

A Novel Parallel Control Method for Continuous-Time Linear Output Regulation With Disturbances

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Abstract—In this article, a novel linear parallel control method is developed for output regulation problems with disturbances. The traditional feedback regulators are passive regulation methods. In order to solve this problem, the parallel controllers are presented, where the time variation of the control is constructed instead of the control value itself, to stabilize the output of the system. The main contributions of the developed method include two aspects: 1) a novel parallel regulator structure is presented, which can provide greater flexibility comparing with traditional methods and 2) the necessary and sufficient conditions for the existence of parallel regulators are analyzed. First, the structure of the linear parallel regulators is provided. Next, considering the situations that the full system information is obtained and the information of error is obtained, respectively, the properties of the parallel regulators are analyzed, and the regulator designs for the two situations are provided. Finally, numerical examples are provided to verify the correctness of the present method.

Index Terms—Continuous-time linear systems, output regulation, parallel control theory, parallel regulators, parallel systems.

I. INTRODUCTION

OVER the past few decades, control theory has been developing rapidly and gained a broad range of applications in industry [1]–[18]. In the study of control theory, the regulating control for the output of the systems (output regulation in brief) is always a research hotspot. The basic idea

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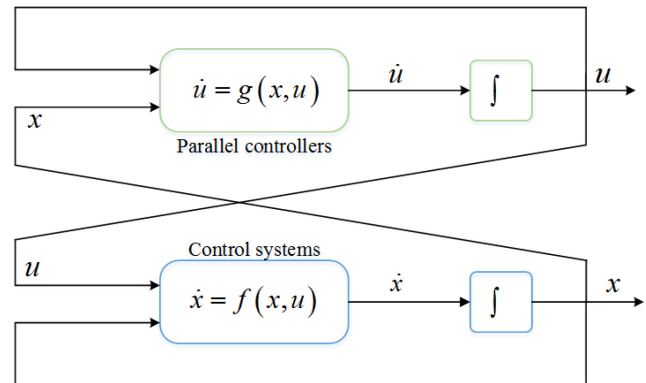


Fig. 1. Structure of parallel controllers.

of output regulation is to design feedback controllers to make the system output track a kind of reference signals [19], [20]. The research of linear output regulation rose in 1970s [21], where two basic methods of linear output regulation were proposed, including the feedforward design method [22] and internal model method [23], [24]. Since then, the output regulation problems have gained tremendous popularity, and a large number of important results were proposed, such as robust output regulation [25]–[28], adaptive output regulation [29]–[32], and others [33]–[39]. In existing output regulation methods, the regulators are generally constructed by system states and outputs, where the control value is directly obtained. However, the traditional feedback control methods exist some disadvantages [40]. For instances, traditional regulators are related to the system states and outputs, which may lead to great changes of control values with the drastic state changes. Furthermore, the state feedback regulators are passive regulation methods, and it is conducive to improving the control performance by introducing the control signals. Therefore, it is necessary to solve the above problems by designing a new controller.

In this article, a novel linear parallel output regulation method is developed inspired by the parallel controller. The basic structure of parallel controllers, which is first proposed by Wang *et al.* [40]–[44], can be shown in Fig. 1. It can be seen that the parallel controller is the function of state and current control, which is different from the feedback control. It is a remarkable feature of parallel control [40]. The studies of the parallel control theory have

gained certain achievement in recent years. In [45], the parallel tracking control problem is studied based on adaptive dynamic programming, which shows that the parallel control can guarantee continuity of control signal for the discontinuous reference signal. In [46], the parallel tracking control method is proposed under event-driven formulation. It should be noted that there are various parallel methods for regulation and tracking problems [47]–[52], which can be divided into the combinations of different control modes [47], [48] and the parallel system architecture [49]–[52] according to the research contents. However, this article presents a novel control structure for the linear output regulation problem, which makes the system and control symmetrical in both form and content mathematically. The contributions of this article can be summarized as follows.

- 1) A novel linear parallel control method is developed for output regulation problems with disturbances, where the time derivative of system control instead of system control itself is modeled to stabilize the system. The present parallel regulators can provide greater flexibility for the improvement of system performance comparing with the traditional feedback regulators.
- 2) The necessary and sufficient conditions for the existence of parallel regulators are provided in the situations that the full information can be obtained and only the tracking error can be obtained, respectively.
- 3) It can be seen that the parallel regulators are able to provide extra design degrees of freedom comparing with the traditional feedback regulators. Simulation results show that the present method has better system performance than the usual feedback regulators.

The remainder of this article is organized as follows. In Section II, the structure of parallel regulators is presented, and the problem formulation for linear parallel regulation is developed. In Section III, considering all the system information can be obtained and only the tracking error can be obtained, respectively, the linear parallel regulation existence theorems are given, and the linear parallel regulator design algorithms are provided. Numerical examples are provided in Section IV, which demonstrate the correctness of the presented method, and the conclusion is finally drawn in Section V.

II. PROBLEM FORMULATIONS

In this section, the basic structure of parallel regulators with disturbances is presented, and the problem formulations of linear parallel regulation problem are given.

A. Basic Structure of Parallel Regulators

The basic structure of parallel regulators is presented in this section. A continuous-time nonlinear system with disturbance can be given as

$$\dot{x} = f(x, u, w) \quad (1a)$$

$$\dot{w} = s(w) \quad (1b)$$

$$e = h(x, w) \quad (1c)$$

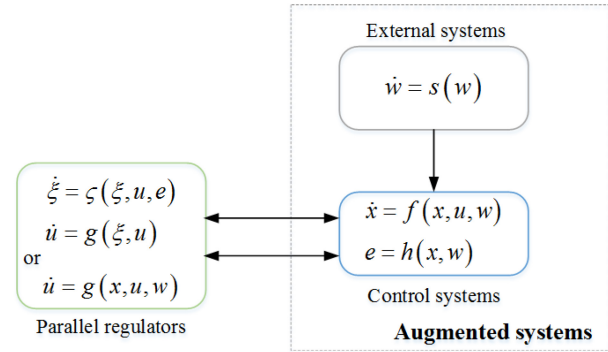


Fig. 2. Structure of parallel regulators.

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $w \in \mathbb{R}^r$, and $e \in \mathbb{R}^l$ are system state, control, external signal, and error, respectively. $f(x, u, w)$, $s(w)$, and $h(x, w)$ are the system function, disturbance function, and error function, respectively. A new type of parallel regulators is presented with the structure shown in Fig. 2. If all the system information can be obtained, the parallel regulator can be established as

$$\dot{u} = g(x, u, w). \quad (2)$$

If only the error e can be obtained, the parallel regulator can be expressed by

$$\dot{\xi} = \varsigma(\xi, u, e) \quad (3a)$$

$$\dot{u} = g(\xi, u) \quad (3b)$$

where $\xi \in \mathbb{R}^{n+r}$ is the observation of system state x and external signal w .

Remark 1: Inspired by the structure of the parallel controller as shown in Fig. 1, the parallel regulator is established where the time derivative of system control instead of system control itself is modeled as shown in Fig. 2, which is the main contribution of this article. The regulator in Fig. 2 not only requires to stabilize the control system but also makes the tracking error e tend to zero. The regulator in Fig. 1 only requires to consider the stability of the control system.

Remark 2: For the control system (1) and the parallel regulators (2), (3), it can be seen that the system and regulators are symmetrical in both form and content mathematically, which is different to traditional feedback regulators. Therefore, the design method of traditional feedback regulators is not suitable for the parallel regulator design. Choosing proper systems ς and g , which make the control system (1) realize output regulation, is a key technical challenge of the parallel regulator design.

Remark 3: It can be seen that the parallel regulators (2) and (3) provide greater flexibility for the improvement of system performance comparing with the traditional feedback regulators. The developed method can not only smooth the control signal [45] but achieve better performance comparing with the existing output regulation framework.

B. Continuous-Time Linear Parallel Regulators

In this section, the continuous-time linear parallel regulators are presented. A continuous-time linear system can be

defined as

$$\dot{x} = Ax + Bu + Pw \quad (4a)$$

$$\dot{w} = Sw \quad (4b)$$

$$e = Cx + Dw \quad (4c)$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $w \in \mathbb{R}^r$, and $e \in \mathbb{R}^l$ are system state, control input, external input, and error, respectively. The matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $P \in \mathbb{R}^{n \times r}$, $S \in \mathbb{R}^{r \times r}$, $C \in \mathbb{R}^{l \times n}$, and $D \in \mathbb{R}^{l \times r}$ are system matrices. Pw and $-Dw$ are system disturbance and reference signal, respectively. Cx is the system output. For the system (4), the following assumptions are made for convenience of analysis [22].

