Distributed Dynamic Event-Triggered Control for Euler–Lagrange Multiagent Systems With Parametric Uncertainties

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***Abstract*—This article investigates the distributed dynamic event-triggered control of networked Euler–Lagrange systems with unknown parameters. Using the designed dynamic event-triggered control algorithm, the leaderless consensus problem and the containment problem of networked Euler–Lagrange systems are solved, and the estimations of unknown parameters are updated by an adaptive updating law as well. The stability analysis is given based on an appropriate Lyapunov function and the distributed control problem is theoretically solved by the designed control algorithm. The Zeno behavior of the designed dynamic event-triggered method is excluded in a finite-time interval. Compared to some existing results for the event-triggered control of networked Euler–Lagrange systems, these event-triggered methods can be seen as the special cases of the dynamic event-triggered method proposed in this article. Simulation results based on UR5 robots of V-rep show that the proposed method can provide an increase ($4.46 \pm 3.36\%$) of the average lengths of event intervals compared to the one of the existing event-triggered methods, which leads to a lower usage of the communication resource. Meanwhile, the time of achieving the consensus/containment and the steady-state control performance are not affected.**

***Index Terms*—Consensus, containment, dynamic event-triggered control, Euler–Lagrange system, multiagent systems (MASs).**

I. Introduction

IN RECENT years, the distributed cooperative control for the multiagent systems (MASs) has drawn considerable attention. The basic principle of MASs is that: every agent plans its own movement based on its local neighbors' information and the entire MAS is driven to achieve desired objectives (i.e., consensus, flocking, formation and containment) via this distributed control mechanism [1], [2]. In general, the MASs can be studied from two aspects: 1) the agent's dynamics and 2) the communication of MASs. From the agent's dynamics aspect, as a representative nonlinear system, the control of MASs whose agents are described by the Euler–Lagrange dynamics has received broad interest due to its various applications in unmanned air vehicles and robot manipulators [3]. Some papers have been published to address the cooperative control [4], to deal with the uncertainty [5], and to achieve finite-time consensus [6] of the networked Euler–Lagrange systems.

From the communication point of view, the agent's inner communication mechanism (the agent's "sensor-to-controller" communication, that is, the periodic sampling, the aperiodic sampling, and the event-triggered sampling) has been extensively investigated. It is noted that the event-triggered control for linear MASs has been demonstrated to be an effective method to reduce the communication resource usage where the communication between the agent's controller and sensor becomes active only when some specific events occur [7]–[11]. Furthermore, reducing the communication resource usage means that the energy for each agent can be saved, and the operational lifespan of MASs can be prolonged.

There have been some publications regarding the event-triggered coordination of some special nonlinear MASs. In [12], an integrated sampled-data-based event-triggered controller was proposed for the nonlinear MASs where the nonlinear term in the agent's dynamics should satisfy the "globally Lipschitz" condition. In [13] and [14], two event-triggered controllers based on function-approximation methods were considered for MASs, and each agent's nonlinear dynamics should take the pure-feedback form and the strict-feedback form, respectively. When the agent is described by the Euler–Lagrange dynamics, few results can be found in [15]–[20]. For example, for the Euler–Lagrange MASs under directed communication networks, an event-triggered strategy was proposed such that all followers asymptotically reached a consensus to one dynamic leader [15]. In [16], an event-triggered control strategy based on the estimation of the referenced trajectory was designed to solve the tracking control problem of Euler–Lagrange MASs, but the final tracking error could only be reduced into a small neighborhood around the origin. Kumari *et al.* [17] employed an event-triggered sliding-mode controller to achieve a robust tracking of Euler–Lagrange MASs with unknown parameters. Liu *et al.* [18], [19] introduced a fully distributed model-independent event-triggered controller without the relative velocity information and an

Manuscript received 2 July 2021; revised 4 September 2021; accepted 5 October 2021. Date of publication 2 November 2021; date of current version 13 January 2023. This work was supported in part by the National Natural Science Foundation of China under Grant 61633016, Grant 62025307, and Grant U1913209; and in part by the Beijing Natural Science Foundation under Grant JQ19020. This article was recommended by Associate Editor Y. Pan. *(Corresponding author: Long Cheng.)*

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Color versions of one or more figures in this article are available at https://doi.org/10.1109/TCYB.2021.3122339.

Digital Object Identifier 10.1109/TCYB.2021.3122339

adaptive controller with the relative velocity information to address the tracking control problem. However, the proposed controllers only work for a special class of Euler–Lagrange systems without the gravitational term, which limits their application. In addition, the agent's regression matrix needs to be globally Lipschitz and absolutely continuous. Xu *et al.* [20] relaxed the condition in [19] to be a differentiable, local Lipschitz regression matrix. The Zeno behavior in a finite-time interval is also excluded in [19] and [20]. The Zeno behavior means that the system is triggered infinite times during a finite-time interval, which cannot be allowed in practice.

Based on the above observations, this article proposes a dynamic event-triggered controller for Euler–Lagrange MASs with unknown parameters. Considering that it is difficult to guarantee the accuracy of the relative velocity information between agents, the designed controller only uses the agents state information and the relative position information with other agents. For the economical usage of communication resources, the event-triggered method is also enrolled in the proposed controller. A novel event-triggered method based on an internal dynamic variable is designed to decide the event occurring time. The controller is designed to tackle two tasks: 1) the leaderless consensus and 2) the containment. The control performance of MASs under two tasks is analyzed by the Lyapunov method, respectively. The theoretical analysis is given to show that the Zeno behavior is excluded in a finite-time interval. Finally, simulations based on UR5 robots of V-rep verify the effectiveness of the proposed algorithm. Compared to [19] and [20], the main features of this article lie in the following three points.

1) The control objective of [19] is to achieve the leader–follower consensus, and the control objective of [20] is to achieve the leaderless consensus control. In contrast, this article considers both the leaderless consensus problem and the containment problem with dynamic leaders.
2) In [19], a model-independent controller is designed for a special class of Euler–Lagrange systems without the gravitational term under undirected communication networks. This article considers a more general Euler–Lagrange dynamics under directed communication networks.
3) Event-triggered methods investigated in [19] and [20] can be seen as the special cases of the event-triggered method proposed in this article. The simulation results imply that the proposed method can allow a longer average length of event intervals, while the time of achieving the consensus/containment and the steady-state control performance are not affected.

Notation: Let $\mathbb{R}^p$ and $\mathbb{R}^{p\times p}$ represent the p-dimensional real column vector and the $p \times p$-dimensional real matrix, respectively. Let I_p denote the p-dimensional identity matrix. Let $\mathbf{0}_p$ and $\mathbf{1}_p$ denote the p-dimensional column vectors with each component being 0 and 1, respectively. Let $\mathbb{N}$ denote the set of natural numbers. Let $\|\cdot\|$ denote the 2-norm of the matrix or the Euclidean norm of the vector. Let $\otimes$ represent the Kronecker product. For two square matrices A and $B \in \mathbb{R}^{n\times n}$, $A \le B$ means $B - A$ is a semi-positive-definite matrix. Let $\sigma_{\max}(P)$ denote the maximum singular value of a matrix $P \in \mathbb{R}^{m\times n}$. Let $\lambda_{\min}(Q)$ denote the minimum eigenvalue of a symmetric matrix $Q \in \mathbb{R}^{n\times n}$. For a vector function $f(t) : \mathbb{R}^+ \to \mathbb{R}$, $f(t) \in \mathcal{L}_2$ if $\|f(t)\|_p = \int_0^\infty f(\tau)^T f(\tau) d\tau \le \infty$ and $f(t) \in \mathcal{L}_\infty$ if for each element of $f(t)$ denoted by $f_i(t)$, $\sup_t |f_i(t)| \le \infty$. For a vector $q \in \mathbb{R}^n$ and a set $\Omega \subseteq \mathbb{R}^n$, let $\text{dis}(q, \Omega) = \inf_{y\in\Omega} \|q - y\|^2$ denote the distance between q and Ω.

II. Problem Formulation

A. Dynamic Model

All agents in Euler–Lagrange MASs follow the same dynamic model, which is described by the following Euler–Lagrange equation:

$$M_i(q_i(t))\ddot{q}_i(t) + C_i(q_i(t), \dot{q}_i(t))\dot{q}_i(t) + g_i(q_i(t)) = \tau_i(t) \quad (1)$$

where $q_i(t) \in \mathbb{R}^p$ is a vector of the generalized coordinates, $M_i(q_i(t)) \in \mathbb{R}^{p\times p}$ is the symmetric positive-definite inertia matrix, $C_i(q_i(t), \dot{q}_i(t))\dot{q}_i(t) \in \mathbb{R}^p$ denotes the Coriolis and centrifugal torque, and $g_i(q_i(t))$ and $\tau_i(t) \in \mathbb{R}^p$ denote the gravitational torque and the generalized control force on the ith agent, respectively. The system model of each agent satisfies the following properties [21].

Property 1 (Parameter Boundness): For the ith agent, there exist positive constants $k_{\underline{m}}$, $k_{\overline{m}}$, k_C, and k_{g_i} such that $k_{\underline{m}} I_p \le M_i(q_i(t)) \le k_{\overline{m}} I_p$, $\|g_i(q_i(t))\| \le k_{g_i}$, and $\|C_i(x, y)\| \le k_C\|y\|$ for any $x, y \in \mathbb{R}^p$.

Property 2 (Skew Symmetric Property): The matrix $\dot{M}_i(q_i(t)) - 2C_i(q_i(t), \dot{q}_i(t))$ is skew-symmetric.

Property 3 (Linearity in the Parameters): $M_i(q_i(t))x + C_i(q_i(t), \dot{q}_i(t))y + g_i(q_i(t)) = Y_i(q_i(t), \dot{q}_i(t), x, y)\Theta_i$ for any $x, y \in \mathbb{R}^p$, where $Y_i(q_i(t), \dot{q}_i(t), x, y)$ is the regression matrix and Θ_i is the constant parameter vector.

Note that Properties 2 and 3 hold for fully actuated systems, for instance, fully actuated robotic manipulators. In addition, an assumption is listed as follows

Assumption 1: Each element of $M_i(q_i(t))$, $C_i(q_i(t), \dot{q}_i(t))$, and $g_i(q_i(t))$ is differentiable and locally Lipschitz.

Remark 1: The same assumption has been investigated in [20], and a stronger version is proposed in [18]. There are many Euler–Lagrange systems satisfy the above assumption. As stated in [20], from Assumption 1 and Property 3, it follows that each element of $Y_i(q_i(t), \dot{q}_i(t), 0, \dot{q}_{ri}(t))$ is differentiable and locally Lipschitz, which means that $\dot{Y}_i(q_i(t), \dot{q}_i(t), 0, \dot{q}_{ri}(t))$ is bounded if $Y_i(q_i(t), \dot{q}_i(t), 0, \dot{q}_{ri}(t))$ is bounded [22].

B. Graph Theory

Consider a MAS consisted of n agents. The network topology is represented by $G = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, 2, \dots, n\}$ is the agent set and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ denotes the edge set [23]. An edge $(i, j) \in \mathcal{E}$ means that agent j can receive information from agent i, and in this case, agent i is called a neighbor of agent j. The agent is called a leader if it has no neighbor; otherwise, it is called a follower. Let $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ represent a weighted adjacency matrix associated with graph G, and the element $a_{ij} > 0$ when $(j, i) \in \mathcal{E}$; otherwise, $a_{ij} = 0$. The

Laplacian matrix $L = [l_{ij}] \in \mathbb{R}^{n\times n}$ associated with graph G is defined as $l_{ii} = \sum_{j=1, j\neq i}^{n} a_{ij}$ and $l_{ij} = -a_{ij}$, $i \neq j$. In a directed graph, a directed path is a sequence of edges of the form $(i_1, i_2), (i_2, i_3), \ldots, (i_{k-1}, i_k)$, where any edge $(i_m, i_{m+1}) \in \mathcal{E}$. A directed graph has a directed spanning tree if there exists at least one agent that has directed paths to any other agents. A directed graph is strongly connected if there is a directed path between any pair of distinct nodes. The following results for the Laplacian matrix hold.

