A Unified Optimization-Based Framework to Adjust Consensus Convergence Rate and Optimize the Network Topology in Uncertain Multi-Agent Systems

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***Abstract*—This paper deals with the consensus problem in an uncertain multi-agent system whose agents communicate with each other through a weighted undirected (primary) graph. The considered multi-agent system is described by an uncertain state-space model in which the involved matrices belong to some matrix boxes. As the main contribution of the paper, a unified optimization-based framework is proposed for simultaneously reducing the weights of the edges of the primary communication graph (optimizing the network topology) and synthesizing a controller such that the consensus in the considered uncertain multi-agent system is ensured with an adjustable convergence rate. Considering the NP-hardness nature of the optimization problem related to the aforementioned framework, this problem is relaxed such that it can be solved by regular LMI solvers. Numerical/practical-based examples are presented to verify the usefulness of the obtained results.**

***Index Terms*—Convergence-rate, element-wise uncertainty, robust controller, topology design.**

I. Introduction

THE distributed consensus is a fundamental issue in multi-agent networks. Over the past few years, there has been considerable interest in developing algorithms to force a multi-agent system to reach a consensus. The survey paper [1] has summarized some new progress in this regard. It is well known that many practical problems in multi-agent networks such as flocking and swarming [2], [3], formation control [4], [5], sensor fusion [6], [7], and synchronization of coupled oscillators [8] can be formulated as a consensus problem. Generally speaking, in a consensus-seeking process, the agents in a given network try to agree on some quantity by communicating what they know only to their neighboring agents. As a particular type of consensus problem, some researchers have studied the multi-agent networks in which the aim is to converge to the average of the involved agents' initial values (e.g., [9]).

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Up to now, various algorithms have been proposed and implemented to reach consensus in multi-agent systems. An important issue regarding the consensus algorithms is their performance. Considering this importance, the performance of consensus algorithms has been evaluated from different aspects. External attack resistance [10], robustness in the case of link failure [11] or the existence of delay [12], robustness to edge weight perturbations [13], and convergence rate are some indicators that determine the effectiveness of these algorithms. In this paper, we focus on the convergence speed in the consensus of multi-agent systems.

Regarding the convergence speed of the consensus, it is intuitively expected that the stronger connections in the communication graph yield a more enhanced convergence rate. Generally speaking, this expectation means that if the number of communication edges between the agents and the weight of these edges increases, the consensus's convergence speed is improved. But, promoting the communication between the agents imposes additional costs, for example, in the viewpoint of energy consumption. In some real-world multi-agent systems, the batteries powering the agents have very low capacity and can not be conveniently recharged/replaced. As a result, reducing energy consumption to extend the agents' battery lifetime has emerged as a critical issue in these networks. That is why various research studies have tried to reduce the energy consumption of communications [14]–[16]. Reviewing the facts mentioned above naturally raises a question about the best network topology for the neighborhood graph and corresponding distributed control law to optimize a cost function considering closed-loop performance (such as convergence speed) and communication costs.

On the other hand, in many practical cases, uncertainty is an inseparable part of modeling, and the controller should be designed to deal with the model's uncertainties. Uncertainties can be generally modeled in different ways in the state-space representation. For example, in some research works, the uncertainty has been modeled via norm-bounded forms (e.g., [17]). As a well-known fact, robust control approaches constructed on basis of norm-bounded modeling of uncertainty often lead to over-conservative results in dealing with uncertainties. In some other cases, uncertainty has been modeled by an affine polytopic structure (e.g., [18]). In polytopic models, system parameters are modeled as uncertain linear combinations of some known quantities. Although the polytopic modeling can cover linear parameters and multi-model uncertainties, it seems that the element-wise (or interval) modeling can provide a richer framework to describe the model in the presence of different sources of uncertainties. Moreover, a great advantage of element-wise modeling is its ability to simply express all systems included by this type of models. Considering these capabilities, element-wise models (or interval matrices in the state-space representation) have received a great attention in control systems literature (for example, in order to benefiting from their abilities in modeling of uncertain plants [19], [20] and stability analysis and stabilization of uncertain interval systems [21]–[23]). Dealing with uncertain multi-agent systems, two approaches are prevalent to model the uncertainties by element-wise structures. The first approach is constructed based on that the uncertainties are assumed to be different for each agent (e.g., [24], [25]). Another approach is based on the assumption that the uncertainties on the dynamic models of the agents are due to the same but uncertain factors. Some research works, such as [26]–[28], have considered this assumption in element-wise modeling of the under-study multi-agent system. In the present paper, we follow the second aforementioned approach.

Related Literature: There are different topology design methods for multi-agent systems in the literature constructed based on performance optimization issues. It has been shown that the eigenvalues of the Laplacian matrix of the communication graph play a significant role in determining the properties and performance of a wide range of multi-agent network systems. Considering this point, [29] has tried to optimize some specific functions of graph Laplacian eigenvalues by appropriately choosing the edge weights via applying semi-definite programming. Some other relevant research works have solved constrained versions of this problem. For example, [30] has introduced an algorithm to find a graph with weighted edges which maximizes the convergence speed under the constraint that the number of edges with nonzero weight is less than or equal to a given positive integer. By considering another case, the mentioned work has solved the problem under the constraint that the Laplacian graph's second eigenvalue is greater than or equal to a given positive value. Also, [31] deals with the problem of finding optimal communication graphs with a fixed number of vertices and edges that maximize the convergence speed. Moreover, [32] has investigated the problem of removing some links such that the largest eigenvalue of the resulting graph's adjacency matrix is minimized. Recently, [33] has introduced a method to optimize any cost function defined based on Laplacian eigenvalues for a directed graph. As another example, [34] has solved the convergence rate problem subject to the constraints limiting the weighted degree of graph nodes.

The majority of the relevant research papers, such as the above-cited works, have focused only on optimizing the network parameters and have not simultaneously optimized the other factors (e.g., free parameters of the controller or the different factors influencing the security issues) which may be effective on the performance of the multi-agent systems. Even though there are a few works in which combined objectives have been satisfied in a unified framework. For example, [35] has introduced an algorithm to balance between convergence rate and security level for an undirected graph. As another example, [36], [37] have considered a discrete-time multi-agent system and proposed an algorithm that minimizes the sum of the quadratic infinite horizon cost and the communication one.