Assumption 1: All the eigenvalues of the matrix S have nonnegative real part.

Assumption 2: The system (A, B) is controllable.

Assumption 3: The system $\begin{pmatrix} A & P \\ 0 & S \end{pmatrix}, [C \quad D]$ is detectable.

We would like to design parallel regulators, such that the system (4) can realize internal stability and output regulation. Internal stability refers to the global asymptotic stability of system (4) and parallel regulators when $w = 0$. Output regulation means that (4) can track the reference signal asymptotically and suppress external disturbance, that is, $\lim_{t \rightarrow \infty} e = 0$.

Remark 4: Assumption 1 is a standard assumption for linear output regulation problem [21]. In fact, if Assumption 1 is violated, that is, the matrix S exists eigenvalues with negative real part, the external signals corresponding to the eigenvalues with negative real part will exponentially converge to zero and will make no effect on the output regulation. Therefore, Assumption 1 loses no generality.

III. MAIN RESULTS

In this section, the parallel regulator design will be discussed in the situations that the full information can be obtained and only the error e can be obtained, respectively.

A. Parallel Output Regulators Under Full System Information

Considering all the system information can be obtained, the parallel regulators design will be analyzed in this section. In this situation, the parallel regulator can be designed as

$$\dot{u} = Kx + Lw + Ju \quad (5)$$

where $K \in \mathbb{R}^{m \times n}$, $L \in \mathbb{R}^{m \times r}$, and $J \in \mathbb{R}^{m \times m}$. We would like to find parallel regulator (5), such that the system (4) can realize internal stability and output regulation.

Remark 5: For the problem of parallel output regulator design under full system information, the external disturbance information w is assumed to be measurable, which is a general assumption for the research of output regulation problems [21], [53], [54]. If the external disturbance information w cannot be measurable, the Luenberger observer can be designed to estimate the information of external disturbance, which is the research content of Section III-B.

The formulations of internal stability and output regulation under full system information are shown as follows.

1) S_{sw} (*Internal Stability*): When $w = 0$, the following system:

$$\dot{x} = Ax + Bu \quad (6a)$$

$$\dot{u} = Kx + Ju \quad (6b)$$

can realize global asymptotic stability.

2) R_{sw} (*Output Regulation*): For any initial conditions $(x(0), w(0), u(0))$, the following system:

$$\dot{x} = Ax + Bu + Pw \quad (7a)$$

$$\dot{w} = Sw \quad (7b)$$

$$e = Cx + Dw \quad (7c)$$

$$\dot{u} = Kx + Lw + Ju \quad (7d)$$

satisfies $\lim_{t \rightarrow \infty} e = 0$.

The following lemmas are introduced to facilitate the analysis.

Lemma 1: Let A' , B' , and Q' be matrices with proper dimensions. The Sylvester equation $XA' - B'X = Q'$ has a unique solution X if and only if A' and B' have no common eigenvalues.

Lemma 2: If system $\dot{x} = Ax + Bu$ is controllable, then there exists parallel controller $\dot{u} = Kx + Ju$, which can make the system have a desired characteristic polynomial.

Proof: Define matrix $G = \begin{pmatrix} A & B \\ K & J \end{pmatrix}$, and the desired characteristic polynomial as

$$|\lambda I - G| = \lambda^{m+n} + \beta_1 \lambda^{m+n-1} + \cdots + \beta_{n+m-1} \lambda + \beta_{n+m}. \quad (8)$$

If system $\dot{x} = Ax + Bu$ is controllable, there exist matrix F_1 and vector v_1 , such that the system $(A + BF_1, Bv_1)$ is controllable [40]. A controllable transformation can be taken for $(A + BF_1, Bv_1)$ as

$$\begin{aligned} & (T'^{-1})(A + BF_1)T' \\ &= \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ -\alpha_n & \cdots & \cdots & \cdots & \cdots & \cdots & -\alpha_1 \end{bmatrix} \end{aligned} \quad (9)$$

and

$$(T'^{-1})Bv_1 = [0 \quad \cdots \quad 0 \quad 1]^T \quad (10)$$

where $\alpha_1, \alpha_2, \dots, \alpha_n$ can be obtained as follows:

$$|\lambda I - A - BF_1| = \lambda^n + \alpha_1 \lambda^{n-1} + \cdots + \alpha_n. \quad (11)$$

Define $\zeta = [\alpha_n \quad \alpha_{n-1} \quad \cdots \quad \alpha_1]$, $\Gamma = F_1 T' + v_1 \zeta$, and $V = [v_1 \quad v_2 \quad \cdots \quad v_m]$, where v_2, \dots, v_m are vectors, which can make V be nonsingular. Then, it can be obtained that

$$\hat{G} = (T_3)^{-1}(T_2)^{-1}(T_1)^{-1}GT_1T_2T_3 = \begin{bmatrix} \hat{A} & \hat{B} \\ \hat{K} & \hat{J} \end{bmatrix} \quad (12)$$

where

$$T_1 = \begin{bmatrix} T' & \\ & I \end{bmatrix}, T_2 = \begin{bmatrix} I & \\ \Gamma & I \end{bmatrix}, T_3 = \begin{bmatrix} I & \\ & V \end{bmatrix}$$

and the matrices \hat{A} and \hat{B} are

$$\hat{A} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \hat{B} = \begin{bmatrix} 0 & \cdot & \cdots & \cdot \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdot & \cdots & \cdot \\ 1 & \cdot & \cdots & \cdot \end{bmatrix} \quad (13)$$

where “ \cdot ” is an unrelated element. Then, let matrices \hat{K} and \hat{J} be expressed as

$$\hat{K} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ \hat{k}_1 & \hat{k}_2 & \cdots & \hat{k}_n \end{bmatrix}, \hat{J} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ \hat{j}_1 & \hat{j}_2 & \cdots & \hat{j}_m \end{bmatrix}. \quad (14)$$

Then, \hat{G} can be repartitioned as

$$\hat{G} = \begin{bmatrix} \hat{A}^* & \hat{B}^* \\ \hat{C}^* & \hat{D}^* \end{bmatrix} \quad (15)$$

where

$$\hat{A}^* = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, \hat{B}^* = \begin{bmatrix} \cdot & \cdot & \cdots & \cdot \\ \vdots & \vdots & \ddots & \vdots \\ \cdot & \cdot & \cdots & \cdot \\ 1 & 0 & \cdots & 0 \end{bmatrix}$$

$$\hat{K}^* = \begin{bmatrix} 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ \hat{k}_1 & \cdots & \hat{k}_n & \hat{j}_1 \end{bmatrix}, \hat{J}^* = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ \hat{j}_2 & \hat{j}_3 & \cdots & \hat{j}_m \end{bmatrix}$$

and matrices \hat{A}^* , \hat{B}^* , \hat{C}^* , and \hat{D}^* have dimension $(n+1) \times (n+1)$, $(n+1) \times (m-1)$, $(m-1) \times (n+1)$, and $(m-1) \times (m-1)$, respectively. From (15), it can be obtained that (\hat{A}^*, \hat{B}^*) is controllable. Then, repeating (9)–(15) $m-2$ times, the matrix \hat{G}_{m-2} can be derived as

$$\hat{G}_{m-2} = \begin{bmatrix} \hat{A}_{m-2}^* & \hat{B}_{m-2}^* \\ \hat{K}_{m-2}^* & \hat{J}_{m-2}^* \end{bmatrix} \quad (16)$$

where

$$\hat{A}_{m-2}^* = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \hat{B}_{m-2}^* = \begin{bmatrix} \cdot \\ \vdots \\ \cdot \\ 1 \end{bmatrix}$$

$$\hat{K}_{m-2}^* = [\hat{k}_{m-2,1} \quad \cdots \quad \hat{k}_{m-2,n+m-2} \quad \hat{j}_{m-2,1}]$$

$$\hat{J}_{m-2}^* = \hat{j}_{m-2,2}.$$

For (16), we execute (9)–(12). The matrix \hat{G}_{m-1} can be obtained as

$$\hat{G}_{m-1} = \begin{bmatrix} \hat{A}_{m-1} & \hat{B}_{m-1} \\ \hat{K}_{m-1} & \hat{J}_{m-1} \end{bmatrix} \quad (17)$$

where

$$\hat{A}_{m-1} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \hat{B}_{m-1} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

For (17), we define \hat{K}_{m-1} and \hat{J}_{m-1} as

$$\hat{K}_{m-1} = [-\beta_{n+m}, \quad -\beta_{n+m-1} \quad \cdots \quad -\beta_2]$$

$$\hat{J}_{m-1} = -\beta_1. \quad (18)$$

Then, it can be seen that \hat{G}_{m-1} , which is similar to G , has characteristic polynomial (8). The proof is complete. ■

Then, the existence of parallel regulators can be analyzed.

