Fuzzy Set-Membership Filtering for Discrete-Time Nonlinear Systems

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***Abstract*—In this article, the problem of state estimation is addressed for discrete-time nonlinear systems subject to additive unknown-but-bounded noises by using fuzzy set-membership filtering. First, an improved T-S fuzzy model is introduced to achieve highly accurate approximation via an affine model under each fuzzy rule. Then, compared to traditional prediction-based ones, two types of fuzzy set-membership filters are proposed to effectively improve filtering performance, where the structure of both filters consists of two parts: prediction and filtering. Under the locally Lipschitz continuous condition of membership functions, unknown membership values in the estimation error system can be treated as multiplicative noises with respect to the estimation error. Real-time recursive algorithms are given to find the minimal ellipsoid containing the true state. Finally, the proposed optimization approaches are validated via numerical simulations of a one-dimensional and a three-dimensional discrete-time nonlinear systems.**

***Index Terms*—Affine model, membership functions, set-membership filtering, stability, Takagi-Sugeno fuzzy modeling.**

I. Introduction

MANY applications involve nonlinear systems and unwanted noises. The noise arises from inputs of the system and outputs derived with the aid of a noisy sensor. Filtering is necessary to obtain information about some quantities that are essentially internal to the system. As a result, an extensive body of theory relating to filter design has grown, such as the famous Kalman filter and its extensions [1]–[4], the $\mathcal{H}_\infty$ filter [5]–[7], and several others [8]–[10]. Kalman filter is a minimum variance estimator, and the $\mathcal{H}_\infty$ filter minimizes the worst-case estimation error. The system noise, including process noise and measurement noise, is normally assumed to be statistically known in the Kalman filtering framework. Nevertheless, the bound of noises can be obtained in many practical applications, such as radar, voltage control [11], system guidance and navigation, and target tracking and attacking. This leads to set-membership filtering [12]–[14]. The idea of set-membership filtering is to give an ellipsoid centered at the state estimate containing the true state, and the size of the ellipsoidal set is subsequently minimized. A reliable localization problem was discussed in [15] for autonomous mobile robots in an unstructured environment. The need of statistical information of the noise was relaxed and linearization errors were taken into account by the proposed set-membership filter. Set-membership filtering was recently implemented in radar applications in [16] to estimate the position of an octorotor. Note that a great many applications involve linear systems, as it spurs linear set-membership filtering. However, most applications involve nonlinear systems. By comparison, nonlinear set-membership filter design is very hard, if not impossible, in many instances. For this reason, there has been a lack of attention for nonlinear set-membership filters. An attempt is made in this article to design fuzzy set-membership filter algorithms for nonlinear systems.

Due to its high-precision modeling and low-complexity computation, we first use a Takagi-Sugeno (T-S) fuzzy model to approximate the nonlinear system. T-S fuzzy modeling is based on a fuzzy partition of the state space, known as IF-THEN rules. Under each fuzzy rule, a basis system is formed. The fuzzy model is given by the aggregation of the basis systems, that is, a convex combination of the basis systems weighted by membership functions. The commonly used basis systems are linear systems [17]–[26]. However, nonlinear systems in general are hard to be approximated with high accuracy by using only the combination of linear systems. To increase the accuracy of the T-S fuzzy model, we use affine basis systems. Based on the obtained T-S fuzzy model, two fuzzy set-membership filters are constructed to find minimal ellipsoids centered at the estimated states that contain the true states. Previous works addressed the problem of set-membership filtering using *prediction*-type observers [19], [27], that is, y_k and $\hat{x}_k$ are used to estimate x_{k+1}. Our constructed filter is comprised of both prediction and filtering. The prediction step will be executed using $\hat{x}_k$, the fuzzy model and the knowledge of the process noise; the filtering step involves updating the predicted value of x_{k+1} with measures y_{k+1}. Furthermore, we exploit properties of membership functions, which is critical to address the stability problem of the estimation error system. However, set-membership

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filtering problems in the literature usually do not include the stability analysis, as we do in this article. The problem of finding a minimal ellipsoid which contains the true system state is eventually transformed into linear objective minimization with linear matrix inequality (LMI) constraints.

The main contributions of this article are summarized as follows.

1) Under each fuzzy subspace, we utilize an affine model to achieve a more accurate approximation compared with linear models when the same fuzzy rules are applied. A complete procedure for identifying the T-S fuzzy model is given.

2) Two fuzzy set-membership filters, namely, FSMF1 and FSMF2, are proposed, where the structure of both filters consists of prediction and filtering. Compared with the *prediction*-type observers, the proposed filters can further reduce the state estimation error by using the most recent measurements.

3) Both the proposed filters are capable of stabilizing the estimation error system and attenuating noises, while filters in most previous studies focus on noise attenuation only. In addition, the design of FSMF2 takes full advantage of the locally Lipschitz continuous condition of the membership function, which has not been explored in previous studies. The simulation example shows a case that FSMF2 completely rejects the measurement noise for a stable nonlinear system.

The remainder of this article is structured as follows. First, the fuzzy set-membership estimation problem is formulated in Section II where the T-S fuzzy model is discussed and two fuzzy set-membership filters are designed to cater to different scenarios. Then, the filter designs are discussed to ensure that the true state is contained in an ellipsoidal set centered at the estimated state in Section III. In Section IV, a one-dimensional nonlinear system and a vertical mass-spring system are given to evaluate the estimation performance of the proposed two filters. Our conclusions and discussions of future research are included in Section V. The proofs of lemmas are given in Appendices.

II. Problem Formulation and Filter Design

A. Nonlinear Plant

Let us study the class of discrete-time systems with a prototype of the nonlinear, finite dimensional system depicted in Fig. 1. Given a finite horizon $[0, T]$, the system is described by state-space equations:

$$x_{k+1} = f(x_k) + B_k w_k \tag{1}$$

$$y_k = C_k x_k + D_k v_k. \tag{2}$$

The n dimensional system state x_k, with the initial state x_0, belongs to a compact set $\mathbb{X}$, y_k is the measurement at time k, $f(\cdot)$ is a known continuous nonlinear function of x_k defined on $\mathbb{X}$, and B_k, C_k, and D_k are known matrices. The input to the system $w_k \in \mathbb{R}^w$ signifies the process noise constrained in the following ellipsoidal set

$$\mathcal{W}_k = \{w_k | w_k^T Q_k^{-1} w_k \le 1\} \tag{3}$$

where $Q_k > 0$ is a known symmetric positive definite matrix with compatible dimension, characterizing the orientation and the size of the ellipsoid, and $v_k \in \mathbb{R}^v$ is the measurement noise constrained in the following ellipsoidal set

$$\mathcal{V}_k = \{v_k | v_k^T R_k^{-1} v_k \le 1\} \tag{4}$$

where $R_k > 0$ is a known symmetric positive definite matrix with compatible dimension.

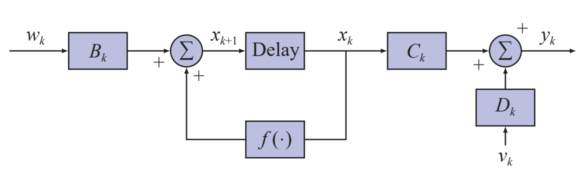

Fig. 1. Finite-dimensional nonlinear systems serving as basic signal model.

B. Fuzzy Model and Fuzzy Filter

We adapt fuzzy filtering algorithms to the nonlinear system (1). Toward this end, the nonlinear model is represented by a set of affine models which are connected via fuzzy membership functions under a set of fuzzy rules.