Various research works have tried to design consensus algorithms for uncertain multi-agent systems to deal with the model uncertainties. In some papers, such as [38], [39] have considered multi-agent models with norm-bounded additive uncertainties in the frequency domain and used analytical tools like small-gain theorem to propose robust consensus algorithms. As another example, [40] has analyzed scalable consensus for a class of scalar uncertain multi-agent systems. Also, some other papers have studied consensus in the multi-agent systems that are modeled with uncertain state-space models. Reference [41] has proposed a consensus algorithm for double integrators' networks with parametric uncertainties. Moreover, [42] has considered a network of scalar uncertain linear time-invariant agents and designed a consensus algorithm for them. Multi-agent models with norm-bounded uncertainties have been considered in [43], [44], and [45] has considered consensus in multi-agent systems with polytopic uncertainty. To the best of the authors' knowledge, multi-agent models with element-wise uncertainty have not received much attention, and consequently, the present paper tries to address this gap. More precisely, in this paper, we will propose a linear matrix inequality (LMI) based method to co-design a network topology and a robust controller for a multi-agent system with element-wise uncertainties such that the preservation of the convergence rate and reducing the communication costs are simultaneously met. The main novelty of the present research work in comparison with the existing literature, including the relevant works discussed above, is that the targets of optimizing the network topology and finding an appropriate controller to ensure consensus with a reasonable convergence rate in uncertain multi-agent systems are simultaneously achieved by solving a single optimization problem. Such a unified optimization-based framework to achieve the aforementioned targets in consensus of multi-agent systems with element-wise uncertainties has not been introduced in previous research works.

Contributions: As discussed, this paper provides a unified computational framework for simultaneously optimizing the network topology and designing robust controllers in uncertain multi-agent systems such that communication costs are reduced with no unfavorable effect on the convergence speed. It is assumed that the uncertainties in the under-study multi-agent systems are modeled in an element-wise form. The main contributions of the paper can be summarized as follows:

1) *Unified Framework for the Design of Robust Controllers and Optimizing the Network Topology:* An optimization-based framework that introduces a sufficient condition that helps us

to find robust controllers for consensus in multi-agent systems with element-wise uncertainties and the weights of the edges in the corresponding optimized communication graph is introduced (Theorem 1).

2) Convex Optimization Relaxation: It is justified that the problem discussed in the previous item (the problem addressed by Theorem 1) is NP-hard. Benefiting from the state-of-the-art in literature, the obtained optimization-based sufficient condition is relaxed such that it can be verified by using regular LMI solvers (Theorem 2).

Organization: The remainder of the paper is organized as follows. Section II briefly introduces the notations and definitions used in this paper. The under-study problem is formulated with some basic assumptions in Section III. In Section IV, firstly, some preliminary results from previous works are restated, and then by using them, the main results of the paper are presented. Examples are given in Section V to illustrate the theoretical results of the paper. Finally, Section VI concludes the paper and suggests some directions for future research works in continuation of the work done in this paper.

II. Notations and Definitions

The symbol $\mathbb{S}^n_{>0}$ ($\mathbb{S}^n_{\geq 0}$) in this paper denotes the set of $n\times n$ positive-definite (semi-definite) symmetric matrices. The notation $A \succ B$ ($A \succeq B$), where A and B are two symmetric matrices, means that $A-B\in\mathbb{S}^n_{>0}$ (respectively, $\mathbb{S}^n_{\geq 0}$). $[A]^\dagger = A + A^\top$, where A is a square matrix. A block diagonal matrix with blocks $A_1, A_2, \ldots, A_n$ is specified by $\mathfrak{Diag}\{A_1, A_2, \ldots, A_n\}$. Also, the Kronecker product of matrices A and B is denoted by $A\otimes B$. Moreover, $\begin{bmatrix} A & B \\ * & C \end{bmatrix}$ denotes the matrix $\begin{bmatrix} A & B \\ B^\top & C \end{bmatrix}$.

In this paper, a weighted graph is denoted by $\mathcal{G} = (\mathcal{V},\mathcal{E},\mathcal{W})$, where $\mathcal{V}$, $\mathcal{E}$, and $\mathcal{W}$ are respectively the sets of graph's vertices, edges, and weights. $\mathcal{G} = (\mathcal{V},\mathcal{E},\mathcal{W})$ is called undirected if $(i, j)\in\mathcal{E} \Longleftrightarrow (j,i)\in\mathcal{E}$. Furthermore, an undirected graph is assumed to be connected provided that there is a path between each pair of distinct vertices in the graph. In graph $\mathcal{G}$, $\mathcal{N}_i \subset \mathcal{V}$ denotes the set of all neighbors of vertex i. If this graph has m edges with positive weights, then the matrix $C = [c_1 \;\; c_2 \;\; \ldots \;\; c_{2m}]$ is called the *incidence matrix* of this graph, and if lth edge of graph $\mathcal{G}$ connects node i to node j then, $(c_{2l})_i = (c_{2l-1})_j = 1$ and $(c_{2l})_j = (c_{2l-1})_i = -1$ and other elements of c_{2l} and c_{2l-1} are equal to zero. $\mathcal{L}$ is used to specify the Laplacian matrix of this graph. Furthermore, in this graph, $0 \leq w_{ij} = w_{ji} \leq 1$ shows the weight of communication edge between vertices i and j. Moreover, the weighted undirected graph $\tilde{\mathcal{G}} = (\mathcal{V},\tilde{\mathcal{E}},\tilde{\mathcal{W}})$ is called subgraph of $\mathcal{G}$ ($\tilde{\mathcal{G}}\subseteq\mathcal{G}$) if and only if $\tilde{\mathcal{E}}\subseteq\mathcal{E}$ and for every $(i, j)\in\tilde{\mathcal{E}}$, we have $0\leq \tilde{w}_{ij} \leq w_{ij}$.

$e^m_1 = (1,0,\ldots,0)^\top, \ldots, e^m_m = (0,0,\ldots,1)^\top$ specify the standard coordinate basis of $\mathbb{R}^m$. Also, $\mathbf{1}_m\in\mathbb{R}^m$ is a vector whose all elements equal 1. The identity matrix in $\mathbb{R}^{n\times n}$ is denoted by I_n (The subscript n is omitted in the cases that the dimension of the identity matrix can be determined with respect to dimensions of the other matrices).

