Fixed-Time Lyapunov Criteria and State-Feedback Controller Design for Stochastic Nonlinear Systems

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***Abstract*—This paper investigates the fixed-time stability theorem and state-feedback controller design for stochastic nonlinear systems. We propose an improved fixed-time Lyapunov theorem with a more rigorous and reasonable proof procedure. In particular, an important corollary is obtained, which can give a less conservative upper-bound estimate of the settling time. Based on the backstepping technique and the addition of a power integrator method, a state-feedback controller is skillfully designed for a class of stochastic nonlinear systems. It is proved that the proposed controller can render the closed-loop system fixed-time stable in probability with the help of the proposed fixed-time stability criteria. Finally, the effectiveness of the proposed controller is demonstrated by simulation examples and comparisons.**

***Index Terms*—Fixed-time stability, Lyapunov theorem, state-feedback control, stochastic nonlinear systems.**

I. Introduction

As is well-known, many practical systems are nonlinear; e.g., robot systems, inverted pendulums, tunnel diode circuits, etc. The controller design of nonlinear systems has attracted increasing attention in the past few decades and many useful tools have emerged [1]−[10]; for example, the backstepping technique [1], the addition of a power integrator method [2], the Lyapunov function criteria method [3], the prescribed performance control (PPC) method [4]−[6], and so on. Since stochastic noises extensively occur in real engineering field, the investigation of stochastic nonlinear systems is necessary and significant. Due to good transient response performance, the backstepping technique has been successfully extended to stochastic nonlinear systems in [11], which together with basic stochastic stability theories [12]−[14] allows for remarkable developments on the controller design and analysis of stochastic nonlinear systems; see, [15]−[25] and the references therein.

Considering the faster convergence speed, higher accuracies, and better disturbance rejection ability of finite-time stability, it is rather meaningful to ensure stochastic nonlinear systems converge in finite time. To this end, [26]−[29] made the first attempt to establish the definitions of finite-time stability for stochastic nonlinear systems and obtained the related Lyapunov theorems. Then, based on the finite-time stability theory, the finite-time controller design has obtained many results in [30]−[35] for stochastic nonlinear systems in various structures. In [36], a general Lyapunov theorem of stochastic finite-time stability and some important corollaries with more general conditions were further presented. Recently, a definition of semi-global finite-time stability in probability was presented in [37] and a related stochastic Lyapunov theorem was established to state-feedback stabilize stochastic nonlinear systems with full-state constraints.

However, the bound estimate of the settling time in finite-time control is dependent on system initial states. This impedes its practical applications since the estimate of the settling time and desirable characteristics cannot be derived without knowledge of initial conditions. As an evolution of the finite-time control, the fixed-time control whose settling time estimate is independent of a system's initial states has gradually attracted scholars' attention. In [38], the fixed-time control technique was used to handle stability issues of linear systems. Since then, fixed-time control was frequently considered for deterministic systems in [39]−[42].

For a stochastic nonlinear system, the fixed-time prescribed performance on the output tracking error was investigated by developing a novel performance function and using traditional Lyapunov bounded in probability stability [43]. Motivated by finite-time stability in probability using the Lyapunov criteria, the authors established the concept of fixed-time stability and used the Lyapunov theorem for stochastic nonlinear systems in [44]. Then, [45] used it to study the global fixed-time stabilization of switched stochastic nonlinear systems and [46] considered stochastic pure-feedback nonlinear systems. In [47], the fixed-time controller was designed for stochastic interconnected nonlinear large-scale systems. However, the obtained upper-bound of the settling time is conservative in [44]−[47]. Motivated by the above discussions, two natural issues arise: Can we further improve the proof of the fixed-time Lyapunov theorem in [44]? How do we use it to stabilize stochastic nonlinear systems with a less conservative settling time?

This paper aims to solve the above two issues. The contributions are listed as follows:

1) An improved fixed-time Lyapunov theorem with more reasonable and rigorous proof is given for stochastic nonlinear systems.

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2) A corollary with a less conservative bound estimate of the settling time is obtained.

3) Based on the backstepping technique and the addition of a power integrator method, a state-feedback controller is skillfully designed for a class of stochastic nonlinear systems. By using the proposed fixed-time stability criteria, it is proved that the proposed controller guarantees the closed-loop system to be fixed-time stable in probability.

The remainder of this paper is organized as follows. Section II gives the preliminaries. A fixed-time stability theorem is given in Section III. In Section IV, the state-feedback controller is designed and analyzed. Section V shows an example, which is followed by Section VI to end this paper.

Notations: Throughout the whole paper, $\mathbb{R}^+$ is the set of the non-negative real numbers; $\mathbb{R}^i$ stands for i-dimensional Euclidean space; X^T denotes the transpose of a given vector or matrix X and $\|X\|$ denotes its Euclidean norm with $\mathrm{Tr}\{X\}$ being its trace when X is square; C^i represents the family of all the functions with continuous ith partial derivatives; a class $\mathcal{K}$ function is continuous, strictly increasing and vanishes at zero; $\mathcal{K}_\infty$ is the set of all functions $\gamma(x)$ which are of class $\mathcal{K}$ and radially unbounded ($\gamma(x)\to\infty$ as $x\to\infty$).

II. Mathematical Preliminaries

Consider stochastic nonlinear system

$$dx = f(x)dt + g(x)d\omega, \quad \forall x(0) = x_0 \in \mathbb{R}^n \tag{1}$$

where $x\in\mathbb{R}^n$ is the system state; ω is an r-dimensional standard Wiener process defined on a complete probability space $\{\Omega,\mathcal{F},P\}$, where Ω is a sample space, $\mathcal{F}$ is a σ-field and P is the probability measure; $f(x)$: $\mathbb{R}^n\to\mathbb{R}^n$ and g: $\mathbb{R}^n\to\mathbb{R}^{n\times r}$ are Borel measurable continuous functions with $f(0)=0$ and $g(0)=0$. This means that system (1) has a trivial zero solution.

Definition 1 [12]: For a C^2 function $V(x)\in\mathbb{R}^n$, denote $\mathcal{L}V$ as the differential operator of V. Then $\mathcal{L}V$ with respect to system (1) is given by $\mathcal{L}V(x)=\frac{\partial V(x)}{\partial x}f(x)+\frac{1}{2}\mathrm{Tr}\{g^T(x)\frac{\partial^2 V(x)}{\partial x^2}g(x)\}$, where the term $\frac{1}{2}\mathrm{Tr}\{g^T\frac{\partial^2 V}{\partial x^2}g\}$ is called the Hessian term.

Definition 2 [30], [36]: The trivial solution of system (1) is said to be finite-time stable in probability, if system (1) admits a solution $x(t;x_0)$ for any initial value $x_0\in\mathbb{R}^n$ and satisfies

i) Finite-time attractiveness in probability: For every initial value $x_0\in\mathbb{R}^n\setminus\{0\}$, the first hitting time $\tau_{x_0}=\inf\{t\ge 0; x(t;x_0)=0\}$, called the stochastic settling time, is almost surely finite, i.e., $P\{\tau_{x_0}<\infty\}=1$;

ii) Stability in probability: For every pair of $\varepsilon\in(0,1)$ and $r>0$, there exists a $\delta(\varepsilon,r)>0$ such that $P\{|x(t;x_0)|<r$, for all $t\ge 0\}\ge 1-\varepsilon$, whenever $|x_0|<\delta$;

iii) $x(t+\tau_{x_0};x_0)=0$, a.s., $\forall t\ge 0$.

Definition 3 [44]: The trivial solution of system (1) is called fixed-time stability in probability, if, i) the trivial solution is finite-time stable in probability; ii) $E(\tau_{x_0})\le T_0$, for $\forall x_0\in\mathbb{R}^n\setminus\{0\}$, where T_0 is a positive constant and independent of the initial values.

Lemma 1 [36]: Assume that there exists a nonnegative, radially unbounded Lyapunov function $V(x)\in C^2$. If $\mathcal{L}V(x)\le 0$, then system (1) has a global solution for any initial data.

Lemma 2 [15]: For system (1), suppose there exists a positive definite, C^2 Lyapunov function $V:\mathbb{R}^n\to\mathbb{R}^+$ and two $\mathcal{K}_\infty$-class functions π_1 and π_2 such that for all $x\in\mathbb{R}^n$ and $t\ge 0$,

$$\pi_1(\|x\|)\le V(x)\le\pi_2(\|x\|), \quad \mathcal{L}V(x)\le -W(x) \tag{2}$$

where $W(x)$ is a nonneagtive continuous function. Then, the trivial solution of system (1) is globally stable in probability and $P\{\lim_{t\to\infty}W(x(t))=0\}=1$, $\forall x_0\in\mathbb{R}^n$.

We present some lemmas to end this section.

Lemma 3 [2]: For any real variables x,y, positive numbers m, n, b and nonnegative continuous function $a(\cdot)$, it holds $a(\cdot)x^m y^n\le b|x|^{m+n}+\frac{n}{m+n}\left(\frac{m+n}{m}\right)^{-\frac{m}{n}}a^{\frac{m+n}{n}}(\cdot)b^{-\frac{m}{n}}|y|^{m+n}$.

Lemma 4 [8]: For $x,y\in\mathbb{R}$, one has $|x+y|^p\le 2^{p-1}|x^p+y^p|$, $(|x|+|y|)^{\frac{1}{p}}\le|x|^{\frac{1}{p}}+|y|^{\frac{1}{p}}\le 2^{\frac{p-1}{p}}(|x|+|y|)^{\frac{1}{p}}$, $|x-y|^p\le 2^{p-1}|x^p-y^p|$, $|x^{\frac{1}{p}}-y^{\frac{1}{p}}|\le 2^{\frac{p-1}{p}}|x-y|^{\frac{1}{p}}$ for $p\ge 1$ and $(x_1+\cdots+x_n)^p\le\max\{n^{p-1},1\}(x_1^p+\cdots+x_n^p)$ for any $p>0$ and $x_1,\ldots,x_n\in\mathbb{R}$.