Lemma 1 [24]: Let G be a directed graph consisting of n nodes and $L \in \mathbb{R}^{n\times n}$ is the Laplacian matrix associated with G. The following two statements hold.

1) The Laplacian matrix L has a single zero eigenvalue and all other eigenvalues have positive real parts if and only if G contains a directed spanning tree.
2) If G is strongly connected, there exists a vector $\xi \triangleq [\xi_1, \ldots, \xi_n]^T \in \mathbb{R}^n$ with $\sum_{i=1}^{n} \xi_i = 1$, and $\xi_i > 0$ $\forall i = 1, \ldots, n$, such that $\xi^T L = 0$.

Lemma 2 [24]: Let G be a directed graph consisting of n nodes and G is strongly connected. Define matrix $B \triangleq \Xi L + L^T \Xi$, where $\Xi \triangleq \mathrm{diag}(\xi_1, \ldots, \xi_n)$ with $\xi = [\xi_1, \ldots, \xi_n]^T$ and L are defined in Lemma 1. Then, B is a symmetric Laplacian matrix associated with an undirected graph. Letting $\vartheta \in \mathbb{R}^n$ be any nonzero vector satisfying $\vartheta^T \xi = 0$, the following inequality holds:

$$a(L) \triangleq \min_{\substack{\vartheta^T \xi = 0 \\ \vartheta \neq \mathbf{0}_n}} \frac{\vartheta^T B \vartheta}{\vartheta^T \Xi \vartheta} > 0. \tag{2}$$

Furthermore, it can be obtained from the above equation that $\vartheta^T B \vartheta \geq a(L)\vartheta^T \Xi \vartheta \geq a(B)\vartheta^T \vartheta$ where $a(B) = a(L)\min(\xi_1, \ldots, \xi_n)$.

When the graph is not strongly connected, the entire MAS is assumed to be consisted of m followers and $n - m$ leaders. It is also assumed as follows.

Assumption 2: For each follower, there exists at least one leader, which has a directed path to this follower.

Assumption 3: Each leader's generalized velocity $\dot{q}_i$ satisfies $\dot{q}_i \in \mathcal{L}_2$, $i = n - m + 1, \ldots, n$.

Then, the following statements hold.

Definition 1 [25]: The set C is convex if and only if the point $(1-t)x + ty \in C$ where $t \in [0, 1]$ and $x, y \in C$. The convex hull $Co(X)$ for a set of points $X = \{x_1, \ldots, x_n\}$ in a real vector space is the minimal convex set containing all points in X, which satisfies $Co(X) = \{\sum_{i=1}^{n} \alpha_i x_i | x_i \in X, \alpha_i \geq 0, \sum_{i=1}^{n} \alpha_i = 1\}$.

Definition 2 [26]: A square matrix $A \in \mathbb{R}^{n\times n}$ is called a nonsingular M-matrix if all of its off-diagonal entries are nonpositive and its eigenvalues have positive real parts. For the nonsingular M-matrix A, there exists a diagonal matrix $D = \mathrm{diag}(d_1, \ldots, d_n)$ with $d_i > 0$ $\forall i = 1, \ldots, n$, such that the matrix $Q = DA + A^T D > 0$.

Lemma 3 [27]: If Assumption 2 holds, then the Laplacian matrix L associated with graph A can be written as

$$L = \begin{bmatrix} L_1 & L_2 \\ \mathbf{0}_{(n-m)\times m} & \mathbf{0}_{(n-m)\times(n-m)} \end{bmatrix}$$

where $L_1 \in \mathbb{R}^{m\times m}$ and $L_2 \in \mathbb{R}^{m\times(n-m)}$. Then, matrix L_1 is a nonsingular M-matrix. Each entry of $-L_1^{-1}L_2$ is non-negative and the row sums of $-L_1^{-1}L_2$ equal to one.

C. Control Objective

The first control objective in this article is to drive the Euler–Lagrange MAS to achieve the leaderless consensus, which is defined as follows.

Definition 3: The Euler–Lagrange MAS is said to achieve the consensus if and only if the state variables q_i of all agents satisfy

$$\lim_{t\to\infty} \|q_i(t) - q_j(t)\| = 0 \quad \forall i, j = 1, 2, \ldots, n.$$

The second control objective in this article is to drive the Euler–Lagrange MASs to achieve the containment, which is defined as follows.

Definition 4: The Euler–Lagrange MAS is said to achieve the containment if and only if all followers go into the convex hull spanned by dynamic leaders as time goes to infinity. That is

$$\lim_{t\to\infty} \mathrm{dis}(q_i(t), Co(q_L(t))) = 0, \ \ i = 1, \ldots, m \tag{3}$$

where $q_L(t) = [q_{n-m}^T(t), \ldots, q_n^T(t)]^T$ denotes the leaders' state variables and $Co(q_L(t))$ denotes the convex hull defined in Definition 1.

III. Distributed Consensus Control

By borrowing a similar idea proposed in [27], design the auxiliary variables for the agent i $(i = 1, \ldots, n)$

$$\begin{aligned} \dot{q}_{ri}(t) &= -\sum_{j=1}^{n} a_{ij}\big(q_i(t) - q_j(t)\big) \\ s_i(t) &= \dot{q}_i(t) - \dot{q}_{ri}(t) = \dot{q}_i(t) + \sum_{j=1}^{n} a_{ij}\big(q_i(t) - q_j(t)\big) \end{aligned} \tag{4}$$

where a_{ij} denotes the $\{i, j\}$-entry of adjacency matrix A.

To achieve the event-triggered communication, each agent of MASs only broadcasts its state information to the controller at a discrete event-triggered time sequence $\{t_k^i\}_{k\in\mathbb{N}}$, where t_k^i denotes the kth event-triggered time of the ith agent. Design the distributed adaptive controller based on the latest state information

$$\begin{aligned} \tau_i(t) &= -k_i s_i\big(t_k^i\big) + Y_i\big(t_k^i\big)\hat{\Theta}_i\big(t_k^i\big), t \in \big[t_k^i, t_{k+1}^i\big) \\ \dot{\hat{\Theta}}_i(t) &= -\Lambda_i Y_i^T(t) s_i(t) \end{aligned} \tag{5}$$

where k_i and Λ_i denote positive control gains, $\hat{\Theta}_i(t)$ is the estimate of the unknown parameter Θ_i, $Y_i(t_k^i)$ is the abbreviation of the regression matrix $Y_i(q_i(t_k^i), \dot{q}_i(t_k^i), 0, \dot{q}_{ri}(t_k^i))$ defined by Property 3, which satisfies the following equation:

$$\begin{aligned} Y_i\big(q_i\big(t_k^i\big), \dot{q}_i\big(t_k^i\big), 0, \dot{q}_{ri}\big(t_k^i\big)\big) = C_i\big(q_i\big(t_k^i\big), \dot{q}_i\big(t_k^i\big)\big)\dot{q}_{ri}\big(t_k^i\big) \\ + g_i\big(q_i\big(t_k^i\big)\big). \end{aligned}$$

Then, the measurement errors $e_i(t)$ and $\epsilon_i(t)$ can be defined by

$$\begin{aligned} e_i(t) &= s_i\big(t_k^i\big) - s_i(t) \\ \epsilon_i(t) &= Y_i\big(t_k^i\big)\hat{\Theta}_i\big(t_k^i\big) - Y_i(t)\hat{\Theta}_i(t). \end{aligned}$$

Define a dynamic internal variable $\chi_i(t)$ of the ith agent that takes the form of

$$\dot{\chi}_i(t) = -\gamma_i\chi_i(t) - \omega_i[\|\epsilon_i(t)\| + k_i\|e_i(t)\| - \beta_i\|s_i(t)\|] \quad (6)$$

where γ_i is a positive constant, ω_i, β_i are non-negative constants, and γ_i and ω_i satisfy $\gamma_i-(1+\beta_i\omega_i)/2 \geq 0$. The dynamic internal variable's initial value $\chi_i(0)$ is a positive constant. To determine the event-triggered time sequence $\{t_k^i\}_{k\in\mathbb{N}}$, the inequality for the triggering condition is proposed as follows:

$$\|\epsilon_i(t)\| + k_i\|e_i(t)\| - \beta_i\|s_i(t)\| \leq \chi_i(t). \quad (7)$$

When (7) is violated, an event is triggered and the event-triggered time is set to be the current time $t_k^i = t$. Then, the controller updates the received state information. Note that the errors $e_i(t)$ and $\epsilon_i(t)$ are also reset to be 0 in accordance with their definitions, which guarantees (7) always holds. From (6) and (7), it can be concluded that

$$\dot{\chi}_i(t) \geq -\gamma_i\chi_i(t) - \omega_i\chi_i(t)$$

which leads to

$$\chi_i(t) \geq \chi_i(0)e^{-(\gamma_i+\omega_i)t}. \quad (8)$$

In the event-triggered control, how to avoid the Zeno behavior is a key point. The definition of Zeno behavior is given as follows.

Definition 5 [28]: During a finite-time interval $t \in [t_1, t_2]$ where $0 \leq t_1 < t_2 < \infty$, the event is triggered infinite times and the event-triggered time sequence is $\{t_k^i, \ldots, t_\infty^i\}$ for some finite $k \geq 0$, then the system has the Zeno behavior.

A. Stability Analysis

Theorem 1: Using the trigger condition (7) and the controller (5) with parameters satisfying the following condition:

$$k_i > \max_{i=1,\ldots,n} \mu_i + k_{\bar{m}}\sigma_{\max}(L) + \frac{\left[k_{\bar{m}}\sigma_{\max}^2(L) + 1\right]^2}{2a(B)} \quad \forall i = 1, \ldots, n \quad (9)$$

where $\mu_i = (1 + 2\beta_i + \beta_i\omega_i)/2$, system (1) can achieve the distributed leaderless consensus and can exclude the possibility that the Zeno behavior occurs in a finite-time interval when the communication network is a strongly connected directed graph.

Proof: Substituting (5) into (1), it can be obtained that

$$M_i(q_i(t))\ddot{q}_i(t) + C_i(q_i(t), \dot{q}_i(t))q_i(t) + g_i(q_i(t)) = -k_is_i(t_k^i) + Y_i(t_k^i)\hat{\Theta}_i(t_k^i).$$

Using the definition of $s_i(t)$ and Property 3, it can be obtained from the above analysis that

$$M_i(q_i(t))\dot{s}_i(t) + C_i(q_i(t), \dot{q}_i(t))s_i(t) = -k_is_i(t_k^i) + Y_i(t_k^i)\hat{\Theta}_i(t_k^i) - Y_i(t)\Theta_i - M_i(q_i(t))\ddot{q}_{ri}(t).$$

Rewrite the above equation into the following vector formulation:

$$M(q(t))\dot{s}(t) + C(q(t), \dot{q}(t))s(t) = -Ks(t_k) + Y(t_k)\hat{\Theta}(t_k) - Y(t)\Theta - M(q(t))\ddot{q}_r(t) \quad (10)$$

where $M(q(t))$, $C(q(t), \dot{q}(t))$, $Y(t_k)$, and $Y(t)$ and the gain matrix K represent block-diagonal matrices of $M_i(q_i(t))$, $C(q_i(t), \dot{q}_i(t))$, $Y_i(t_k^i)$, $Y_i(t)$, and k_iI_p, respectively. Let $\hat{\Theta}(t_k)$, Θ, and $\ddot{q}_r(t)$ denote the column stack vectors of $\hat{\Theta}_i(t_k)$, Θ_i, and $\ddot{q}_{ri}(t)$, respectively.