III. Problem Statement

Consider a multi-agent system with n agents such that each agent updates its state vector by the dynamic model

$$\dot{x}_i(t) = A^\star x_i(t) + B^\star u_i(t); \quad i = 1,2,\ldots,n \tag{1}$$

where $x_i(t)\in\mathbb{R}^{n_x}$ and $u_i(t)\in\mathbb{R}^{n_u}$ are respectively the state vector and the control input of the ith agent. It is assumed that there is a predefined weighted graph $\mathcal{G} = (\mathcal{V},\mathcal{E},\mathcal{W})$ for the communication between the agents of this multi-agent system, where $\mathcal{V} = \{1,2,\ldots,n\}$. In this paper, the main objective is to simultaneously find a sub-graph $\tilde{\mathcal{G}}\subseteq\mathcal{G}$ with minimum weights for its edges (optimizing the network topology) and the control signals $\{u_i(t)\}_{1\leq i\leq n}$ such that consensus is achieved with a suitable convergence rate in the multi-agent system (1). Formally speaking, we aim to ensure that for any initial condition $x(0) = [x_1(0) \;\; x_2(0) \;\; \cdots \;\; x_n(0)]\in\mathbb{R}^{n\times n_x}$, reducing the communication graph to sub-graph $\tilde{\mathcal{G}}$ and applying the obtained control signals can guarantee

$$\lim_{t\to\infty} \|x_i(t) - x_j(t)\| = 0; \quad \forall 1\leq i, j\leq n. \tag{2}$$

Also, it is assumed that the matrices $A^\star$ and $B^\star$ in (1) are not precisely known, and all the information we have about them is described as follows.

Assumption 1 (Uncertainty Characterizations): Consider the multi-agent system (1). There are known nominal matrices A, B such that

$$|A^\star - A| \leq A^b, \quad |B^\star - B| \leq B^b$$

where the above inequalities are interpreted in an element-wise form, and matrices $A^b = \begin{bmatrix} a^b_{ij} \end{bmatrix}_{i,j}$ and $B^b = \begin{bmatrix} b^b_{ik} \end{bmatrix}_{i,k}$ denote the uncertainty bounds.

Assumption 1 describes the model uncertainties and available information about the nominal values and uncertainty bounds in system (1). The next assumption is related to the primary communication graph of the considered multi-agent system.

Assumption 2 (Communication Characterizations): Consider multi-agent system (1). For this multi-agent system, there is a weight graph $\mathcal{G} = (\mathcal{V},\mathcal{E},\mathcal{W})$, that determines the quality of communication between agents in system (1). In this graph, $\mathcal{V} = \{1,2,\ldots,n\}$. Moreover, $W = \{w_{ij}\}_{1\leq i\neq j\leq n}$, where $0\leq w_{ij} = w_{ji}\leq 1$ is the weight of the edge which connects the agent i to agent j. Also, it is assumed that the graph $\mathcal{G}$ has a spanning tree, and m denotes the number of its edges which have positive weights.

In this paper, we intend to simultaneously find a simple controller and a subgraph for graph $\mathcal{G}$ under which system (1) reaches to consensus at an appropriate convergence rate. Naturally, we expect a trade-off between the communication graph edges' weight values and the convergence speed. Generally speaking, this means that the convergence rate can be improved for a given controller if the weights of the graph edges are increased. Finding the optimal subgraph and the robust controller guaranteeing consensus is the main focus of this paper in Section IV. The corresponding problem can be formulated as follows.

Problem 1: Consider multi-agent system (1) under Assumptions 1 and 2. Determine the continuous-time controller $\{x_j(t)\}_{j\in\mathcal{N}_i} \mapsto u_i(t)$ and the subgraph $\tilde{\mathcal{G}} \subseteq \mathcal{G}$ for the communication between agents such that the consensus with an arbitrary convergence rate is guaranteed, whereas the minimum weights for edges in subgraph $\tilde{\mathcal{G}}$ are conceived.

Due to the trade-off nature of Problem 1, by defining a balance parameter, the importance of two objective factors in this problem (i.e., the convergence speed and the weights of the edges in the graph) can be relatively rated. By changing such a parameter, the answer of the Problem 1 moves between the answer yielding in the highest convergence speed and the solution resulting in the edges having the smallest weights guaranteeing consensus.

IV. Main Results

The main focus of this section is to find a solution for Problem 1. To formulate the concept of convergence rate (speed), it is necessary to provide a precise mathematical definition for this concept before trying to solve Problem 1. The next definition clarifies this point.

Definition 1 (Convergence Rate) [46]: Consider a dynamic system in the form $\dot{x} = f(x)$. Assume that the origin is a globally asymptotically stable fixed point for this system. Also, assume that there exists Lyapunov function $V(x)$ such that $V(x) > 0$ and $\dot{V}(x) < 0$ for all $x \neq 0$. Based on this Lyapunov function, the convergence rate of the system can be defined in following form:

$$\beta := \frac{1}{2}\inf\left\{-\frac{\dot{V}(x)}{V(x)}\middle|\forall x \neq 0\right\}. \tag{3}$$

Definition 1 introduced the convergence rate for a general nonlinear system. In the special case of facing with a stable linear system in the form $\dot{x} = Ax$, by simple calculations it can be shown that the convergence rate, which is generally expressed by (3), can be found for this linear system by solving an optimization program in the following form:

$$\begin{cases} \beta = \sup\limits_{\lambda,P} \lambda \\ \text{s.t.} \quad P = P^\top > 0, \lambda \in \mathbb{R}, [PA]^\dagger \leq -\lambda P. \end{cases} \tag{4}$$

Now that the convergence rate is mathematically defined, we can begin to solve Problem 1. For this purpose, consider a simple proportional controller and assume that each agent updates its control effort as

$$u_i(t) = K\sum_{j\in\mathcal{N}_i}\tilde{w}_{ij}(x_j - x_i); \qquad 1 \leq i \leq n \tag{5}$$

where, $K \in \mathbb{R}^{n_u\times n_x}$ is the controller gain and $\tilde{w}_{ij}$ is the weight of the edge that connects the agent i to the agent j in the subgraph $\tilde{\mathcal{G}}$. The next theorem provides an optimization based framework to obtain the controller gain K in (5) along with a suitable communication graph $\tilde{\mathcal{G}}$ and the corresponding Lyapunov function.