Theorem 1: Give the matrices K and J , which can realize condition S_{sw} . Then, there exists a parallel regulator (5) for the system (4), which can realize internal stability S_{sw} and output regulation R_{sw} , if and only if there exist matrices Π_1 , Π_2 , and L , which satisfy the following parallel regulation equations:

$$A\Pi_1 + B\Pi_2 + P = \Pi_1 S \quad (19)$$

$$K\Pi_1 + L + J\Pi_2 = \Pi_2 S \quad (20)$$

$$C\Pi_1 + D = 0. \quad (21)$$

Proof: First, consider the internal stability S_{sw} of the system (4). When $w = 0$, we can obtain the system (6). Based on Lemma 2, the matrices K and J can be obtained to make (6) have desired dynamic characteristics [40].

Then, consider the problem of output regulation R_{sw} of system (4). Rewrite (19), (20) as

$$\begin{pmatrix} A & B \\ K & J \end{pmatrix} \begin{pmatrix} \Pi_1 \\ \Pi_2 \end{pmatrix} - \begin{pmatrix} \Pi_1 \\ \Pi_2 \end{pmatrix} S = - \begin{pmatrix} P \\ L \end{pmatrix}. \quad (22)$$

The matrix $\begin{pmatrix} A & B \\ K & J \end{pmatrix}$ has eigenvalues with negative real parts, and matrix S has no eigenvalues with negative real parts as Assumption 1. Therefore, for a given matrix L , the Sylvester equation (22) has a unique solution $\begin{pmatrix} \Pi_1 \\ \Pi_2 \end{pmatrix}$. There always exist matrices Π_1 , Π_2 , and L , satisfying (22). Then the solvability of (19)–(21) is equivalent to the feasibility of the solution Π_1 of (22) for (21).

For the system (7), take coordinate transforms as

$$\tilde{x} = x - \Pi_1 w \quad (23a)$$

$$\tilde{w} = w \quad (23b)$$

$$\tilde{u} = u - \Pi_2 w. \quad (23c)$$

Then, it can be derived that

$$\begin{aligned} \dot{\tilde{x}} &= \dot{x} - \Pi_1 \dot{w} \\ &= A(\tilde{x} + \Pi_1 \tilde{w}) + B(\tilde{u} + \Pi_2 \tilde{w}) + P\tilde{w} - \Pi_1 S\tilde{w} \\ &= A\tilde{x} + B\tilde{u} + (A\Pi_1 + B\Pi_2 + P - \Pi_1 S)\tilde{w} \\ &= A\tilde{x} + B\tilde{u} \end{aligned} \quad (24)$$

and

$$\begin{aligned} \dot{\tilde{u}} &= \dot{u} - \Pi_2 \dot{w} \\ &= K(\tilde{x} + \Pi_1 \tilde{w}) + L\tilde{w} + J(\tilde{u} + \Pi_2 \tilde{w}) - \Pi_2 S\tilde{w} \\ &= K\tilde{x} + J\tilde{u} + (K\Pi_1 + L + J\Pi_2 - \Pi_2 S)\tilde{w} \\ &= K\tilde{x} + J\tilde{u}. \end{aligned} \quad (25)$$

Thus, we can obtain

$$\begin{aligned} e &= Cx + Dw \\ &= C(\tilde{x} + \Pi_1 \tilde{w}) + D\tilde{w} \\ &= C\tilde{x} + (C\Pi_1 + D)\tilde{w}. \end{aligned} \quad (26)$$

Since $\begin{pmatrix} A & B \\ K & J \end{pmatrix}$ is stable, if (22) exists solutions Π_1, Π_2 , and L , which satisfy (21), then we can obtain that $\lim_{t \rightarrow \infty} e = \lim_{t \rightarrow \infty} C\tilde{x} = 0$. Then, the sufficiency can be proven.

To show the necessity, assume the parallel regulator (5) can realize internal stability S_{sw} and output regulation R_{sw} . Due to Assumption 1, \tilde{w} cannot decay to zero for $\tilde{w}(0) \neq 0$. Then, for (24)–(26), it can be seen that

$$\begin{aligned} \lim_{t \rightarrow \infty} e &= \lim_{t \rightarrow \infty} (C\tilde{x} + (C\Pi_1 + D)\tilde{w}) \\ &= \lim_{t \rightarrow \infty} (C\Pi_1 + D)\tilde{w} \\ &= 0. \end{aligned} \quad (27)$$

Therefore, the necessity can be proven. ■

An algorithm is presented to find parallel regulator in the case of full information as Algorithm 1.

B. Parallel Output Regulators Under the Error e

In this part, considering only the error e can be obtained, the design of the parallel regulator are analyzed. First, we transform the system (4) as

$$\begin{pmatrix} \dot{x} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} A & P \\ 0 & S \end{pmatrix} \begin{pmatrix} x \\ w \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} u \quad (28a)$$

$$e = (C, D) \begin{pmatrix} x \\ w \end{pmatrix}. \quad (28b)$$

Then, for the system (28), the Luenberger observer is designed as

$$\begin{aligned} \begin{pmatrix} \dot{\hat{x}} \\ \dot{\hat{w}} \end{pmatrix} &= \left(\begin{pmatrix} A & P \\ 0 & S \end{pmatrix} - \begin{pmatrix} G_1 \\ G_2 \end{pmatrix} (C, D) \right) \begin{pmatrix} \hat{x} \\ \hat{w} \end{pmatrix} \\ &+ \begin{pmatrix} B \\ 0 \end{pmatrix} u + \begin{pmatrix} G_1 \\ G_2 \end{pmatrix} e. \end{aligned} \quad (31)$$

In this situation, the parallel regulator can be designed as

$$\dot{u} = (H_1, H_2) \begin{pmatrix} \hat{x} \\ \hat{w} \end{pmatrix} + J'u. \quad (32)$$

We would like to find parallel regulator (32), such that the system (4) can realize internal stability and output regulation. The formulations of internal stability and output regulation under the error are shown as follows.

1) S_e (Internal Stability): When $w = 0$, the following system:

$$\dot{x} = Ax + Bu \quad (33a)$$

$$\begin{aligned} \begin{pmatrix} \dot{\hat{x}} \\ \dot{\hat{w}} \end{pmatrix} &= \begin{pmatrix} A - G_1C & P - G_1D \\ -G_2C & S - G_2D \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{w} \end{pmatrix} \\ &+ \begin{pmatrix} B \\ 0 \end{pmatrix} u + \begin{pmatrix} G_1 \\ G_2 \end{pmatrix} e \end{aligned} \quad (33b)$$

$$e = Cx \quad (33c)$$

$$\dot{u} = (H_1, H_2) \begin{pmatrix} \hat{x} \\ \hat{w} \end{pmatrix} + J'u \quad (33d)$$

can realize global asymptotic stability.

2) R_e (Output Regulation): For any initial conditions $(x(0), w(0), u(0), \hat{x}(0), \hat{w}(0))$, the following system:

$$\dot{x} = Ax + Bu + Pw \quad (34a)$$

$$\dot{w} = Sw \quad (34b)$$

Algorithm 1 Parallel Regulator Design Algorithm in the Case of Full Information

Initialization:

Give matrices A, B, P, S, C, D .

Give characteristic polynomial (8).

Iteration:

1: If $m = 1$, then

1) Find matrix T' , and take controllable transformation for (A, B) .

2) Define $\Gamma = [\alpha_n \quad \alpha_{n-1} \quad \cdots \quad \alpha_1]$.