Plant Rule i: If $x_{1,k}$ is $M_{1,i}$, $x_{2,k}$ is $M_{2,i}$, $\ldots$, and $x_{n,k}$ is $M_{n,i}$, then

$$f_i(x_k) = A_i x_k + b_i, \quad i \in \{1, \ldots, N\} \tag{5}$$

where $x_k = [x_{1,k}, \ldots, x_{n,k}]^T$, $M_{h,i}$ for $h \in \{1, \ldots, n\}$ are fuzzy sets representing a fuzzy subset where fuzzy rules can be applied for reasoning, N represents the number of fuzzy rules, and A_i and b_i are constant matrices and vectors, respectively. Assume that A_i is nonsingular and the pair $[A_i, C_{k+1}]$ is completely observable for $i \in \{1, \ldots, N\}$. The dimensions of A_i and b_i are determined by the nonlinear function f. The fuzzy approximation of the nonlinear function f inferred from (5) for all N rules is obtained as

$$f(x_k) = \sum_{i=1}^{N} \beta_i(x_k) f_i(x_k) \tag{6}$$

with the normalized weight $\beta_i(x_k)$ for Rule i defined by

$$\beta_i(x_k) = \frac{\mu_i(x_k)}{\sum_{i=1}^{N} \mu_i(x_k)} \tag{7}$$

where $\mu_i(x_k) \ge 0$ are fuzzy membership functions. Thus, we obtain:

$$\beta_i \ge 0, \quad \sum_{i=1}^{N} \beta_i = 1. \tag{8}$$

In the following example, we will show how to obtain the fuzzy model for a given nonlinear function.

Example 1: Consider the nonlinear function $f(x_1, x_2) = x_1 x_2$. It is aimed to approximate the function $f(x_1, x_2)$ in the intervals $x_1 \in [-1.5, 1.5]$ and $x_2 \in [0.05, 0.15]$ using the T-S fuzzy modeling method. Let us define four fuzzy sets with their corresponding normalized weights as follows:

$$\beta_1(x) = \frac{(1.5 - x_1)(0.15 - x_2)}{0.3},$$
$$\beta_2(x) = \frac{(1.5 - x_1)(x_2 - 0.05)}{0.3},$$
$$\beta_3(x) = \frac{(x_1 + 1.5)(0.15 - x_2)}{0.3},$$
$$\beta_4(x) = \frac{(x_1 + 1.5)(x_2 - 0.05)}{0.3}.$$

In order to identify the parameters A_i and b_i for $i \in \{1, \ldots, 4\}$, we take 1000 points for x_1 and x_2 uniformly distributed in the interval $[-1.5, 1.5]$ and $[0.05, 0.15]$, respectively. By applying the weighting parameters approach with $\gamma = 0.001$ [28], the fuzzy system becomes

Rule 1*: If* x_1 *is around* –1.5, *and* x_2 *is around* 0.05, *then*

$$f_1(x_1, x_2) = 0.0345x_1 - 0.0057x_2 - 0.023; \tag{9}$$

Rule 2*: If* x_1 *is around* –1.5, *and* x_2 *is around* 0.15, *then*

$$f_2(x_1, x_2) = -0.069 + 0.1034x_1 - 0.0057x_2; \tag{10}$$

Rule 3: *If* x_1 *is around* 1.5, *and* x_2 *is around* 0.05, *then*

$$f_3(x_1, x_2) = 0.023 + 0.0345x_1 + 0.0057x_2; \tag{11}$$

Rule 4: *If* x_1 *is around* 1.5, *and* x_2 *is around* 0.15, *then*

$$f_4(x_1, x_2) = 0.069 + 0.1034x_1 + 0.0057x_2. \tag{12}$$

The error between the nonlinear system and the identified fuzzy model is within $1.857 * 10^{-4}$. As can be seen, this fuzzy model is obviously very close to the nonlinear function and presents a good approximation. Besides, the occurrence of ill-conditioned matrices during modeling is avoided.

Here we propose two T-S fuzzy set-membership filters, which will be introduced below. We define $\hat{x}_0$ to be the estimated value of x_0 given no measurements. Then $\hat{x}_k \in \mathbb{X}$ can be obtained recursively for $k \in \{1, \ldots, T\}$.

1) FSMF1: The first T-S fuzzy set-membership filter, named FSMF1, comprises the systems depicted in Fig. 2 and Fig. 3 and is described for $k \geq 0$ by the equations:

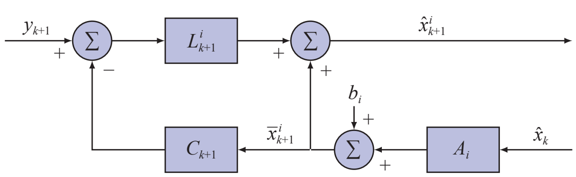

Fig. 2. Fuzzy filter under filter rule *i*.

Filter Rule i: If $\hat{x}_{1,k}$ is $M_{1,i}$, $\hat{x}_{2,k}$ is $M_{2,i}, \ldots$, and $\hat{x}_{n,k}$ is $M_{n,i}$, then

$$\bar{x}^i_{k+1} = A_i\hat{x}_k + b_i, \quad i \in \{1, \ldots, N\} \tag{13}$$

$$\hat{x}^i_{k+1} = \bar{x}^i_{k+1} + L^i_{k+1}(y_{k+1} - C_{k+1}\bar{x}^i_{k+1}) \tag{14}$$

where the fuzzy sets $M_{h,i}$ for $h \in \{1, \ldots, n\}$ are the same as the ones used in the plant rules, A_i and b_i are defined in (5) and can be derived by the T-S fuzzy modeling, and the gain matrices L^i_{k+1} are to be determined.

One obtains the filtered estimate $\hat{x}_{k+1}$ from (14) of all N rules as follows

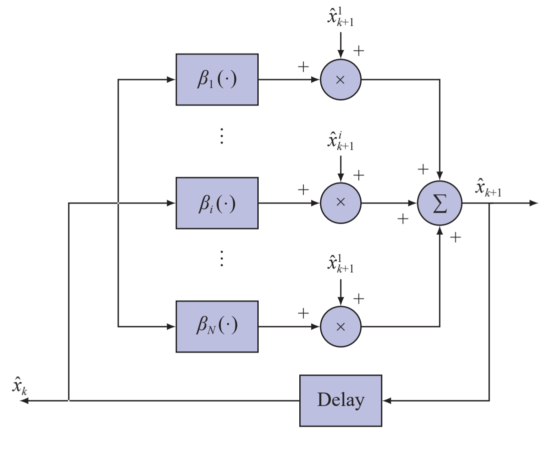

Fig. 3. Fuzzy filtering fusion.

$$\hat{x}_{k+1} = \sum_{i=1}^{N} \beta_i(\hat{x}_k)\hat{x}^i_{k+1} \tag{15}$$

where the normalized weight β_i is defined in (7). Note that $\beta_i(\hat{x}_k)$ is defined from $\beta_i(x_k)$.

The filter depicted in Fig. 2 is designed under each rule according to (13) and (14). The defuzzification process is shown in Fig. 3 according to (15) which is the aggregate fuzzy estimate. The output $\hat{x}^i_{k+1}$ in Fig. 2 is the input $\hat{x}^i_{k+1}$ in Fig. 3, and one of the outputs $\hat{x}_k$ in Fig. 3 is the input $\hat{x}_k$ in Fig. 2.

2) FSMF2: The second T-S fuzzy set-membership filter, named to FSMF2, comprises the systems depicted in Fig. 4 and Fig. 5 and is described for $k \geq 0$ by the equations:

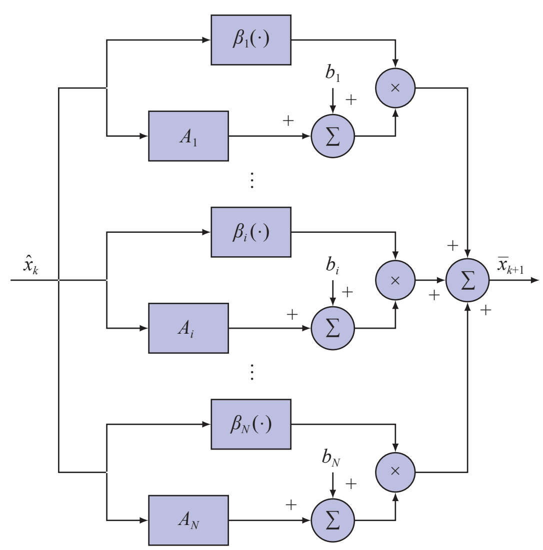

Fig. 4. Fuzzy prediction.