By Lemma 1, there exists a vector $\xi \triangleq [\xi_1, \ldots, \xi_n] \in \mathbb{R}^n$ with $\xi^TL = 0$, where $\sum_{i=1}^n \xi_i = 1$ and $\xi_i > 0\ \forall i = 1, \ldots, n$. Define a reference vector as $\bar{q}(t) \triangleq \sum_{i=1}^n \xi_iq_i(t) = (\xi^T \otimes I_p)q(t)$, where $q(t)$ is the column stack vector of $q_i(t)$. Let $\tilde{q}_i(t) \triangleq q_i(t) - \bar{q}(t)$. Define $\tilde{q}(t)$ as the column stack vector of $\tilde{q}_i(t)$, $i = 1, \ldots, n$. Note that $\tilde{q}(t) = q(t) - \mathbf{1}_n \otimes \bar{q}(t)$. By the above definition, it can be obtained that

$$\begin{aligned}(L \otimes I_p)\tilde{q}(t) &= (L \otimes I_p)(q(t) - \mathbf{1}_n \otimes \bar{q}(t)) \\ &= (L \otimes I_p)q(t) - (L\mathbf{1}_n \otimes \bar{q}(t)) \\ &= (L \otimes I_p)q(t)\end{aligned}$$

where the fact that all row sums of the Laplacian matrix equal to 0 is used, that is, $L\mathbf{1}_n = \mathbf{0}_n$. By the above equation and (4), it can be obtained that

$$\begin{aligned}s(t) &= \dot{q}(t) + (L \otimes I_p)q(t) \\ &= \dot{q}(t) + (L \otimes I_p)\tilde{q}(t)\end{aligned} \quad (11)$$

where $s(t)$ and $\dot{q}(t)$ are the column stack variables of $s_i(t)$ and $\dot{q}_i(t)$, $i = 1, \ldots, n$, respectively.

Consider the following Lyapunov candidate:

$$V(t) = \frac{1}{2}s^T(t)M(q(t))s(t) + \frac{1}{2}\tilde{\Theta}^T(t)\Lambda^{-1}\tilde{\Theta}(t) + \tilde{q}^T(t)(\Xi \otimes I_p)\tilde{q}(t) + \frac{1}{2}\sum_{i=1}^n \chi_i^2(t)$$

where Λ is the block-diagonal matrix of Λ_iI_p, Ξ follows the definition in Lemma 2, and $\tilde{\Theta}(t)$ is the column stack vector of the estimate error $\tilde{\Theta}_i(t)$ which is defined by $\tilde{\Theta}_i(t) = \hat{\Theta}_i(t)-\Theta_i$. Let $V_1(t) = (1/2)s^T(t)M(q(t))s(t) + (1/2)\tilde{\Theta}^T(t)\Lambda^{-1}\tilde{\Theta}(t) + \sum_{i=1}^n \chi_i^2(t)$ and $V_2(t) = \tilde{q}^T(t)(\Xi \otimes I_p)\tilde{q}(t)$. Then, the time derivative of $V_1(t)$ along the trajectory of system (10) is

$$\dot{V}_1(t) = \frac{1}{2}\left[s^T(t)\dot{M}(q(t))s(t) + 2s^T(t)M(q(t))\dot{s}(t)\right] + \tilde{\Theta}^T(t)\Lambda^{-1}\dot{\tilde{\Theta}}(t) + \sum_{i=1}^n \dot{\chi}_i(t)\chi_i(t).$$

Using (5) and (10), the above equation can be rewritten as follows:

$$\begin{aligned}\dot{V}_1(t) = &\frac{1}{2}s^T(t)\dot{M}(q(t))s(t) + s^T(t)M(q(t))\dot{s}(t) \\ &- \tilde{\Theta}^T(t)\Lambda^{-1}\Lambda Y^T(t)s(t) + \sum_{i=1}^n \dot{\chi}_i(t)\chi_i(t) \\ = &\frac{1}{2}s^T(t)\dot{M}(q(t))s(t) + s^T(t)C(q(t), \dot{q}(t))s(t) \\ &- s^T(t)Ks(t_k) + s^T(t)Y(t_k)\hat{\Theta}(t_k) - s^T(t)Y(t)\Theta(t) \\ &- s^T(t)M(q(t))\ddot{q}_r(t) - s^T(t)Y(t)\tilde{\Theta}(t) + \sum_{i=1}^n \dot{\chi}_i(t)\chi_i(t).\end{aligned}$$

Using the definitions of $e_i(t)$, $\epsilon_i(t)$, and $\tilde{\Theta}_i(t)$ and the fact that $(1/2)M(q(t)) - C(q(t), \dot{q}(t))$ is skew-symmetric, it can be obtained that

$$\begin{aligned}\dot{V}_1(t) &= -s^T(t)Ks(t_k) + s^T(t)Y(t_k)\hat{\Theta}(t_k) - s^T(t)Y(t)\hat{\Theta}(t)\\&\quad - s^T(t)M(q(t))\ddot{q}_r(t) + \sum_{i=1}^{n}\dot{\chi}_i(t)\chi_i(t)\\&= -s^T(t)Ks(t) - s^T(t)Ke(t) + s^T(t)\epsilon(t)\\&\quad - s^T(t)M(q(t))\ddot{q}_r(t) + \sum_{i=1}^{n}\dot{\chi}_i(t)\chi_i(t)\\&= -s^T(t)Ks(t) + \sum_{i=1}^{n}s_i^T(t)[\epsilon_i(t) - k_ie_i(t)]\\&\quad - s^T(t)M(q(t))\ddot{q}_r(t) + \sum_{i=1}^{n}\dot{\chi}_i(t)\chi_i(t) \qquad (12)\end{aligned}$$

where $e(t)$ and $\epsilon(t)$ are column stack vectors of $e_i(t)$ and $\epsilon_i(t)$, respectively. Note that

$$\begin{aligned}&\sum_{i=1}^{n}s_i^T(t)[\epsilon_i(t) - k_ie_i(t)] + \sum_{i=1}^{n}\dot{\chi}_i(t)\chi_i(t)\\&\le \sum_{i=1}^{n}\{\|s_i(t)\|[\|\epsilon_i(t)\| + k_i\|e_i(t)\| - \beta_i\|s_i(t)\| + \beta_i\|s_i(t)\|]\\&\qquad + \dot{\chi}_i(t)\chi_i(t)\}\\&\le \sum_{i=1}^{n}\Big\{\|s_i(t)\|[\|\epsilon_i(t)\| + k_i\|e_i(t)\| - \beta_i\|s_i(t)\|]\\&\qquad + \beta_i\|s_i(t)\|^2 + \dot{\chi}_i(t)\chi_i(t)\Big\}\\&\le \sum_{i=1}^{n}\Big\{\beta_i\|s_i(t)\|^2 + \|s_i(t)\|\chi_i(t) - \gamma_i\chi_i^2(t)\\&\qquad - \omega_i[\|\epsilon_i(t)\| + k_i\|e_i(t)\| - \beta_i\|s_i(t)\|]\chi_i(t)\Big\}\\&\le \sum_{i=1}^{n}\Big\{\beta_i\|s_i(t)\|^2 + \|s_i(t)\|\chi_i(t) - \gamma_i\chi_i^2(t)\\&\qquad + \beta_i\omega_i\|s_i(t)\|\chi_i(t)\Big\}\\&\le \sum_{i=1}^{n}\left[\frac{1 + 2\beta_i + \beta_i\omega_i}{2}\|s_i(t)\|^2 - \left(\gamma_i - \frac{1 + \beta_i\omega_i}{2}\right)\chi_i^2(t)\right]\end{aligned} \qquad (13)$$

where the third inequality is obtained based on (6) and (7). Note that it can be inferred from $(x-y)^2 = x^2 - 2xy + y^2 \ge 0$ that $xy \le x^2/2 + y^2/2$, where x and y are scalars. Then, the last inequality in (13) can be obtained by the fact that

$$\begin{aligned}(1 + \beta_i\omega_i)\|s_i(t)\|\chi_i(t) \le \frac{(1 + \beta_i\omega_i)}{2}\|s_i(t)\|^2\\ + \frac{(1 + \beta_i\omega_i)}{2}\chi_i^2(t).\end{aligned} \qquad (14)$$

Considering the fact that $x^TPy \le \|x\|\|P\|\|y\| = \sigma_{\max}(P)\|x\|\|y\|$ and $\ddot{q}_r(t) = -(L \otimes I_p)\dot{q}(t) = -(L \otimes I_p)[s(t) - (L \otimes I_p)\tilde{q}(t)]$, it has

$$\begin{aligned}s^T(t)M(q(t))\ddot{q}_r(t) &\le k_{\bar{m}}\|s(t)\|\|\ddot{q}_r(t)\|\\&\le k_{\bar{m}}\sigma_{\max}^2(L)\|s(t)\|\|\tilde{q}(t)\|\\&\quad + k_{\bar{m}}\sigma_{\max}(L)\|s(t)\|^2.\end{aligned} \qquad (15)$$

Taking (13) and (15) into (12), it has

$$\begin{aligned}\dot{V}_1(t) &\le -k_{\min}\|s(t)\|^2 + \sum_{i=1}^{n}\left[-\left(\gamma_i - \frac{1 + \beta_i\omega_i}{2}\right)\chi_i^2(t)\right.\\&\qquad\left. + \mu_i\|s_i(t)\|^2\right]\\&\quad + k_{\bar{m}}\sigma_{\max}^2(L)\|s(t)\|\|\tilde{q}(t)\|\\&\quad + k_{\bar{m}}\sigma_{\max}(L)\|s(t)\|^2\\&\le -\left[k_{\min} - \max_{i=1,\ldots,n}\mu_i - k_{\bar{m}}\sigma_{\max}(L)\right]\|s(t)\|^2\\&\quad + k_{\bar{m}}\sigma_{\max}^2(L)\|s(t)\|\|\tilde{q}(t)\|\\&\quad - \sum_{i=1}^{n}\left(\gamma_i - \frac{1 + \beta_i\omega_i}{2}\right)\chi_i^2(t)\end{aligned} \qquad (16)$$

where $k_{\min} = \min\{k_1, \ldots, k_n\}$.