3) Take matrix transformation (12). Define \hat{K}, \hat{J} be

$$\hat{K} = [-\beta_{n+m} \quad -\beta_{n+m-1} \quad \cdots \quad -\beta_2], \hat{J} = -\beta_1$$

then matrices K, J can be obtained from

$$G = T_1 T_2 T_3 \hat{G}(T_3)^{-1} (T_2)^{-1} (T_1)^{-1}. \quad (29)$$

where $T_3 = I$.

2: If $m > 1$, then

1) Find F_1 and v_1 , which make $(A + BF_1, Bv_1)$ controllable.

2) Find T' , and take matrix transformation for $(A + BF_1, Bv_1)$ as (9)–(10).

3) Define $\zeta = [\alpha_n \quad \alpha_{n-1} \quad \cdots \quad \alpha_1]$, $\Gamma = F_1 P + v_1 \zeta$, $V = [v_1 \quad v_2 \quad \cdots \quad v_m]$.

4) Take matrix transformation as (12). Define matrices \hat{K} and \hat{J} be (14). Then, execute (15) for \hat{G} .

5) Repeat 1)–4) $m - 2$ times, and obtain \hat{G}_{m-2} as (16).

6) For \hat{G}_{m-2} , do steps 1)–3), take matrix transformation (12). Then, the matrix \hat{G}_{m-1} can be derived as (17).

7) Let \hat{K}_{m-1} and \hat{J}_{m-1} be as (18), then K, J can be obtained as

$$\begin{aligned} G &= T_1 T_2 T_3 \cdots T_{m-1,1} T_{m-1,2} T_{m-1,3} \hat{G}_{m-1} (T_{m-1,3})^{-1} \\ &\cdot (T_{m-1,2})^{-1} (T_{m-1,1})^{-1} \cdots (T_3)^{-1} (T_2)^{-1} (T_1)^{-1}. \end{aligned} \quad (30)$$

3: According to Theorem 1, determine the solvability of the parallel regulation equations (19)–(21), and obtain the matrix L .

4: **return** K, J, L .

$$\begin{aligned} \begin{pmatrix} \dot{\hat{x}} \\ \dot{\hat{w}} \end{pmatrix} &= \begin{pmatrix} A - G_1C & P - G_1D \\ -G_2C & S - G_2D \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{w} \end{pmatrix} \\ &+ \begin{pmatrix} B \\ 0 \end{pmatrix} u + \begin{pmatrix} G_1 \\ G_2 \end{pmatrix} e \end{aligned} \quad (34c)$$

$$e = Cx + Dw \quad (34d)$$

$$\dot{u} = (H_1, H_2) \begin{pmatrix} \hat{x} \\ \hat{w} \end{pmatrix} + J'u \quad (34e)$$

can satisfy $\lim_{t \rightarrow \infty} e = 0$.

Then, the existence of parallel regulators can be analyzed.

Theorem 2: Give the matrices H_1, J', G_1 , and G_2 , which can realize condition S_e . Then, there exists a parallel regulator (32) for system (4), which can realize internal stability S_e and output regulation R_e , if and only if there exist matrices Π'_1, Π'_2, Π'_3 , and H_2 , which satisfy the following parallel regulation equations:

$$A\Pi'_1 + B\Pi'_2 + P = \Pi'_1 S \quad (35)$$

$$(H_1, H_2)\Pi'_3 + J'\Pi'_2 = \Pi'_2 S \quad (36)$$

$$\begin{pmatrix} A - G_1 C & P - G_1 D \\ -G_2 C & S - G_2 D \end{pmatrix} \Pi'_3 + \begin{pmatrix} B \\ 0 \end{pmatrix} \Pi'_2 = \Pi'_3 S \quad (37)$$

$$C\Pi'_1 + D = 0. \quad (38)$$

Proof: First, consider the problem of internal stability S_e of system (4). When $w = 0$, we can obtain the system (33). Then, we rewrite the system (33) as

$$\dot{z}' = \begin{pmatrix} A & B & 0 & 0 \\ 0 & J & H_1 & H_2 \\ G_1 C & B & A - G_1 C & P - G_1 D \\ G_2 C & 0 & -G_2 C & S - G_2 D \end{pmatrix} \begin{pmatrix} x \\ u \\ \hat{x} \\ \hat{w} \end{pmatrix} \triangleq G' z'. \quad (39)$$

For the system (39), we take coordinate transforms as

$$z' = T\tilde{z} = \begin{pmatrix} I & & & \\ & I & & \\ & & I & \\ & & & I \end{pmatrix} \tilde{z}.$$

Then, we can obtain the system as

$$\dot{\tilde{z}} = T^{-1}G'T\tilde{z} \triangleq \tilde{G}\tilde{z} \quad (40)$$

where

$$\tilde{G} = \begin{pmatrix} A & B & 0 & 0 \\ H_1 & J' & H_1 & H_2 \\ 0 & 0 & A - G_1 C & P - G_1 D \\ 0 & 0 & -G_2 C & S - G_2 D \end{pmatrix} = \begin{bmatrix} \tilde{G}_1 & \tilde{G}_{12} \\ & \tilde{G}_2 \end{bmatrix}.$$

Based on Assumption 2 and Lemma 2, it can be derived that there exist H_1 and J' to make \tilde{G}_1 have desired poles [40]. Based on Assumption 3, there exist G_1 and G_2 to make \tilde{G}_2 have desired poles [55]. We can obtain the matrices H_1 , J' , G_1 , and G_2 , which can satisfy the condition S_e .

Then, considering the problem R_e of the system (4), we rewrite (35)–(37) as

$$G' \begin{pmatrix} \Pi'_1 \\ \Pi'_2 \\ \Pi'_3 \end{pmatrix} - \begin{pmatrix} \Pi'_1 \\ \Pi'_2 \\ \Pi'_3 \end{pmatrix} S = - \begin{pmatrix} P \\ 0 \\ G_1 D \\ G_2 D \end{pmatrix}. \quad (41)$$

It can be seen that G' has eigenvalues with negative real parts, and S has no eigenvalues with negative real parts as Assumption 1. Therefore, for a given matrix H_2 , the Sylvester

equation (41) has a unique solution $\begin{pmatrix} \Pi'_1 \\ \Pi'_2 \\ \Pi'_3 \end{pmatrix}$. There always

exist matrices Π'_1 , Π'_2 , Π'_3 , and H_2 , satisfying (41). Then, the solvability of (35)–(38) is equivalent to the feasibility of the solution Π'_1 of (41) for (38).

For the system (34), defining $\xi = \begin{pmatrix} \hat{x} \\ \hat{w} \end{pmatrix}$, we take coordinate transforms as

$$x' = x - \Pi'_1 w \quad (42a)$$

$$w' = w \quad (42b)$$

$$u' = u - \Pi'_2 w \quad (42c)$$

$$\xi' = \xi - \Pi'_3 w. \quad (42d)$$

Then, we can derive

$$\begin{aligned} \dot{x}' &= \dot{x} - \Pi'_1 \dot{w} \\ &= A(x' + \Pi'_1 w') + B(u' + \Pi'_2 w') + Pw' - \Pi'_1 S w' \\ &= Ax' + Bu' \end{aligned} \quad (43)$$

and

$$\begin{aligned} \dot{u}' &= \dot{u} - \Pi'_2 \dot{w} \\ &= (H_1 \ H_2)(\xi' + \Pi'_3 w') + J(u' + \Pi'_2 w') - \Pi'_2 S w' \\ &= (H_1 \ H_2)\xi' + J'u'. \end{aligned} \quad (44)$$

Thus, we can obtain

$$\begin{aligned} \dot{\xi}' &= \dot{\xi} - \Pi'_3 \dot{w} \\ &= \begin{pmatrix} A - G_1 C & P - G_1 D \\ -G_2 C & S - G_2 D \end{pmatrix} (\xi' + \Pi'_3 w') \\ &\quad + \begin{pmatrix} B \\ 0 \end{pmatrix} (u' + \Pi'_2 w') \\ &\quad + \begin{pmatrix} G_1 \\ G_2 \end{pmatrix} (C(x' + \Pi'_1 w') + Dw') - \Pi'_3 S w' \\ &= \begin{pmatrix} A - G_1 C & P - G_1 D \\ -G_2 C & S - G_2 D \end{pmatrix} \xi' + \begin{pmatrix} B \\ 0 \end{pmatrix} u' \\ &\quad + \begin{pmatrix} G_1 \\ G_2 \end{pmatrix} C x' \end{aligned} \quad (45)$$

and

$$\begin{aligned} e &= Cx + Dw \\ &= C(x' + \Pi'_1 w') + Dw' \\ &= Cx' + (C\Pi'_1 + D)w'. \end{aligned} \quad (46)$$

Since G' is stable, to show the sufficiency, if (41) has the solutions Π'_1 , Π'_2 , Π'_3 , and H_2 , which satisfy (38), then we can obtain that systems (43)–(45) are asymptotic stability, and $\lim_{t \rightarrow \infty} e = \lim_{t \rightarrow \infty} Cx' = 0$.