Lemma 5 [26]: Assume that $\eta(\cdot):\mathbb{R}\to\mathbb{R}$ and $\phi(\cdot):\mathbb{R}^n\to\mathbb{R}$ are two smooth functions and $x(t):=x(t;x_0)$ is the solution of system (1), then, it holds

$$d[\eta(\phi(x))]=\frac{d\eta}{d\phi}d(\phi(x))+\frac{1}{2}\frac{d^2\eta}{d\phi^2}\mathrm{Tr}\Big\{(\frac{\partial\phi}{\partial x}g)^T(\frac{\partial\phi}{\partial x}g)\Big\}dt. \tag{3}$$

III. Lyapunov Theorem of Fixed-Time Stability

A Lyapunov theorem on fixed-time stability with a rigorous proof is given for stochastic nonlinear systems.

Theorem 1: Consider system (1). Suppose there exists a continuous differentiable function $\gamma>0$, $\int_0^\epsilon\frac{1}{\gamma(s)}ds\le M$ for any $0<\epsilon<+\infty$ and $\gamma'(s)\ge 0$ for any $s>0$, and there also exists a positive definite, C^2 and radially unbounded Lyapunov function $V:\mathbb{R}^n\to\mathbb{R}^+$ such that

$$\mathcal{L}V(x)\le -\gamma(V(x)) \tag{4}$$

where M is a positive constant, then, the trivial solution of system (1) is fixed-time stable in probability and the stochastic settling time satisfies $E(\tau_{x_0})\le M$, for $\forall x_0\in\mathbb{R}^n\setminus\{0\}$.

Proof: Firstly, from (4), one gets $\mathcal{L}V(x)\le 0$, which together with Lemma 1 shows that for each $x_0\in\mathbb{R}^n$, there exists a global continuous solution $x(t;x_0)$ to system (1). Since $V(x)$ is positive definite and radially unbounded, according to Lemma 4.3 of [3], one can find two class $\mathcal{K}_\infty$ functions π_1 and π_2 such that $\pi_1(\|x\|)\le V(x)\le\pi_2(\|x\|)$. By defining $W(x)=\gamma(V(x))$ in Lemma 2, one can get that the trivial solution of system (1) is globally stable in probability.

Now, we aim to prove the trivial solution is finite-time attractive in probability.

When $x(0)=x_0=0$, it follows from $\mathcal{L}V(x)\le 0$ and $V(x)\ge 0$ that $V(x(t;x_0))$ is a nonnegative continuous supermartingale with filtration $\{\mathcal{F}_t\}_{t\ge 0}$. This together with Lemma 3 in [36] and $\pi_1(\|x\|)\le V(x)$ implies $x(t;x_0)=0$ a.s., for all $t\ge 0$.

When $x_0\in\mathbb{R}^n\setminus\{0\}$, there exists $k\in\{2,3,4,\ldots\}$ such that $x_0\in(\frac{1}{k},k)$. Define the stopping time sequence as

$$\tau_k=\inf\Big\{t\ge 0:\|x(t;x_0)\|\notin(\frac{1}{k},k)\Big\} \tag{5}$$

$$\tau_{1k}=\inf\Big\{t\ge 0:\|x(t;x_0)\|\in[0,\frac{1}{k}]\Big\}. \tag{6}$$

It is clear that when the solution is global, one has $\tau_\infty = \tau_{1\infty}$ a.s., with $\tau_\infty = \lim_{k\to\infty}\tau_k$ and $\tau_{1\infty} = \lim_{k\to\infty}\tau_{1k}$. Define a function

$$\eta(V(x)) = \int_0^{V(x)} \frac{1}{\gamma(s)}ds, \quad V(x) \in [0, +\infty). \tag{7}$$

According to Lemma 5, one has

$$d[\eta(V(x))] = \frac{d\eta}{dV}d(V(x)) + \frac{d^2\eta}{2dV^2}\mathrm{Tr}\Big\{(\frac{\partial\phi}{\partial x}g)^T(\frac{\partial\phi}{\partial x}g)\Big\}dt \tag{8}$$

for $x \in \mathbb{R}^n$. Moreover, from system (1) and Itô's formula, it holds

$$dV(x) = \mathcal{L}V(x)dt + \frac{\partial V}{\partial x}g(x)d\omega \tag{9}$$

which together with $\gamma'(s) \ge 0$, (4) and (8) shows

$$\begin{aligned} d[\eta(V(x))] &= \frac{d\eta}{dV}\mathcal{L}V(x)dt + \frac{d\eta}{dV}\frac{\partial V}{\partial x}g(x)d\omega \\ &\quad + \frac{d^2\eta}{2dV^2}\mathrm{Tr}\Big\{(\frac{\partial\phi}{\partial x}g)^T(\frac{\partial\phi}{\partial x}g)\Big\}dt \\ &= \frac{\mathcal{L}V(x)}{\gamma(V(x))}dt + \frac{1}{\gamma(V(x))}\frac{\partial V}{\partial x}g(x)d\omega \\ &\quad - \frac{\gamma'(V(x))}{2\gamma^2(V(x))}\mathrm{Tr}\Big\{(\frac{\partial\phi}{\partial x}g)^T(\frac{\partial\phi}{\partial x}g)\Big\}dt \\ &\le \frac{\mathcal{L}V(x)}{\gamma(V(x))}dt + \frac{1}{\gamma(V(x))}\frac{\partial V}{\partial x}g(x)d\omega \\ &\le -dt + \frac{1}{\gamma(V(x))}\frac{\partial V}{\partial x}g(x)d\omega. \end{aligned} \tag{10}$$

For any k and $\|x\| \in (\frac{1}{k}, k)$, integrating (10) on $[0, t\wedge\tau_k]$ and taking expectations on both sides, one obtains

$$\begin{aligned} &E[\eta(V(x(t\wedge\tau_k)))] - E[\eta(V(x_0))] \le -E(t\wedge\tau_k) \\ &\quad + E\int_0^{t\wedge\tau_k}\frac{1}{\gamma(V(x(s;x_0)))}\frac{\partial V}{\partial x}g(x(s;x_0))d\omega(s). \end{aligned} \tag{11}$$

Noting that $\frac{1}{\gamma(V(x))}\frac{\partial V}{\partial x}g(x)$ is bounded in the domain $\frac{1}{k} < \|x\| < k$, we have

$$E\int_0^t \frac{I\{s\le\tau_k\}}{\gamma^2(V(x(s;x_0)))}\left|\frac{\partial V}{\partial x}g(x(s;x_0))\right|^2 ds < \infty$$

where $I\{\cdot\}$ denotes the indicator function. This together with

$$\begin{aligned} &E\int_0^{t\wedge\tau_k}\frac{1}{\gamma(V(x(s;x_0)))}\frac{\partial V}{\partial x}g(x(s;x_0))d\omega(s) \\ &\quad = E\int_0^t \frac{I\{s\le\tau_k\}}{\gamma(V(x(s;x_0)))}\frac{\partial V}{\partial x}g(x(s;x_0))d\omega(s) \end{aligned}$$

implies

$$E\int_0^{t\wedge\tau_k}\frac{1}{\gamma(V(x(s;x_0)))}\frac{\partial V}{\partial x}g(x(s;x_0))d\omega(s) = 0 \tag{12}$$

from Theorem 3.2.1 of [13].

Considering (11) and (12), one has

$$E[\eta(V(x(t\wedge\tau_k)))] \le E[\eta(V(x_0))] - E(t\wedge\tau_k). \tag{13}$$

Since $\eta(V(x)) \ge 0$, one gets from (13) that

$$E(t\wedge\tau_k) \le E[\eta(V(x_0))] = \eta(V(x_0)).$$

Letting $t, k \to \infty$, by Fatou Lemma and $\tau_\infty = \tau_{1\infty}$ a.s., one has $E(\tau_\infty) = E(\tau_{1\infty}) \le \eta(V(x_0))$. Considering the definition $\tau_{x_0} = \inf\{t \ge 0; x(t;x_0) = 0\} = \lim_{k\to\infty}\tau_{1k} = \tau_{1\infty}$ a.s., it is obvious that

$$E(\tau_{x_0}) = E(\tau_{1\infty}) \le \eta(V(x_0)) = \int_0^{V(x_0)}\frac{1}{\gamma(s)}ds \le M \tag{14}$$

which shows $P\{\tau_{x_0} < \infty\} = 1$; i.e., the solution is finite-time attractive in probability.

Recalling $V(x(t;x_0))$ is a nonnegative continuous supermartingale with $V_{\tau_{x_0}} = V(x(\tau_{x_0};x_0)) = 0$ and Lemma 3 in [36], it is easy to get $V_{t+\tau_{x_0}} = 0$ a.s., $\forall t \ge 0$. On the other hand, since $0 \le \pi_1(\|x\|) \le V(x)$, it holds

$$x(t+\tau_{x_0};x_0) = 0, \text{ a.s.}, \quad \forall t \ge 0$$

which means Condition iii) in Definition 2 holds.