Theorem 1 (Solving Problem 1): Consider the multi-agent system (1) satisfying Assumptions 1 and 2. Also, consider the optimization problem

$$\begin{cases} \max\limits_{\tilde{w}_{ij},\bar{K},\beta} \quad f(\beta,\{w_{ij}\}_{i,j}) \\ \qquad \text{s.t.} \quad Q \in \mathbb{S}^{n_x}_{>0}, \lambda_2, \mu, \{\tilde{w}_{ij}\}_{ij} \in \mathbb{R}_{\geq 0}, \beta \in \mathbb{R} \\ \qquad \lambda_2 I_n \leq \mu \mathbf{1}_n\mathbf{1}_n^\top + C\,\mathfrak{Diag}\{\tilde{w}_{ij},\tilde{w}_{ij}\}C^\top \\ \qquad 0 \leq \tilde{w}_{ij} \leq w_{ij} \\ \qquad \left[(A^\star - \lambda_2 B^\star \bar{K})\right]^\dagger \leq -\beta Q, \\ \qquad \forall |A^\star - A| \leq A^b, |B^\star - B| \leq B^b \end{cases} \tag{6}$$

where $C \in \mathbb{R}^{n_x\times m}$ is the incidence matrix of graph $\mathcal{G}$, and $f : \mathbb{R}^{n^2-n+1} \to \mathbb{R}$ is an arbitrary function that makes a trade-off between the weight of edges in subgraph $\tilde{\mathcal{G}}$ and convergence rate β. Also, assume that $\bar{K}^\star$,$\beta^\star$,$Q^\star$, and $\{\tilde{w}^\star_{ij}\}_{i,j}$ are the optimal solutions of decision variables in the optimization problem (6), where $\beta^\star \geq 0$. Then, the system (1) with the communication graph $\tilde{\mathcal{G}}$ and the controller (5) reaches to consensus with convergence rate $\beta^\star$ if the controller gain is chosen as $K = \bar{K}^\star Q^{\star-1}$.

Proof: Define $e_i(t)$ as

$$e_i(t) := x_1(t) - x_i(t); \qquad 2 \leq i \leq n \tag{7}$$

which can be used to verify the consensusability in the considered multi-agent system. From definition (7) and dynamic model (1), it is obtained that

$$\dot{e}_i(t) = A^\star e_i(t) - B^\star K\left[\sum_{j\in\mathcal{N}_1\cup\mathcal{N}_i}(w_{1j} - w_{ij})e_j(t) + \sum_{j\in\mathcal{N}_i} w_{ij}e_i(t)\right]$$

for $2 \leq i \leq n$. Defining $e(t) := \begin{bmatrix} e_2^\top(t) & e_3^\top(t) & \cdots & e_n^\top(t) \end{bmatrix}^\top$, it can be concluded that

$$\dot{e}(t) = \left[I_{n-1}\otimes A^\star - (\tilde{\mathcal{L}}_{22} + \mathbf{1}_{n-1}.\tilde{\nu})\otimes B^\star K\right]e(t)$$

where $\tilde{\mathcal{L}}$ is the Laplacian matrix of graph $\tilde{\mathcal{G}}$ and is in the form

$$\tilde{\mathcal{L}} = \begin{bmatrix} \tilde{\nu}_{11} & -\tilde{\nu} \\ * & \tilde{\mathcal{L}}_{22} \end{bmatrix}.$$

According to the fact that

$$\begin{bmatrix} 1 & 0 \\ \mathbf{1}_{n-1} & I_{n-1} \end{bmatrix}^{-1} \tilde{\mathcal{L}} \begin{bmatrix} 1 & 0 \\ \mathbf{1}_{n-1} & I_{n-1} \end{bmatrix} = \begin{bmatrix} 0 & -\tilde{\nu} \\ 0 & \tilde{\mathcal{L}}_{22} + \mathbf{1}_{n-1}.\tilde{\nu} \end{bmatrix}$$

it can be deduced that the eigenvalues of $\tilde{\mathcal{L}}_{22} + \mathbf{1}_{n-1}.\tilde{\nu}$ are equal to non-zero eigenvalue of $\tilde{\mathcal{L}}$. Without loss of generality, assume that $\tilde{\mathcal{L}}_{22} + \mathbf{1}_{n-1}.\tilde{\nu}$ is diagonalizable. Hence, there exists matrix $T \in \mathbb{R}^{(n-1)\times(n-1)}$, such that

$$T^{-1}(\tilde{\mathcal{L}}_{22} + \mathbf{1}_{n-1}.\tilde{\nu})T = J$$

where $J \in \mathbb{R}^{(n-1)\times(n-1)}$ is a diagonal matrix. Using this equation and doing some straightforward computations, it is deduced that

$$(T\otimes I_n)^{-1}F_e^\star(T\otimes I_n) = I_{n-1}\otimes A^\star - J\otimes B^\star K \tag{8}$$

where $F_e^\star = I_{n-1}\otimes A^\star - (\tilde{\mathcal{L}}_{22} + \mathbf{1}_{n-1}.\tilde{\nu})\otimes B^\star K$. From (8), we can conclude that the stability and convergence rate of system $\dot{e}(t) = F_e^\star e(t)$ is similar to those of system $\dot{\bar{e}}(t) = (I_{n-1}\otimes A^\star - J\otimes B^\star K)\bar{e}(t)$. According to the diagonalizability assumption for matrix $\tilde{\mathcal{L}}_{22} + \mathbf{1}_{n-1}.\tilde{\nu}$, it can be concluded that the matrix

$I_{n-1} \otimes A^\star - J \otimes B^\star K$ is a diagonal matrix. Therefore, the convergence speed of $\dot{\bar{e}}(t) = (I_{n-1} \otimes A^\star - J \otimes B^\star K)\bar{e}(t)$ is determined by the minimum of the convergence rate of the following systems:

$$\dot{y}_j(t) = (A^\star - \lambda_j B^\star K) y_j(t); \qquad 2 \le j \le n \tag{9}$$

where $\lambda_2 \le \cdots \le \lambda_n$ are diagonal elements of matrix J, or equivalently the eigenvalues of matrix $\tilde{\mathcal{L}}_{22} + \mathbf{1}_{n-1}.\tilde{\nu}$, which are equal to nonzero eigenvalues of $\tilde{\mathcal{L}}$. According to (4), we can find the convergence rate of systems in (9) from solving the optimization problem

$$\begin{cases} \sup\limits_{\alpha_j, P_j, K} \quad \alpha_j \\ \qquad \text{s.t.} \quad P_j \in \mathbb{S}^{n_x}_{>0}, K \in \mathbb{R}^{n_u \times n_x}, \alpha_j \in \mathbb{R} \\ \qquad \left[P_j(A^\star - \lambda_j B^\star K)\right]^\dagger \le -\alpha_j P_j \\ \qquad \forall |A^\star - A| \le A^b, |B^\star - B| \le B^b. \end{cases} \tag{10}$$