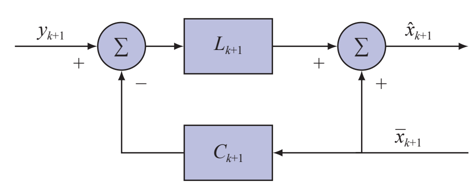

Fig. 5. Fuzzy filtering.

Filter Rule i: If $\hat{x}_{1,k}$ is $M_{1,i}$, $\hat{x}_{2,k}$ is $M_{2,i}, \ldots$, and $\hat{x}_{n,k}$ is $M_{n,i}$, then

$$\bar{x}_{k+1} = \sum_{i=1}^{N} \beta_i(\hat{x}_k)(A_i\hat{x}_k + b_i) \tag{16}$$

$$\hat{x}_{k+1} = \bar{x}_{k+1} + L_{k+1}(y_{k+1} - C_{k+1}\bar{x}_{k+1}) \tag{17}$$

where the fuzzy sets $M_{h,i}$ for $h \in \{1, \ldots, n\}$ are the same as the ones used in the plant rules, A_i and b_i are defined in (5) and can be derived by the T-S fuzzy modeling for $i \in \{1, \ldots, N\}$, and the gain matrix L_{k+1} is to be determined.

The predictor depicted in Fig. 4 is designed according to (16) using the T-S fuzzy model. The filter depicted in Fig. 5 is designed according to (17). The delayed output $\hat{x}_{k+1}$ in Fig. 5 is the input $\hat{x}_k$ in Fig. 4, and the output $\bar{x}_{k+1}$ in Fig. 4 is one of the inputs in Fig. 5.

C. Objective

The estimation error at time k is defined as

$$e_k = x_k - \hat{x}_k. \tag{18}$$

The initial state x_0 and initial estimate $\hat{x}_0$ are assumed to be confined in the following ellipsoid

$$e_0^T P_0^{-1} e_0 \leq 1 \tag{19}$$

where P_0 is a given symmetric positive-definite matrix.

The objective of this article is to find a minimal ellipsoid centered at $\hat{x}(k)$ which contains $x(k)$, that is,

$$e_k^T P_k^{-1} e_k \leq 1 \tag{20}$$

by optimizing L_k^i for $i \in \{1, \ldots, N\}$ in FSMF1, and L_k in FSMF2, respectively, for $k \geq 1$.

III. Main Results

A. Preliminaries

Lemma 1: If the pair $[A, C]$ is completely observable and A is nonsingular, then there exists an L such that an arbitrary characteristic polynomial $(I - LC)A$ can be obtained.

The proof of Lemma 1 is presented in Appendix A.

Lemma 2: Given matrices $\Phi_i \in \mathbb{R}^{n\times(n+m)}$, $P = P^T \in \mathbb{R}^{n\times n}$, $Y \in \mathbb{R}^{p\times(n+m)}$, $Z = Z^T \in \mathbb{R}^{p\times p}$, $X = X^T \in \mathbb{R}^{(n+m)\times(n+m)}$, the following inequality

$$\begin{bmatrix} \sum_{i=1}^N \beta_i \Phi_i^T P^{-1} \sum_{i=1}^N \beta_i \Phi_i + X & * \\ Y & Z \end{bmatrix} \leq 0 \tag{21}$$

holds for any β_i satisfying (8) if the following N inequalities

$$\begin{bmatrix} X & * & * \\ Y & Z & * \\ \Phi_i & 0 & -P \end{bmatrix} \leq 0 \tag{22}$$

hold for $i \in \{1, 2, \ldots, N\}$.

The proof of Lemma 2 is presented in Appendix B.

B. Design of FSMF1

Define the estimation error under fuzzy rule i as

$$e_{k+1}^i = x_{k+1} - \hat{x}_{k+1}^i, \quad i \in \{1, \ldots, N\}$$

Then, from (15) and (18),

$$e_{k+1} = \sum_{i=1}^{N} \beta_i(\hat{x}_k) e_{k+1}^i. \tag{23}$$

From (14), we can obtain the state estimation error e_{k+1}^i as

$$\begin{aligned} e_{k+1}^i &= (I - L_{k+1}^i C_{k+1})(x_{k+1} - \bar{x}_{k+1}^i) \\ &\quad - L_{k+1}^i D_{k+1} v_{k+1}. \end{aligned} \tag{24}$$

Let us calculate $x_{k+1} - \bar{x}_{k+1}^i$ using (1), (5), (6), (13) and (18)

$$\begin{aligned} x_{k+1} - \bar{x}_{k+1}^i &= \sum_{j=1}^{N} \beta_j(x_k)(A_j x_k + b_j) + B_k w_k - A_i \hat{x}_k - b_i \\ &= \sum_{j=1}^{N} \beta_j(x_k) A_j e_k + \sum_{j=1}^{N} \beta_j(x_k)(A_j - A_i)\hat{x}_k \\ &\quad + \sum_{j=1}^{N} \beta_j(x_k)(b_j - b_i) + B_k w_k. \end{aligned} \tag{25}$$

Based on (24)–(25), we can obtain

$$\begin{aligned} e_{k+1}^i &= \sum_{j=1}^{N} \beta_j(x_k)(I - L_{k+1}^i C_{k+1}) \\ &\quad \times (A_j e_k + (A_j - A_i)\hat{x}_k + b_j - b_i + B_k w_k) \\ &\quad - \sum_{j=1}^{N} \beta_j(x_k) L_{k+1}^i D_{k+1} v_{k+1}. \end{aligned} \tag{26}$$

Let us define η_k as

$$\eta_k = \begin{bmatrix} e_k^T & w_k^T & v_{k+1}^T & 1_{n\times 1}^T \end{bmatrix}^T. \tag{27}$$

Then, from (23) the estimation error can be rewritten as

$$e_{k+1} = \sum_{j=1}^{N} \beta_j(x_k) \Phi_{j,k} \eta_k \tag{28}$$

where

$$\begin{aligned} \Phi_{j,k} &= [\, \Phi_{j1,k} \;\; \Phi_{j2,k} \;\; \Phi_{j3,k} \;\; \Phi_{j4,k} \,] \\ \Phi_{j1,k} &= \sum_{i=1}^{N} \beta_i(\hat{x}_k)(I - L_{k+1}^i C_{k+1}) A_j \\ \Phi_{j2,k} &= \sum\nolimits_{i=1}^{N} \beta_i(\hat{x}_k)(I - L_{k+1}^i C_{k+1}) B_k \\ \Phi_{j3,k} &= \sum\nolimits_{i=1}^{N} \beta_i(\hat{x}_k) L_{k+1}^i D_{k+1} \\ \Phi_{j4,k} &= \sum\nolimits_{i=1}^{N} \beta_i(\hat{x}_k)(I - L_{k+1}^i C_{k+1}) \\ &\quad \times ((A_j - A_i)\hat{x}_k + b_j - b_i). \end{aligned}$$

Given P_0 and k, the minimal ellipsoid P_{k+1} can be found recursively for $k \geq 0$ by solving the following optimization problem:

$$\begin{aligned} \min_{P_{k+1},\, L_{k+1}^i} \quad & \mathrm{tr}\{P_{k+1}\} \\ \text{s.t.} \quad & e_{k+1}^T P_{k+1}^{-1} e_{k+1} \leq 1 \end{aligned} \tag{29}$$

$$e_k^T P_k^{-1} e_k \leq 1 \tag{30}$$

$$w_k^T Q_k^{-1} w_k \leq 1 \tag{31}$$

$$v_{k+1}^T R_{k+1}^{-1} v_{k+1} \leq 1. \tag{32}$$

It is worth mentioning that the true values of the state x_k and $\beta_j(x_k)$ are unknown for $j \in \{1,\ldots,N\}$. Thus, the problem becomes how we guarantee constraint (29) with an unknown x_k. This challenge is overcome by taking advantage of (8), (28) and constraints (30)–(32). The main result is shown in the following theorem.