Thus, from Definition 2, one obtains that the trivial solution of system (1) is finite-time stable in probability with $E(\tau_{x_0}) \le M$, $\forall x_0 \in \mathbb{R}^n$. Since M is a positive constant independent of initial values, one gets from Definition 4 that the trivial solution is fixed-time stable in probability. ■

Remark 1: For the fixed-time Lyapunov theorem of stochastic nonlinear systems, Theorem 1 in this paper improves the unreasonable parts of the proof procedure given by [44]. Firstly, only when $\int_0^{T(x_0)}\frac{\partial V}{\partial x}\frac{g(x)}{\gamma(V(x))}d\omega$ is a martingale, $E[\int_0^{T(x_0)}\frac{\partial V}{\partial x}\frac{g(x)}{\gamma(V(x))}d\omega] = 0$. But in fact, $\int_0^{T(x_0)}\frac{\partial V}{\partial x}\frac{g(x)}{\gamma(V(x))}d\omega$ is only a local martingale. Hence, without proving $\int_0^{T(x_0)}\frac{\partial V}{\partial x}\frac{g(x)}{\gamma(V(x))}d\omega$ a martingale, it is not correct and reasonable to use $E[\int_0^{T(x_0)}\frac{\partial V}{\partial x}\frac{g(x)}{\gamma(V(x))}d\omega] = 0$ to prove $E[F(V(x(T(x_0))))] = 0$. Secondly, the authors in [44] obtained the proof by integrating the related equation from 0 to $T(x_0)$, which leads to the conclusion cannot hold generally; see [27] for a counter example, where $T(x_0)$ is the settling time used in [44].

We give a useful corollary of Theorem 1 as follows.

Corollary 1: For system (1), if there exists a positive definite, C^2 and radially unbounded Lyapunov function $V(x)$, real numbers $\alpha > 0$, $\beta > 0$, $0 < p < 1$ and $q > 1$ such that

$$\mathcal{L}V(x) \le -\alpha V^p(x) - \beta V^q(x), \quad \forall x \in \mathbb{R}^n \tag{15}$$

then, the trivial solution of system (1) is fixed-time stable in probability and the stochastic settling time satisfies

$$E(\tau_{x_0}) \le \frac{(\alpha/\beta)^{\frac{1-p}{q-p}}}{\alpha(1-p)} + \frac{(\alpha/\beta)^{\frac{1-q}{q-p}}}{\beta(q-1)}, \quad \forall x_0 \in \mathbb{R}^n \setminus \{0\}. \tag{16}$$

Proof: Choose $\gamma(V(x))$ in Theorem 1 as $\gamma(V(x)) = \alpha V^p + \beta V^q \ge 0$, then, $\gamma'(V(x)) = \alpha p V^{p-1} + \beta q V^{q-1} \ge 0$. Furthermore, for $0 < \epsilon < +\infty$, it can be verified that

$$\begin{aligned} E(\tau_{x_0}) &\le \int_0^\epsilon \frac{1}{\gamma(V)}dV = \int_0^\epsilon \frac{1}{\alpha V^p + \beta V^q}dV \\ &\le \int_0^\varepsilon \frac{1}{\alpha V^p + \beta V^q}dV + \int_\varepsilon^\infty \frac{1}{\alpha V^p + \beta V^q}dV \\ &\le \frac{1}{\alpha(1-p)}V^{1-p}|_0^\varepsilon + \frac{-1}{\beta(q-1)}V^{q-1}|_\varepsilon^\infty \\ &= \frac{\varepsilon^{1-p}}{\alpha(1-p)} + \frac{\varepsilon^{1-q}}{\beta(q-1)} \end{aligned} \tag{17}$$

where $\varepsilon > 0$ is a design constant. Thus, considering Theorem 1 and (15), one says the trivial solution of system (1) is fixed-time stable in probability. By choosing $\varepsilon = (\alpha/\beta)^{\frac{1}{q-p}}$, one can get a settling time that satisfies (16). ■

Now, we aim to prove that the upper-bound estimation of the settling time τ_{x_0} is less conservative than that given by [44]. To prove this point, one needs to prove $T_0 = \frac{(\alpha/\beta)^{\frac{1-p}{q-p}}}{\alpha(1-p)} + \frac{(\alpha/\beta)^{\frac{1-q}{q-p}}}{\beta(q-1)} \le \frac{1}{\alpha(1-p)} + \frac{1}{\beta(q-1)} = T_1$.

Set

$$f(\varepsilon) = \frac{\varepsilon^{1-p}}{\alpha(1-p)} + \frac{\varepsilon^{1-q}}{\beta(q-1)}$$

then, $f'(\varepsilon) = \frac{\varepsilon^{-p}}{\alpha} + \frac{\varepsilon^{-q}}{\beta}$. When letting $f'(\varepsilon) = 0$, one gets the extreme point $\varepsilon^* = (\alpha/\beta)^{\frac{1}{q-p}}$. It is obvious from Fig. 1 that for $\varepsilon < (\alpha/\beta)^{\frac{1}{q-p}}$, $f'(\varepsilon) < 0$ and for $\varepsilon > (\alpha/\beta)^{\frac{1}{q-p}}$, $f'(\varepsilon) > 0$. This implies that the minimum value of $f(\varepsilon)$ is $f_{\min} = f(\varepsilon = (\alpha/\beta)^{\frac{1}{q-p}}) = T_0$. Hence, $T_0 \le T_1 = f(\varepsilon = 1)$. For example, choosing $\alpha = \frac{1}{4}$, $\beta = 1$, $p = \frac{2}{3}$ and $q = 2$, one can obtain $T_0 = 11.3137 < 13 = T_1$; see, Fig. 1 for details.

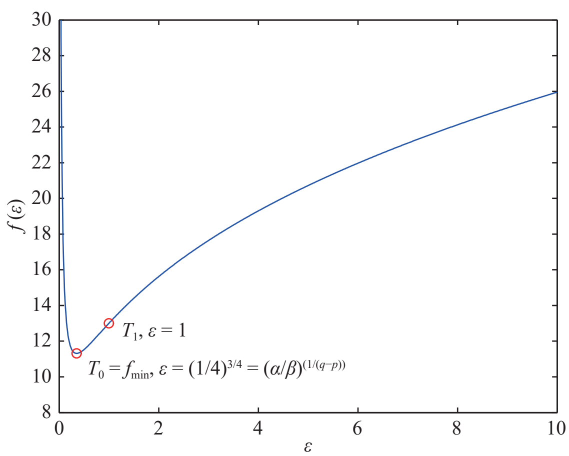


Fig. 1. The response of $f(\varepsilon)$.

IV. State-Feedback Control for Stochastic Nonlinear Systems

Consider the stochastic nonlinear systems

$$\begin{aligned} dx_i &= x_{i+1}dt + f_i(\bar{x}_i)dt + g_i(\bar{x}_i)d\omega \\ dx_n &= udt + f_n(\bar{x}_n)dt + g_n(\bar{x}_n)d\omega \end{aligned} \tag{18}$$

where $i = 1,2,\ldots,n-1$; $x_i \in \mathbb{R}$ and $u \in \mathbb{R}$ are system measurable states and control input, respectively; $\bar{x}_i = (x_1, x_2,\ldots,x_i)^T$; ω is defined as those in (1); the drift terms $f_i(\bar{x}_i) : \mathbb{R}^i \to \mathbb{R}$ and the diffusion terms $g_i(\bar{x}_i) : \mathbb{R}^i \to \mathbb{R}^{1\times r}$ are unknown smooth functions with $f_i(0) = 0$ and $g_i(0) = 0$.

For system (18), impose the assumption as:

Assumption 1: For smooth functions $f_i(\bar{x}_i)$ and $g_i(\bar{x}_i)$ in system (18), there exist known nonnegative smooth functions $\bar{f}_i(\bar{x}_i)$ and $\bar{g}_i(\bar{x}_i)$ such that

$$|f_i(\bar{x}_i)| \le \bar{f}_i(\bar{x}_i)\sum_{j=1}^{i}|x_j|, \quad \|g_i(\bar{x}_i)\| \le \bar{g}_i(\bar{x}_i)\sum_{j=1}^{i}|x_j|.$$

Remark 2: The conditions that guarantee the existence and uniqueness of the solution to system (18) are: $f(x)$ and $g(x)$ satisfies the linear growth condition and the local Lipschitz condition. For system (18), $f(x)$ and $g(x)$ are smooth functions, which naturally satisfy the local Lipschitz condition. Assumption 1 is similar to the linear growth condition, which together with the local Lipschitz condition guarantees that the considered system has an unique solution. This is the basis of investigating the controller design and stability analysis.

A. Controller Design

The following coordinate transformation is given to start the design procedure:

$$z_1 = x_1^{\frac{1}{r_1}}, \quad z_i = x_i^{\frac{1}{r_i}} - x_i^{*\frac{1}{r_i}}, \quad i = 2,3,\ldots,n \tag{19}$$

where $x_2^*, x_3^*,\ldots,x_n^*$ are virtual control laws to be determined; $r_j := \frac{2\mu-2j+3}{4\mu+5}$ $(j = 1,2,\ldots,n+1)$ are parameters with $\mu \ge 1$ being an integer. The properties of r_j are listed as follows:

$$\begin{cases} 0 < r_{n+1} < r_n < \cdots < r_2 < r_1 < \dfrac{1}{2} \\ \iota = r_1 - r_2 = \cdots = r_n - r_{n+1} = \dfrac{2}{4\mu+5} < 1 \\ \dfrac{r_{i+1}}{r_i} = \dfrac{2\mu-2i+1}{2\mu-2i+3} > \dfrac{1}{2}, \quad i = 1,\ldots,n. \end{cases} \tag{20}$$

For better reading, we denote $p_0 = 4 - r_1 + r_2$ and $\sigma_i > \iota = r_1 - r_2$ in the following parts and show the design procedure through n recursive steps.