Next, the time derivative of $V_2(t)$ along the trajectory of system (10) is

$$\dot{V}_2(t) = 2\tilde{q}^T(t)(\Xi \otimes I_p)\dot{\tilde{q}}(t). \qquad (17)$$

Note that $\dot{\bar{q}}(t) = (\xi^T \otimes I_p)\dot{q}(t) = (\xi^T \otimes I_p)s(t) - (\xi^TL \otimes I_p)\tilde{q}(t) = (\xi^T \otimes I_p)s(t)$. Using the definitions of $s(t)$ and $\tilde{q}(t)$, the time derivative of $\tilde{q}(t)$ can be written as

$$\begin{aligned}\dot{\tilde{q}}(t) &= \dot{q}(t) - 1_n \otimes \dot{\bar{q}}(t)\\&= \dot{q}(t) - 1_n \otimes \left[(\xi^T \otimes I_p)\dot{q}(t)\right]\\&= s(t) - (L \otimes I_p)\tilde{q}(t) - 1_n \otimes \left[(\xi^T \otimes I_p)s(t)\right].\end{aligned} \qquad (18)$$

Note that

$$\begin{aligned}&(\Xi \otimes I_p)\left\{s(t) - \mathbf{1}_n \otimes \left[(\xi^T \otimes I_p)s(t)\right]\right\}\\&= (\Xi \otimes I_p)\left\{s(t) - \left[(\mathbf{1}_n \otimes \xi^T) \otimes I_p\right]s(t)\right\}\\&= (\Xi \otimes I_p)s(t) - \left\{\left[\Xi(\mathbf{1}_n \otimes \xi^T)\right] \otimes I_p\right\}s(t)\\&= \left[(\Xi - \xi\xi^T) \otimes I_p\right]s(t).\end{aligned}$$

Then, it can be obtained by (17) that

$$\begin{aligned}\dot{V}_2(t) &= 2\tilde{q}^T(t)(\Xi \otimes I_p)\left\{s(t) - (L \otimes I_p)\tilde{q}\right.\\&\qquad\left. - 1_n \otimes \left[(\xi^T \otimes I_p)s(t)\right]\right\}\\&= -\tilde{q}^T(t)\left[(\Xi L + L^T\Xi) \otimes I_p\right]\tilde{q}(t)\\&\quad + 2\tilde{q}^T(t)\left[(\Xi - \xi\xi^T) \otimes I_p\right]s(t).\end{aligned} \qquad (19)$$

Let $B = \Xi L + L^T\Xi$. Note that $(\xi^T \otimes I_p)\tilde{q}(t) = (\xi^T \otimes I_p)q(t) - (\xi^T\mathbf{1}_n) \otimes \bar{q}(t) = \mathbf{0}_p$ and the communication network associated with L is strongly connected. It follows from Lemma 2 that:

$$\begin{aligned}\tilde{q}^T(t)\left[(\Xi L + L^T\Xi) \otimes I_p\right]\tilde{q}(t) &= \tilde{q}^T(t)(B \otimes I_p)\tilde{q}(t)\\&\le a(B)\|\tilde{q}(t)\|^2\end{aligned} \qquad (20)$$

where $a(B)$ follows the same definition in Lemma 2. The matrix $\Xi - \xi\xi^T$ is diagonally dominant and positive semidefinite. According to the Gersgorin theorem [29], it can be obtained that $\sigma_{\max}(\Xi - \xi\xi^T) \le 1/2$. Then, it can be concluded from (19) and (20) that

$$\dot{V}_2(t) \le -a(B)\|\tilde{q}(t)\|^2 + \|\tilde{q}(t)\|\|s(t)\|. \qquad (21)$$

By (16) and (21), the time derivative of $V(t)$ is

$$\begin{aligned}\dot{V}(t) \le& -\left[k_{\min} - \max_{i=1,\dots,n} \mu_i - k_{\bar{m}}\sigma_{\max}(L)\right]\|s(t)\|^2\\ &- a(B)\|\tilde{q}(t)\|^2 + \left[k_{\bar{m}}\sigma_{\max}^2(L) + 1\right]\|s(t)\|\|\tilde{q}(t)\|\\ &-\sum_{i=1}^{n}\left(\gamma_i - \frac{1+\beta_i\omega_i}{2}\right)\chi_i^2(t).\end{aligned}$$

Using the fact that

$$\begin{aligned}\left[k_{\bar{m}}\sigma_{\max}^2(L) + 1\right]\|\tilde{q}(t)\|\|s(t)\| \le& \frac{\left[k_{\bar{m}}\sigma_{\max}^2(L) + 1\right]^2}{2a(B)}\|s(t)\|^2\\ &+ \frac{a(B)}{2}\|\tilde{q}(t)\|^2\end{aligned}$$

it can be obtained that

$$\begin{aligned}\dot{V}(t) \le& -\left[k_{\min} - \max_{i=1,\dots,n} \mu_i - k_{\bar{m}}\sigma_{\max}(L)\right]\|s(t)\|^2\\ &- a(B)\|\tilde{q}(t)\|^2 + \frac{\left[k_{\bar{m}}\sigma_{\max}^2(L) + 1\right]^2}{2a(B)}\|s(t)\|^2\\ &+ \frac{a(B)}{2}\|\tilde{q}(t)\|^2 - \sum_{i=1}^{n}\left(\gamma_i - \frac{1+\beta_i\omega_i}{2}\right)\chi_i^2(t)\\ \le& -k_0\|s(t)\|^2 - \frac{a(B)}{2}\|\tilde{q}(t)\|^2\\ &- \sum_{i=1}^{n}\left(\gamma_i - \frac{1+\beta_i\omega_i}{2}\right)\chi_i^2(t)\end{aligned} \tag{22}$$

where

$$k_0 = k_{\min} - \max_{i=1,\dots,n} \mu_i - k_{\bar{m}}\sigma_{\max}(L) - \frac{\left[k_{\bar{m}}\sigma_{\max}^2(L) + 1\right]^2}{2a(B)}.$$

Obviously, k_0 is a positive constant and $\dot{V}(t)$ is negative definite when the control gain k_i satisfies condition (9) for $i = 1, \dots, n$.

From the fact that $V(t) \ge 0$ and $\dot{V}(t) \le 0$, it follows that $s(t), \tilde{q}(t) \in \mathcal{L}_\infty$. From (18), it is obvious that $\dot{\tilde{q}}(t) \in \mathcal{L}_\infty$. Integrating both sides of (22), it gives

$$\begin{aligned}V(0) - V(\infty) \ge& k_0\int_0^\infty \|s(t)\|^2 dt + \frac{a(B)}{2}\int_0^\infty \|\tilde{q}(t)\|^2 \mathrm{d}t\\ &+ \sum_{i=1}^{n}\left(\gamma_i - \frac{1+\beta_i\omega_i}{2}\right)\int_0^\infty \chi_i^2(t)\mathrm{d}t.\end{aligned} \tag{23}$$

In (23), $V(0) - V(\infty)$ is bounded due to $\dot{V}(t) \le 0$, which implies that the sum of the three terms on the right-hand side of (23) is bounded. Note that all three terms on the right-hand side are positive, which implies that any of them including $\int_0^\infty \|\tilde{q}(t)\|^2 dt$ has to be bounded when their sum is bounded. Hence, it can be obtained that $\tilde{q}(t) \in \mathcal{L}_2$. Using Barbalat's lemma, it can be concluded that $\tilde{q}(t) \to 0_p$ as $t \to \infty$. Consider the definition of $\tilde{q}(t)$, $\|q_i(t) - q_j(t)\| \to \mathbf{0}$ holds as $t \to \infty$, which implies that the system can achieve the leaderless consensus under the designed control law. ■

B. Zeno Behavior

First, define $f_i(t)$ as $f_i(t) = \|\epsilon_i(t)\| + k_i\|e_i(t)\|$ $\forall i = 1, \dots, n$. The time derivative of $k_i\|e_i(t)\|$ is

$$\begin{aligned}\frac{d}{\mathrm{dt}}k_i\|e_i(t)\| =& \frac{d}{\mathrm{dt}}\left[k_i\left(e_i^T(t)e_i(t)\right)^{\frac{1}{2}}\right] = k_i\frac{e_i^T(t)\dot{e}_i(t)}{\|e_i(t)\|}\\ \le& k_i\frac{\|e_i(t)\|\|\dot{e}_i(t)\|}{\|e_i\|} \le k_i\|\dot{s}_i(t)\|\end{aligned} \tag{24}$$

where the fact that $\|\dot{e}_i(t)\| = \|\dot{s}_i(t)\|$ is used. From the previous stability analysis, it can be concluded that $s_i(t)$, $\dot{q}_i(t)$, and $\hat{\Theta}_i(t)$ are bounded $\forall i = 1, \dots, n$. Then, from the system model (1) and the controller (5), it gives that $\ddot{q}_i(t)$ is bounded. It follows from $\dot{s}_i(t) = \ddot{q}_i(t) + (L \otimes I_p)\dot{q}(t)$ that $\dot{s}_i(t)$ is bounded. Let B_e denote the positive upper bound of $\dot{s}_i(t)$. From (24), the following inequalities hold:

$$\frac{d}{dt}k_i\|e_i(t)\| \le k_i\|\dot{s}_i(t)\| \le k_iB_e. \tag{25}$$

The time derivative of $\|\epsilon_i(t)\|$ is

$$\begin{aligned}\frac{d}{\mathrm{dt}}\|\epsilon_i(t)\| =& \frac{d}{\mathrm{dt}}\left[\epsilon_i^T(t)\epsilon_i(t)\right]^{\frac{1}{2}}\\ \le& k_i\frac{\|\epsilon_i(t)\|\|\dot{\epsilon}_i(t)\|}{\|\epsilon_i(t)\|} \le \left\|\frac{d}{dt}Y_i(t)\hat{\Theta}_i(t)\right\|\\ \le& \left\|\dot{Y}_i(t)\hat{\Theta}_i(t)\right\| + \left\|Y_i(t)\dot{\hat{\Theta}}_i(t)\right\|.\end{aligned} \tag{26}$$

It can be concluded from (4) that $\dot{q}_{ri}(t)$ is bounded. Then, by Property 3, $Y_i(t)$ is bounded. Using Assumption 1 and Remark 1, it can be obtained that $\dot{Y}_i(t)$ is bounded. Meanwhile, $\dot{\hat{\Theta}}_i(t)$ is bounded according to the controller (5). Hence, it can be concluded that $\|\dot{Y}_i(t)\hat{\Theta}_i(t)\| + \|Y_i(t)\dot{\hat{\Theta}}_i(t)\|$ is bounded by positive constant B_ϵ. From (26), it gives

$$\frac{d}{\mathrm{dt}}\|\epsilon_i(t)\| \le B_\epsilon. \tag{27}$$

Combining (25) and (27), it gives that

$$\dot{f}_i(t) = \frac{d}{\mathrm{dt}}\|\epsilon_i(t)\| + \frac{d}{dt}k_i\|e_i(t)\| \le B_\epsilon + k_iB_e.$$

It is obvious that $f_i(t) \le \chi_i(t)$ is the sufficient condition to guarantee the trigger condition (7) holds. For any $t \in [t_k^i, t_{k+1}^i)$, it gives

$$\begin{aligned}f_i(t) =& \int_{t_k^i}^{t}\dot{f}_i(\tau)d\tau + f_i\left(t_k^i\right) = \int_{t_k^i}^{t}\dot{f}_i(\tau)d\tau\\ \le& (B_\epsilon + k_iB_e)\left(t - t_k^i\right)\end{aligned} \tag{28}$$

where the fact that $e_i(t_k^i)$ and $\epsilon(t_k^i)$ are reset to be 0 at any triggering time t_k^i is used. From (8) and (28), a sufficient condition that makes $f_i(t) \le \chi_i(t)$ hold is given as follows:

$$(B_\epsilon + k_iB_e)\left(t - t_k^i\right) \le \chi_i(0)e^{-(\gamma_i+\omega_i)t}. \tag{29}$$

Note that the next triggering time t_{k+1}^i decided by (29) is lower than t_{k+1}^i decided by (7), which means the Zeno behavior under the condition (7) can be excluded if there is no Zeno behavior under the condition (29). Define the minimum length of event intervals between t_{k+1}^i and t_k^i as l, then l is the solution of the following equation:

$$(B_\epsilon + k_iB_e)l = \chi_i(0)e^{-(\gamma_i+\omega_i)\left(l+t_k^i\right)}. \tag{30}$$

Note that the solution of (30) is always strictly positive in a finite-time interval; hence, the Zeno behavior can be excluded.