Theorem 1: For the discrete-time nonlinear system (1), the system output (2), and FSMF1 (13)–(15), a feasible P_{k+1} in (29) can be found by solving the following N LMIs:

$$\begin{bmatrix} -P_{k+1} & \Phi_{j,k} \\ * & -\Omega_{j,k} \end{bmatrix} \leq 0, \quad j \in \{1,\ldots,N\} \tag{33}$$

where P_{k+1} determines the minimal ellipsoid (20), $\Phi_{j,k}$ is defined in (28), $\sigma_{c,j,k}$ for $c \in \{1,2,3\}$ are positive scalars, and

$$\Omega_{j,k} = \mathrm{diag}\{\sigma_{1,j,k} P_k^{-1}, \sigma_{2,j,k} Q_k^{-1}, \sigma_{3,j,k} R_{k+1}^{-1}, \Omega_{4,j,k}\}$$
$$\Omega_{4,j,k} = 1 - \sigma_{1,j,k} - \sigma_{2,j,k} - \sigma_{3,j,k}.$$

Proof: According to (28), (29) can be rewritten as

$$\eta_k^T \sum_{j=1}^{N} \beta_j(x_k) \Phi_{j,k}^T P_{k+1}^{-1} \sum_{j=1}^{N} \beta_j(x_k) \Phi_{j,k} \eta_k \leq 1 \tag{34}$$

and the constraints (30)–(32) can be written as

$$\begin{cases} \eta_k^T \mathrm{diag}\{P_k^{-1}, 0, 0, -1\} \eta_k \leq 0 \\ \eta_k^T \mathrm{diag}\{0, Q_k^{-1}, 0, -1\} \eta_k \leq 0 \\ \eta_k^T \mathrm{diag}\{0, 0, R_{k+1}^{-1}, -1\} \eta_k \leq 0. \end{cases} \tag{35}$$

Then, by applying S-Procedure [29], (35) implies (34) if there exists a positive scalar $\sigma_{c,j,k}$ for $c \in \{1,2,3\}$, $j \in \{1,\ldots,N\}$ such that

$$\sum_{j=1}^{N} \beta_j(x_k) \Phi_{j,k}^T P_{k+1}^{-1} \sum_{j=1}^{N} \beta_j(x_k) \Phi_{j,k} - \Omega_{j,k} \leq 0. \tag{36}$$

By taking Lemma 2 into consideration with $Y = 0$, the inequality (36) holds if the following N inequalities

$$\begin{bmatrix} -P_{k+1} & \Phi_{j,k} \\ * & -\Omega_{j,k} \end{bmatrix} \leq 0 \tag{37}$$

hold for $j \in \{1,\ldots,N\}$, which are the conditions in (33).

Based on the above discussion, we conclude that the constraint (29) is satisfied if the inequality (33) holds. By induction, the true state is guaranteed to be within the ellipsoids determined by (20) for all $k \geq 1$ if (19) holds. ■

Based on Lemma 1, a sequence of L_{k+1}^i for $i \in \{1,\ldots,N\}$ always exist to make $(I - L_{k+1}^i C_{k+1}) A_j$ for $j \in \{1,\ldots,N\}$ stable. However, finding a necessary and sufficient condition to guarantee that $\sum_{j=1}^{N} \beta_j(x_k)(I - L_{k+1}^i C_{k+1}) A_j$ in (26) is stable for unknown x_k is an open problem. A sufficient condition can be found through (33) by deleting unrelated terms. Theorem 1 shows a feasibility condition of P_{k+1}. In what follows, the minimal ellipsoid will be given through the solution of the following convex optimization problem.

For the discrete-time nonlinear system (1), the system output (2) and FSMF1 (13)–(15), the minimal ellipsoidal set parameterized by P_{k+1} in (20) can be found by solving the following optimization problem:

$$\min_{P_{k+1}, L_k, \sigma_{c,j,k}} \mathrm{tr}\{P_{k+1}\} \tag{38}$$

$$\text{s.t.} \quad \begin{bmatrix} -P_{k+1} & \Phi_{j,k} \\ * & -\Omega_{j,k} \end{bmatrix} \leq 0 \tag{39}$$

where $\Omega_{j,k}$ for $j \in \{1,\ldots,N\}$ is defined in Theorem 1 and $\Phi_{j,k}$ is defined in (28).

The equations of FSMF1 for the discrete-time nonlinear system (1) are summarized as follows:

- **Fuzzy modeling:**

$$f(x_k) = \sum_{i=1}^{N} \beta_i(x_k)(A_i x_k + b_i)$$

- **Fuzzy prediction:**

$$\bar{x}_{k+1}^i = A_i \hat{x}_k + b_i, \ i \in \{1,\ldots,N\}$$

- **Fuzzy filtering:**

$$\hat{x}_{k+1}^i = \bar{x}_{k+1}^i + L_{k+1}^i (y_{k+1} - C_{k+1} \bar{x}_{k+1}^i), \ i \in \{1,\ldots,N\}$$

- **Defuzzy filtering:**

$$\hat{x}_{k+1} = \sum_{i=1}^{N} \beta_i(\hat{x}_k) \hat{x}_{k+1}^i.$$

C. Design of FSMF2

To use the fuzzy filter FSMF2, the membership function $\mu(\cdot)$ is specially chosen such that the normalized weight $\beta_i(\cdot)$ satisfies the locally Lipschitz continuous condition

$$\|\beta_i(\alpha) - \beta_i(\gamma)\| \leq F_i \|\alpha - \gamma\| \tag{40}$$

for all $\alpha \in \mathbb{X}$ and $\gamma \in \mathbb{X}$. From (17), we can obtain the state estimation error e_{k+1} as

$$\begin{aligned} e_{k+1} = &(I - L_{k+1} C_{k+1})(x_{k+1} - \bar{x}_{k+1}) \\ &- L_{k+1} D_{k+1} v_{k+1}. \end{aligned} \tag{41}$$

We calculate $x_{k+1} - \bar{x}_{k+1}$ using (1), (5), (6), (16) and (18)

$$\begin{aligned} x_{k+1} - \bar{x}_{k+1} &= \sum_{i=1}^{N} \beta_i(x_k)(A_i x_k + b_i) + B_k w_k \\ &\quad - \sum_{i=1}^{N} \beta_i(\hat{x}_k)(A_i \hat{x}_k + b_i) \\ &= \sum_{i=1}^{N} (\beta_i(x_k) - \beta_i(\hat{x}_k))(A_i \hat{x}_k + b_i) \\ &\quad + \sum_{i=1}^{N} \beta_i(x_k) A_i e_k + B_k w_k. \end{aligned} \tag{42}$$

We denote

$$g(x_k, \hat{x}_k) \triangleq \sum_{i=1}^{N} (\beta_i(x_k) - \beta_i(\hat{x}_k))(A_i \hat{x}_k + b_i). \tag{43}$$

Notice that $\hat{x}_k$, $\beta_i(\hat{x}_k)$, A_i and b_i for $i \in \{1,2,\dots,N\}$ are known to the filter at time k. Since β_i satisfies locally Lipschitz continuous condition, there exists a function $h(\hat{x}_k)$ such that

$$\|g(x_k,\hat{x}_k)\| \le h(\hat{x}_k)\|e_k\|. \tag{44}$$

The value of $h(\hat{x}_k)$ can be calculated by solving the following optimization problem:

$$\begin{aligned} \max_{x_k \in \mathbb{X}} \quad & \frac{\|g(x_k,\hat{x}_k)\|}{\|e_k\|} \\ \text{s.t.} \quad & \|e_k\| \neq 0 \\ & (x_k-\hat{x}_k)^T P_k^{-1}(x_k-\hat{x}_k) \le 1. \end{aligned}$$

Based on (41)–(43), we get

$$\begin{aligned} e_{k+1} = & \sum_{i=1}^{N} \beta_i(x_k)(I-L_{k+1}C_{k+1})(A_i e_k + B_k w_k) \\ & + (I-L_{k+1}C_{k+1})g(x_k,\hat{x}_k) - L_{k+1}D_{k+1}v_{k+1}. \end{aligned} \tag{45}$$