Step 1: For system (18), consider the Lyapunov function candidate $V_1(x_1) = \frac{r_1}{4}x_1^{\frac{4}{r_1}}$. In terms of (18), Definition 1, (19), (20), Assumption 1, Lemma 3 and Itô's rule, one can verify that

$$\begin{aligned} \mathcal{L}V_1 &\le x_1^{\frac{4-r_1}{r_1}}(x_2 + f_1) + \frac{4-r_1}{2r_1}x_1^{\frac{4-2r_1}{r_1}}\|g_1g_1^T\| \\ &\le z_1^{4-r_1}x_2 + |z_1|^{4-r_1}\bar{f}_1(x_1)|x_1| \\ &\quad + \frac{4-r_1}{2r_1}|z_1|^{4-2r_1}\bar{g}_1^2(x_1)|x_1|^2 \\ &\le z_1^{4-r_1}(x_2 - x_2^*) + z_1^{4-r_1}x_2^* + (\phi_1 + \varphi_1)|z_1|^{p_0} \end{aligned} \tag{21}$$

where $\phi_1 = \bar{f}_1|z_1|^{r_1-r_2}$ and $\varphi_1 = \frac{4-r_1}{2r_1}\bar{g}_1^2|z_1|^{r_1-r_2}$.

Then, we choose the first virtual control law as

$$\begin{aligned} x_2^*(x_1) &= -z_1^{r_2}\beta_1(x_1) \\ \beta_1(x_1) &= c_{11} + \phi_1(x_1) + \varphi_1(x_1) + b_1z_1^{\sigma_1} \end{aligned} \tag{22}$$

which changes (21) into

$$\mathcal{L}V_1 \le -c_{11}|z_1|^{p_0} - b_1|z_1|^{p_0+\sigma_1} + z_1^{4-r_1}(x_2 - x_2^*) \tag{23}$$

where $c_{11} > 0$ and $b_1 > 0$ are design constants.

Step i $(2 \le i \le n-1)$: We summarize the inductive step in a proposition.

Proposition 1: If at Step $(i-1)$, there exist a series of virtual control laws $x_2^*(x_1) = -z_1^{r_2}\beta_1(x_1)$, $x_3^*(\bar{x}_2) = -z_2^{r_3}\beta_2(\bar{x}_2),\ldots,$

$x_i^*(\bar{x}_{i-1}) = -z_{i-1}^{r_i}\beta_{i-1}(\bar{x}_{i-1})$ such that the C^2 Lyapunov functions $V_{i-1} = V_1 + \sum_{j=2}^{i-1} U_j$ with $U_j = \int_{x_j^*}^{x_j} (s^{\frac{1}{r_j}} - x_j^{*\frac{1}{r_j}})^{4-r_j} ds$ are positive definite, proper and satisfy

$$V_{i-1} \leq 2\left(z_1^4 + \cdots + z_{i-1}^4\right) \tag{24}$$

$$\mathcal{L}V_{i-1} \leq -\sum_{j=1}^{i-1} c_{j,i-1}|z_j|^{p_0} - \sum_{j=1}^{i-1} b_j|z_j|^{p_0+\sigma_j} + z_{i-1}^{4-r_{i-1}}(x_i - x_i^*) \tag{25}$$

where $\beta_1,\beta_2,\ldots,\beta_{i-1}$ are nonnegative continuous functions; $c_{j,i-1}$ and b_j are positive design constants. Then, for the ith Lyapunov function

$$V_i = V_{i-1} + U_i,\ U_i = \int_{x_i^*}^{x_i} \left(s^{\frac{1}{r_i}} - x_i^{*\frac{1}{r_i}}\right)^{4-r_i} ds \tag{26}$$

it can be verified that

$$V_i \leq 2\left(z_1^4 + \cdots + z_i^4\right) \tag{27}$$

$$\mathcal{L}V_i \leq -\sum_{j=1}^{i} c_{j,i}|z_j|^{p_0} - \sum_{j=1}^{i} b_j|z_j|^{p_0+\sigma_j} + z_i^{4-r_i}(x_{i+1} - x_{i+1}^*) \tag{28}$$

where the virtual control law is designed as

$$\begin{aligned} x_{i+1}^*(\bar{x}_i) &= -z_i^{r_{i+1}}\beta_i(\bar{x}_i) \\ \beta_i(\bar{x}_i) &= c_{ii} + \phi_i(\bar{x}_i) + \varphi_i(\bar{x}_i) + b_i z_i^{\sigma_i} \end{aligned} \tag{29}$$

where ϕ_i and φ_i are nonnegative continuous design functions; c_{ii}, b_i, ε_{ijk}, $\varepsilon_{j+1,1}$, and $c_{ji} = c_{jj} - \varepsilon_{j+1,1} - \sum_{k=2}^{4}\varepsilon_{ijk}$ $(j = 1,\ldots, i-1)$ are positive design constants.

Proof: See the Appendix.

Step n: By exactly following the design procedure at step i, for Lyapunov function:

$$V_n = V_1 + \sum_{i=2}^{n} U_i,\ U_i = \int_{x_i^*}^{x_i} \left(s^{\frac{1}{r_i}} - x_i^{*\frac{1}{r_i}}\right)^{4-r_i} ds \tag{30}$$

one can construct the controller as

$$u(\bar{x}_n) = -z_n^{r_{n+1}}\beta_n(\bar{x}_n) \tag{31}$$

such that

$$V_n \leq 2\sum_{i=1}^{n} z_i^4,\ \mathcal{L}V_n \leq -\sum_{i=1}^{n} c_{in}|z_i|^{p_0} - \sum_{i=1}^{n} b_i|z_i|^{p_0+\sigma_i} \tag{32}$$

where $\beta_n = c_{nn} + \phi_n + \varphi_n + b_n z_n^{\sigma_n}$; $c_{in} = c_{i,i} - \varepsilon_{i+1,1} - \Sigma_{k=2}^{4}\varepsilon_{nik} > 0$ for $i = 1,2,\ldots,n-1$, $c_{nn}>0$, $b_n > 0$, and $c_{nn} > 0$ are design constants; ϕ_n and φ_n are nonnegative continuous functions.

B. Stability Analysis

The following theorem shows the stability given by the designed controller.

Theorem 2: For the stochastic nonlinear system (18), under controller (31), the solution of the closed-loop system consisting of (18), (19), (22), (29), and (31) can be ensured to be fixed-time stable in probability and the settling time satisfies

$$E(\tau_{x_0}) \leq \min_{i=1,2,\ldots,n}\left\{\frac{4(c/b)^{\frac{4-p_0}{\sigma_i}}}{c(4-p_0)} + \frac{4(c/b)^{\frac{4-p_0-\sigma_i}{\sigma_i}}}{b(p_0+\sigma_i-4)}\right\} \tag{33}$$

where $b \geq \max_{i=1,2,\ldots,n}\{n^{1-\frac{p_0+\sigma_i}{4}}2^{-\frac{p_0+\sigma_i}{4}}b_0\}$ and $c \geq 2^{-\frac{p_0}{4}}c_0$ are positive constants with $c_0 = \min_{i=1,2,\ldots,n}\{c_{in}\}$ and $b_0 = \min_{i=1,2,\ldots,n}\{b_i\}$.

Proof: Considering (30) and (32), it is obvious that V_n is C^2, nonnegative, radially unbounded. In addition, from $V_n \leq 2\sum_{i=1}^{n} z_i^4$ and Lemma 4, one verifies that

$$\begin{aligned} \mathcal{L}V_n &\leq -2^{-\frac{p_0}{4}}c_0 V_n^{\frac{p_0}{4}} - n^{1-\frac{p_0+\sigma_i}{4}}2^{-\frac{p_0+\sigma_i}{4}}b_0 V_n^{\frac{p_0+\sigma_i}{4}} \\ &\leq -cV_n^{\frac{p_0}{4}} - bV_n^{\frac{p_0+\sigma_i}{4}} \end{aligned} \tag{34}$$

where $c_0 = \min_{i=1,2,\ldots,n}\{c_{in}\}$, $b_0 = \min_{i=1,2,\ldots,n}\{b_i\}$, $c \geq 2^{-\frac{p_0}{4}}c_0$ and $b \geq \max_{i=1,2,\ldots,n}\{n^{1-\frac{p_0+\sigma_i}{4}}2^{-\frac{p_0+\sigma_i}{4}}b_0\}$ are positive constants.

Since $\sigma_i > r_1 - r_2$, $p_0 = 4 - r_1 + r_2$ and $r_1 - r_2 > 0$, it is obvious that $0 < \frac{p_0}{4} < 1$ and $\frac{p_0+\sigma_i}{4} > 1$. Thus, using Corollary 1 and letting $\alpha = c$, $\beta = b$, $p = \frac{p_0}{4}$ and $q = \frac{p_0+\sigma_i}{4}$, one can obtain that the solution of the closed-loop system is fixed-time stable in probability and (33) holds. ■

V. Simulation Examples

Example 1: Consider the stochastic simple pendulum system established by [32] as

$$\begin{aligned} dx_1 &= x_2 dt \\ dx_2 &= u dt + (-9.8\sin(x_1) - 0.5x_2)dt - 0.1x_2 d\omega. \end{aligned} \tag{35}$$

It can be verified that Assumption 1 is satisfied with $\bar{f}_2 = 10$ and $\bar{g}_2 = 0.1$. By following the design procedure in Section V exactly, one gets the state-feedback controller:

$$z_1 = x_1^{\frac{1}{r_1}},\quad z_2 = x_2^{\frac{1}{r_2}} - x_2^{*\frac{1}{r_2}} \tag{36}$$

$$x_2^* = -(c_{11} + b_1 z_1^{\sigma_1})z_1^{r_2} \tag{37}$$

$$u = -z_2^{r_3}(c_{22} + \phi_2 + \varphi_2 + b_2 z_2^{\sigma_2}) \tag{38}$$

where the parameters are chosen as $r_1 = \frac{41}{85}$, $r_2 = \frac{39}{85}$, $r_3 = \frac{37}{85}$, $\sigma_1 = \frac{1}{5}$, $\sigma_2 = \frac{1}{3}$, $c_{11} = 5$, $c_{22} = 2$, $b_1 = 1$, $b_2 = 1$ and the other constants during the controller design procedure are given by 1 in simulation. Then, it can be verified that $\phi_2 = \phi_{21} + \phi_{22} + \phi_{23}$ with $\phi_{21} = 0.0451$; $\phi_{22} = 4.1971\lambda_{11}^{\frac{338}{255}}|\beta_1|^{\frac{338}{255}} + 4.22\lambda_{11}^{\frac{169}{147}}$ and $\phi_{23} = 9.0049(1 + \sqrt{1+\beta_1^2})^{\frac{338}{301}}(|z_1|^{\frac{1352}{25585}} + |z_1|^{\frac{676}{25585}}) + 10\sqrt{1+\beta_1^2}|z_2|^{\frac{2}{85}}$; in addition, $\varphi_2 = 0.081|\beta_1|^{\frac{14365}{4212}} + 0.3859|z_2|^{\frac{122}{85}}$, where $\beta_1 = c_{11} + z_1^{\sigma_1}$ and $\lambda_{11} = 1.5061|\beta_1|^{\frac{47}{39}}|z_1|^{\frac{1}{3}} + \frac{85}{41}|\beta_1|^{\frac{85}{39}}$. In addition, it can be verified that $b \geq 0.47$ and $c \geq 0.502$. Then, one can calculate $E(\tau_{x_0}) \leq 308.8$.