Remark 2: Note that γ_i and ω_i determine the decay rate of the dynamic variable $\chi_i(t)$. Lower γ_i and ω_i can reduce the decay velocity of $\chi_i(t)$, which may lead to a larger length of the event interval. Meanwhile, it may have a negative effect on the system performance because the controller cannot obtain the latest state information timely. In conclusion, how to tune γ_i and ω_i is a tradeoff between the event frequency and the system performance.

In some recent papers [19], [20], another triggering condition has been proposed for the distributed cooperative control and its inequality takes the following form:

$$\|\epsilon_i(t)\| + k_i\|e_i(t)\| - \frac{\eta_i}{2}k_i\|s_i(t)\| \le \alpha_i e^{-\rho_i t} \tag{31}$$

where $0 < \eta_i < 1$, $\alpha_i, \rho_i > 0$, $\|\epsilon_i(t)\|$, k_i, $\|e_i(t)\|$, and $\|s_i(t)\|$ are defined the same as the ones in this article.

If parameters in (7) are chosen as $\beta_i = \eta_i k_i/2$, $\omega_i = 0$, $\chi_i(0) = \alpha_i$, and $\gamma_i = \rho_i$, (7) is equivalent to (31). So, (31) can be seen as a special case of (7) proposed in this article. In fact, if parameters are chosen as $\beta_i = \eta_i k_i/2$, $\omega_i > 0$, and $\gamma_i + \omega_i = \rho_i$, it can be obtained from (8) that $\chi_i(t) \ge \alpha_i e^{-\rho_i t}$, which implies that the next event trigger time determined by (7) is larger than the one determined by (31). Hence, although a thorough analysis to exclude the Zeno behavior as time goes to infinity is still an open challenge [19], the proposed algorithm has a better performance in avoiding the Zeno behavior.

IV. Distributed Containment Control

In this section, the distributed containment control problem with multiple followers and dynamic leaders is investigated. Define the follower set and the leader set as $\mathcal{V}_F = \{1, \dots, m\}$ and $\mathcal{V}_L = \{m+1, \dots, n\}$, respectively.

Design the auxiliary variables for the ith agent as follows:

$$\begin{aligned}
\dot{\hat{q}}_{ri}(t) &= \hat{v}_i - \sum_{j\in\mathcal{V}_L\cup\mathcal{V}_F} a_{ij}\big(q_i(t) - q_j(t)\big)\\
\ddot{\hat{q}}_{ri}(t) &= \hat{a}_i - \sum_{j\in\mathcal{V}_L\cup\mathcal{V}_F} a_{ij}\big(\dot{q}_i(t) - \dot{q}_j(t)\big)\\
\hat{s}_i(t) &= \dot{q}_i(t) - \dot{\hat{q}}_{ri}(t)\\
&= \dot{q}_i(t) - \hat{v}_i + \sum_{j\in\mathcal{V}_L\cup\mathcal{V}_F} a_{ij}\big(q_i(t) - q_j(t)\big),\ i \in \mathcal{V}_F.
\end{aligned} \tag{32}$$

Define all followers' desired generalized coordinate positions as $q_d(t) = [q_{d1}^T(t), \dots, q_{dm}^T(t)]^T = -(L_1^{-1}L_2 \otimes I_p)q_L(t)$. Note that $q_d(t)$ is in the convex hull spanned by dynamic leaders. The variables $\hat{v}_i(t)$ and $\hat{a}_i(t)$ in (32) are the ith follower's estimates of the desired generalized coordinate derivative $\dot{q}_{di}(t)$ and acceleration $\ddot{q}_{di}(t)$, respectively.

The distributed event-triggered containment controller is proposed as follows:

$$\begin{aligned}
\tau_i(t) &= -k_i\hat{s}_i\big(t_k^i\big) + \hat{Y}_i\big(t_k^i\big)\hat{\Theta}_i\big(t_k^i\big),\ t \in \big[t_k^i, t_{k+1}^i\big)\\
\dot{\hat{v}}_i(t) &= -b_1\operatorname{sgn}\left[\sum_{j\in\mathcal{V}_F} a_{ij}\big(\hat{v}_i - \hat{v}_j\big) + \sum_{j\in\mathcal{V}_L} a_{ij}\big(\hat{v}_i - \dot{q}_j\big)\right]\\
\dot{\hat{a}}_i(t) &= -b_2\operatorname{sgn}\left[\sum_{j\in\mathcal{V}_F} a_{ij}\big(\hat{a}_i - \hat{a}_j\big) + \sum_{j\in\mathcal{V}_L} a_{ij}\big(\hat{a}_i - \ddot{q}_j\big)\right]\\
\dot{\hat{\Theta}}_i(t) &= -\Lambda_i\hat{Y}_i^T(t)\hat{s}_i(t),\ i \in \mathcal{V}_F
\end{aligned} \tag{33}$$

where Λ_i and t_k^i are designed as the ones in (5), and b_1 and b_2 are positive constants which are defined later. The regression matrix $\hat{Y}_i(t_k^i)$ is defined as the abbreviation of $Y(q_i(t_k^i), \dot{q}_i(t_k^i), 0, \hat{\dot{q}}_{ri}(t_k^i))$. The estimator variables $\hat{v}_i$ and $\hat{a}_i$ satisfy the following lemma.

Lemma 4 [30]: Assume that Assumption 2 holds. If $b_1 > \|\ddot{q}_d(t)\|$, there must exist a positive constant T_1 such that $\|\hat{v}_i(t) - \dot{q}_{di}(t)\| = 0\ \forall i \in \mathcal{V}_F$, when $t > T_1$. If $b_2 > \|\dddot{q}_d(t)\|$, there must exist a positive constant T_2 such that $\|\hat{a}_i(t) - \ddot{q}_{di}(t)\| = 0\ \forall i \in \mathcal{V}_F$, when $t > T_2$.

The measurement errors $e_i(t)$ and $\epsilon_i(t)$ can be defined by

$$\begin{aligned}
e_i(t) &= s_i\big(t_k^i\big) - s_i(t)\\
\epsilon_i(t) &= Y_i\big(t_k^i\big)\hat{\Theta}_i\big(t_k^i\big) - Y_i(t)\hat{\Theta}_i(t),\ i \in \mathcal{V}_F.
\end{aligned}$$

Design the dynamic internal variable $\chi_i(t)$ of the ith agent, which takes the following form:

$$\dot{\chi}_i(t) = -\gamma_i\chi_i(t) - \omega_i[\|\epsilon_i(t)\| + k_i\|e_i(t)\| - \beta_i\|s_i(t)\|] \tag{34}$$

where γ_i and ω_i are defined as the ones in (6), which satisfy the conditions that $\gamma_i > 0$, $\omega_i \ge 0$ and $\gamma_i - (1 + \beta_i\omega_i)/2 \ge 0\ \forall i \in \mathcal{V}_F$. The inequality of the triggering condition is proposed as

$$\|\epsilon_i(t)\| + k_i\|e_i(t)\| - \beta_i\|s_i(t)\| \le \chi_i(t),\ i \in \mathcal{V}_F. \tag{35}$$

Theorem 2: Using the trigger condition (35) and the controller (33) with parameters satisfying the following conditions:

$$\begin{aligned}
k_i &> \max_{i\in\mathcal{V}_F}\mu_i + k_{\bar{m}}\sigma_{\max}(L) + \frac{\big[k_{\bar{m}}\sigma_{\max}^2(L) + 1\big]^2}{2a(B)}\\
b_1 &> \|\ddot{q}_d(t)\|,\ b_2 > \big\|\dddot{q}_d(t)\big\| \quad \forall i = 1, \dots, n
\end{aligned}$$

where μ_i follows the same definition in (9), the system (1) can achieve the containment and exclude the Zeno behavior in a finite-time interval when Assumption 2 holds.

Proof: When $t < \max\{T_1, T_2\}$ where T_1 and T_2 are defined in Lemma 4, it can be obtained from (33) that $\hat{v}_i(t)$ and $\hat{a}_i(t)$ are bounded for finite initial values $\hat{v}_i(0)$ and $\hat{a}_i(0)\ \forall i \in \mathcal{V}_F$. Furthermore, from (32), it can be concluded that $\hat{s}_i(t)$, $\hat{\dot{q}}_{ri}(t)$ and $\hat{\ddot{q}}_{ri}(t)$ are bounded for bounded states $q_i(t)$ and $\dot{q}_i(t)\ \forall i \in \mathcal{V}_F$. According to Property 3, $\hat{Y}(t_k^i)$ is bounded for bounded states $q_i(t)$ and $\dot{q}_i(t)\ \forall i \in \mathcal{V}_F$. Then, $\hat{\Theta}(t)$ is also bounded for finite initial value $\hat{\Theta}(0)$. According to (32), $\tau_i(t)$ is bounded. Then, from the system dynamics (1), $\ddot{q}_i(t)$ is bounded. Finally, it can be concluded that the state variables $q_i(t)$ and $\dot{q}_i(t)$ remain bounded for finite initial values $q_i(0)$ and $\dot{q}_i(0)\ \forall i \in \mathcal{V}_F$, $t < \max\{T_1, T_2\}$.

Then, define

$$\begin{aligned}
\dot{q}_{ri}(t) &= \dot{q}_{di} - \sum_{j\in\mathcal{V}_L\cup\mathcal{V}_F} a_{ij}\big(q_i(t) - q_j(t)\big)\\
s_i(t) &= \dot{q}_i(t) - \dot{q}_{ri}(t)\\
&= \dot{q}_i(t) - \dot{q}_{di} + \sum_{j\in\mathcal{V}_L\cup\mathcal{V}_F} a_{ij}\big(q_i(t) - q_j(t)\big).
\end{aligned} \tag{36}$$

According to Lemma 4, when $t \geq \max\{T_1, T_2\}$, $\hat{v}_i(t) = \dot{q}_{di}(t)$ and $\hat{a}_i(t) = \ddot{q}_{di}(t)$ hold. Hence, it can be obtained that $\hat{\dot{q}}_{ri}(t) = \dot{q}_{ri}(t)$, $\hat{\ddot{q}}_{ri}(t) = \ddot{q}_{ri}(t)$ and $\hat{s}_i(t) = s_i(t)$. Taking (33) into (1) and using the above equalities, the system dynamics when $t \geq \max\{T_1, T_2\}$ can be written as follows:

$$\begin{aligned} M(q_F(t))\dot{s}_F(t) + C(q_F(t), \dot{q}_F(t))s_F(t) \\ = -K_F s_F(t_k) + Y_F(t_k)\hat{\Theta}_F(t_k) - Y_F(t)\Theta_F(t) \\ - M(q_F(t))\ddot{q}_r(t) \end{aligned} \tag{37}$$

where $M(q_F(t))$, $C(q_F(t), \dot{q}_F(t))$, $Y_F(t_k)$, and $Y_F(t)$ and the gain matrix K_F represent block-diagonal matrices of $M_i(q_i(t))$, $C(q_i(t), \dot{q}_i(t))$, $Y_i(t_k^i)$, $Y_i(t)$, and $k_i I_p$, respectively. Let $\hat{\Theta}_F(t_k)$, $\Theta_F(t)$, $s_F(t)$, $q_F(t)$, and $\ddot{q}_r(t)$ denote the column stack vectors of $\hat{\Theta}_i(t_k)$, $\Theta_i(t)$, $s_i(t)$, $q_i(t)$, and $\ddot{q}_{ri}(t)$ $\forall i \in \mathcal{V}_F$, respectively.