To show the necessity, assume parallel regulator (32) can realize internal stability S_e and output regulation R_e . Due to Assumption 1, w' cannot decay to zero for $w'(0) \neq 0$. Then, for (43)–(46), it can be seen that

$$\begin{aligned} \lim_{t \rightarrow \infty} e &= \lim_{t \rightarrow \infty} (Cx' + (C\Pi'_1 + D)w') \\ &= \lim_{t \rightarrow \infty} (C\Pi'_1 + D)w' \\ &= 0. \end{aligned} \quad (47)$$

Therefore, necessarily, $C\Pi'_1 + D = 0$. ■

We can obtain Algorithm 2 to find parallel regulators in the case of partial information.

Remark 6: For the parallel output regulation problems, only internal stability with $w = 0$ needs to be considered rather than external stability with $w \neq 0$. The external stability means that designing parallel regulator (5) or (32), which makes the system output satisfy $\lim_{t \rightarrow \infty} Cx = 0$. Then, it can be obtained that $\lim_{t \rightarrow \infty} e = \lim_{t \rightarrow \infty} (Cx + Dw) = \lim_{t \rightarrow \infty} Dw \neq 0$, which conflicts with output regulation condition.

Algorithm 2 Parallel Regulator Design Algorithm in the Case That Only Error e Can Be Obtained

Initialization:

- Give matrices A, B, P, S, C, D .
- Give characteristic polynomial of \tilde{G}_1 .
- Give characteristic polynomial of \tilde{G}_2 .

Iteration:

- 1: If $m = 1$, then
 - 1) Find matrix T' , and take controllable transformation for (A, B) .
 - 2) Define $\Gamma = [\alpha_n \alpha_{n-1} \cdots \alpha_1]$.
 - 3) Take matrix transformation (12). Define \hat{K}, \hat{J} be

$$\hat{K} = [-\beta_{n+m} -\beta_{n+m-1} \cdots -\beta_2], \hat{J} = -\beta_1$$

and then matrices K, J are obtained from

$$G = T_1 T_2 T_3 \hat{G}(T_3)^{-1} (T_2)^{-1} (T_1)^{-1}. \quad (48)$$

where $T_3 = I$.

- 2: If $m > 1$, then
 - 1) Find F_1 and v_1 , which make $(A + BF_1, Bv_1)$ controllable.
 - 2) Find T' , and take matrix transformation for $(A + BF_1, Bv_1)$ as (9)–(10).
 - 3) Define $\zeta = [\alpha_n \alpha_{n-1} \cdots \alpha_1]$, $\Gamma = F_1 P + v_1 \zeta$, and $V = [v_1 v_2 \cdots v_m]$.
 - 4) Take matrix transformation as (12). Define \hat{K} and \hat{J} be (14). Then execute (15) for \hat{G} .
 - 5) Repeat 1)–4) $m - 2$ times, and obtain \hat{G}_{m-2} as (16).
 - 6) For \hat{G}_{m-2} , do steps 1)–3), take matrix transformation (12). Then, the matrix \hat{G}_{m-1} can be derived as (17).
 - 7) Let \hat{K}_{m-1} and \hat{J}_{m-1} be as (18), and then K, J can be obtained as

$$G = T_1 T_2 T_3 \cdots T_{m-1,1} T_{m-1,2} T_{m-1,3} \hat{G}_{m-1} (T_{m-1,3})^{-1} \cdot (T_{m-1,2})^{-1} (T_{m-1,1})^{-1} \cdots (T_3)^{-1} (T_2)^{-1} (T_1)^{-1}. \quad (49)$$

- 3: Let $H_1 = K, J' = J$.
- 4: Find matrices G_1, G_2 according to [55], which make \tilde{G}_2 have desired characteristic polynomial.
- 5: Determine the solvability of the parallel regulation equations (35)–(38), and obtain the matrix H_2 .
- 6: **return** H_1, H_2, J', G_1, G_2 .

Remark 7: From Algorithms 1 and 2, it is shown that the parallel regulators provide extra design degrees of freedom comparing with the traditional feedback regulators. Therefore, the parallel control can provide greater flexibility for the improvement of system performance.

IV. NUMERICAL ANALYSIS

In this section, numerical examples are provided to verify the correctness of parallel regulators.

A. Parallel Output Regulators Under Full System Information

In this example, an LCL coupled inverter-based distributed generation system with modifications is investigated, whose

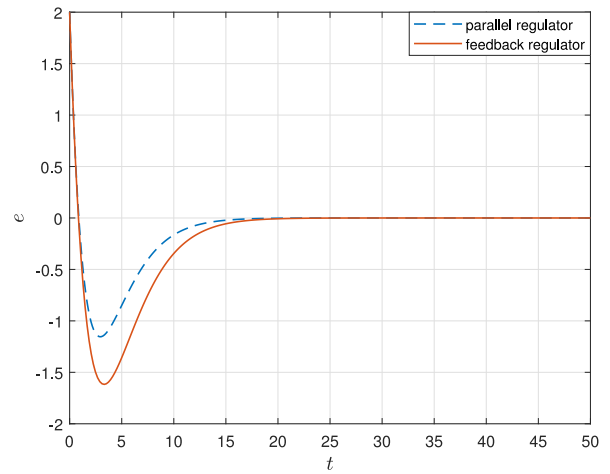


Fig. 3. Error curves of Example 1.

dynamic model can be described as follows [34], [56]:

$$\begin{aligned} \dot{x} &= \begin{bmatrix} -R_1/L_1 & -1/L_1 & 0 \\ 1/C_1 & 0 & -1/C_1 \\ 0 & 1/L_2 & -R_2/L_2 \end{bmatrix} x \\ &\quad + \begin{bmatrix} 1/L_1 \\ 0 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} w \\ \dot{w} &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} w \\ e &= [0 \quad 0 \quad 1]x + [1 \quad 0]w \end{aligned} \quad (50)$$

where the values of C_1, R_1, R_2, R_2 , and L_2 can be found in [56]. It is easy to derive that Assumptions 1–3 are satisfied. In order to verify the effectiveness of the present method, comparisons are constructed with the traditional feedback regulator. The feedback regulator can be obtained as

$$u = [-2.4136 \quad -19.1550 \quad 1.4385]x + [567.1814, \quad -105.6275]w. \quad (51)$$

The poles are assigned at $-0.5, -0.5$, and -0.5 by the traditional feedback regulator. Next, the parallel regulator, which assigns the poles of system at $-0.5, -0.5, -0.5$, and -3 , is designed for the system. According to Algorithm 1, we can obtain the parallel regulator as

$$\begin{aligned} \dot{u} &= [-7.6866 \quad -56.2246 \quad 4.1931]x - 3.9655u \\ &\quad + [1595.9, \quad -886.5457]w. \end{aligned} \quad (52)$$

Simulation is performed with $x(0) = (2 \ -4 \ 3)^T$, $w(0) = (-1 \ 1)^T$, and $u(0) = 2$. We can obtain the results as shown in Figs. 3–5. The error curves are shown in Fig. 3, the control curves are shown in Fig. 4, and the output and reference signal curves are shown in Fig. 5. It can be seen that the parallel regulators (52) can realize the internal stability S_{sw} and output regulation R_{sw} . It is noted that the system poles assigned by feedback regulator are contained in the set of the poles by parallel regulator, and the parallel regulator can achieve better results as shown in Figs. 3–5.