To give the comparison, we show a simulation from two cases.

Case 1: Different initial values $x_1(0) = 0.6$, $x_2(0) = -1$ and $x_1(0) = 0.6$, $x_2(0) = 10$.

When $x_1(0) = 0.6$ and $x_2(0) = -1$, the convergence time of Example 1 is almost 6 s from Figs. 2 and 3, which satisfies $E(\tau_{x_0}) \leq 308.8$; when $x_1(0) = 0.6$ and $x_2(0) = 10$, the convergence time of Example 1 is also almost 6 s from Figs. 4

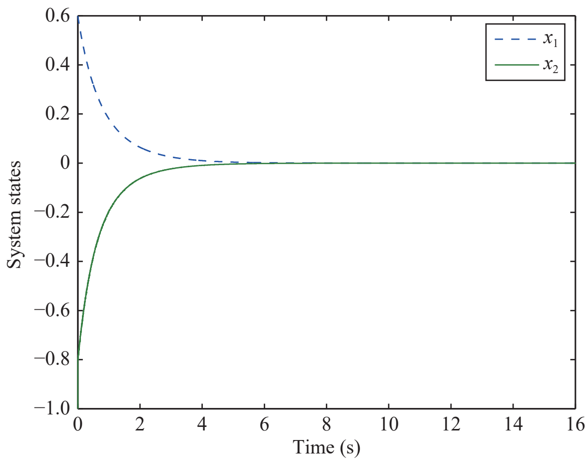


Fig. 2. The response of states with $x_1(0) = 0.6$ and $x_2(0) = -1$.

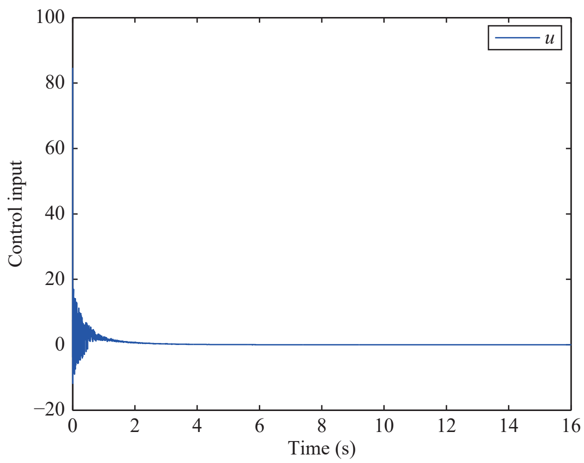


Fig. 3. The response of control input with $x_1(0) = 0.6$ and $x_2(0) = -1$.

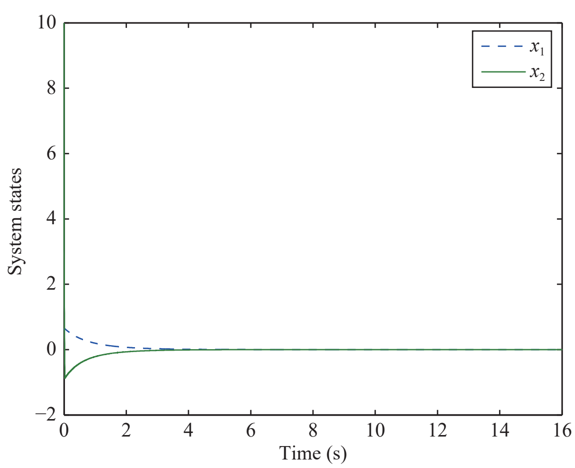


Fig. 4. The response of states with $x_1(0) = 0.6$ and $x_2(0) = 10$.

and 5. This means that fixing $x_1(0)$, the convergence time of system states and controller under different values of $x_2(0)$ are almost the same. Similarly, when fixing $x_2(0)$, the convergence time of system states and controller under different values of $x_1(0)$ will not change too.

Case 2: To show the difference between fixed-time (FxT) control and finite-time control, we show the finite-time (FT)

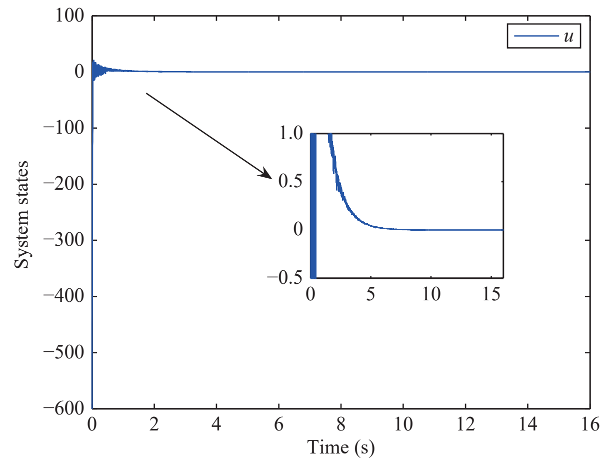


Fig. 5. The response of controller with $x_1(0) = 0.6$ and $x_2(0) = 10$.

controller given by [32] as

$$
\begin{aligned}
& z_1 = x_1^{\frac{1}{r_1}}, \quad z_2 = x_2^{\frac{1}{r_2}} - x_2^{*\frac{1}{r_2}} \\
& x_2^* = -c_{11} z_1^{r_2} \\
& u = -z_2^{r_3}(c_{22} + \phi_2 + \varphi_2)
\end{aligned} \tag{39}
$$

where ϕ_2 and φ_2 are shown as in [32]. In the comparison, we choose all the constants to be the same as those in this paper. The initial values are given by $x_1(0) = 0.6$, $x_2(0) = -1$ and $x_1(0) = 0.6$ and $x_2(0) = 10$, respectively. Figs. 6 and 7 show the effectiveness of the proposed controller. In particular, it is clear that when fixing x_1 and increasing the initial values from $x_2(0) = -1$ to $x_2(0) = 10$, the settling time of FxT controller in this paper is almost the same and around 6 s. But the settling time of FT controller in [32] grows from 6 s to 12 s as the the initial values increase.

Example 2: Consider

$$dx = (-\frac{1}{2}x^3 - x^{\frac{1}{3}})dx + \frac{1}{2}x^2 d\omega.$$

If we choose $V(x) = \frac{1}{2}x^2$, we have

$$\mathcal{L}V = -x^{\frac{4}{3}} - \frac{3}{8}x^4 = -2^{\frac{2}{3}}V^{\frac{2}{3}} - \frac{3}{2}V^2.$$

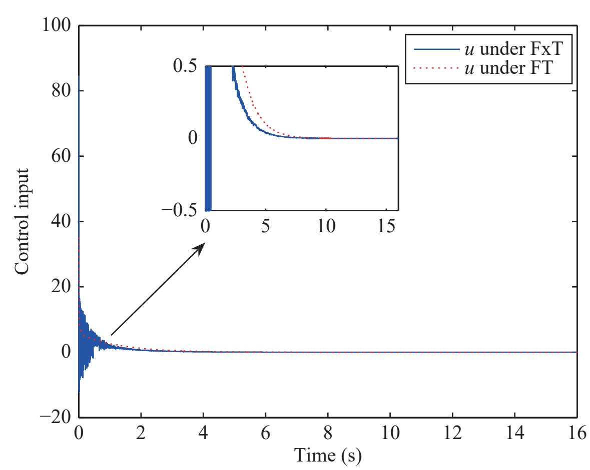


Fig. 6. The response of controller with $x_1(0) = 0.6$ and $x_2(0) = -1$.

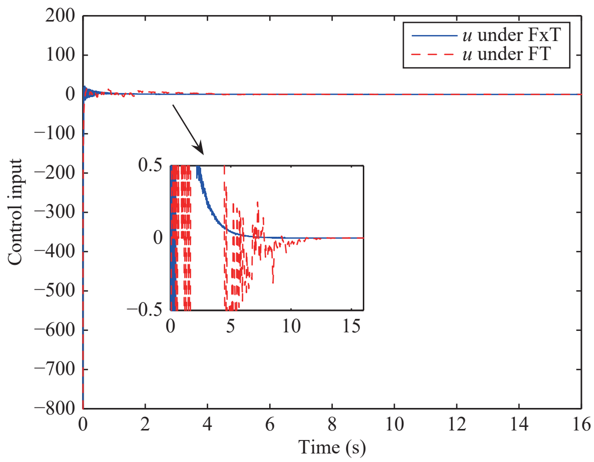


Fig. 7. The response of controller with $x_1(0) = 0.6$ and $x_2(0) = 10$.