Rewrite (32) into the vector form as follows:

$$\dot{q}_r(t) = -(L_1 \otimes I_p)q_F(t) - (L_2 \otimes I_p)q_L(t). \tag{38}$$

Defining $\bar{q}_F(t) = q_F(t) + (L_1^{-1}L_2 \otimes I_p)q_L(t)$, then it has

$$\begin{aligned} \dot{\bar{q}}_F(t) &= \dot{q}_F(t) + \left(L_1^{-1}L_2 \otimes I_p\right)\dot{q}_L(t) \\ &= s_F(t) - (L_1 \otimes I_p)\bar{q}_F(t) + \left(L_1^{-1}L_2 \otimes I_p\right)\dot{q}_L(t). \end{aligned} \tag{39}$$

From (38), the time derivative of $\dot{q}_r(t)$ can be obtained that

$$\begin{aligned} \ddot{q}_r(t) &= -(L_1 \otimes I_p)\dot{q}_F(t) - (L_2 \otimes I_p)\dot{q}_L(t) \\ &= -(L_1 \otimes I_p)\left[s_F(t) - (L_1 \otimes I_p)\bar{q}_F(t)\right] \\ &\quad - (L_2 \otimes I_p)\dot{q}_L(t). \end{aligned} \tag{40}$$

When $t \geq \max\{T_1, T_2\}$, constitute a Lyapunov candidate as follows:

$$\begin{aligned} V(t) &= \frac{1}{2}s_F^T(t)M(q(t))s_F(t) + \frac{1}{2}\tilde{\Theta}^T(t)\Lambda^{-1}\tilde{\Theta}(t) \\ &\quad + \bar{q}_F^T(t)(D \otimes I_p)\bar{q}_F(t) + \frac{1}{2}\sum_{i=1}^{n}\chi_i^2(t). \end{aligned} \tag{41}$$

The time derivative of $V(t)$ along the trajectory of system (37) is

$$\begin{aligned} \dot{V}(t) &= s_F^T(t)M(q(t))\dot{s}_F(t) + \frac{1}{2}s_F^T(t)\dot{M}(q(t))s_F(t) \\ &\quad + \frac{1}{2}\tilde{\Theta}^T(t)\Lambda^{-1}\dot{\tilde{\Theta}}(t) + \sum_{i=1}^{n}\dot{\chi}_i(t)\chi_i(t) \\ &\quad + \bar{q}_F^T(t)(D \otimes I_p)\dot{\bar{q}}_F(t) + \dot{\bar{q}}_F^T(t)(D \otimes I_p)\bar{q}_F(t). \end{aligned} \tag{42}$$

Note that

$$\begin{aligned} &\bar{q}_F^T(t)(D \otimes I_p)\dot{\bar{q}}_F(t) + \dot{\bar{q}}_F^T(t)(D \otimes I_p)\bar{q}_F(t) \\ &\leq \bar{q}_F^T(t)(D \otimes I_p)\Big[s_F(t) - (L_1 \otimes I_p)\bar{q}_F(t) \\ &\qquad + \left(L_1^{-1}L_2 \otimes I_p\right)\dot{q}_L(t)\Big] \\ &\quad + \Big[s_F(t) - (L_1 \otimes I_p)\bar{q}_F(t) \\ &\qquad + \left(L_1^{-1}L_2 \otimes I_p\right)\dot{q}_L(t)\Big]^T(D \otimes I_p)\bar{q}_F(t) \\ &\leq 2\bar{q}_F^T(t)(D \otimes I_p)s_F(t) - \bar{q}_F^T(t)\left[(DL_1 + L_1^T D) \otimes I_p\right]\bar{q}_F(t) \\ &\quad + 2\bar{q}_F^T(t)\left(DL_1^{-1}L_2 \otimes I_p\right)\dot{q}_L(t) \\ &\leq 2d_{\max}\|\bar{q}_F(t)\|\|s_F(t)\| - \lambda_{\min}(Q)\|\bar{q}_F(t)\|^2 \\ &\quad + 2\sigma_{\max}\left(DL_1^{-1}L_2\right)\|\bar{q}_F(t)\|\|\dot{q}_L(t)\| \end{aligned} \tag{43}$$

where $d_{\max} = \max\{d_1, \ldots, d_n\}$ and $Q = DL_1 + L_1^T D$. The second inequality in (43) is obtained by using (39). Substituting (43) into (42) gives

$$\begin{aligned} \dot{V}(t) &\leq s_F^T(t)M(q(t))\dot{s}_F(t) + \frac{1}{2}s_F^T(t)\dot{M}(q(t))s_F(t) \\ &\quad + \frac{1}{2}\tilde{\Theta}^T(t)\Lambda^{-1}\dot{\tilde{\Theta}}(t) + \sum_{i \in \mathcal{V}_F}\dot{\chi}_i(t)\chi_i(t) \\ &\quad + 2d_{\max}\|\bar{q}_F(t)\|\|s_F(t)\| - \lambda_{\min}(Q)\|\bar{q}_F(t)\|^2 \\ &\quad + 2\sigma_{\max}\left(DL_1^{-1}L_2\right)\|\bar{q}_F(t)\|\|\dot{q}_L(t)\| \\ &\leq -s_F^T(t)K_F s_F(t) - s_F^T(t)K_F e_F(t) + s_F^T(t)\epsilon_F(t) \\ &\quad - s_F^T(t)M(q_F(t))\ddot{q}_r(t) + \sum_{i \in \mathcal{V}_F}\dot{\chi}_i(t)\chi_i(t) \\ &\quad + 2d_{\max}\|\bar{q}_F(t)\|\|s_F(t)\| - \lambda_{\min}(Q)\|\bar{q}_F(t)\|^2 \\ &\quad + 2\sigma_{\max}\left(DL_1^{-1}L_2\right)\|\bar{q}_F(t)\|\|\dot{q}_L(t)\|. \end{aligned} \tag{44}$$

Using (40), it can be obtained that

$$\begin{aligned} &-s_F^T(t)M(q_F(t))\ddot{q}_r(t) \\ &\leq s_F^T(t)M(q_F(t))(L_1 \otimes I_p)\left[s_F(t) - (L_1 \otimes I_p)\bar{q}_F(t)\right] \\ &\quad + s_F^T(t)M(q_F(t))(L_2 \otimes I_p)\dot{q}_L(t) \\ &\leq k_{\bar{m}}\sigma_{\max}(L_1)\|s_F(t)\|^2 + k_{\bar{m}}\sigma_{\max}^2(L_1)\|s_F(t)\|\|\bar{q}_F(t)\| \\ &\quad + k_{\bar{m}}\sigma_{\max}^2(L_2)\|s_F(t)\|\|\dot{q}_L(t)\|. \end{aligned} \tag{45}$$

Note that

$$\begin{aligned} &-s_F^T(t)K_F e_F(t) + s_F^T(t)\epsilon_F(t) + \sum_{i=1}^{n}\dot{\chi}_i(t)\chi_i(t) \\ &\leq \sum_{i \in \mathcal{V}_F}\{\|s_i(t)\|[\|\epsilon_i(t)\| + k_i\|e_i(t)\| - \beta_i\|s_i(t)\| + \beta_i\|s_i(t)\|] \\ &\qquad - \gamma_i\chi_i^2(t) - \omega_i[\|\epsilon_i(t)\| + k_i\|e_i(t)\| \\ &\qquad\qquad - \beta_i\|s_i(t)\|]\chi_i(t)\} \\ &\leq \sum_{i \in \mathcal{V}_F}\Big[\beta_i\|s_i(t)\|^2 + \|s_i(t)\|\chi_i(t) - \gamma_i\chi_i^2(t) \\ &\qquad + \omega_i\beta_i\|s_i(t)\|\chi_i(t)\Big] \\ &\leq \sum_{i \in \mathcal{V}_F}\Big[\beta_i\|s_i(t)\|^2 + \frac{1 + \omega_i\beta_i}{2}\|s_i(t)\|^2 + \frac{1 + \omega_i\beta_i}{2}\chi_i^2(t) \\ &\qquad - \gamma_i\chi_i^2(t)\Big] \\ &\leq \max_{i \in \mathcal{V}_F}\mu_i\|s_F(t)\|^2 - \sum_{i \in \mathcal{V}_F}\left(\gamma_i - \frac{1 + \omega_i\beta_i}{2}\right)\chi_i^2(t) \end{aligned} \tag{46}$$

where the first inequality is obtained by (34) and the third inequality is obtained by (14). Taking (45) and (46) into (44) gives

$$\begin{aligned} \dot{V}(t) &\leq -s_F^T(t)K_F s_F(t) + \max_{i \in \mathcal{V}_F}\mu_i\|s_F(t)\|^2 \\ &\quad - \sum_{i \in \mathcal{V}_F}\left(\gamma_i - \frac{1 + \omega_i\beta_i}{2}\right)\chi_i^2(t) \end{aligned}$$

$$
\begin{aligned}
&+ k_{\bar{m}}\sigma_{\max}(L_1)\|s_F(t)\|^2 - \lambda_{\min}(Q)\|\bar{q}_F(t)\|^2 \\
&+ \left[k_{\bar{m}}\sigma_{\max}^2(L_1) + 2d_{\max}\right]\|\bar{q}_F(t)\|\|s_F(t)\| \\
&+ 2\sigma_{\max}\left(DL_1^{-1}L_2\right)\|\bar{q}_F(t)\|\|\dot{q}_L(t)\| \\
&+ k_{\bar{m}}\sigma_{\max}^2(L_2)\|s_F(t)\|\|\dot{q}_L(t)\| \\
\le &-\left[k_{\min} - \max_{i\in\mathcal{V}_F}\mu_i - k_{\bar{m}}\sigma_{\max}(L_1)\right]\|s_F(t)\|^2 \\
&- \sum_{i\in\mathcal{V}_F}\left(\gamma_i - \frac{1+\omega_i\beta_i}{2}\right)\chi_i^2(t) \\
&- \lambda_{\min}(Q)\|\bar{q}_F(t)\|^2 + \frac{\lambda_{\min}(Q)}{2}\|\bar{q}_F(t)\|^2 \\
&+ \frac{\left[k_{\bar{m}}\sigma_{\max}^2(L_1) + 2d_{\max}\right]^2}{2\lambda_{\min}(Q)}\|s_F(t)\|^2 \\
&+ \frac{\lambda_{\min}(Q)}{4}\|\bar{q}_F(t)\|^2 + \frac{4\sigma_{\max}^2\left(DL_1^{-1}L_2\right)}{\lambda_{\min}(Q)}\|\dot{q}_L(t)\|^2 \\
&+ \|s_F(t)\|^2 + \frac{k_{\bar{m}}^2\sigma_{\max}^4(L_2)}{4}\|\dot{q}_L(t)\|^2 \\
\le &-k_0\|s_F(t)\|^2 - \frac{\lambda_{\min}(Q)}{4}\|\bar{q}_F(t)\|^2 + \omega_1\|\dot{q}_L(t)\|^2 \\
&- \sum_{i\in\mathcal{V}_F}\left(\gamma_i - \frac{1+\omega_i\beta_i}{2}\right)\chi_i^2(t) \qquad (47)
\end{aligned}
$$

where the fact that

$$
\begin{aligned}
&\left[k_{\bar{m}}\sigma_{\max}^2(L_1) + 2d_{\max}\right]\|\bar{q}_F(t)\|\|s_F(t)\| \\
&\quad\le \frac{\left[k_{\bar{m}}\sigma_{\max}^2(L_1) + 2d_{\max}\right]^2}{2\lambda_{\min}(Q)}\|s_F(t)\|^2 + \frac{\lambda_{\min}(Q)}{2}\|\bar{q}_F(t)\|^2 \\
&2\sigma_{\max}\left(DL_1^{-1}L_2\right)\|\bar{q}_F(t)\|\|\dot{q}_L(t)\| \\
&\quad\le \frac{\lambda_{\min}(Q)}{4}\|\bar{q}_F(t)\|^2 + \frac{4\sigma_{\max}^2\left(DL_1^{-1}L_2\right)}{\lambda_{\min}(Q)}\|\dot{q}_L(t)\|^2 \\
&k_{\bar{m}}\sigma_{\max}^2(L_2)\|s_F(t)\|\|\dot{q}_L(t)\| \\
&\quad\le \|s_F(t)\|^2 + \frac{k_{\bar{m}}^2\sigma_{\max}^4(L_2)}{4}\|\dot{q}_L(t)\|^2
\end{aligned}
$$

is used to obtain the second inequality. The constants k_0 and ω_1 satisfy

$$
\begin{aligned}
k_0 &= k_{\min} - \max_{i\in\mathcal{V}_F}\mu_i - k_{\bar{m}}\sigma_{\max}(L_1) \\
&\quad - \frac{\left[k_{\bar{m}}\sigma_{\max}^2(L_1) + 2d_{\max}\right]^2}{2\lambda_{\min}(Q)} - 1 \\
\omega_1 &= \frac{k_{\bar{m}}^2\sigma_{\max}^4(L_2)}{4} + \frac{4\sigma_{\max}^2\left(DL_1^{-1}L_2\right)}{\lambda_{\min}(Q)}.
\end{aligned}
$$

