Adaptive Control With Guaranteed Transient Behavior and Zero Steady-State Error for Systems With Time-Varying Parameters

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***Abstract*—It is nontrivial to achieve global zero-error regulation for uncertain nonlinear systems. The underlying problem becomes even more challenging if mismatched uncertainties and unknown time-varying control gain are involved, yet certain performance specifications are also pursued. In this work, we present an adaptive control method, which, without the persistent excitation (PE) condition, is able to ensure global zero-error regulation with guaranteed output performance for parametric strict-feedback systems involving fast time-varying parameters in the feedback path and input path. The development of our control scheme benefits from generalized t-dependent and x-dependent functions, a novel coordinate transformation and "congelation of variables" method. Both theoretical analysis and numerical simulation verify the effectiveness and benefits of the proposed method.**

***Index Terms*—Adaptive control, global property, guaranteed performance, uncertain nonlinear systems.**

I. Introduction

We consider the following SISO nonlinear systems with fast time-varying parameters [1]:

$$\begin{cases} \dot{x}_1 = \phi_1^T(x_1)\theta(t) + x_2 \\ \quad\vdots \\ \dot{x}_i = \phi_i^T(\underline{x}_i)\theta(t) + x_{i+1} \\ \quad\vdots \\ \dot{x}_n = \phi_n^T(\underline{x}_n)\theta(t) + b(t)u \\ y = x_1 \end{cases} \tag{1}$$

where $\underline{x}_i = [x_1, \dots, x_i]^T \in \mathbb{R}^i$ is the state vector, $u \in \mathbb{R}$ is the input, and $y \in \mathbb{R}$ is the output. The regressors $\phi_i : \mathbb{R}^i \to \mathbb{R}^q$, $i = 1, \dots, n$, are smooth mappings and satisfy $\phi_i(0) = 0$. $\theta(t) \in \mathbb{R}^q$ and $b(t) \in \mathbb{R}$ satisfy the following assumptions [1].

Assumption 1 (Bounded parameters): The parameter $\theta(t)$ is piecewise continuous and $\theta(t) \in \Theta_0$, for all $t \geq 0$, where Θ_0 is a compact set. The "radius" of Θ_0 is assumed to be known, while Θ_0 can be unknown.

Assumption 2 (Sign-definite parameter): The control gain $b(t)$ is bounded away from zero in the sense that there exists a constant ℓ_b, such that $\mathrm{sgn}(\ell_b) = \mathrm{sgn}(b(t)) \neq 0$ and $0 < |\ell_b| \leq |b(t)|$, for all $t \geq 0$. The sign of $b(t)$ is known and does not change.

Stabilization of system (1) satisfying Assumptions 1 and 2 is originally investigated in [1]–[3], where it is shown that asymptotic stability can be achieved by the so-called congelation of variables method and both full state feedback and partial state feedback approaches are considered. By "congelation of variables" it means that the time-varying $\theta(t)$ can be substituted by a constant ℓ_θ (ℓ_θ can be regarded as the average of $\theta(t)$) to avoid unnecessary time derivatives while not destroying the certainty equivalence principle [4]. It is noted that if the parameter $\theta(t)$ in (1) is unknown but constant, numerous adaptive control results have been reported in the literature during the past decades, including the well-known adaptive backstepping control, robust and adaptive control, adaptive observers, immersion and invariance adaptive control, and neural adaptive control, etc. (see [4]–[10] and the references therein).

However, real-word engineering systems with fast time-varying parameters are frequently encountered. For instance, the value of a circuit resistor might change with temperature, and some morphing aerial vehicles are normally designed with varying structures and parameters in order to complete some specific tasks, where the parameters might change with time or system states swiftly [11], [12]. For these types of systems, traditional adaptive methods might not be able to ensure desired control performance in terms of transient behavior and convergence accuracy, or even maintain the stability of the system. Efforts have been made (see, for instance [13] and [14]) in developing adaptive control methods with the aid of persistence of excitation to achieve exponential stability of linear time-varying systems. In [15], it is shown that the PE condition is not necessary to stabilize a linear time-varying system. The results in [16] and [17] implement the asymptotic/exponential tracking of robotic systems with/without time-varying parameters. In [18]–[20], along with observer based adaptive control, a projection algorithm is proposed to ensure the boundedness of the slow time-varying

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parameter estimate. In the context of adaptive control for time-varying nonlinear systems, [21] explores a soft sign function based approach to deal with unknown time-varying parameters. Recently, an elegant method based on the "congelation of variables" method is proposed in [1]–[3] to asymptotically stabilize a class of nonlinear system with fast time-varying parameters, which is further extended to address multi-agent systems in [22] and [23]. Thus far, meaningful results on adaptive control of systems with unknown and fast time-varying parameters are still limited, rendering the underlying problem interesting yet challenging.