Using Corollary 1 and substituting $\alpha = 2^{\frac{2}{3}}$, $\beta = \frac{3}{2}$, $p = \frac{2}{3}$ and $q = 2$ in (16), the system is fixed-time stable in probability and $E(\tau_{x_0}) \leq 2.46$. Choosing $x_0 = -6$, $x_0 = 1$, and $x_0 = 10$, we can see from Fig. 8 that, regardless of the initial values, the settling time is around 1.8 s satisfies $E(\tau_{x_0}) \leq 2.46$.

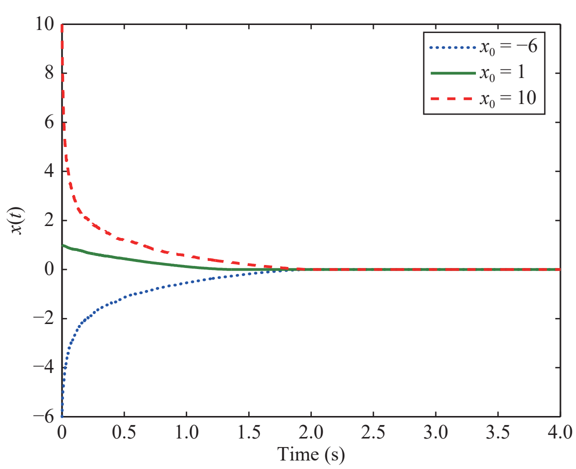


Fig. 8. The response of Example 2.

VI. Conclusions

In this paper, the Lyapunov criteria of fixed-time stability for stochastic nonlinear systems has been improved. Compared with existing results, the upper-bound estimate of the settling time is less conservative with rigorous and reasonable analysis. In addition, with a backstepping design, a state-feedback controller is designed for stochastic nonlinear systems, which ensures the closed-loop system to be fixed-time stable in probability. There are still problems to be investigated: i) Determining how to use the proposed stability to give the output-feedback controller design for more general stochastic nonlinear systems. ii) For stochastic nonlinear systems with inverse dynamics, determining how to obtain the fixed-time stability theorem and design the controller.

Appendix

Proof of Proposition 1: Firstly, from (19), (26) and Itô's rule, one has

$$\frac{\partial U_i}{\partial x_i} = z_i^{4-r_i}, \quad \frac{\partial^2 U_i}{\partial x_i^2} = \frac{4-r_i}{r_i} z_i^{3-r_i} x_i^{\frac{1}{r_i}-1}$$

$$\frac{\partial U_i}{\partial x_j} = -(4-r_i)\frac{\partial x_i^{*\frac{1}{r_i}}}{\partial x_j}\int_{x_i^*}^{x_i}\left(s^{\frac{1}{r_i}} - x_i^{*\frac{1}{r_i}}\right)^{3-r_i} ds$$

$$\frac{\partial^2 U_i}{\partial x_i \partial x_j} = \frac{\partial^2 U_i}{\partial x_j \partial x_i} = -(4-r_i)\frac{\partial x_i^{*\frac{1}{r_i}}}{\partial x_j} z_i^{3-r_i}$$

$$\begin{aligned}\frac{\partial^2 U_i}{\partial x_j^2} &= -(4-r_i)\frac{\partial^2 x_i^{*\frac{1}{r_i}}}{\partial x_j^2}\int_{x_i^*}^{x_i}\left(s^{\frac{1}{r_i}} - x_i^{*\frac{1}{r_i}}\right)^{3-r_i} ds \\ &\quad + (4-r_i)(3-r_i)\Big(\frac{\partial x_i^{*\frac{1}{r_i}}}{\partial x_j}\Big)^2 \\ &\quad \times \int_{x_i^*}^{x_i}\left(s^{\frac{1}{r_i}} - x_i^{*\frac{1}{r_i}}\right)^{2-r_i} ds \\ \frac{\partial^2 U_i}{\partial x_j \partial x_k} &= -(4-r_i)\frac{\partial^2 x_i^{*\frac{1}{r_i}}}{\partial x_j \partial x_k}\int_{x_i^*}^{x_i}\left(s^{\frac{1}{r_i}} - x_i^{*\frac{1}{r_i}}\right)^{3-r_i} ds \\ &\quad + (4-r_i)(3-r_i)\frac{\partial x_i^{*\frac{1}{r_i}}}{\partial x_j}\frac{\partial x_i^{*\frac{1}{r_i}}}{\partial x_k} \\ &\quad \times \int_{x_i^*}^{x_i}\left(s^{\frac{1}{r_i}} - x_i^{*\frac{1}{r_i}}\right)^{2-r_i} ds \end{aligned} \tag{40}$$

where $j, k = 1, \ldots, i-1$ and $j \neq k$. Obviously, V_i is C^2, positive definite and proper. Moreover, by means of (24), (26) and Lemma 4, one yields

$$\begin{aligned} V_i &\leq 2\sum_{j=1}^{i-1} z_j^4 + z_i^{4-r_i}(x_i - x_i^*) \\ &\leq 2\sum_{j=1}^{i-1} z_j^4 + |z_i|^{4-r_i} 2^{1-r_i}(x_i^{\frac{1}{r_i}} - x_i^{*\frac{1}{r_i}})^{r_i} \\ &\leq 2\sum_{j=1}^{i} z_j^4 \end{aligned} \tag{41}$$

which means (27) holds.

On the other hand, from (18), Definition 1, (19), (25), (26) and Itô's rule, it can be verified that

$$\begin{aligned} \mathcal{L}V_i \leq &-\sum_{j=1}^{i-1} c_{j,i-1}|z_j|^{p_0} - \sum_{j=1}^{i-1} b_j |z_j|^{p_0+\sigma_j} + \frac{\partial U_i}{\partial x_i}x_{i+1} \\ &+ z_{i-1}^{4-r_{i-1}}(x_i - x_i^*) + \sum_{j=1}^{i-1}\frac{\partial U_i}{\partial x_j}(x_{j+1} + f_j) \\ &+ \frac{\partial U_i}{\partial x_i} f_i + \mathrm{Tr}\left\{\bar{G}_i^T \frac{\partial^2 U_i}{\partial \bar{x}_i^2}\bar{G}_i\right\}, \end{aligned} \tag{42}$$

where $\mathrm{Tr}\{\bar{G}_i^T \frac{\partial^2 U_i}{\partial \bar{x}_i^2}\bar{G}_i\} = \frac{1}{2}\sum_{j,k=1, j\neq k}^{i-1}\frac{\partial^2 U_i}{\partial x_j \partial x_k}\|g_j g_k^T\| + \frac{1}{2}\sum_{j=1}^{i-1} \times \frac{\partial^2 U_i}{\partial x_j^2}\|g_j g_j^T\| + \sum_{j=1}^{i-1}\frac{\partial^2 U_i}{\partial x_i \partial x_j}\|g_i g_j^T\| + \frac{1}{2}\frac{\partial^2 U_i}{\partial x_i^2}\|g_i g_i^T\|$.

Now, we focus on estimating the terms in (42).

Firstly, from (19), Lemmas 3 and 4, one arrives at

$$\begin{aligned} z_{i-1}^{4-r_{i-1}}(x_i-x_i^*) &= z_{i-1}^{4-r_{i-1}}\left((x_i-x_i^*)^{\frac{1}{r_i}}\right)^{r_i} \\ &\le 2^{1-r_i}|z_{i-1}|^{4-r_{i-1}}z_i^{r_i} \\ &\le \varepsilon_{i1}|z_{i-1}|^{p_0}+\phi_{i1}|z_i|^{p_0} \end{aligned} \tag{43}$$

where $\phi_{i1}=\frac{r_i}{p_0}(\frac{p_0}{(4-r_{i-1})}\varepsilon_{i1})^{-\frac{4-r_{i-1}}{r_i}}$ and $\varepsilon_{i1}>0$ is a design constant.

Secondly, from (19), Lemmas 3 and 4, one obtains

$$\begin{aligned} |x_{i+1}| &\le (|z_{i+1}+|z_i||\beta_i|^{\frac{1}{r_{i+1}}})^{r_{i+1}} \\ &\le |z_{i+1}|^{r_{i+1}}+|\beta_i||z_i|^{r_{i+1}}. \end{aligned} \tag{44}$$

Together with Assumption 1, it can be verified that

$$\begin{aligned} |f_i| &\le \bar{f}_i(\bar{x}_i)\sum_{j=1}^{i}\left(|z_j|^{r_j}+|\beta_{j-1}||z_{j-1}|^{r_j}\right) \\ &\le \bar{f}_i(\bar{x}_i)\sum_{j=1}^{i}l_j(\bar{x}_j)\left(|z_j|^{r_j}+|z_j|^{r_{j+1}}\right) \end{aligned} \tag{45}$$

where

$$l_j(\bar{x}_j)=\begin{cases} s_j+s_{j+1}, & j=1,\ldots,i-1 \\ s_j, & j=i \end{cases}$$

with $s_j=\max\{1,|\beta_{j-1}|\}$ and $s_0=1$. Furthermore, by applying a similar induction argument as those in [9], it holds for $j=1,\ldots,i-1$ that

$$\left|\frac{\partial x_i^{*\frac{1}{r_i}}}{\partial x_j}\right| \le \lambda_{i1}(\bar{x}_{i-1})\sum_{j=1}^{i-1}|z_j|^{1-r_j} \tag{46}$$

where $\lambda_{i1}\ge 0$ is a continuous design function. Then, by means of (40), (44)−(46) and Lemma 3, one has

$$\begin{aligned} &\left|\sum_{j=1}^{i-1}\frac{\partial U_i}{\partial x_j}(x_{j+1}+f_j)\right| \\ &\le \sum_{j=1}^{i-1}(4-r_i)\left|\frac{\partial x_i^{*\frac{1}{r_i}}}{\partial x_j}\right|z_i^{3-r_i}(x_i-x_i^*)(x_{j+1}+f_j) \\ &\le \sum_{j=1}^{i-1}\bar{\lambda}_{i1}|z_i|^3\left(\sum_{j=1}^{i-1}|z_j|^{1-r_j}\right)\Bigg(|z_{j+1}|^{r_{j+1}}+|\beta_j||z_j|^{r_{j+1}} \\ &\quad +\bar{f}_j\sum_{k=1}^{j}l_k(|z_k|^{r_k}+|z_k|^{r_{k+1}})\Bigg) \\ &\le \sum_{j=1}^{i-1}\varepsilon_{ij2}|z_j|^{p_0}+\phi_{i2}(\bar{x}_i,\hat{\theta})|z_i|^{p_0} \end{aligned} \tag{47}$$

where $\bar{\lambda}_{i1}(\bar{x}_{i-1})=2(4-r_i)\lambda_{i1}$ and ϕ_{i2} are nonnegative continuous functions.