By integrating both sides of the last inequality of (47), it can be obtained that

$$
\begin{aligned}
V(t) - V(0) \le &-k_0\int_0^\infty \|s_F(t)\|^2 dt + \omega_1\int_0^\infty \dot{q}_L(t) \\
&- \frac{\lambda_{\min}(Q)}{2}\int_0^\infty \|\tilde{q}_F(t)\|^2 \mathrm{dt} \\
&- \sum_{i=1}^n\left(\gamma_i + \omega_i - \frac{1}{2}\right)\int_0^\infty \chi_i(t)\mathrm{dt}. \qquad (48)
\end{aligned}
$$

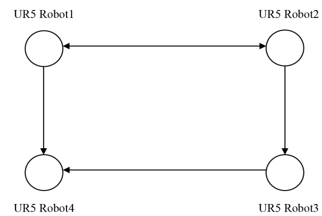

Fig. 1. Interaction graph of UR5 robots in the leaderless consensus case.

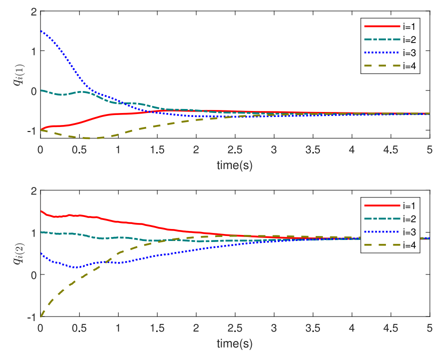

Fig. 2. Angle trajectories of UR5 robots using (5) and (7) in the leaderless consensus case.

From Assumption 3, $\dot{q}_L(t) \in \mathcal{L}_2$. Hence, the last term on the right-hand side of (48) is bounded, which means that $V(t)$ is upper bounded. Then, it can be obtained from (41) that $s_F(t), \bar{q}_F(t) \in \mathcal{L}_\infty$. From (39), it can be concluded that $\bar{q}_F(t) \in \mathcal{L}_2$. Then, using Barbalat's lemma, it can be obtained that $\bar{q}_F(t) \to 0_p$ as $t \to \infty$, which implies that $q_F(t) \to -(L_1^{-1}L_2 \otimes I_p)q_L(t)$. According to Lemma 3, each element of the matrix $-L_1^{-1}L_2$ is non-negative and the all row sums of $-L_1^{-1}L_2$ equal to one, which implies that $-(L_1^{-1}L_2 \otimes I_p)q_L(t)$ is within the convex hull spanned by the dynamic leaders. Therefore, $\mathrm{dis}(q_i(t), Co(q_L(t))) \to 0$ as $t \to \infty$.

The Zeno behavior analysis is the same as the one in the proof of Theorem 1. ■

V. Simulation Results

In this section, simulations regarding the leaderless consensus and the containment based on UR5 robots of V-rep are presented to show the effectiveness of the proposed control algorithm. The consensus controller runs in the joint space while the containment controller runs in the end-effector space.

In the leaderless consensus case, consider the MASs with four UR5 robot of V-rep described by the form of Euler–Lagrange dynamics (1). For each robot, the controller's parameters of (5) are $k_i = 5.0$ and $\Lambda_i = 0.5$. The parameters of the triggering condition (7) are $\gamma_i = 1.0$, $\beta_i = 1.8$, $\omega_i = 0.1$, and $\chi_i(0) = 5$. The initial angles of four robots are $[-1.0, 1.5]^T$ rad, $[0.0, 1.0]^T$ rad, $[1.5, 0.5]^T$ rad, and $[-1.0, -1.0]^T$ rad, respectively. The initial angle velocities of four robots are, respectively, $[1.25, 0.3]^T$ rad/s, $[-0.25, 0.75]^T$ rad/s, $[-1.5, 0.5]^T$ rad/s, and $[0.2, 0.3]^T$ rad/s. The interaction graph is given in Fig. 1, which is strongly connected and contains a directed spanning tree.

Fig. 2 shows the angle trajectories of the robots. Fig. 3 shows the event occurrence instants of the robots. Snapshots

TABLE I
CONSENSUS CASE: COMPARISON RESULTS OF THE AVERAGE LENGTHS OF EVENT INTERVALS BASED ON 20 GROUPS OF INITIAL ANGLE POSITIONS UNDER THE METHOD PROPOSED IN THIS ARTICLE AND THE METHODS PROPOSED IN [19] AND [20]

ave_len_i	1	2	3	4	5	6	7	8	9	10
The proposed method	0.0219	0.0233	0.0185	0.0201	0.0205	0.0221	0.0209	0.0262	0.0229	0.0234
Methods in [19], [20]	0.0212	0.0234	0.0177	0.0183	0.0193	0.0208	0.0193	0.0255	0.0219	0.0227
	11	12	13	14	15	16	17	18	19	20
The proposed method	0.0274	0.0232	0.0239	0.0255	0.0170	0.0250	0.0228	0.0307	0.0199	0.0200
Methods in [19], [20]	0.0272	0.0216	0.0237	0.0237	0.0168	0.0237	0.0223	0.0277	0.0190	0.0203

TABLE II
CONSENSUS CASE: COMPARISON RESULTS OF THE MINIMUM LENGTHS OF EVENT INTERVALS BASED ON 20 GROUPS OF INITIAL ANGLE POSITIONS UNDER THE METHOD PROPOSED IN THIS ARTICLE AND THE METHODS PROPOSED IN [19] AND [20]

min_time_i	1	2	3	4	5	6	7	8	9	10
The proposed method	0.006	0.005	0.005	0.003	0.005	0.006	0.008	0.004	0.005	0.006
Methods in [19], [20]	0.006	0.004	0.002	0.003	0.005	0.005	0.008	0.004	0.004	0.006
	11	12	13	14	15	16	17	18	19	20
The proposed method	0.004	0.004	0.005	0.006	0.002	0.006	0.004	0.006	0.005	0.004
Methods in [19], [20]	0.002	0.002	0.005	0.005	0.005	0.005	0.0001	0.006	0.006	0.005

TABLE III
CONSENSUS CASE: COMPARISON RESULTS OF THE STANDARD DEVIATIONS OF ANGLE POSITIONS AT THE STEADY STATE BASED ON 20 GROUPS OF INITIAL ANGLE POSITIONS UNDER THE METHOD PROPOSED IN THIS ARTICLE AND THE METHODS PROPOSED IN [19] AND [20]

sd_pos_i	1	2	3	4	5	6	7	8	9	10
The proposed method	0.0040	0.0094	0.0092	0.0052	0.0056	0.0036	0.0009	0.0046	0.0023	0.0128
Methods in [19], [20]	0.0039	0.0091	0.0092	0.0053	0.0058	0.0037	0.0007	0.0046	0.0025	0.0119
	11	12	13	14	15	16	17	18	19	20
The proposed method	0.0259	0.0050	0.0054	0.0037	0.0118	0.0030	0.0086	0.0016	0.0037	0.0148
Methods in [19], [20]	0.0266	0.0050	0.0057	0.0035	0.0136	0.0031	0.0088	0.0014	0.0033	0.0147

TABLE IV
CONSENSUS CASE: COMPARISON RESULTS OF THE CONVERGENCE TIME BASED ON 20 GROUPS OF INITIAL ANGLE POSITIONS UNDER THE METHOD PROPOSED IN THIS ARTICLE AND THE METHODS PROPOSED IN [19] AND [20]

conv_time_i	1	2	3	4	5	6	7	8	9	10
The proposed method	4.0570	4.8410	4.9250	4.5180	4.7130	4.1980	2.7390	4.4590	4.0890	4.7490
Methods in [19], [20]	4.0560	4.8450	4.9400	4.4470	4.7600	4.1920	1.9520	4.4810	4.1530	4.9360
	11	12	13	14	15	16	17	18	19	20
The proposed method	4.0530	4.9130	4.6710	4.4620	3.5210	4.4530	4.6520	4.0720	4.4830	1.8620
Methods in [19], [20]	4.4630	4.8710	4.5580	4.4640	3.5630	4.4390	4.0800	4.0380	4.4990	1.7950

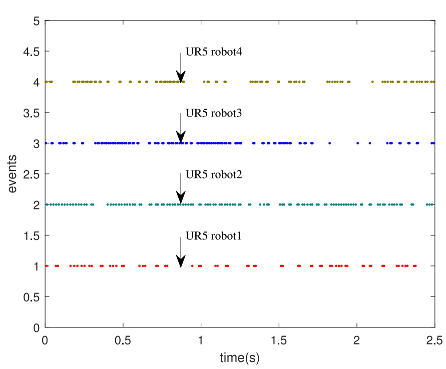

Fig. 3. Event occurrence instants during the time interval [0, 2.5] using (5) and (7) in the leaderless consensus case.

during the leaderless consensus tasks are shown in Fig. 4. The numerical solution of the minimum length of event intervals l in (30) is 0.0045 s. From simulation results, the designed controller (5) with the event-triggered condition (7) achieve the consensus of MASs without the Zeno behavior.

In the comparison simulation, this article employs the event-triggered method proposed in [19] and [20] to achieve the leaderless consensus under the same simulation environment. The event-triggered condition (31) is employed to determine the event-triggered time sequence $\{t_k^i\}_{k\in\mathbb{N}}$, and the distributed controller takes the form of (5). The controller parameters and the initial angle velocities are also set as the same as the ones chosen before. The parameters of the triggering condition are $\rho_i = 1.1$, $\eta_i = 0.72$, and $\alpha_i = 5$. The 20 groups of initial angle positions are generated by the Monte Carlo sampling method. The average lengths of event intervals under all initial values are calculated to evaluate the communication resource usage. To verify that the proposed method has a better performance in avoiding the Zeno behavior, the minimum lengths of event intervals are employed to show whether the Zeno behavior occurs. The standard derivations of angle positions at the steady state and the average time of achieving the consensus are employed to evaluate the steady-state control performance. The comparison results are listed in Tables I–IV.