The convergence rate of system $\dot{e}(t) = F_e^\star e(t)$ will be equal to $\min\{\alpha_2^\star, \ldots, \alpha_n^\star\}$, where $\alpha_i^\star$ denotes the optimal solution of optimization problem (10). According to [46, Proposition 2], it can be concluded that if $\alpha_i^\star$ is the solution of optimization problem (10), then for all $\varepsilon > 0$, there exists matrix $K = -lB^{\star\top} P_j$ with $l > 0$, such that the solution of optimization problem

$$\begin{cases} \sup\limits_{\beta_j, P_j} \quad \beta_j \\ \text{s.t.} \quad \left[P_j(A^\star - l\lambda_j B^\star B^{\star\top} P_j)\right]^\dagger \le -\beta_j P_j \\ \qquad \forall |A^\star - A| \le A^b, |B^\star - B| \le B^b \end{cases} \tag{11}$$

is greater than or equal to $\alpha_j^\star - \varepsilon$. This fact helps to show that $K^\star = -l^\star B^{\star\top} P_j$ can be the optimal solution for the optimization problem (10). Assume that $\alpha_2^\star$ is the solution of the optimization problem (10) for $j = 2$. According to (11),

$$\alpha_2^\star I_{n_x} + 2(\lambda_i - \lambda_2) l^\star B^\star B^{\star\top} P_2 \le \alpha_i^\star I_{n_x}; \qquad 3 \le i \le n$$

where $l^\star > 0$ is a constant guaranteeing that the solution of the optimization problem (11) is greater than $\alpha_2^\star - \varepsilon$, for an arbitrary small $\varepsilon > 0$. The above-mentioned inequality and result related to optimization problem (11) show that the minimum convergence rate between systems in (9) belongs to the system which is related to λ_2. Now, we reformulate the inequality $\lambda_2 \le \lambda_j$ for all $3 \le j$. It can be replaced by the inequality

$$\lambda_2 I_n \le \mu \mathbf{1}_n \mathbf{1}_n^\top + \tilde{\mathcal{L}}. \tag{12}$$

On the other hand, the Laplacian matrix of graph $\tilde{\mathcal{G}}$ can be written in the following form:

$$\tilde{\mathcal{L}} = C\, \mathfrak{Diag}\{\tilde{w}_{ij}, \tilde{w}_{ij}\} C^\top \tag{13}$$

where C is the incidence matrix of graph $\mathcal{G}$. The proof is then completed by considering the equations (10), (12), and (13) with variables $Q := P_2^{-1}$, $\bar{K} := KQ$, and $\beta = \alpha_2$. ■

To justify the feasibility of the optimization problem (6), we can find a feasible point satisfying the constraints of this problem. For this purpose, assume that $\bar{K}_0 = 0$, $Q_0 = I_{n_x}$, and $\tilde{w}_{ij_0} = w_{ij}$. Now if λ_{2_0} is the second smallest eigenvalue of $\mathcal{L}$ and β_0 is the minimum of the eigenvalues of the matrices in the convex set $\{-[A^\star]^\dagger\}_{|A^\star - A| \le A^b}$, then $\left(\bar{K}_0, Q_0, \lambda_{2_0}, \beta_0, \tilde{w}_{ij_0}\right)$ specifies a feasible point for the optimization problem (6).

As mentioned before, by determining the objective function $f(\beta, \{\tilde{w}_{ij}\}_{i,j})$ in the optimization problem (6) the trade-off between the weight of edges in the communication graph and the convergence rate can be appropriately balanced. The following remark investigates special forms for this objective function.

Remark 1 (Special Objective Functions): As discussed before, increasing the weights of the edges in the neighborhood graph yields in improving data sharing between the agent, and this generally results in speeding up the convergence. This fact reveals that there is a trade-off between convergence rate and graph's edge weights. Such a trade-off can be considered in various forms in the objective function $f(\cdot)$ of the optimization problem (6) in order to meet the design objective. As an example, consider the case that the exchange of data between two agents costs proportional to the weight of the edge connecting these two agents. In this case, the concave objective function can be considered as follows:

$$f(\beta, \{\tilde{w}_{ij}\}_{i,j}) = \ln(\beta) - \zeta \ln\left(\sum_{i,j} \tilde{w}_{ij}\right). \tag{14}$$

In the function (14), by properly setting the parameter $\zeta \in \mathbb{R}$, the mentioned trade-off can be appropriately modeled. As a special case of this function, if we set $\zeta = 0$, then the optimization problem (6) reduces to the classic problem of optimizing the convergence rate in consensus of a linear multi-agent system (e.g., [47]). As mentioned, in objective function (14), the parameter ζ balances the weight of the convergence rate and that of the communication load. In particular, if in the design process the reduction of the communication load is a higher priority in comparison to the convergence speed, the designer should accomplish this priority by choosing a large value for parameter ζ.

The optimization problem (6) in Theorem 1, in its general form, is non-convex. The non-convexity originates from the fact that the last constraint in this optimization problem should be satisfied for a family of matrices. It is worth noting that, as a special case, if the system matrices ($A^\star$ and $B^\star$) are fixed and known, then the problem reduces an LMI and can be solved by using regular LMI solvers. But, unfortunately, when these matrices are uncertain, finding the exact solution of the aforementioned optimization problem is provably intractable. In order to clarify the point, let us consider the problem of checking the stability of system

$$\dot{z}(t) = (A^\star - \lambda_2 B^\star K) z(t) \tag{15}$$

for all possible values of $A^\star$ and $B^\star$, where K is known. Checking the stability of system (15) is equivalent to finding $P \in S^{n_x}_{>0}$ (for constructing the Lyapunov function $V(z) = z^\top P z$), such that it can guarantee

$$\begin{aligned} &\forall |A^\star - A| \le A^b, \forall |B^\star - B| \le B^b \\ &\exists P_{A^\star, B^\star} \in S^{n_x}_{>0} : \left[P_{A^\star, B^\star}\left(A^\star + B^\star K\right)\right]^\dagger \le 0. \end{aligned} \tag{16}$$

The problem (16) is a special case of the problem of stability checking of an interval matrix [48], which is strongly

NP-hard [49, Corollary 2.6]. It is clear that the problem (16) has a solution if and only if the solution of the optimization problem (10) for $j=2$ and a fixed K is non-negative. Consequently, it is expected that there is no polynomial-time algorithm for solving problem (16) (or problem (6)). A conservative approach for relaxing the situation is to use a common Lyapunov function in (16), which yields in the following problem:

$$\exists P\in S^{n_x}_{>0}: \forall|A^\star - A|\le A^b,\, \forall|B^\star - B|\le B^b,$$
$$\left[P\left(A^\star + B^\star K\right)\right]^\dagger \le 0. \tag{17}$$

Inequality (17) is a special form of the problems, which are known as the "matrix cube problems" in [50]. Unfortunately it has been proved that this class of problems are also NP-hard [50, Proposition 4.1]. Hence, we have to resort to approximation methods for solving such problems. Existing results in this area offer effective conservative approximations such that the resulted conservatism is bounded independently of the size of the problem [51]. Benefiting from such developments, the following theorem provides a conservative (but computationally tractable) convex optimization to be solved instead of the optimization problem (6).