Thirdly, by combining (40), (45) with Lemma 3, one yields

$$\begin{aligned} \left|\frac{\partial U_i}{\partial x_i}f_i\right| &\le |z_i|^{4-r_i}\left(\bar{f}_i\sum_{j=1}^{i-1}l_j(|z_j|^{r_j}+|z_j|^{r_{j+1}})+\bar{f}_il_i|z_i|^{r_i}\right) \\ &\le \sum_{j=1}^{i-1}\varepsilon_{ij3}|z_j|^{p_0}+\phi_{i3}(\bar{x}_i)|z_i|^{p_0} \end{aligned} \tag{48}$$

where ε_{ij3} is an any positive design parameter to be chosen and $\phi_{i3}=\sum_{j=1}^{i-1}\frac{4-r_i}{p_0}(\frac{p_0}{r_{i+1}}\varepsilon_{ij3})^{-\frac{r_{i+1}}{4-r_i}}(\bar{f}_il_j|z_j|^{r_j-r_{i+1}})^{\frac{p_0}{4-r_i}}+\sum_{j=1}^{i-1}\frac{4-r_i}{p_0}\times(\frac{p_0}{r_{i+1}}\varepsilon_{ij3})^{-\frac{r_{i+1}}{4-r_i}}(\bar{f}_il_j|z_j|^{r_{j+1}-r_{i+1}})^{\frac{p_0}{4-r_i}}+\bar{f}_il_i|z_i|^{r_i-r_{i+1}}$.

Finally, using similar induction proof of (46), one gets

$$\left|\frac{\partial^2 x_i^{*\frac{1}{r_i}}}{\partial x_j^2}\right| \le \lambda_{i3}(\bar{x}_{i-1})\sum_{j=1}^{i-1}|z_j|^{1-2r_{\lambda_{i,j}}} \tag{49}$$

$$\left|\frac{\partial^2 x_i^{*\frac{1}{r_i}}}{\partial x_j\partial x_k}\right| \le \lambda_{i4}(\bar{x}_{i-1})\sum_{j=1}^{i-1}|z_j|^{1-2r_{\lambda_{i,j}}} \tag{50}$$

where $\lambda_{i,j}=\max\{i,j\}$; λ_{i3} and λ_{i4} are nonnegative smooth functions. In addition, from Assumption 1 and (44), it holds

$$\|g_i\| \le \bar{g}_i(\bar{x}_i)\sum_{j=1}^{i}l_j(\bar{x}_j)(|z_j|^{r_j}+|z_j|^{r_{j+1}}) \tag{51}$$

where l_j has been defined in (45).

Then, by adopting (40), (46), (49)−(51), Lemmas 3 and 4, one can verify that

$$\begin{aligned} &\frac{1}{2}\sum_{j=1}^{i-1}\frac{\partial^2 U_i}{\partial x_j^2}\|g_jg_j^T\| \\ &\le \sum_{j=1}^{i-1}\left(2(4-r_i)\left|\frac{\partial^2 x_i^{*\frac{1}{r_i}}}{\partial x_j^2}\right||z_i|^3+2(4-r_i)(3-r_i)\right. \\ &\quad \left.\cdot\left|\frac{\partial x_i^{*\frac{1}{r_i}}}{\partial x_j}\right|^2|z_i|^2\right)\bar{g}_j^2(\sum_{k=1}^{j}l_k(|z_k|^{r_k}+|z_k|^{r_{k+1}}))^2 \\ &\le \sum_{j=1}^{i-1}\bar{\lambda}_{i5}(\bar{x}_i)\left(\sum_{j=1}^{i-1}|z_j|^{1-2r_{\lambda_{i,j}}}\right)\sum_{k=1}^{j}l_k^2|z_k|^{2r_k}|z_i|^3 \\ &\quad +\sum_{j=1}^{i-1}\tilde{\lambda}_{i5}(\bar{x}_i)\left(\sum_{j=1}^{i-1}|z_j|^{1-r_j}\right)^2\sum_{k=1}^{j}l_k^2|z_k|^{2r_k}|z_i|^2 \\ &\le \sum_{j=1}^{i-1}\varepsilon_{ij41}|z_j|^{p_0}+\varphi_{i1}(\bar{x}_i)|z_i|^{p_0} \end{aligned} \tag{52}$$

where $\bar{\lambda}_{i5}=2(4-r_i)\max\{\bar{g}_j^2, j=1,\ldots,i-1\}$, $\tilde{\lambda}_{i5}=2(4-r_i)(3-r_i)\max\{\bar{g}_j^2, j=1,\ldots,i-1\}$ and φ_{i1} are nonnegative continuous functions with ε_{ij51} being a positive design constant. Thus, similarly, based on (40), (46), (49)−(51), Lemmas 3 and 4, one can further get

$$\left|\sum_{j=1}^{i-1}\frac{\partial^2 U_i}{\partial x_i\partial x_j}\|g_ig_j^T\|\right| \le \sum_{j=1}^{i-1}\varepsilon_{ij42}|z_j|^{p_0}+\varphi_{i2}(\bar{x}_i)|z_i|^{p_0} \tag{53}$$

$$\left|\sum_{j,k=1,j\neq k}^{i-1}\frac{\partial^2 U_i}{2\partial x_j\partial x_k}\|g_j g_k^T\|\right|\leq\sum_{j=1}^{i-1}\varepsilon_{ij43}|z_j|^{p_0}+\varphi_{i3}|z_i|^{p_0}\tag{54}$$

$$\left|\frac{1}{2}\frac{\partial^2 U_i}{\partial x_i^2}\|g_i g_i^T\|\right|\leq\sum_{j=1}^{i-1}\varepsilon_{ij44}|z_j|^{p_0}+\varphi_{i4}(\bar{x}_i)|z_i|^{p_0}\tag{55}$$

where ε_{ij52}, ε_{ij53} and ε_{ij54} are positive design constants; φ_{i2}, φ_{i3} and φ_{i4} are nonnegative continuous functions. By substituting (52)–(55) into $\mathrm{Tr}\{\bar{G}_i^T\frac{\partial^2 W_i}{\partial\bar{x}_i^2}\bar{G}_i\}$, letting $\varepsilon_{ij4}=\sum_{k=1}^4\varepsilon_{ij4k}$ and $\varphi_i=\sum_{k=1}^4\varphi_{ik}$, one can finally get

$$\left|\mathrm{Tr}\left\{\bar{G}_i^T\frac{\partial^2 U_i}{\partial\bar{x}_i^2}\bar{G}_i\right\}\right|\leq\sum_{j=1}^{i-1}\varepsilon_{ij4}|z_j|^{p_0}+\varphi_i(\bar{x}_i)|z_i|^{p_0}.\tag{56}$$

Substituting (43), (47), (48) and (56) into (42) leads to

$$\begin{aligned}\mathcal{L}V_i\leq&-\sum_{j=1}^{i-1}c_{j,i-1}|z_j|^{p_0}+\sum_{j=1}^{i-1}\sum_{k=2}^{4}\varepsilon_{ijk}|z_j|^{p_0}+\varepsilon_{i1}|z_{i-1}|^{p_0}\\&-\sum_{j=1}^{i-1}b_j|z_j|^{p_0+\sigma_j}+z_i^{4-r_i}(x_{i+1}-x_{i+1}^*)\\&+z_i^{4-r_i}x_{i+1}^*+\phi_i(\bar{x}_i)|z_i|^{p_0}+\varphi_i(\bar{x}_i)|z_i|^{p_0}\end{aligned}\tag{57}$$

where $\phi_i=\sum_{k=1}^3\phi_{ik}$. Thus, by constructing x_{i+1}^* as (29), one can obtain (28). ■

REFERENCES

[1] M. Krstić, I. Kanellakopoulos, and P. V. Kokotović, *Nonlinear and Adaptive Control Design*, New York: Wiley, 1995.

[2] W. Lin and C. J. Qian, "Adding one power integrator: A tool for global stabilization of high-order lower triangular systems," *Syst. Control Lett.*, vol. 39, no. 5, pp. 339–351, 2000.

[3] H. K. Khalil. *Nonlinear Systems*. Prentice-Hall, New Jersey, 1996.

[4] C. P. Bechlioulis and G. A. Rovithakis, "Robust adaptive control of feedback linearizable MIMO nonlinear systems with prescribed performance," *IEEE Trans. Autom. Control*, vol. 53, no. 9, pp. 2090–2099, 2008.

[5] J. X. Zhang and G. H. Yang, "Prescribed performance fault-tolerant control of uncertain nonlinear systems with unknown control directions," *IEEE Trans. Autom. Control*, vol. 62, no. 12, pp. 6529–6535, 2017.

[6] J. X. Zhang and G. H. Yang, "Low-complexity tracking control of strict-feedback systems with unknown control directions," *IEEE Trans. Autom. Control*, vol. 64, no. 12, pp. 5175–5182, 2019.

[7] A. K. Jain and S. Bhasin, "Tracking control of uncertain nonlinear systems with unknown constant input delay," *IEEE/CAA J. Autom. Sinica*, vol. 7, no. 2, pp. 420–425, 2020.