In this note, we address the stabilization problem of fast time-varying system as described in (1) and our goal is to achieve zero-error full state regulation while at the same time maintaining global output performance, i.e., regulating each state to zero asymptotically and meanwhile confining the convergence process of the output within a prescribed boundary. Our development consists of three major steps: i) disassociating the recursive controller design from the initial condition of system (1) via two generalized functions and a novel coordinate transformation, thereby obtaining a global prescribed performance controller; ii) designing adaptive laws via the "congelation of variables" method to estimate fast time-varying parameters involved in the constrained systems; and iii) separating the lumped nonlinear terms and exploiting additional nonlinear damping terms in each virtual control input to completely offset the undesired perturbations caused by unknown time-varying control gain to achieve asymptotic convergence. With this comprehensive treatment, output convergence transient behavior is well preset and asymptotic (zero-error) regulation is achieved in the presence of mismatched time-varying uncertainties.

Unlike most prescribed performance control methods that only achieve uniformly ultimately bounded (UUB) stability for nonlinear systems with unknown but constant parameters [24]–[32], the proposed method ensures zero-error stabilization and global output performance for systems with fast time-varying parameters and mismatched uncertainties.

II. Preliminaries

A. Two Useful Functions and Coordinate Transformation

Before presenting the control algorithm, we introduce two useful functions and a novel coordinate transformation, which plays important roles in control design.

Definition 1: The generalized performance function $\beta(t)$ satisfies the following properties:

1) $\beta(t):[0,\infty)\to\mathbb{R}^+$ is an n-times differentiable function;

2) $\beta(0)=1$ and $\lim_{t\to+\infty}\beta(t)<1$;

3) $\beta(t)\in\mathcal{L}_\infty$ and $\dot\beta(t)\in\mathcal{L}_\infty$, $\forall t\in[0,+\infty)$.

Remark 1: There are many (in fact, an infinite number of) functions that satisfy the aforementioned properties. For example,

$$\beta(t)=\begin{cases}(1-\beta_\infty)\left(\dfrac{T-t}{T}\right)^n+\beta_\infty, & 0\le t<T\\ \beta_\infty, & t\ge T\end{cases}\tag{2}$$

where $\beta_\infty=\lim_{t\to+\infty}\beta(t)$, $T>0$ is a constant and n is the system order. Note that the performance function is not necessarily monotonically decreasing, which might be advantageous in various applications, e.g., when the system time-varying parameter changes strongly or the system is perturbed by some calibration so that a large error would enforce a large input action.

Definition 2: The generalized normalized function $\psi(x)$ satisfies the following properties:

1) $\psi(x):\mathbb{R}\to(-1,1)$ is a monotonically increasing and n-times differentiable function;

2) $\lim_{x\to\pm\infty}\psi(x)=\pm1$ and $\psi(0)=0$;

3) $\psi'(x)$ is bounded below by a positive constant over $[0,\infty)$, where $\psi'(x)=\frac{d\psi}{dx}$.

Remark 2: We list two choices for $\psi(x)$ as follows:

$$\psi(x)=\frac{x}{\sqrt{x^2+1}},\quad \psi(x)=\tanh(x)\tag{3}$$

and for the above two choices, we have

$$\psi'(x)=\frac{1}{(\sqrt{x^2+1})^{3/2}},\quad \psi'(x)=\operatorname{sech}^2(x).\tag{4}$$

Denoting the inverse function by ψ^{-1}, it is seen that[1]

$$\psi'(x)>0,\quad \psi_x=\frac{\psi}{x}>0$$
$$\psi^{-1}(\beta(0))=\psi^{-1}(1)=+\infty.\tag{5}$$

Making use of such $\beta(t)$ and $\psi(x)$, we construct the following coordinate transformation function to enable the properties on z and x as stated in Lemma 1

$$z(\beta,\psi)=\frac{\beta(t)\psi(x)}{\beta^2(t)-\psi^2(x)}.\tag{6}$$

Lemma 1: For any $\beta(t)$ as defined in Section II and z as defined in (6), if $\forall t\ge0$, $z\in\mathcal{L}_\infty$, then it holds that $-\psi^{-1}(\beta)<x<\psi^{-1}(\beta)$.