Theorem 2 (Convex Optimization Framework): Consider the multi-agent system (1) under Assumptions 1 and 2. Assume that C is the incidence matrix of the graph $\mathcal{G}$ and f is a concave function with respect to its variables. The system (1) with the controller (5) reaches to consensus under the communication graph $\mathcal{G}=(\mathcal{V},\mathcal{E},\tilde{\mathcal{W}})$ and with the controller gain $K=\bar{K}^\star Q^{\star -1}$, if $\beta^\star\ge 0$, where $\beta^\star,\bar{K}^\star,Q^\star$ and the weight of graph edges ($\{\tilde{w}_{ij}\}_{i,j}$) are the solution of convex optimization problem

$$\begin{cases}\max\limits_{\tilde{w}_{ij},\bar{K},\beta} & f(\beta,\{w_{ij}\}_{i,j})\\ & \text{s.t. } \; Q\in\mathbb{S}^{n_x}_{>0},\, \lambda_2,\mu,\{\tilde{w}_{ij}\}_{ij}\in\mathbb{R}_{\ge 0},\, \beta\in\mathbb{R}\\ & \kappa_{ij},\gamma_{ik}\in\mathbb{R}_{\ge 0},\, H_1=\mathbf{1}_{n_x}\otimes I_{n_x}\\ & H_2=\lambda_2\bar{K}\left[\;\mathbf{1}_{n_u}\otimes e_1^{n_x}\ldots\mathbf{1}_{n_u}\otimes e_{n_x}^{n_x}\;\right]\\ & G_1=-\mathfrak{Diag}\{\kappa_{ij}I\}_{i,j},\, G_2=-\mathfrak{Diag}\{\gamma_{ik}I\}_{i,k}\\ & \lambda_2 I_n\le \mu\mathbf{1}_n\mathbf{1}_n^\top + C\,\mathfrak{Diag}\{\tilde{w}_{ij},\tilde{w}_{ij}\}C^\top\\ & 0\le\tilde{w}_{ij}\le w_{ij}\\ & \begin{bmatrix} H_0 & \star & \star\\ H_1^\top & G_1 & \star\\ H_2^\top & 0 & G_2\end{bmatrix}\le 0\end{cases} \tag{18}$$

with[1]

$$H_0=\left[A-\lambda_2 B\bar{K}\right]^\dagger - \beta Q+\sum_{i,j}\left\{\kappa_{ij}(a_{ij}^b)^2 e_i^{n_x}[e_i^{n_x}]^\top\right\}$$
$$+\sum_{i,k}\left\{\gamma_{ik}(b_{ij}^b)^2 e_i^{n_x}[e_i^{n_x}]^\top\right\}. \tag{19}$$

[1] Formally speaking, there is a bilinear term in the constraint of the optimization (18) which can cause non-convexity. However, since the only source of non-convexity is the scalar variable β, a straightforward approach is to adjust this variable through a grid-search or some other efficient methods like as that proposed in [52].

Proof: To approximate problem (6) by a conservative one that can be solved by off-the-shelves convex optimization solvers, we need to use a conservative relaxation for the constraint $\left[(A^\star-\lambda_2B^\star\bar{K})\right]^\dagger\le -\beta Q$. According to the mentioned definitions,

$$A^\star = A+\sum_{i=1}^{n_x}\sum_{j=1}^{n_x} e_i^{n_x}\delta a_{ij}[e_j^{n_x}]^\top,\quad |\delta a_{ij}|\le a_{ij}^b$$
$$B^\star = B+\sum_{i=1}^{n_x}\sum_{k=1}^{n_u} e_i^{n_x}\delta b_{ik}[e_k^{n_u}]^\top,\quad |\delta b_{ik}|\le b_{ik}^b \tag{20}$$

where $\{\delta a_{ij}\}$ and $\{\delta b_{ik}\}$ denote the elements of matrices $A^\star - A$ and $B^\star - B$, respectively. From (20), the constraint $\left[(A^\star-\lambda_2B^\star\bar{K})\right]^\dagger\le -\beta Q$ can be represented as

$$\left[\lambda_2 B\bar{K}-A\right]^\dagger+\beta Q+\sum_{i=1}^{n_x}\sum_{j=1}^{n_x}\left[e_i^{n_x}(-\delta a_{ij})[e_j^{n_x}]^\top\right]^\dagger$$
$$+\sum_{i=1}^{n_x}\sum_{k=1}^{n_u}\left[e_i^{n_x}\delta b_{ik}\lambda_2[e_k^{n_u}]^\top\bar{K}\right]^\dagger\ge 0. \tag{21}$$

According to [51, Theorem 3.1], it can concluded that the constraint (21) holds if there exist parameters $M_{ij},N_{ik},\kappa_{ij},\gamma_{ik}$, where $i,j\in\{1,\ldots,n_x\}$ and $k\in\{1,\ldots,n_u\}$, such that

$$\kappa_{ij},\gamma_{ij}\ge 0$$
$$\begin{bmatrix} M_{ij}-\kappa_{ij}(a_{ij}^b)^2 e_i^{n_x}[e_i^{n_x}]^\top & \star\\ [e_j^{n_x}]^\top & \kappa_{ij}I\end{bmatrix}\ge 0$$
$$\begin{bmatrix} N_{ik}-\gamma_{ik}(b_{ij}^b)^2 e_i^{n_x}[e_i^{n_x}]^\top & \star\\ \lambda_2[e_k^{n_u}]^\top\bar{K} & \gamma_{ik}I\end{bmatrix}\ge 0$$
$$\left[\lambda_2B\bar{K}-A\right]^\dagger+\beta Q\ge\sum\nolimits_{i,j}M_{ij}+\sum\nolimits_{i,k}N_{ik}. \tag{22}$$

Using the Schur complement in the first two inequalities of (22) yields

$$\kappa_{ij},\gamma_{ik}\ge 0$$
$$M_{ij}-\kappa_{ij}(a_{ij}^b)^2e_i^{n_x}[e_i^{n_x}]^\top-\kappa_{ij}^{-1}e_j^{n_x}[e_j^{n_x}]^\top\ge 0$$
$$N_{ik}-\gamma_{ik}(b_{ij}^b)^2e_i^{n_x}[e_i^{n_x}]^\top-\gamma_{ik}^{-1}\lambda_2^2\bar{K}^\top e_k^{n_u}[e_k^{n_u}]^\top\bar{K}\ge 0$$
$$\left[\lambda_2B\bar{K}-A\right]^\dagger+\beta Q\ge\sum\nolimits_{i,j}M_{ij}+\sum\nolimits_{i,k}N_{ik}.$$