[8] C. J. Qian, "A homogeneous domination approach for global output feedback stabilization of a class of nonlinear systems," in *Proc. American Control Conf.*, USA, 2005, pp. 4708–715.

[9] X. Q. Huang, W. Lin, and B. Yang, "Global finite-time stabilization of a class of uncertain nonlinear systems," *Automatica*, vol. 41, no. 5, pp. 881–888, 2005.

[10] L. Liu, T. Gao, Y. J. Liu, and S. Tong, "Time-varying asymmetrical BLFs based adaptive finite-time neural control of nonlinear systems with full state constraints," *IEEE/CAA J. Autom. Sinica*, vol. 7, no. 5, pp. 1335–1343, 2020.

[11] H. Deng and M. Krstić, "Stochastic nonlinear stabilization, part I: A backstepping design," *Syst. Control Lett.*, vol. 32, no. 3, pp. 143–150, 1997.

[12] R. Z. Has'minskii, *Stochastic Stability of Diffrential Equations*. Norwell, MA: Kluwer Academic Publishers, 1980.

[13] B. Øksendal, *Stochastic Differential Equations: An Introduction with Applications* (6th ed.). New York, Springer, 2010.

[14] X. R. Mao, *Stochastic Differential Equations and Their Applications*, Chichester: Horwood Publishing, 2007.

[15] H. Deng, M. Krstić, and R. J. Williams, "Stabilization of stochastic nonlinear systems driven by noise of unknown covariance," *IEEE Trans. Autom. Control*, vol. 46, no. 8, pp. 1237–1253, 2001.

[16] H. Min, S. Xu, B. Zhang, and Q. Ma, "Output-feedback control for stochastic nonlinear systems subject to input saturation and time-varying delay," *IEEE Trans. Autom. Control*, vol. 64, no. 1, pp. 359–364, 2019.

[17] X. J. Xie, N. Duan, and C. R. Zhao, "A combined homogeneous domination and sign function approach to output feedback stabilization of stochastic high-order nonlinear systems," *IEEE Trans. Autom. Control*, vol. 59, no. 5, pp. 1303–1309, 2014.

[18] H. Min and N. Duan, "Neural network-based adaptive state-feedback control for high-order stochastic nonlinear systems," *Acta Autom. Sin.*, vol. 40, no. 12, pp. 2976–2980, 2014.

[19] X. J. Xie and M. Jiang, "Dynamic state feedback stabilization of stochastic cascade nonlinear time-delay systems with SISS inverse dynamics," *IEEE Trans. Autom. Control*, vol. 64, no. 12, pp. 5132–5139, 2019.

[20] X. Fu, Y. Kang, and P. Li, "Sampled-data observer design for a class of stochastic nonlinear systems based on the approximate discrete-time models," *IEEE/CAA J. Autom. Sinica*, vol. 4, no. 3, pp. 507–511, 2017.

[21] W. Li, X. Yao, and M. Krstić, "Adaptive-gain observer-based stabilization of stochastic strict-feedback systems with sensor uncertainty," *Automatica*, vol. 120, pp. 109–112, 2020.

[22] S. Liu, S. Ge, and J. Zhang, "Adaptive output-feedback control for a class of uncertain stochastic non-linear systems with time delays," *Int. J. Control*, vol. 81, no. 8, pp. 1210–1220, 2008.

[23] T. Jiao, W. X. Zheng, and S. Xu, "Unified stability criteria of random nonlinear time-varying impulsive switched systems," *IEEE Trans. Circuits Syst. I.*, vol. 67, no. 9, pp. 3099–2112, 2020.

[24] M. Wang, Z. Wang, Y. Chen, *et al*., "Event-based adaptive neural tracking control for discrete-time stochastic nonlinear systems: A triggering threshold compensation strategy," *IEEE Trans. Neural Netw. Learning Syst.*, vol. 31, no. 6, pp. 1968–1981, 2019.

[25] W. Sun, S. F. Su, G. Dong, and W. Bai, "Reduced adaptive fuzzy tracking control for high-order stochastic nonstrict feedback nonlinear system with full-state constraints," *IEEE Trans. Syst. Man Cybern. Syst.*, DOI: 10.1109/TSMC.2019.2898204, 2019.

[26] W. S. Chen and L. C. Jiao, "Finite-time stability theorem of stochastic nonlinear systems," *Automatica*, vol. 46, pp. 2105–2108, 2010.

[27] J. Yin and S. Khoo, "Correspondence: Comments on finite-time stability theorem of stochastic nonlinear systems [Automatica 46 (2010) 2105–2108]," *Automatica*, vol. 47, no. 7, pp. 1542–1543, 2011.

[28] W. S. Chen and L. C. Jiao, "Correspondence: Authors' reply to comments on finite-time stability theorem of stochastic nonlinear systems [Automatica 46 (2010) 2105–2108]," *Automatica*, vol. 47, pp. 1544–1545, 2011.

[29] J. Yin, S. Khoo, Z. Man, and X. Yu, "Finite-time stability and instability of stochastic nonlinear systems," *Automatica*, vol. 47, no. 12, pp. 2671–2677, 2011.

[30] S. Y. Khoo, J. L. Yin, Z. H. Man, and X. H. Yu, "Finite-time stabilization of stochastic nonlinear systems in strict feedback form," *Automatica*, vol. 49, no. 5, pp. 1403–1410, 2013.

[31] J. L. Yin and S. Y. Khoo, "Continuous finite-time state feedback stabilizers for some nonlinear stochastic systems," *Int. J. Robust Nonlinear Control*, vol. 25, no. 11, pp. 1581–1600, 2015.

[32] H. Min, S. Xu, Y. Li, Y. Chu, Y. Wei, and Z. Zhang, "Adaptive finite-time control for stochastic nonlinear systems subject to unknown covariance noise," *J. Franklin Institute*, vol. 355, pp. 2645–2661, 2018.

[33] W. T. Zha, J. Y. Zhai, S. M. Fei, and Y. J. Wang, "Finite-time

stabilization for a class of stochastic nonlinear systems via output feedback," *ISA Trans.*, vol. 53, no. 3, pp. 709–716, 2014.

[34] S. P. Huang and Z. R. Xiang, "Finite-time stabilization of a class of switched stochastic nonlinear systems under arbitrary switching," *Int. J. Robust Nonlin. Control*, vol. 26, no. 10, pp. 2136–2152, 2016.

[35] M. M. Jiang, X. J. Xie, and K. Zhang, "Finite-time stabilization of stochastic high-order nonlinear systems with FT-SISS inverse dynamics," *IEEE Trans. Autom. Control*, vol. 64, no. 1, pp. 313–320, 2019.

[36] X. Yu, J., J. L. Yin, and S.Y. Khoo, "Generalized Lyapunov criteria on finite-time stability of stochastic nonlinear systems," *Automatica*, vol. 107, pp. 183–189, 2019.

[37] H. Min, S. Xu, and Z. Zhang, "Adaptive finite-time stabilization of stochastic nonlinear systems subject to full-state constraints and input saturation," *IEEE Trans. Autom. Control*, vol. 66, no. 3, pp. 1306–1313, 2021.

[38] A. Polyakov, "Nonlinear feedback design for fixed-time stabilization of linear control systems," *IEEE Trans. Autom. Control*, vol. 57, no. 8, pp. 2106–2110, 2012.

[39] Z. Zuo, "Non-singular fixed-time terminal sliding mode control of nonlinear systems," *IET Control Theory Appl.*, vol. 9, no. 4, pp. 545–552, 2015.

[40] S. Shi, J. Gu, S. Xu, and H. Min, "Globally fixed-time high-order sliding mode control for new sliding mode systems subject to mismatched terms and its application," *IEEE Trans. Industrial Electron.*, vol. 67, no. 12, pp. 10776–10786, 2020.

[41] C. Hu, J. Yu, Z. Chen, H. Jiang, and T. Huang, "Fixed-time stability of dynamical systems and fixed-time synchronization of coupled discontinuous neural networks," *Neural Netw.*, vol. 89, pp. 74–83, 2017.

[42] H. Yao and D. Ye, "Fixed-time stabilization of uncertain strict-feedback nonlinear systems via a bi-limit-like strategy," *Int. J. Robust Nonlin. Control*, vol. 28, no. 17, pp. 5531–5544, 2018.

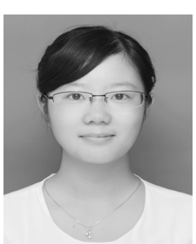
[43] C. Hua, P. Ning, K. Li, and X. Guan, "Fixed-time prescribed tracking control for stochastic nonlinear systems with unknown measurement sensitivity," *IEEE Trans. Cybern.*, 2020. DOI: 10.1109/TCYB.2020.3012560

[44] J. Yu, S. Yu, J. Li, and Y. Yan, "Fixed-time stability theorem of stochastic nonlinear systems," *Int. J. Control*, vol. 92, no. 9, pp. 2194–2220, 2019.

[45] Z. Song, P. Li, J. Zhai, Z. Wang, and X. Huang, "Global fixed-time stabilization for switched stochastic nonlinear systems under rational switching powers," *Appl. Math. Comput.*, vol. 387, 2020.

[46] Y. Liang, Y. X. Li, and Z. Hou, "Adaptive fixed-time tracking control for stochastic pure-feedback nonlinear systems," *Int. J. Adaptive Control Signal Processing*, 2021. DOI: 10.1002/acs.3285

[47] K. Li, Y. M. Li, and G. Zong, "Adaptive fuzzy fixed-time decentralized control for stochastic nonlinear systems," *IEEE Trans. Fuzzy Syst.*, 2020. DOI: 10.1109/TFUZZ.2020.3022570



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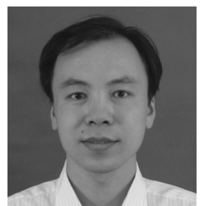

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