Proof: We first consider the moment when $t=0$. According to $\beta(0)=1$ and $\psi(x)\in(-1,1)$, we know that $\beta(0)-\psi(|x(0)|)>0$, i.e., $|x(0)|<\psi^{-1}(\beta(0))$. Next, we continue the proof by contradiction. Note that $z\in\mathcal{L}_\infty$ implies $\beta(t)-\psi(x)\neq0$. Assume that $\exists\, t\in(0,\infty)$ such that $|x(t)|\ge\psi^{-1}(\beta(t))$, i.e., $\beta(t)-\psi(|x(t)|)\le0$. As a result, by recalling that $\beta(0)-\psi(x(0))>0$, we have $\exists\, t_1\in(0,t]$ causes $\psi(|x(t_1)|)=\beta(t_1)$, and therefore yields an unbounded z_1, which, however, contradicts the premise $z\in\mathcal{L}_\infty$. ■

This coordinate transformation introduced in (6) appears as a more straightforward approach compared to the tuning function modified transformation [27] and the multiple cascade transformation [33], by reason of its simple structure, smoothness and nonsingularity.

B. Control Objective

The control objective is to design an adaptive control law such that the closed-loop system is asymptotically stable, while the system output is always confined within a prescribed performance funnel $F_{\beta(t)}$. Furthermore, the boundary of $F_{\beta(t)}$ is $\beta(t)$, which can be pre-defined at user's will, irrespective of initial conditions.

[1] Properties 1 and 2 of $\psi(x)$ ensure that ψ_x is positive and invertible for all $x\in\mathbb{R}$.

Remark 3: If we choose a function $\beta(t)$ with an exponential decay rate, e.g., $\beta(t)=(1-\beta_\infty)e^{-t}+\beta_\infty$. By qualitative analysis, $\psi^{-1}(\beta)$ is a function that increases monotonically as $\beta\to\infty$, and $\beta(t)$ is a function that decays exponentially as $t\to\infty$, thus $\psi^{-1}(\beta)$ is a function that decays exponentially as $t\to\infty$ and $\psi^{-1}(\beta(0))\to\infty$. Therefore, $|x|<\psi^{-1}(\beta)$ implies that there exist some positive constants l_1, l_2 and ϵ such that $|x(t)|<l_1e^{-l_2t}+\epsilon$ for any $x(0)$, resulting in the fact that the system output converges at least e^{-l_2t} exponentially fast to the corresponding set. Similarly, if we choose $\beta(t)$ as defined in (2), one can find that the system output converges to a prescribed set at a prescribed time T, a favorable feature in practice.

III. Motivating Example

Consider the following first-order system[2]:

$$\dot{x}=b(t)u+\theta(t)x \tag{7}$$

where x is the state, u is the control input, $\theta(t)\in\mathbb{R}$ satisfies Assumption 1, and $b(t)\in\mathbb{R}$ satisfies Assumption 2.

By using the coordinate transformation (6), we can convert (7) into the following z-dynamics:

$$\dot{z}=\Pi(x,t)\dot{x}+\Psi(x,t) \tag{8}$$

with

$$\Pi(x,t)=\frac{(\beta^2(t)-\psi^2(x))\beta(t)\psi'(x)+2\psi^2(x)\psi'(x)\beta(t)}{(\beta^2(t)-\psi^2(x))^2}$$

$$\Psi(x,t)=\frac{\dot{\beta}(t)\psi(x)(\beta^2(t)-\psi^2(x))-2\beta^2(t)\dot{\beta}(t)\psi(x)}{(\beta^2(t)-\psi^2(x))^2}$$

where Ψ and Π are known time-varying smooth functions and are bounded as long as z is bounded. In addition, $\Pi>0$ for $\forall z\in\mathcal{L}_\infty$. These facts ensure the controllability of (8). Motivated by [1], we design $u=\hat{\rho}\bar{u}$, with $\hat{\rho}$ being an "estimate" of $1/\ell_b$, and $\bar{u}$ being the compensating signal to be specified later. Then, (8) can be written as

$$\begin{aligned}\dot{z}=\Pi\Big(&\bar{u}+\hat{\theta}x+(\theta(t)-\ell_\theta)x+(b(t)-\ell_b)\hat{\rho}\bar{u}\\&+(\ell_\theta-\hat{\theta})x-\ell_b\left(\frac{1}{\ell_b}-\hat{\rho}\right)\bar{u}+\frac{\Psi}{\Pi}\Big)\end{aligned} \tag{9}$$