By doing some simple computations and by eliminating $\{M_{ij}\}_{i,j},\{N_{ik}\}_{i,k}$ in the above inequalities, it is obtained that

$$\kappa_{ij},\gamma_{ik}\ge 0$$
$$\left[A-\lambda_2B\bar{K}\right]^\dagger-\beta Q$$
$$+\sum\nolimits_{i,j}\left\{\kappa_{ij}(a_{ij}^b)^2e_i^{n_x}[e_i^{n_x}]^\top+\kappa_{ij}^{-1}e_j^{n_x}[e_j^{n_x}]^\top\right\}$$
$$+\sum\nolimits_{i,k}\left\{\gamma_{ik}(b_{ij}^b)^2e_i^{n_x}[e_i^{n_x}]^\top+\gamma_{ik}^{-1}\lambda_2^2\bar{K}^\top e_k^{n_u}[e_k^{n_u}]^\top\bar{K}\right\}\le 0. \tag{23}$$

Reusing the Schur complement for (23) leads to the conclusion that these inequalities are equivalent to

$$\begin{bmatrix} H_0 & \star & \star \\ H_1^\top & G_1 & \star \\ H_2^\top & 0 & G_2 \end{bmatrix} \leq 0 \tag{24}$$

where, H_0, H_1, H_2, G_1, and G_2 were defined in (18) and (19). Using the result of (24), we can conclude that (6) can be replaced with the conservative version (18). ∎

To confirm the feasibility of optimization problem (18), assume that $\bar{K}_1 = 0$, $Q_1 = I_{n_x}$, $\tilde{w}_{ij_1} = w_{ij}$, and $\kappa_{ij_1} = 1$. Let λ_{2_1} be the second smallest eigenvalue of $\mathcal{L}$ and β_1 and γ_{ij_1} be a feasible point in the convex set

$$\begin{cases} \gamma_{ij} \in \mathbb{R}_{>0}, \beta \in \mathbb{R}, W_{ij} \in \mathbb{R}^{n_x \times n_x} \\ \begin{bmatrix} W_{ij} - \gamma_{ij}(a_{ij}^b)^2 e_i^{n_x} e_i^{n_x \top} J & * \\ e_j^{n_x \top} J & \gamma_{ij} \end{bmatrix} \geq 0 \\ 0 \geq \beta I + A + A^\top + \sum_{i,j} W_{ij}. \end{cases}$$

Then, $\left(\bar{K}_1, Q_1, \lambda_{2_1}, \tilde{w}_{ij_1}, \beta_1, \gamma_{ij_1}, \kappa_{ij_1}\right)$ is a feasible point for optimization problem (18).

Remark 2 (About Solutions of the Optimization Problems): In [47, Corollary 1], it has been proved that the certain version of system (1) (i.e., in the case $A^b = 0$ and $B^b = 0$) is consensusable if the communication graph $\mathcal{G}$ has a spanning tree and the pair $(A^\star, B^\star)$ is stabilizable. This result reveals that if the system matrices are known and the pair $(A^\star, B^\star)$ is stabilizable, then the optimization problem (6) has a positive solution for $\beta^\star$ (because we can set $\tilde{\mathcal{G}} = \mathcal{G}$ and according to [47], the system (1) with this communication graph has a stabilizer in the form (5)). On the other hand, in the case that the system matrices are uncertain, if the optimization problem (18) is infeasible, there is no β with the condition $\beta \geq 0$ in the feasible set of this optimization problem. Considering the conservative nature of problem (18), the absence of non-negative β in the mentioned feasible set does not mean that the problem (6) is necessarily infeasible (i.e., in this case it cannot be necessarily deduced that the controller gain K and graph weights $\{\tilde{w}_{ij}\}_{i,j}$ are not found to stabilize system (1) for all possible values of $A^\star$ and $B^\star$). But thanks to the results of [51], we can conclude that if the optimization problem (18) is infeasible for some bounds A^b and B^b, then the optimization (6) will be infeasible for optimization bounds $A_n^b = \frac{\pi}{2} A^b$ and $B_n^b = \frac{\pi}{2} B^b$. This means that in this case matrix K in controller (5) cannot be found to guarantee the consensus in uncertain system (1) with uncertainty bounds $A^b = A_n^b$ and $B^b = B_n^b$.

V. Numerical Examples

In this section, numerical examples illustrating the applicability of the main results of the paper are presented.

Example 1: Consider a group of 5 homogeneous uncertain agents described by dynamic model (1), which satisfies Assumption 1, with

$$A = \begin{bmatrix} 1.40 & -0.21 & 6.71 & -5.68 \\ -0.58 & -4.29 & 0 & 0.67 \\ 1.07 & 4.27 & -6.65 & 5.89 \\ 0.05 & 4.27 & 1.34 & -2.10 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 5.68 & 0 \\ 1.14 & -3.15 \\ 1.14 & 0 \end{bmatrix}.$$

$A^b = 0.1(\mathbf{1}_4^\top \otimes \mathbf{1}_4)$ and $B^b = 0.1(\mathbf{1}_2^\top \otimes \mathbf{1}_4)$ (The nominal matrices are selected from "REA2" example in *Complib* library of MATLAB[2]).

Suppose that the primary communication graph of the mentioned multi-agent system is as that shown in Fig. 1 and the weight of all edges in this graph is equal to 1. In this example, we consider the objective function for the optimization problem (18) in the form $f(\beta, \{\tilde{w}_{ij}\}_{i,j}) = \ln(\beta) - \zeta \ln\left(\sum_{i,j} \tilde{w}_{ij}\right)$. By solving this optimization problem for various random multi-agent systems with dynamic matrices in the aforementioned intervals, the trade-off between the weight of edges and the convergence speed for different values of parameter ζ is revealed in the numerical simulation results presented in Fig. 2. These results have been derived by solving the optimization problem (18) for different values of ζ, and then finding the convergence speed and the weight of edges based on the obtained solutions. The red (blue) plot in Fig. 2 shows the variations of the optimal convergence speed (sum of the weight of edges as an indicator for the communication load) with respect to ζ. Confirming through the numerical simulation results of Fig. 2, by increasing the tuning parameter ζ, the weight of the communication load in comparison to that of the convergence speed in the optimization problem increases. As a result, in solution of the optimization problem (6) (or in that of its conservative version given by (18)), the communication load decreases by increasing ζ. This causes that the convergence rate decreases and the consensus time increases.