Define

$$\eta_k = \begin{bmatrix} e_k^T & g^T(x_k,\hat{x}_k) & w_k^T & v_{k+1}^T \end{bmatrix}^T. \tag{46}$$

Then, (45) can be rewritten as

$$e_{k+1} = \sum_{i=1}^{N} \beta_i(x_k)\Phi_{i,k}\eta_k \tag{47}$$

where

$$\begin{aligned} \Phi_{i,k} = [\, & (I-L_{k+1}C_{k+1})A_i \quad I-L_{k+1}C_{k+1} \\ & (I-L_{k+1}C_{k+1})B_k \quad -L_{k+1}D_{k+1}\,]. \end{aligned}$$

Given P_0 and k, the minimal ellipsoid P_{k+1} can be found recursively for $k \ge 0$ by solving the following optimization problem:

$$\min_{P_{k+1},\,L_{k+1}} \ \mathrm{tr}\{P_{k+1}\}$$

$$\text{s.t.} \quad e_{k+1}^T P_{k+1}^{-1} e_{k+1} \le 1 \tag{48}$$

$$e_k^T P_k^{-1} e_k \le 1 \tag{49}$$

$$g^T(x_k,\hat{x}_k)g(x_k,\hat{x}_k) \le h^2(\hat{x}_k)e_k^T e_k \tag{50}$$

$$w_k^T Q_k^{-1} w_k \le 1 \tag{51}$$

$$v_{k+1}^T R_{k+1}^{-1} v_{k+1} \le 1. \tag{52}$$

It is worth mentioning that the true value of the state x_k is unknown, as is the value of $\beta_i(x_k)$ for $i \in \{1,\dots,N\}$. The problem becomes how to guarantee the constraint (48) with unknown x_k. This challenge is overcome by taking advantage of (8), (47) and constraints (49)–(52), and the main result is shown in the following theorem.

Theorem 2: For the discrete-time nonlinear system (1), the system output (2) and FSMF2 (16)–(17), a feasible P_{k+1} in (48) can be found by solving the following N LMIs:

$$\begin{bmatrix} -\Omega_{i,k} & * & * \\ 0 & -\Psi_{i,k} & * \\ \Phi_{i,k} & 0 & -P_{k+1} \end{bmatrix} \le 0, \quad i \in \{1,\dots,N\} \tag{53}$$

where P_{k+1} determines the minimal ellipsoid (20), $\Phi_{i,k}$ is defined in (47), $\sigma_{d,k}$ for $d \in \{1,\dots,4\}$ are positive scalars, and

$$\begin{aligned} \Omega_{i,k} &= \mathrm{diag}\{\Omega_{1,i,k}, \sigma_{2,k}I, \sigma_{3,k}Q_k^{-1}, \sigma_{4,k}R_{k+1}^{-1}\} \\ \Omega_{1,i,k} &= \sigma_{1,k}P_k^{-1} - \sigma_{2,k}h^2(\hat{x}_k)I \\ \Psi_{i,k} &= 1-\sigma_{1,k}-\sigma_{3,k}-\sigma_{4,k}. \end{aligned}$$

Proof: According to (47), (48) can be written as

$$\eta_k^T \sum_{i=1}^{N} \beta_i(x_k)\Phi_{i,k}^T P_{k+1}^{-1} \sum_{i=1}^{N} \beta_i(x_k)\Phi_{i,k}\eta_k \le 1 \tag{54}$$

and the constraints (49)–(52) can be written as

$$\begin{cases} \eta_k^T \mathrm{diag}\{P_k^{-1},0,0,0\}\eta_k \le 1 \\ \eta_k^T \mathrm{diag}\{-h^2(\hat{x}_k)I, I, 0, 0\}\eta_k \le 0 \\ \eta_k^T \mathrm{diag}\{0,0,Q_k^{-1},0\}\eta_k \le 1 \\ \eta_k^T \mathrm{diag}\{0,0,0,R_k^{-1}\}\eta_k \le 1. \end{cases} \tag{55}$$

Then, applying S-Procedure, (55) implies (54) if there exist positive scalars $\sigma_{d,k}$ for $d \in \{1,\dots,4\}$ such that

$$\begin{bmatrix} \sum_{i=1}^{N} \beta_i(x_k)\Phi_{i,k}^T P_{k+1}^{-1} \sum_{i=1}^{N} \beta_i(x_k)\Phi_{i,k} - \Omega_{i,k} & * \\ 0 & -\Psi_{i,k} \end{bmatrix} \le 0.$$

According to Lemma 2, the above inequality holds if the following N inequalities

$$\begin{bmatrix} -\Omega_{i,k} & * & * \\ 0 & -\Psi_{i,k} & * \\ \Phi_{i,k} & 0 & -P_{k+1} \end{bmatrix} \le 0 \tag{56}$$

hold for $i \in \{1,\dots,N\}$, which are the conditions in (53).

By induction, we infer that the true state is guaranteed to be within the ellipsoids determined by (20) for all $k \ge 1$ if (19) holds. ■

Based on Lemma 1, a sequence of L_{k+1} always exist to make $(I-L_{k+1}C_{k+1})A_i$ for $i \in \{1,\dots,N\}$ stable. It is challenging to find a common L_{k+1} to simultaneously stabilize N systems. In addition, finding a necessary and sufficient condition to guarantee that $\sum_{i=1}^{N}\beta_i(x_k)(I-L_{k+1}C_{k+1})A_i$ in (45) is stable for unknown x_k is an open problem. A sufficient condition can be found through (53) by deleting unrelated terms. Theorem 2 shows a feasibility condition of P_{k+1}. In what follows, the minimal ellipsoid will be given through the solution of the following convex optimization problem.

For the discrete-time nonlinear system (1), the system output (2) and FSMF2 (16)–(17), the minimal ellipsoidal set parameterized by P_{k+1} in (20) can be derived by solving the following optimization problem:

$$\min_{P_{k+1},\,L_k,\,\sigma_{d,k}} \ \mathrm{tr}\{P_{k+1}\} \tag{57}$$

$$\text{s.t.} \quad \begin{bmatrix} -\Omega_{i,k} & * & * \\ 0 & -\Psi_{i,k} & * \\ \Phi_{i,k} & 0 & -P_{k+1} \end{bmatrix} \le 0 \tag{58}$$

where $\Omega_{i,k}$ and $\Psi_{i,k}$ for $i \in \{1,\dots,N\}$ are defined in Theorem 2, and $\Phi_{i,k}$ is defined in (47).

The equations of FSMF2 for the discrete-time nonlinear system (1) is summarized as follows:

- **Fuzzy modeling:**

$$f(x_k)=\sum_{i=1}^{N}\beta_i(x_k)(A_ix_k+b_i)$$

- **Fuzzy prediction:**

$$\bar{x}_{k+1}=\sum_{i=1}^{N}\beta_i(\hat{x}_k)(A_i\hat{x}_k+b_i)$$

- **Fuzzy filtering:**

$$\hat{x}_{k+1}=\bar{x}_{k+1}+L_{k+1}(y_{k+1}-C_{k+1}\bar{x}_{k+1}).$$

Remark 1: The differences between FSMF1 and FSMF2 are compared from three different aspects.

- **Complexity:** There are N filter gains L^i_{k+1} for $i\in\{1,\dots,N\}$ in FSMF1, while there is only one filter gain L_{k+1} in FSMF2.
- **Stability:** Using FSMF1, the estimation error cannot converge to zero even when there are no process and measurement noises due to the constant terms $(A_j-A_i)\hat{x}_k$ and b_j-b_i for $i,j\in\{1,\dots,N\}$ in (28); using FSMF2, it is possible that the estimation error converges to zero when there are no process and measurement noises in view of (44) and (45).
- **Performance:** The estimation performance of FSMF1 is in general better than that of FSMF2 since there are more parameters to optimize. This is especially true when the number of fuzzy rules is large and the process and measurement noises are present.