where $\hat{\theta}$ is an "estimate" of ℓ_θ, $\Psi/\Pi\in\mathcal{L}_\infty$ for $\forall z\in\mathcal{L}_\infty$. Note that ℓ_θ and ℓ_b are unknown constants, which can be regarded as the "average" of $\theta(t)$ and $b(t)$, respectively. Consider the Lyapunov function candidate

$$V=\frac{1}{2}z^2+\frac{1}{2\gamma_\theta}(\ell_\theta-\hat{\theta})^2+\frac{|\ell_b|}{2\gamma_\rho}\left(\frac{1}{\ell_b}-\hat{\rho}\right)^2 \tag{10}$$

where $\gamma_\rho>0$. Then, the derivative of (10) along the trajectory of (7) becomes

$$\begin{aligned}\dot{V}=\;&\Pi\left(z\bar{u}+z\hat{\theta}x+z\Delta_\theta x+\frac{\Psi}{\Pi}z\right)\\&+\Pi z\Delta_b\hat{\rho}\bar{u}+\frac{1}{\gamma_\theta}(\ell_\theta-\hat{\theta})(\gamma_\theta z\Pi x-\dot{\hat{\theta}})\\&-\frac{|\ell_b|}{\gamma_\rho}\left(\frac{1}{\ell_b}-\hat{\rho}\right)\left(\gamma_\rho\,\mathrm{sgn}(\ell_b)z\Pi\bar{u}+\dot{\hat{\rho}}\right)\end{aligned} \tag{11}$$

[2] For simplicity, arguments of functions are sometimes omitted if no confusion occurs.

where $\Delta_\theta=\theta(t)-\ell_\theta$ and $\Delta_b=b(t)-\ell_b$. The last two lines of (11) will be canceled by the following adaptive laws:

$$\dot{\hat{\theta}}(x,\beta)=\gamma_\theta z\Pi x \tag{12}$$

$$\dot{\hat{\rho}}=-\gamma_\rho\,\mathrm{sgn}(\ell_b)z\Pi\bar{u}. \tag{13}$$

Remark 4: Note that $z(x,\beta)$ as defined in (6) is a smooth function and $x=0\Leftrightarrow z=0$. Thus we can directly express x as $x=W(x,\beta)z$ by using Hadamard's lemma (see [1]–[3], [34]), where $W(x,\beta)$ is a bounded smooth mapping for every bounded z. As a matter of fact, here $W=\frac{x}{z}=\frac{x(\beta^2-\Psi^2)}{\beta\Psi}=\frac{\beta^2-\Psi^2}{\beta\Psi_x}\in\mathbb{R}^+$.

According to Remark 1 and (5), the perturbation terms in the first line of (11) can be rewritten as

$$\begin{aligned}&z\hat{\theta}x+\frac{\Psi}{\Pi}z=z\hat{\theta}x+\frac{\Psi_x}{\Pi}zx=\left(\hat{\theta}+\frac{\Psi_x}{\Pi}\right)W(x,\beta)z^2\\&z\Delta_\theta x=\Delta_\theta W(x,\beta)z^2.\end{aligned} \tag{14}$$

By applying Young's inequality, then

$$\left(\hat{\theta}+\frac{\Psi_x}{\Pi}\right)W(x,\beta)z^2\le\frac{1}{2}\left(\hat{\theta}+\frac{\Psi_x}{\Pi}\right)^2W^2z^2+\frac{1}{2}z^2 \tag{15}$$

$$\Delta_\theta W(x,\beta)z^2\le\frac{1}{2}\delta_{\Delta_\theta}W^2z^2+\frac{\delta_{\Delta_\theta}}{2}z^2 \tag{16}$$

where $\delta_{\Delta_\theta}\ge|\Delta_\theta|$ is the "radius" of the compact set of $\theta(t)$. Now consider $\bar{u}$ with a nonpositive nonlinear gain as

$$\begin{aligned}\bar{u}&=-\left(\frac{k}{\Pi}+\frac{1}{2}\left(\delta_{\Delta_\theta}+1\right)+\frac{W^2}{2}\left(\hat{\theta}+\frac{\Psi_x}{\Pi}\right)^2\right)z\\&=-\kappa(x,\beta,\hat{\theta})z\end{aligned} \tag{17}$$

where $k>0$.

We are now in the position to state the following theorem.