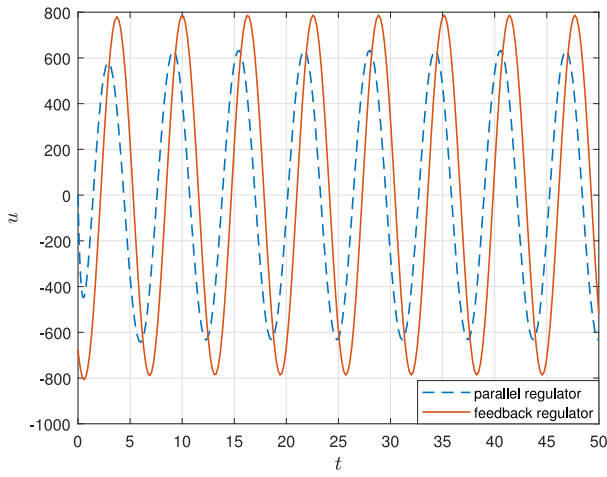


Fig. 4. Control curves of Example 1.

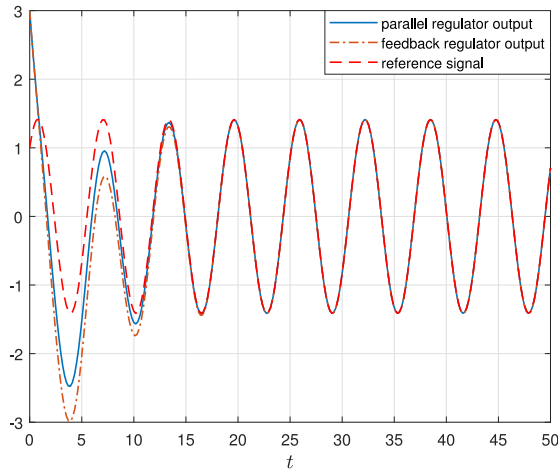


Fig. 5. Output and reference curves of Example 1.

B. Parallel Output Regulators Under the Error e

Consider the F16 aircraft system as follows [57]:

$$\begin{aligned} \dot{x} &= \begin{bmatrix} -1.019 & 0.905 & -0.002 \\ 0.822 & -1.077 & -0.176 \\ 0 & 0 & -1 \end{bmatrix} x \\ &+ \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix} u + \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} w \\ \dot{w} &= \begin{bmatrix} 0 & 1 \\ -0.01 & 0 \end{bmatrix} w \\ e &= [1 \ 0 \ 0]x + [-1 \ 0]w. \end{aligned} \quad (53)$$

It is easy to derive that Assumptions 1–3 are satisfied. The traditional feedback regulator is also employed for comparisons as (54), shown at the top of the next page, which assigns the system poles at $-0.5, -0.5, -0.5, -0.5, -0.5, -0.5$, and -0.5 . According to Algorithm 2, we can obtain the parallel regulator, which can assign the system poles at $-0.5, -0.5, -0.5, -0.5, -0.5, -0.5, -0.5, -0.5$, and -3 as (55), shown at the top of the next page. To further verify the advantages of the developed method, adaptive regulator is employed for comparisons, where the specific control

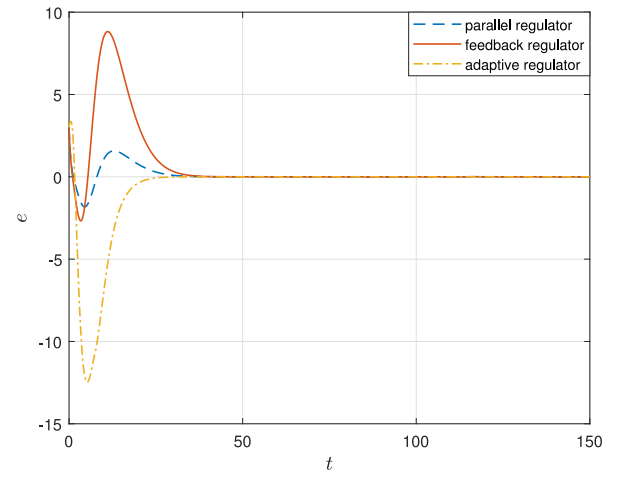


Fig. 6. Error curves of Example 2.

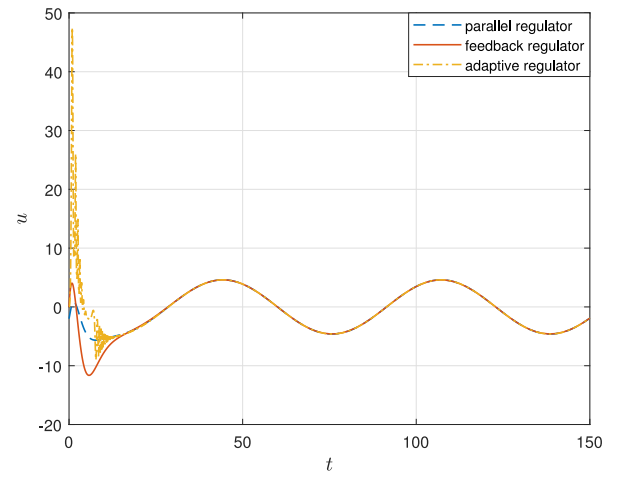


Fig. 7. Control curves of Example 2.

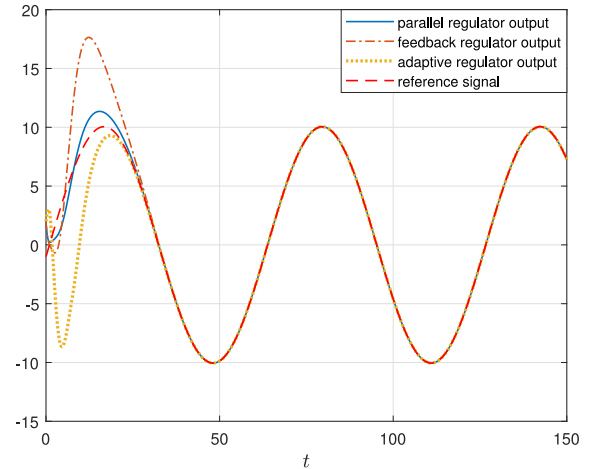


Fig. 8. Output and reference curves of Example 2.

model can be found in [58]. Simulation is performed with $x(0) = (2 \ -4 \ 3)^T$, $w(0) = (-1, \ 1)^T$, $u(0) = -2$, $\hat{x}(0) = (0 \ 0 \ 0)^T$, and $\hat{w}(0) = (0, \ 0)^T$. Then, we can obtain the results as shown in Figs. 6–8. The error curves are shown in Fig. 6, the control curves are shown in Fig. 7, and

$$\begin{pmatrix} \dot{\hat{x}} \\ \dot{\hat{w}} \end{pmatrix} = \begin{pmatrix} 0.1085 & 0.9050 & -0.0020 & -1.1275 & 1.0000 \\ 0.0075 & -1.0770 & -0.1760 & 0.8145 & 0 \\ -8.7061 & 9.4470 & 0.5960 & -0.5164 & 1.6860 \\ 0.5315 & 0 & 0 & -0.5315 & 1.0000 \\ 0.0766 & 0 & 0 & -0.0866 & 0 \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{w} \end{pmatrix} + \begin{pmatrix} -1.1275 \\ 0.8145 \\ 0.1941 \\ -0.5315 \\ -0.0766 \end{pmatrix} e$$

$$u = (-1.7024 \quad 1.8894 \quad 0.3192 \quad -0.1421 \quad 0.3372) \begin{pmatrix} \hat{x} \\ \hat{w} \end{pmatrix} \quad (54)$$

$$\begin{pmatrix} \dot{\hat{x}} \\ \dot{\hat{w}} \end{pmatrix} = \begin{pmatrix} 0.1085 & 0.9050 & -0.0020 & -1.1275 & 1.0000 \\ 0.0075 & -1.0770 & -0.1760 & 0.8145 & 0 \\ -0.1941 & 0 & -1 & 0.1941 & 0 \\ 0.5315 & 0 & 0 & -0.5315 & 1.0000 \\ 0.0766 & 0 & 0 & -0.0866 & 0 \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{w} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 5 \\ 0 \\ 0 \end{pmatrix} u + \begin{pmatrix} -1.1275 \\ 0.8145 \\ 0.1941 \\ -0.5315 \\ -0.0766 \end{pmatrix} e$$

$$\dot{u} = (-1.8193 \quad 2.0927 \quad 0.3093 \quad -0.4298 \quad -0.8326) \begin{pmatrix} \hat{x} \\ \hat{w} \end{pmatrix} - 1.404u \quad (55)$$

the output and reference signal curves are shown in Fig. 8. From Figs. 6–8, it can be seen that the parallel regulator can realize internal stability S_e and output regulation R_e , and the parallel regulators can achieve better results comparing with traditional feedback regulator and adaptive regulator.