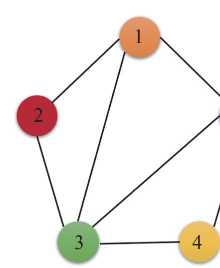

Fig. 1. Communication topology in Example 1.

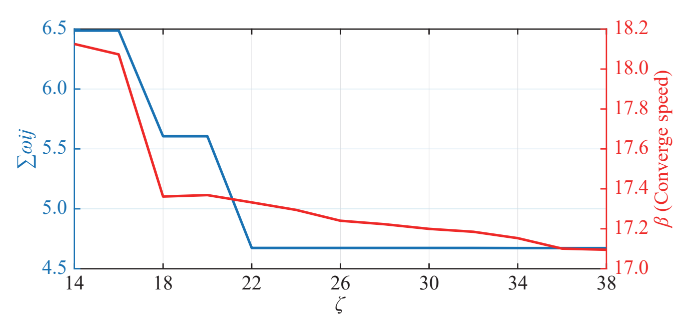

Fig. 2. The trade-off between the weight of edges and the convergence speed for different values of parameter ζ.

The following example, borrowed from [28] with some modifications, verifies the applicability of the obtained results in consensus analysis in a platoon of homogeneous vehicles

[2] http://www.complib.de/

with boundary uncertainties.

Example 2: Consider a platoon of 10 automated vehicles, as a multi-agent system with 10 agents. Using an inverse model compensation based approach, such agents have been modeled by uncertain linear models in [28]. The reduced uncertain linear dynamic model of agent $i \in \{1,\dots,10\}$ is of the form

$$\dot{x}_i = (A+\Delta A)x_i + (B+\Delta B)u_i \tag{25}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -\bar{k} \end{bmatrix}, \quad \Delta A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\Delta k \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 & -\bar{k} \end{bmatrix}^\top, \quad \Delta B = \begin{bmatrix} 0 & 0 & -\Delta k \end{bmatrix}^\top.$$

$x_i = \begin{bmatrix} p_i & v_i & a_i \end{bmatrix}^\top$ is the state vector, in which p_i, v_i, and a_i are the ith vehicle's position, velocity and acceleration, respectively. In this example, it is assumed that $1.21 \le \bar{k} + \Delta k \le 2.95$. Suppose that the primary communication graph of this multi-vehicular system is as that shown in Fig. 3. The sum of the weight of the primary graph's edges is equal to 17, and the Laplacian matrix of this graph is in the form

$$\mathcal{L} = \begin{bmatrix} 2 & \star & \star & \cdots & \star \\ -1 & 3 & \star & \cdots & \star \\ -1 & -1 & 4 & \cdots & \star \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 2 \end{bmatrix}.$$

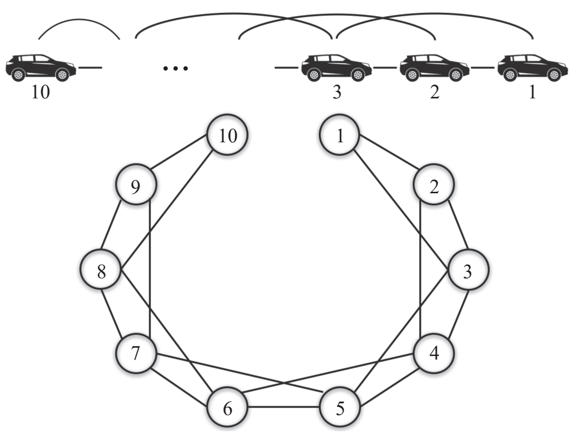

Fig. 3. Communication topology in Example 2.

Let the aim in this problem be to optimize the network topology such that the convergence rate in consensus of the agents is not considerably affected. For this purpose, the optimization problem (18) with the objective function in the form (14) and $\zeta = 30$ has been solved. By solving the optimization problem with the specified objective function, the sum of the weight of the graph edges is reduced to 1.9541. In addition, starting from the initial positions $[1\ 2\ 3\ \dots\ 10]^\top$, the agents converge to consensus after about 7 seconds, as that shown in Fig. 4.

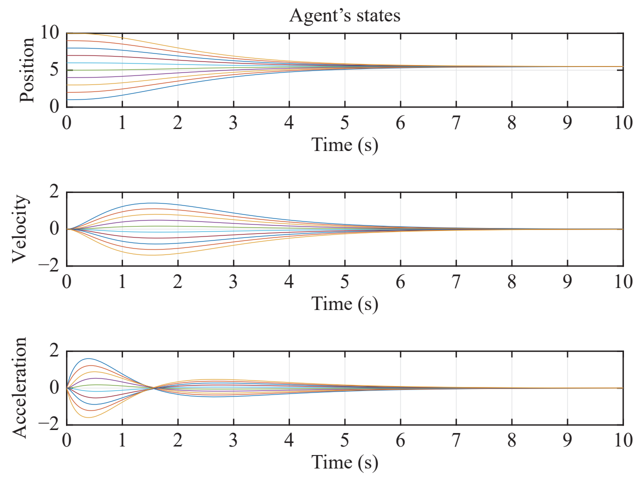

Fig. 4. The position, velocity, and acceleration of the vehicles in Example 2 under the controller and the optimized network topology obtained by solving the optimization program (18).

VI. Conclusion

In this paper, an optimization-based framework was introduced to solve the consensus problem for an uncertain homogeneous linear multi-agent system. The main advantage of this framework, introduced in Theorem 1, is that in addition to synthesizing a robust controller for ensuring consensus with an adjustable convergence rate, it can simultaneously reduce the communications between the agents. Furthermore, it was shown that the optimization problem that was introduced in Theorem 1 is NP-hard. To deal with this challenge, the corresponding optimization problem was conservatively relaxed in the Theorem 2 such that the alternative conservative problem can be solved by applying the regular LMI solvers. There are some interesting lines, which invite further research works in the continuation of the research done in this paper. For instance, a relevant research work in future can be to propose a distributed algorithm for solving the optimization problem (18). It seems that the idea of *Projected Primal-Dual Gradient Flow*, which has been introduced in [53], can be helpful to propose such a distributed algorithm. As another relevant future work, similarly to the frameworks surveyed in [54], the obtained results can be extended to balance the convergence rate and the communication load in consensus of heterogeneous multi-agent systems.

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