IV. Simulation

In this section, two nonlinear systems are given to show how the proposed two filters work and to prove the validity.

A. One-dimensional Nonlinear System

Consider the discrete-time nonlinear process on the time horizon $[0,1.5]$ given by

$$x_{k+1}=\rho\times(1-0.05T)x_k+0.04Tx_k^2$$

with corresponding measurement model

$$y_k=x_k+v_k$$

where $x_0=2.3$, $T=0.01$ sec is the inter-sample interval, ρ is a coefficient, and v_k is the measurement noise. Note that x_0 is unknown to the filter. In what follows, FSMF2 will be designed to estimate the state of the above nonlinear system.

We consider three cases: (1) $\rho=1$ and $R=0$; (2) $\rho=1$ and $R=0.09$; and (3) $\rho=0.9$ and $R=100$.

1) Noise-free conditions: The first step is to form a fuzzy model by using a set of fuzzy rules to represent the nonlinear function $f(x)=x^2$ as a set of affine models connected by fuzzy membership functions. Let us define two fuzzy sets for $x\in\left[\underline{x},\bar{x}\right]$ with corresponding membership functions:

$$\mu_1(x)=\frac{\bar{x}-x}{\bar{x}-\underline{x}},\quad \mu_2(x)=\frac{x-\underline{x}}{\bar{x}-\underline{x}}.$$

The corresponding fuzzy model is

Rule 1: If x is around $\underline{x}$, then

$$f_1(x)=\underbrace{(\underline{x}-\alpha)}_{A_1}x+\underbrace{\alpha\underline{x}}_{b_1};$$

Rule 2: If x is around $\bar{x}$, then

$$f_2(x)=\underbrace{(\bar{x}-\alpha)}_{A_2}x+\underbrace{\alpha\bar{x}}_{b_2}.$$

Then, we can derive

$$f(x)=\mu_1(x)f_1(x)+\mu_2(x)f_2(x)$$

where the error between the nonlinear function and the fuzzy model is 0. Notice that the membership functions μ_i for $i\in\{1,2\}$ are linear functions. Then (43) can be written as $g(x_k,\hat{x}_k)=h(\hat{x}_k)e_k$, where $h(\hat{x}_k)=\sum_{i=1}^{N}\gamma_i(A_i\hat{x}_k+b_i)$, $\gamma_1=\frac{-1}{\bar{x}-\underline{x}}$ and $\gamma_2=\frac{1}{\bar{x}-\underline{x}}$. In this case, $\underline{x}=2$ and $\bar{x}=3$. We set $\alpha=2.5$. Then, $A_1=-0.5$, $A_2=0.5$, $b_1=5$, and $b_2=7.5$.

Case 1. $\rho=1$, $R=0$

This case is dedicated to the stability analysis of the estimation error system with $v_k=0$ when FSMF2 is applied. The estimation error is

$$e_{k+1}=\sum_{i=1}^{2}\beta_i(x_k)(I-L_{k+1})(A_i+h(\hat{x}_k))e_k$$

where $h(\hat{x}_k)=\frac{1}{3}\hat{x}_k+\frac{5}{6}$.

To guarantee the stability of the above estimation error system, the following condition

$$\left|\sum_{i=1}^{2}\beta_i(x_k)(I-L_{k+1})(A_i+h(\hat{x}_k))\right|<1$$

must be satisfied. Note that $\beta_i(x_k)$ is unknown. A sufficient condition to satisfy the above inequality is:

$$\left|(I-L_{k+1})(A_i+h(\hat{x}_k))\right|<1,\quad i\in\{1,2\}$$

and a condition on L_{k+1} which stabilizes the estimation error system is

$$1-|A_i+h(\hat{x}_k)|^{-1}<L_{k+1}<1+|A_i+h(\hat{x}_k)|^{-1}. \tag{59}$$

Regardless of the initial estimation $\hat{x}_0$, we can choose $L_1=1$, then the estimation error e_k is zero for all $k\geq 1$.

Fig. 6 depicts the true state x_k and its estimate $\hat{x}_k$ (upper plot) and the estimation error (lower plot) with $\hat{x}_0=2.8$, $P_0=0.25$, from which we can see that the estimation error e_k converges to zero in one step. This example shows the validity of FSMF2 in the stabilization of the estimation error system in noise-free conditions.

2) Noisy conditions: In the following, we show the performance of FSMF2 in noisy conditions. The assumption that $v_k=0$ is no longer used. Thus, we have the estimation error e_{k+1}

$$e_{k+1}=\sum_{i=1}^{N}\beta_i(x_k)(I-L_{k+1})(A_i+h(\hat{x}))e_k-L_{k+1}v_{k+1}.$$

Then, the augmented form of e_k and v_{k+1} is denoted by

$$\eta_k=\left[\begin{array}{cc} e_k^T & v_{k+1}^T\end{array}\right]^T.$$

Thus, the state estimation error e_{k+1} can be rewritten as

$$e_{k+1}=\sum_{i=1}^{2}\beta_i(x_k)\Phi_{i,k}\eta_k$$

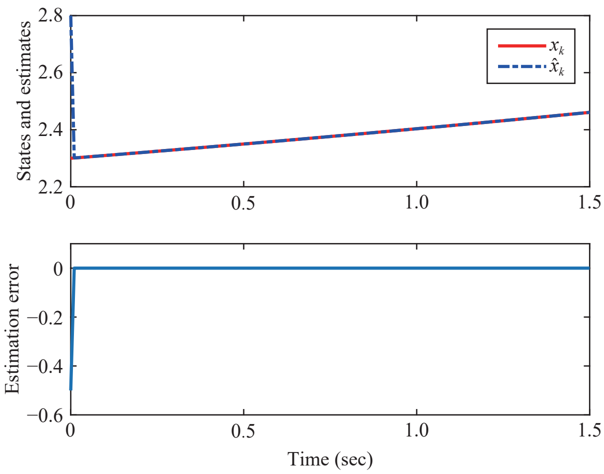

Fig. 6. System states and their estimates without noises.

where $\Phi_{i,k} = [\,(I - L_{k+1})(A_i + h(\hat{x}_k)) \;\; -L_{k+1}\,]$.

The minimal ellipsoid centered at $\hat{x}_{k+1}$ parameterized by P_{k+1} can be found by solving the following optimization problem with LMI constraints:

$$\min_{P_{k+1},\, L_k,\, \sigma_{j,k}} \operatorname{tr}\{P_{k+1}\}$$

$$\text{s.t.} \quad \begin{bmatrix} -\Omega_{i,k} & * & * \\ 0 & -\Psi_{i,k} & * \\ \Phi_{i,k} & 0 & -P_{k+1} \end{bmatrix} \le 0$$

where $\Phi_{i,k}$ for $i \in \{1, 2\}$ is defined above, and

$$\Omega_{i,k} = \operatorname{diag}\{\sigma_{1,k} P_k^{-1}, \sigma_{2,k} R_{k+1}^{-1}\}$$
$$\Psi_{i,k} = 1 - \sigma_{1,k} - \sigma_{2,k}.$$

In what follows, we will show the performance of the filter for divergent states and convergent states, respectively.

Case 2. $\rho = 1$, $R = 0.09$

Let us consider $\mathbb{X} = [1.5, 3.5]$. By employing the T-S fuzzy modeling method in Section II-B, we obtain $A_1 = 1.0008$, $A_2 = 1.0016$, $b_1 = -0.0010$, and $b_2 = -0.0023$. Set $\hat{x}_0 = 2.6$, $P_0 = 0.16$. The measurement noise sequence $\{v_k\}$ is generated from the uniform distribution in $[-0.3, 0.3]$. Fig. 7 shows the true state x_k and its estimate $\hat{x}_k$ (upper plot), where true states are within the bounds (the green lines) determined by P_k and R_k, respectively. Fig. 8 shows P_k and $e_k^T P_k^{-1} e_k$. It can be seen from the equation of e_{k+1}, there is a trade-off between the stabilization of the estimation error and the attenuation of the measurement noises. It is shown in Fig. 8 that P_k converges to 0.09. The main role of FSMF2 is to stabilize the estimation error system by taking values very close to 1 so that P_k converges to R_k. Besides, since $e_k^T P_k^{-1} e_k$ is kept below 1, then inequality (20) is guaranteed.