V. CONCLUSION

A novel linear parallel regulation method is developed in this article. The basic design method of parallel regulators is presented. Considering the full system information can be obtained and only the error can be obtained, respectively, the existence of parallel regulators and the parallel regulator design algorithms is developed. Finally, numerical simulations are given to verify the correctness of the parallel regulators. Some interesting future works are to extent the parallel regulation results to other output regulation problems, such as robust linear output regulation, adaptive linear output regulation, and nonlinear output regulation.

REFERENCES

- [1] F.-Y. Wang, *Shadow Systems: A New Concept for Nested and Embedded Co-Simulation for Intelligent Systems*, Univ. Arizona, Tucson, AZ, USA, 1994.
- [2] D. Liu, S. Xue, B. Zhao, B. Luo, and Q. Wei, "Adaptive dynamic programming for control: A survey and recent advances," *IEEE Trans. Syst., Man, Cybern., Syst.*, vol. 51, no. 1, pp. 142–160, Jan. 2021.
- [3] Z. Liu, G. Lai, Y. Zhang, and C. L. P. Chen, "Adaptive neural output feedback control of output-constrained nonlinear systems with unknown output nonlinearity," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 26, no. 8, pp. 1789–1802, Aug. 2015.
- [4] D. Liu, Q. Wei, D. Wang, X. Yang, and H. Li, *Adaptive Dynamic Programming With Applications in Optimal Control*. Cham, Switzerland: Springer, 2017.
- [5] H. Li, Y. Wu, and M. Chen, "Adaptive fault-tolerant tracking control for discrete-time multiagent systems via reinforcement learning algorithm," *IEEE Trans. Cybern.*, vol. 51, no. 3, pp. 1163–1174, Mar. 2021.
- [6] Q. Wei, X. Wang, X. Zhong, and N. Wu, "Consensus control of leader-following multi-agent systems in directed topology with heterogeneous disturbances," *IEEE/CAA J. Automatica Sinica*, vol. 8, no. 2, pp. 423–431, Feb. 2021.
- [7] H. Zhang, Y. Liu, and Y. Wang, "Observer-based finite-time adaptive fuzzy control for nontriangular nonlinear systems with full-state constraints," *IEEE Trans. Cybern.*, vol. 51, no. 3, pp. 1110–1120, Mar. 2021.
- [8] Y. Hao, T. Wang, G. Li, and C. Wen, "Linear quadratic optimal control of time-invariant linear networks with selectable input matrix," *IEEE Trans. Cybern.*, vol. 51, no. 9, pp. 4743–4754, Sep. 2021, doi: [10.1109/TCYB.2019.2953218](https://doi.org/10.1109/TCYB.2019.2953218).
- [9] Q. Wei, L. Zhu, T. Li, and L. Derong, "A new approach to finite-horizon optimal control for discrete-time affine nonlinear systems via a pseudo-linear method," *IEEE Trans. Autom. Control*, early access, Jun. 8, 2021, doi: [10.1109/TAC.2021.3087452](https://doi.org/10.1109/TAC.2021.3087452).
- [10] Q. Wei, L. Han, and T. Zhang, "Spiking adaptive dynamic programming based on poisson process for discrete-time nonlinear systems," *IEEE Trans. Neural Netw. Learn. Syst.*, early access, Jun. 18, 2021, doi: [10.1109/TNNLS.2021.3085781](https://doi.org/10.1109/TNNLS.2021.3085781).
- [11] S. Sui, C. L. P. Chen, and S. Tong, "Event-trigger-based finite-time fuzzy adaptive control for stochastic nonlinear system with unmodeled dynamics," *IEEE Trans. Fuzzy Syst.*, vol. 29, no. 7, pp. 1914–1926, Jul. 2021, doi: [10.1109/TFUZZ.2020.2988849](https://doi.org/10.1109/TFUZZ.2020.2988849).
- [12] Q. Wei, L. Zhu, R. Song, P. Zhang, D. Liu, and J. Xiao, "Model-free adaptive optimal control for unknown nonlinear multiplayer nonzero-sum game," *IEEE Trans. Neural Netw. Learn. Syst.*, early access, Oct. 27, 2020, doi: [10.1109/TNNLS.2020.3030127](https://doi.org/10.1109/TNNLS.2020.3030127).
- [13] Q. Wei, D. Liu, Y. Liu, and R. Song, "Optimal constrained self-learning battery sequential management in microgrid via adaptive dynamic programming," *IEEE/CAA J. Automatica Sinica*, vol. 4, no. 2, pp. 168–176, Apr. 2017.
- [14] L. Wang, C. L. P. Chen, and H. Li, "Event-triggered adaptive control of saturated nonlinear systems with time-varying partial state constraints," *IEEE Trans. Cybern.*, vol. 50, no. 4, pp. 1485–1497, Apr. 2020.
- [15] L. Wang and C. L. P. Chen, "Reduced-order observer-based dynamic event-triggered adaptive NN control for stochastic nonlinear systems subject to unknown input saturation," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 32, no. 4, pp. 1678–1690, Apr. 2021.
- [16] K. J. Hunt, D. Sbarbaro, R. Żbikowski, and P. J. Gawthrop, "Neural networks for control systems: A survey," *Automatica*, vol. 28, no. 6, pp. 1083–1112, 1992.
- [17] Q. Wei, D. Liu, and H. Lin, "Value iteration adaptive dynamic programming for optimal control of discrete-time nonlinear systems," *IEEE Trans. Cybern.*, vol. 46, no. 3, pp. 840–853, Mar. 2016.
- [18] Q. Wei, H. Li, X. Yang, and H. He, "Continuous-time distributed policy iteration for multicontroller nonlinear systems," *IEEE Trans. Cybern.*, vol. 51, no. 5, pp. 2372–2383, May 2021.
- [19] L. Li, L. Song, T. Li, and J. Fu, "Event-triggered output regulation for networked flight control system based on an asynchronous switched system approach," *IEEE Trans. Syst., Man, Cybern., Syst.*, early access, Apr. 21, 2020, doi: [10.1109/TSMC.2020.2981192](https://doi.org/10.1109/TSMC.2020.2981192).
- [20] Z. Liu, G. Lai, Y. Zhang, and C. L. P. Chen, "Adaptive fuzzy tracking control of nonlinear time-delay systems with dead-zone output mechanism based on a novel smooth model," *IEEE Trans. Fuzzy Syst.*, vol. 23, no. 6, pp. 1998–2011, Dec. 2015.
- [21] W. Liu and J. Huang, "Output regulation of linear systems via sampled-data control," *Automatica*, vol. 113, Mar. 2020, Art. no. 108684.
- [22] B. A. Francis, "The linear multivariable regulator problem," *SIAM J. Control Optim.*, vol. 15, no. 3, pp. 486–505, 1977.