In Table I, the abbreviation ave_len_i denotes the average length of event intervals under the ith group. Table I indicates that the proposed event-triggered method provides an increase $(4.46 \pm 3.36\%)$ of the average length of event

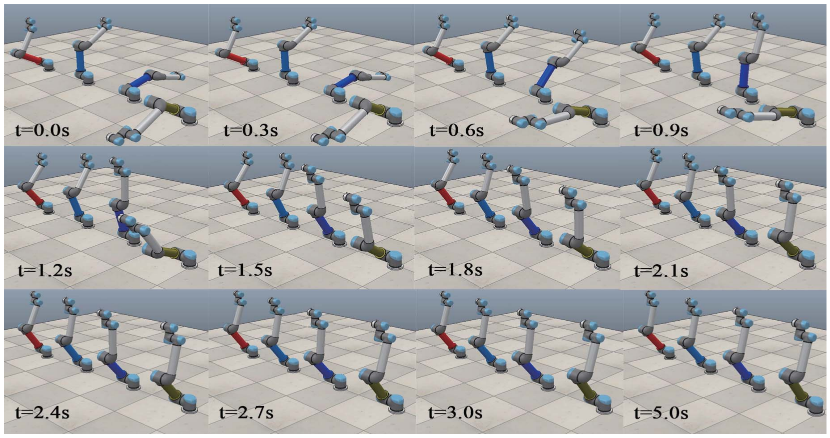

Fig. 4. Snapshots of four UR5 robots using (5) and (7) in the consensus case.

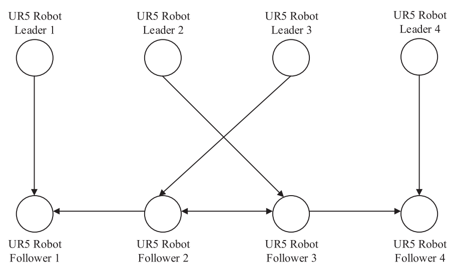

Fig. 5. Interaction graph of UR5 robots in the containment case.

intervals compared to the one of the existing methods, which leads to a lower usage of the communication resource. The minimum lengths of event intervals denoted by min_len_i are shown in Table II. Note that the minimum length of event intervals of the 17th group under the existing methods equals to the MATLAB simulations' fixed-time step size of 0.0001 s, and the one under the proposed method is 0.004 s. This result indicates that the proposed method avoids the Zeno behavior while the Zeno behavior may occur under the existing methods. In Table III, the abbreviation sd_pos_i denotes the standard derivation of angle positions at the steady state under the ith group. Table III indicates that the average standard deviations at the steady state under the two methods are 0.00706 and 0.00712 rad, respectively. In Table IV, the abbreviation conv_time_i denotes the average time of achieving the consensus under the ith group. Table IV shows that the average time of achieving the consensus under the proposed method is 4.2215 s while it under the existing method is 4.1766 s. Hence, it can be concluded that the proposed algorithm reduces the communication resource usage and has a better performance in avoiding the Zeno behavior without affecting the time of achieving the consensus and the steady-state control performance.

In the containment case, the MAS consists of four virtual leaders and four UR5 robots as the followers. The interaction graph of the MAS is given in Fig. 5 which satisfies Assumption 2. The initial positions of all followers' end-effectors are $[-0.5, 0.7]^T$ m, $[-0.7, 0.5]^T$ m, $[0.7, 0.5]^T$ m, and $[0.6, 0.3]^T$ m, respectively. The followers' initial velocities are $[0.75, -0.5]^T$ m/s, $[-1.25, 1.0]^T$ m/s, $[-0.5, 0.75]^T$ m/s,

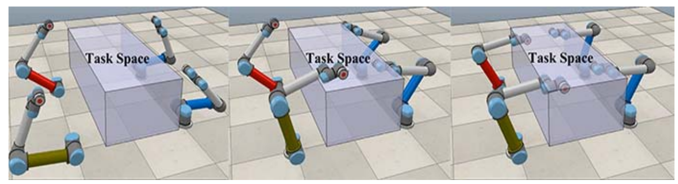

Fig. 6. Snapshots of four UR5 Robots using (33) and (35) in the containment case under stationary leaders.

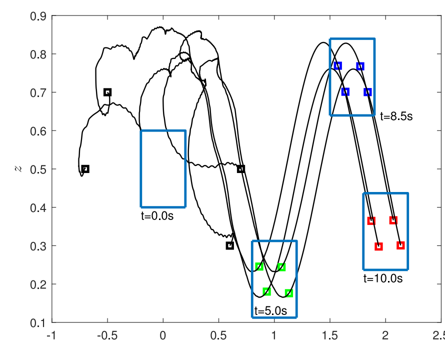

Fig. 7. $y - z$ plane movement trajectories using (33) and (35) in the containment case under dynamic leaders.

and $[1.0, 0.5]^T$ m/s, respectively. The controller parameters of (33) are $k_i = 4.8$, $b_1 = 0.4$, $b_2 = 0.4$, and $\Lambda_i = 0.5$. The parameters of the triggering condition (35) are $\gamma_i = 0.7$, $\beta_i = 2.0$, $\omega_i = 0.1$, and $\chi_i(0) = 4$.

First, a simulation is given to verify the proposed algorithm under stationary leaders. The stationary leaders' initial positions are $[-0.2, 0.6]^T$ m, $[0.2, 0.6]^T$ m, $[-0.2, 0.4]^T$ m, and $[0.2, 0.4]^T$ m, respectively. In Fig. 6, the snapshots of UR5 robots are shown. It is obvious that the end effectors of four UR5 robots move into the task space spanned by leaders.

Second, a simulation is given to verify the proposed algorithm under dynamic leaders. The dynamic leaders' initial positions are set as the same as the ones of stationary leaders. The dynamic leaders' velocities are $[0.2, 0.3\cos(t)]^T$ m/s. In Fig. 7, the motion trajectories of UR5 robots are shown. Fig. 8 displays the event occurrence instants of UR5 robots. The numerical solution of the minimum length of event intervals l in (30) is 0.0055 s. The results indicate that the robots achieve the

TABLE V
CONTAINMENT CASE OF DYNAMIC LEADERS: COMPARISON RESULTS OF THE AVERAGE LENGTHS OF EVENT INTERVALS BASED ON 20 GROUPS OF INITIAL ANGLE POSITIONS UNDER THE METHOD PROPOSED IN THIS ARTICLE AND THE METHODS PROPOSED IN [19] AND [20]

ave_len_i	1	2	3	4	5	6	7	8	9	10
The proposed method	0.0148	0.0147	0.0173	0.0164	0.0159	0.0181	0.0148	0.0166	0.0161	0.0169
Methods in [19], [20]	0.0137	0.0143	0.0159	0.0167	0.0151	0.0176	0.0138	0.0158	0.0164	0.0159
	11	12	13	14	15	16	17	18	19	20
The proposed method	0.0149	0.0157	0.0170	0.0142	0.0150	0.0159	0.0152	0.0177	0.0147	0.0144
Methods in [19], [20]	0.0146	0.0155	0.0175	0.0141	0.0142	0.0152	0.0148	0.0163	0.0144	0.0140

TABLE VI
CONTAINMENT CASE OF DYNAMIC LEADERS: COMPARISON RESULTS OF THE TIME OF ACHIEVING THE CONTAINMENT BASED ON 20 GROUPS OF INITIAL ANGLE POSITIONS UNDER THE METHOD PROPOSED IN THIS ARTICLE AND THE METHODS PROPOSED IN [19] AND [20]

cont_time_i	1	2	3	4	5	6	7	8	9	10
The proposed method	5.4130	4.3030	2.9050	4.8900	4.5280	4.8510	4.7250	4.1390	4.8780	4.8900
Methods in [19], [20]	5.4580	4.2410	2.8570	5.0340	4.4190	5.1730	4.7720	4.2790	5.2560	5.0630
	11	12	13	14	15	16	17	18	19	20
The proposed method	5.4430	5.3660	5.2310	4.3140	3.5460	4.7720	5.4420	5.4630	4.4410	5.1960
Methods in [19], [20]	5.4170	5.1680	5.2310	4.3020	3.3500	4.7660	5.3940	5.4760	4.5400	5.1740

TABLE VII
CONTAINMENT CASE OF ONE DYNAMIC LEADER: COMPARISON RESULTS OF THE RMSES AT THE STEADY STATE BASED ON 20 GROUPS OF INITIAL ANGLE POSITIONS UNDER THE METHOD PROPOSED IN THIS ARTICLE AND THE METHODS PROPOSED IN [19] AND [20]

cont_time_i	1	2	3	4	5	6	7	8	9	10
The proposed method	0.0109	0.0118	0.0108	0.0109	0.0118	0.0114	0.0104	0.0113	0.0115	0.0101
Methods in [19], [20]	0.0112	0.0112	0.0101	0.0106	0.0122	0.0111	0.0101	0.0110	0.0110	0.0103
	11	12	13	14	15	16	17	18	19	20
The proposed method	0.0103	0.0115	0.0101	0.0129	0.0106	0.0094	0.0130	0.0107	0.0099	0.0112
Methods in [19], [20]	0.0107	0.0106	0.0092	0.0132	0.0107	0.0089	0.0135	0.0105	0.0101	0.0109

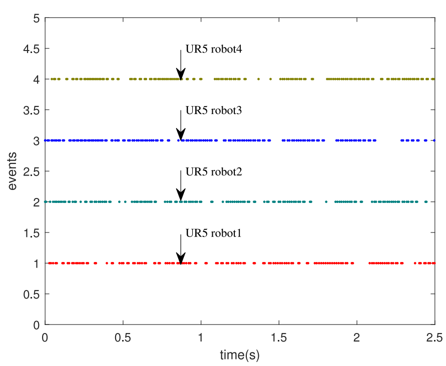

Fig. 8. Event occurrence instants during the time interval [0, 2.5] using (33) and (35) in the containment case under dynamic leaders.

containment, and the Zeno behavior is avoided. To verify the advantages of the proposed algorithm in the containment case, this article compares the length of event intervals and the time of achieving the containment between the proposed method and the methods proposed in [19] and [20]. The simulation results are shown in Tables V and VI. Table V shows that 17/20 simulations under the proposed method have a longer length of event intervals compared to the one of the existing methods proposed in [22] and [23]. Table VI indicates that the average time of achieving the containment under the proposed method is 4.7380 s while the corresponding time under the existing methods is 4.7685 s. In conclusion, the proposed method reduces the communication resource usage in the containment case, and the time of achieving the containment is not affected.

The above comparison only considers the time of achieving the containment, however, whether the steady-state control performance is affected is not taken into account. Note that the standard deviations at the steady state in the consensus case cannot be used as an index to evaluate the containment problem. Hence, this article provides a simulation with only one dynamic leader and four followers, and the root mean-squared error (RMSE) at the steady state between the leader and the followers are employed to indicate the steady-state control performance in the containment case. The weighted adjacency matrix associated with the communication graph of the MASs is

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

The comparison results are shown in Table VII. The average RMSEs at the steady state under two methods are 0.0110 and 0.0109 m, respectively, which means that the proposed method has similar RMSEs at the steady state as the one of the existing methods. Hence, the proposed algorithm does not affect the steady-state control performance either.

VI. CONCLUSION

This article studied the distributed event-triggered control of networked Euler–Lagrange systems with parameter uncertainties. The event-triggered time sequence was determined by a novel dynamic event-triggered method, and the Zeno behavior was excluded by the theoretical analysis. The proposed method can lead to a longer average length of event intervals. Hence, the usage of the communication resource can be further reduced. By the Lyapunov stability analysis, sufficient conditions were obtained to guarantee the performance of the MASs. Finally, simulations verified that the proposed controllers achieve the control objectives: the leaderless consensus and the leader-following containment. The comparison results showed that the proposed method prolongs the lengths of event intervals while the time of achieving the consensus/containment and the steady-state control performance are not affected. In future work, the authors are going to further consider the self-triggering control problem for Euler–Lagrange MASs to avoid the continuous communication among the agents.

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