Case 3. $\rho = 0.9$, $R = 100$

Let us consider $\mathbb{X} = [-1, 10]$. By employing the T-S fuzzy modeling method in Section II-B, we obtain $A_1 = 0.8997$, $A_2 = 0.9041$, $b_1 = 0.005$, $b_2 = -0.0051$. We set $\hat{x}_0 = 8.3$, $P_0 = 36$. The measurement noise sequence $\{v_k\}$ is generated from the uniform distribution in $[-10, 10]$. Fig. 9 shows the true state x_k and its estimate $\hat{x}_k$ (upper plot), where true states are within the bounds (the green lines) determined by P_k and

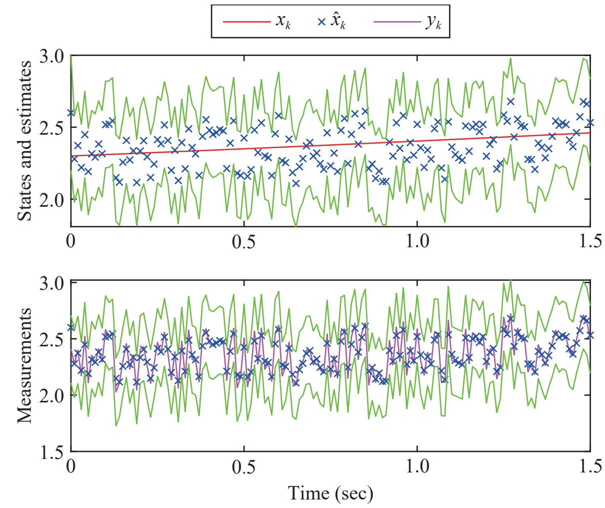

Fig. 7. Bounds of x_k and y_k when $\rho = 1$ and $R = 0.09$.

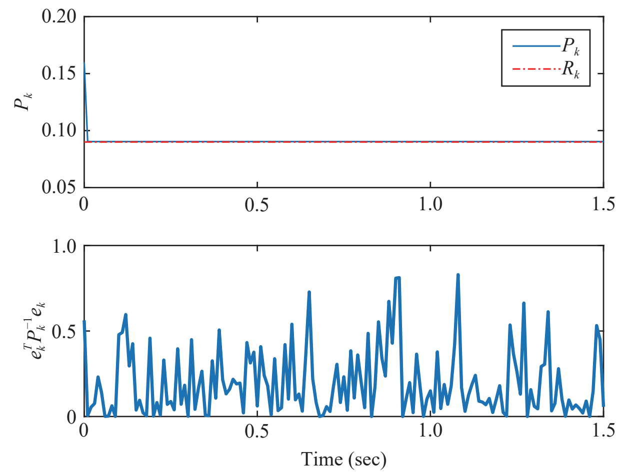

Fig. 8. P_k and $e_k^T P_k^{-1} e_k$ when $\rho = 1$ and $R = 0.09$.

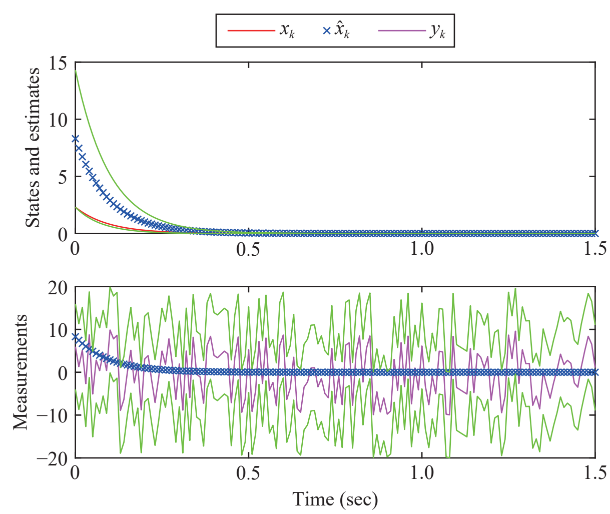

Fig. 9. Bounds of x_k and y_k when $\rho = 1$ and $R = 100$.

R_k, respectively. Fig. 10 shows P_k and $e_k^T P_k^{-1} e_k$. It should be mentioned that P_k converges to 0 even with measurement noises present. Besides, since $e_k^T P_k^{-1} e_k$ is kept below 1, then

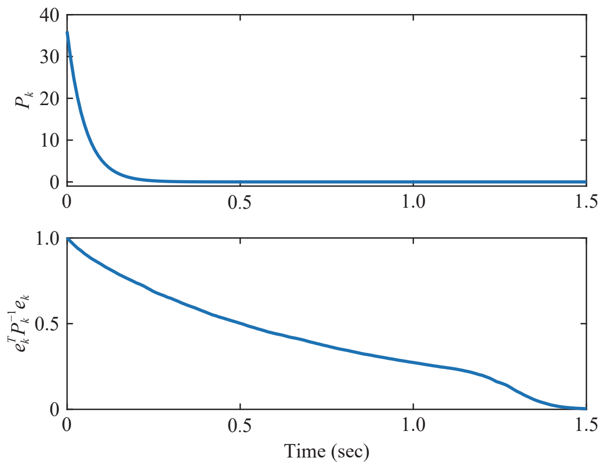

Fig. 10. P_k and $e_k^T P_k^{-1} e_k$ when $\rho = 1$ and $R = 100$.

inequality (20) is guaranteed.

Remark 2: We adopt the T-S fuzzy modeling method in Section II-B for cases with noise. The T-S fuzzy modeling method shown in the noise-free case can also be used, but it leads to bounded estimation error for Case 3. Different combinations of A_i and b_i for $i \in \{1, \ldots, N\}$ have substantial influence over the convergence of the estimation error system.

B. Vertical Mass-Spring System

This example uses a vertical mass-spring system. As a nonlinear model parameter estimation problem, we assume that $x_{3,k} = 0.1$ is an unknown constant. The discrete-time nonlinear plant and linear observation equations for this model are

$$\begin{aligned} x_{1,k+1} &= x_{1,k} + T x_{2,k} \\ x_{2,k+1} &= -\omega^2 T x_{1,k} + x_{2,k} - 2\omega T x_{2,k} x_{3,k} + 12T + w_k \\ x_{3,k+1} &= x_{3,k} \\ z_k &= x_{1,k} + v_k \end{aligned}$$

where $x_k = [x_{1,k} \ \ x_{2,k} \ \ x_{3,k}]^T$ are the position, velocity, and damping factor states, respectively, $T = 0.02$ sec is the inter-sample interval, $\omega = 10$ rad/s is the undamped resonant frequency, and

$$w_k^T w_k \le 4.47T \text{ (ft/s)}^2, v_k^T v_k \le 0.001 \text{ (ft)}^2.$$

The initial condition is $x_0 = [0\, 0\, 0.1]^T$.

Let us define two fuzzy sets in the interval $x_{l,k} \in \left[\underline{x}_l, \bar{x}_l\right]$ for $l \in \{2, 3\}$ with their corresponding membership functions

$$\mu_{l,1}(x_{l,k}) = \frac{\bar{x}_l - x_{l,k}}{\bar{x}_l - \underline{x}_l}, \quad \mu_{l,2}(x_{l,k}) = \frac{x_{l,k} - \underline{x}_l}{\bar{x}_l - \underline{x}_l}$$

and normalized weights as follows:

$$\begin{aligned} \beta_1(x_{2,k}, x_{3,k}) &= \mu_{2,1} \times \mu_{3,1}, \quad \beta_2(x_{2,k}, x_{3,k}) = \mu_{2,1} \times \mu_{3,2} \\ \beta_3(x_{2,k}, x_{3,k}) &= \mu_{2,2} \times \mu_{3,1}, \quad \beta_4(x_{2,k}, x_{3,k}) = \mu_{2,2} \times \mu_{3,2} \end{aligned}$$

where $\underline{x}_2 = -2$, $\bar{x}_2 = 2$, $\underline{x}_3 = -0.1$, $\bar{x}_3 = 0.3$.