- [23] B. A. Francis and W. M. Wonham, "The internal model principle of control theory," *Automatica*, vol. 12, no. 5, pp. 457–469, 1976.
- [24] E. J. Davison, "The robust control of a servomechanism problem for linear time-invariant multivariable systems," *IEEE Trans. Autom. Control*, vol. 21, no. 1, pp. 25–34, Feb. 1976.
- [25] Z. Chen and J. Huang, "Robust output regulation with nonlinear exosystems," *Automatica*, vol. 41, no. 8, pp. 1447–1454, 2005.
- [26] A. Serrani and A. Isidori, "Global robust output regulation for a class of nonlinear systems," *Syst. Control Lett.*, vol. 39, no. 2, pp. 133–139, 2000.
- [27] P. Wang, S. S. Ge, X. Zhang, and D. Yu, "Output-based event-triggered cooperative robust regulation for constrained heterogeneous multiagent systems," *IEEE Trans. Cybern.*, early access, Dec. 30, 2020, doi: [10.1109/TCYB.2020.3041267](https://doi.org/10.1109/TCYB.2020.3041267).
- [28] X. Chen and Z. Chen, "Robust perturbed output regulation and synchronization of nonlinear heterogeneous multiagents," *IEEE Trans. Cybern.*, vol. 46, no. 12, pp. 3111–3122, Dec. 2016.
- [29] Z. Ding, "Adaptive consensus output regulation of a class of nonlinear systems with unknown high-frequency gain," *Automatica*, vol. 51, pp. 348–355, Jan. 2015.
- [30] H. Lei and W. Lin, "Adaptive regulation of uncertain nonlinear systems by output feedback: A universal control approach," *Syst. Control Lett.*, vol. 56, nos. 7–8, pp. 529–537, 2007.
- [31] H. Su, J. Chen, X. Chen, and H. He, "Adaptive observer-based output regulation of multiagent systems with communication constraints," *IEEE Trans. Cybern.*, vol. 51, no. 11, pp. 5259–5268, Nov. 2021, doi: [10.1109/TCYB.2020.2995147](https://doi.org/10.1109/TCYB.2020.2995147).
- [32] M. Meng, L. Liu, and G. Feng, "Adaptive output regulation of heterogeneous multiagent systems under Markovian switching topologies," *IEEE Trans. Cybern.*, vol. 48, no. 10, pp. 2962–2971, Oct. 2018.
- [33] H. Cai, F. L. Lewis, G. Hu, and J. Huang, "The adaptive distributed observer approach to the cooperative output regulation of linear multiagent systems," *Automatica*, vol. 75, pp. 299–305, Jan. 2017.
- [34] W. Gao and Z.-P. Jiang, "Adaptive dynamic programming and adaptive optimal output regulation of linear systems," *IEEE Trans. Autom. Control*, vol. 61, no. 12, pp. 4164–4169, Dec. 2016.
- [35] H. Zhang, J. Chen, Z. Wang, C. Fu, and S. Song, "Distributed event-triggered control for cooperative output regulation of multiagent systems with an online estimation algorithm," *IEEE Trans. Cybern.*, early access, Jun. 8, 2020, doi: [10.1109/TCYB.2020.2991761](https://doi.org/10.1109/TCYB.2020.2991761).
- [36] J. Wang, A. Sheng, D. Xu, Z. Chen, and Y. Su, "Event-based practical output regulation for a class of multiagent nonlinear systems," *IEEE Trans. Cybern.*, vol. 49, no. 10, pp. 3689–3698, Oct. 2019.
- [37] B. Luo, Y. Yang, and D. Liu, "Adaptive Q -learning for data-based optimal output regulation with experience replay," *IEEE Trans. Cybern.*, vol. 48, no. 12, pp. 3337–3348, Dec. 2018.
- [38] J. Fan, Q. Wu, Y. Jiang, T. Chai, and F. L. Lewis, "Model-free optimal output regulation for linear discrete-time lossy networked control systems," *IEEE Trans. Syst., Man, Cybern., Syst.*, vol. 50, no. 11, pp. 4033–4042, Nov. 2020.
- [39] Y. Wu and L. Liu, "Distributed output regulation for a class of nonlinear multiagent systems with dynamic edges," *IEEE Trans. Cybern.*, early access, Nov. 5, 2020, doi: [10.1109/TCYB.2020.3028504](https://doi.org/10.1109/TCYB.2020.3028504).
- [40] Q. Wei, H. Li, and F.-Y. Wang, "Parallel control for continuous-time linear systems: A case study," *IEEE/CAA J. Automatica Sinica*, vol. 7, no. 4, pp. 919–928, Jul. 2020.
- [41] F.-Y. Wang, "Computational experiments for behavior analysis and decision evaluation of complex systems," *J. Syst. Simulat.*, vol. 16, no. 5, pp. 893–897, May 2004.
- [42] F.-Y. Wang, "Parallel system methods for management and control of complex systems," *Control Decis.*, vol. 19, no. 5, pp. 485–489, 2004.
- [43] F.-Y. Wang, "On the modeling, analysis, control and management of complex systems," *Complex Syst. Complex. Sci.*, vol. 3, no. 2, pp. 26–34, Jan. 2006.
- [44] Q. Wei, L. Wang, J. Lu, and F.-Y. Wang, "Discrete-time self-learning parallel control," *IEEE Trans. Syst., Man, Cybern., Syst.*, early access, Jun. 9, 2020, doi: [10.1109/TSMC.2020.2995646](https://doi.org/10.1109/TSMC.2020.2995646).
- [45] J. Lu, Q. Wei, and F.-Y. Wang, "Parallel control for optimal tracking via adaptive dynamic programming," *IEEE/CAA J. Automatica Sinica*, vol. 7, no. 6, pp. 1662–1674, Nov. 2020.
- [46] J. Lu, Q. Wei, Y. Liu, T. Zhou, and F.-Y. Wang, "Event-triggered optimal parallel tracking control for discrete-time nonlinear systems," *IEEE Trans. Syst., Man, Cybern., Syst.*, early access, Apr. 27, 2021, doi: [10.1109/TSMC.2021.3073429](https://doi.org/10.1109/TSMC.2021.3073429).
- [47] Y. Dong and S. Xu, "Rendezvous with connectivity preservation problem of linear multiagent systems via parallel event-triggered control strategies," *IEEE Trans. Cybern.*, early access, Oct. 1, 2020, doi: [10.1109/TCYB.2020.3021788](https://doi.org/10.1109/TCYB.2020.3021788).
- [48] H. Ding, J. Zhao, and Q. Zhu, "A new hydraulic speed regulation scheme: Valve-pump parallel variable mode control," *IEEE Access*, vol. 6, pp. 55257–55263, 2018.
- [49] B. Fan, J. Peng, J. Duan, Q. Yang, and W. Liu, "Distributed control of multiple-bus microgrid with paralleled distributed generators," *IEEE/CAA J. Automatica Sinica*, vol. 6, no. 3, pp. 676–684, May 2019.
- [50] M. Kordestani, A. A. Safavi, and M. Saif, "Recent survey of large-scale systems: Architectures, controller strategies, and industrial applications," *IEEE Syst. J.*, early access, Jan. 21, 2021, doi: [10.1109/JSYST.2020.3048951](https://doi.org/10.1109/JSYST.2020.3048951).
- [51] M. S. Sadabadi, "A distributed control strategy for parallel DC-DC converters," *IEEE Control Syst. Lett.*, vol. 5, no. 4, pp. 1231–1236, Oct. 2021.
- [52] Z. He and Y. Xing, "Distributed control for UPS modules in parallel operation with RMS voltage regulation," *IEEE Control Syst. Lett.*, vol. 55, no. 8, pp. 2860–2869, Aug. 2008.
- [53] Y. Jiang, B. Kiumarsi, J. Fan, T. Chai, J. Li, and F. L. Lewis, "Optimal output regulation of linear discrete-time systems with unknown dynamics using reinforcement learning," *IEEE Trans. Cybern.*, vol. 50, no. 7, pp. 3147–3156, Jul. 2020.
- [54] C. I. Byrnes, I. G. Lauko, D. S. Gilliam, and V. I. Shubov, "Output regulation for linear distributed parameter systems," *IEEE Trans. Autom. Control*, vol. 45, no. 12, pp. 2236–2252, Dec. 2000.
- [55] G. F. Franklin, J. D. Powell, and A. Emami-Naeini, *Feedback Control of Dynamic Systems*. 6th ed. Upper Saddle River, NJ, USA: Prentice Hall, 2009.
- [56] K. H. Ahmed, A. M. Massoud, S. J. Finney, and B. W. Williams, "A modified stationary reference frame-based predictive current control with zero steady-state error for LCL coupled inverter-based distributed generation systems," *IEEE Trans. Ind. Electron.*, vol. 58, no. 4, pp. 1359–1370, Apr. 2011.
- [57] F. A. Yaghmaie, S. Gunnarsson, and F. L. Lewis, "Output regulation of unknown linear systems using average cost reinforcement learning," *Automatica*, vol. 110, Dec. 2019, Art. no. 108549.
- [58] R. Marino and P. Tomei, "Output regulation for linear systems via adaptive internal model," *IEEE Trans. Autom. Control*, vol. 48, no. 12, pp. 2199–2202, Dec. 2003.



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