<0$$

for all $x\in\mathbb{R}^n\backslash\mathcal{E}_s$, and hence, (24) holds by noting that $0<v(h+T(x))\le v(0)$ and $0<c_1\le w_p(h+T(x))\le c_2$. ■

Lemma 12 is almost like saying that $\nabla U(x)A_px<-\varepsilon(x)U(x)$ except that $U(x)$ is not smooth. On the other hand, since Lemma 11 shows that $U(\cdot)$ is locally Lipschitz continuous, it is possible to approximate $U(\cdot)$ by a smooth (i.e., infinitely many times differentiable) function without significantly changing its Dini derivative (24) via some smoothing procedures introduced by [32]–[34].

Proof of Theorem 1: It follows from Lemma 11 that $U(\cdot)$ is locally Lipschitz continuous on $\mathbb{R}^n$. Furthermore, it follows from Lemma 12 that the upper Dini derivative of $U(\cdot)$ is strictly less than zero and bounded above by $-\varepsilon(x)U(x)$, which is positive for all $x\in\mathbb{R}^n\backslash\mathcal{E}_s$. Hence, by Theorem 2.5 of [33], there exists a smooth function $W:\mathbb{R}^n\backslash\mathcal{E}_s\to\mathbb{R}$ such that $W(x)>0$ for $x\in\mathbb{R}^n\backslash\mathcal{E}_s$ and $\limsup_{h\to 0^+}(1/h)[W(e^{A_ph}x)-W(x)]<0$ for $x\in\mathbb{R}^n\backslash\mathcal{E}_s$. Next, we extend $W(\cdot)$ to all of $\mathbb{R}^n$ by taking $W(z)=0$ for $z\in\mathcal{E}_s$. Then $W(\cdot)$ is a continuous Lyapunov function that is smooth on $\mathbb{R}^n\backslash\mathcal{E}_s$. Now letting $V(x)=W(x)e^{-(W(x))^{-2}}$ for $x\in\mathbb{R}^n\backslash\mathcal{E}_s$ and $V(x)=0$ for $x\in\mathcal{E}_s$. Then $V(\cdot)$ satisfies all the conditions stated in Theorem 1. ■

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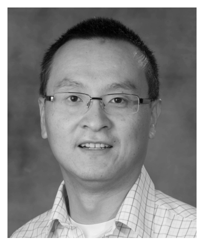

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