By employing the T-S fuzzy modeling method in Section II-B, we can obtain the following fuzzy model:

Rule 1: If $x_{2,k}$ is around –2, and $x_{3,k}$ is around –0.1, then

$$x_{k+1} = \begin{bmatrix} 1 & 0.02 & 0 \\ -2 & 1.0317 & 0.0079 \\ 0 & 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 0.2242 \\ 0 \end{bmatrix};$$

Rule 2: If $x_{2,k}$ is around –2, and $x_{3,k}$ is around 0.3, then

$$x_{k+1} = \begin{bmatrix} 1 & 0.02 & 0 \\ -2 & 0.9050 & 0.0079 \\ 0 & 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 0.2875 \\ 0 \end{bmatrix};$$

Rule 3: If $x_{2,k}$ is around 2, and $x_{3,k}$ is around –0.1, then

$$x_{k+1} = \begin{bmatrix} 1 & 0.02 & 0 \\ -2 & 1.0317 & -0.0079 \\ 0 & 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 0.2558 \\ 0 \end{bmatrix};$$

Rule 4: If $x_{2,k}$ is around 2, and $x_{3,k}$ is around 0.3, then

$$x_{k+1} = \begin{bmatrix} 1 & 0.02 & 0 \\ -2 & 0.9050 & -0.0079 \\ 0 & 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 0.1925 \\ 0 \end{bmatrix}.$$

Next, we can further derive

$$x_{k+1} = \sum_{i=1}^{4} \beta_i(x_{2,k}, x_{3,k})(A_i x_k + b_i).$$

The discussed system is unstable. The initial estimate is chosen as $\hat{x}_0 = [0\ \ 0\ \ 0]^T$, $P_0 = \mathrm{diag}\{2, 2, 0.1\}$. In Fig. 11, estimates $\hat{x}_{1,k}$ and $\hat{x}_{2,k}$ approach the system states well while $\hat{x}_{3,k}$ approaches gradually. Estimating the parameter $x_{3,k}$ is a notorious nonlinear problem. The problem becomes even more difficult due to the fact that only the first state of the system is measurable. Fig. 12 shows the trace of the matrix P_k. Considering that both the process noises and the measurement noises are unknown-but-bounded, the estimation error at every step will accumulate to make sure that the true state is contained in the minimal ellipsoid even in the worst case scenario. Similar problem is mentioned in the simulation of [12]. Meanwhile, since $e_k^T P_k^{-1} e_k$ is kept below 1, then the minimal ellipsoid condition in (20) is guaranteed. In view that the system is unstable, the estimation performance is acceptable.

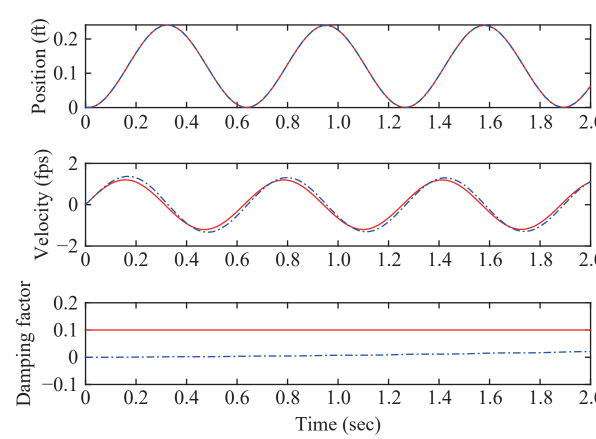

Fig. 11. System states and their estimates of the vertical spring-mass system.

In summary, both filters can be implemented online based on real-time data except the T-S fuzzy modeling, which is

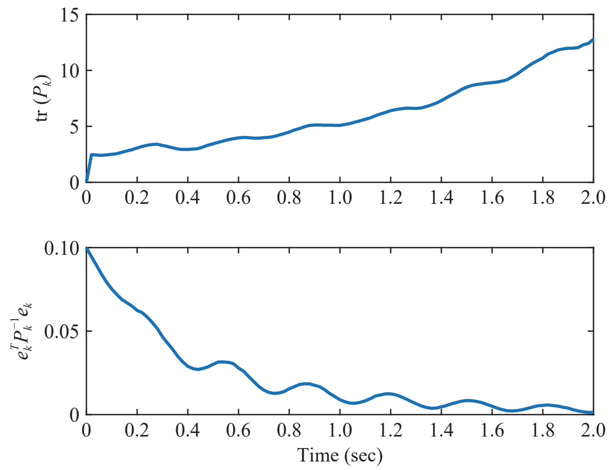

Fig. 12. $\mathrm{tr}(P_k)$ and $e_k^T P_k^{-1} e_k$ of the vertical spring-mass system.

done offline. We can use Theorem 1 to derive L_{k+1}^i for $i \in \{1, \ldots, N\}$ and Theorem 2 to obtain L_{k+1} at each time instant. The estimate $\hat{x}_{k+1}$ can be computed once y_{k+1} is obtained.

V. Conclusions and Future Research

In this article, the state estimation problem was studied for discrete-time nonlinear systems subject to unknown-but-bounded noises. An improved T-S fuzzy model was introduced to achieve highly accurate approximation. Two fuzzy set-membership filters, namely, FSMF1 and FSMF2, were proposed that consider both the prediction and the filtering. Some features of the membership functions were utilized in the filter design so that the stability of the estimation error system can be ensured. Computational procedures were given for finding the minimal ellipsoid. Both filters can run online recursively to provide the state estimate. The methods were validated in simulation. FSMF2 showed the ability to stabilize the estimation error system and reject measurement noises.

The estimation performance is very sensitive to the parameters in the fuzzy model. Finding a satisfactory filter becomes challenging when the number of fuzzy rules increases. This is especially true when the state of the nonlinear system does not converge to zero. Possible directions for future work are the co-design of the fuzzy model and the fuzzy filter and to employ nonnegative polynomials to obtain a less conservative version of Lemma 2. In addition, we plan to extend our results to networked control systems by taking into account event-triggered communication, packet loss, etc.

Appendix A
Proof of Lemma 1

The n-dimensional pair $[A, C]$ is observable if and only if the matrix

$$O = [C^T (CA)^T \cdots (CA^{n-1})]^T$$

has rank n. Since A is nonsingular,

$$\mathrm{rank}\, OA = \mathrm{rank}\, O = n.$$

Therefore, the pair $[A, CA]$ is observable, that is, there exists an L such that an arbitrary characteristic polynomial $A - LCA$ can be obtained.

Appendix B
Proof of Lemma 2

Multiplying (22) by β_i for $i \in \{1, 2, \ldots, N\}$ leads to

$$\begin{bmatrix} \beta_i X & * & * \\ \beta_i Y & \beta_i Z & * \\ \beta_i \Phi_i & 0 & -\beta_i P \end{bmatrix} \leq 0 \tag{60}$$

Summing (60) over i, $i \in \{1, 2, \ldots, N\}$ yields to

$$\begin{bmatrix} \sum_{i=1}^N \beta_i X & * & * \\ \sum_{i=1}^N \beta_i Y & \sum_{i=1}^N \beta_i Z & * \\ \sum_{i=1}^N \beta_i \Phi_i & 0 & -\sum_{i=1}^N \beta_i P \end{bmatrix} \leq 0 \tag{61}$$

that is,

$$\begin{bmatrix} X & * & * \\ Y & Z & * \\ \sum_{i=1}^N \beta_i \Phi_i & 0 & -P \end{bmatrix} \leq 0. \tag{62}$$

Utilizing Schur Complement to (62), we have

$$\begin{bmatrix} \sum_{i=1}^N \beta_i \Phi_i^T P^{-1} \sum_{i=1}^N \beta_i \Phi_i + X & * \\ Y & Z \end{bmatrix} \leq 0 \tag{63}$$

which is the same as (21).

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