Theorem 1: Under Assumptions 1 and 2, when the fast time-varying parameters appear in the feedback path and input path, the nonlinear system (7) is globally asymptotically stable by using the control law (17) and the parameter update laws (12) and (13). Furthermore, the state $x(t)$ is always confined within the prescribed performance funnel $F_\beta=\{(t,x)\in\mathbb{R}_{\ge0}\times\mathbb{R}\big||x(t)|/\psi^{-1}(\beta(t))<1\big\}$, and ultimately converges to zero. Furthermore, $\lim_{t\to\infty}\hat{\theta}$ and $\lim_{t\to\infty}\hat{\rho}$ exist (although not necessarily equal to ℓ_θ and $1/\ell_b$, respectively). In addition, the control input and update laws remain uniformly bounded over $[0,\infty)$.

Proof: Substituting (17) into (11), yields

$$\begin{aligned}\dot{V}\le&-kz^2+\Pi\left(z\Delta_\theta x-\frac{1}{2}\delta_{\Delta_\theta}W^2z^2-\frac{\delta_{\Delta_\theta}}{2}z^2\right)\\&+\Pi\left(z\hat{\theta}x-\frac{\dot{\beta}}{\beta}zx-\frac{W^2}{2}\left(\hat{\theta}-\frac{\dot{\beta}}{\beta}\right)^2z^2-\frac{1}{2}z^2\right)\\&+\Pi z\Delta_b\hat{\rho}\bar{u}\\\le&-kz^2-\Pi\kappa(x,\beta,\hat{\theta})\Delta_b\hat{\rho}z^2.\end{aligned} \tag{18}$$

Then, substituting (17) into (13) yields $\dot{\hat{\rho}}(t)=\gamma_\rho\Pi\,\mathrm{sgn}(\ell_b)\kappa z^2$, where $\Pi>0$ and $\kappa(x,\beta,\hat{\theta})>0$. When $b(t)>0$, according to Assumption 2, we can obtain $0<\ell_b<b(t)$ and thus

$\mathrm{sgn}(\ell_b) > 0$ and $\Delta_b > 0$, implying that $\dot{\hat{\rho}}(t) \geq 0$. It follows from $\hat{\rho}(0) > 0$ that $\hat{\rho}(t) > 0$, and therefore $z\Delta_b\hat{\rho}\bar{u} = -\kappa\Delta_b\hat{\rho}(t)z^2 \leq 0$. Similarly, when $b(t) < 0$, according to Assumption 2, we can obtain $\Delta_b < 0$, $\mathrm{sgn}(\ell_b) < 0$, $\dot{\hat{\rho}}(t) \leq 0$ and therefore $z\Delta_b\hat{\rho}\bar{u} = -\kappa\Delta_b\hat{\rho}(t)z^2 \leq 0$ by selecting the initial condition $\hat{\rho}(0) < 0$. Recalling (12) and (13), and noting the fact $-\Pi\kappa\Delta_b\hat{\rho}z^2 \leq 0$, it can be concluded that for any bounded initial $z(0)$, $V(t) \leq V(0)$, which yields $z(t)$, $\hat{\theta}$, $\hat{\rho}$, and $W(x,\beta)$ are bounded.

The boundedness of Π, $1/\Pi$, Ψ and κ is guaranteed by the boundedness of z and $\beta(t)$, and it follows from (9) and (18) that $\dot{z} \in \mathcal{L}_\infty$ and $z \in \mathcal{L}_2$. Therefore, invoking Barbalat's lemma one can conclude that $\lim_{t\to\infty} z(t) = 0$, which further indicates that $\lim_{t\to\infty} x(t) = 0$. Hence the closed-loop system (7) is asymptotically stable. Furthermore, by using Lemma 1, we have $x(t) \in F_\beta = \{(t,x) \in \mathbb{R}_{\geq 0} \times \mathbb{R} \big| |x(t)|/\psi^{-1}(\beta(t)) < 1\}$.

To show the asymptotic constancy of $\hat{\theta}$ and $\hat{\rho}$, recalling (12), (13), (18) and the fact that $z \in \mathcal{L}_2$, we have $\dot{\hat{\theta}} \in \mathcal{L}_1$ and $\dot{\hat{\rho}} \in \mathcal{L}_1$. Then, by using the argument similar to Theorem 3.1 in [35], it is concluded that $\hat{\theta}$ and $\hat{\rho}$ have a limit as $t \to \infty$. Furthermore, it is seen from (12), (13) and (17) that the update laws $\dot{\hat{\theta}} \in \mathcal{L}_\infty$, $\dot{\hat{\rho}} \in \mathcal{L}_\infty$, and the control input $u = \hat{\rho}\bar{u} \in \mathcal{L}_\infty$. ■

IV. Design for High-Order Time-Varying Systems

Motivated by the design process for the first-order system, we now explore its applicability to a general higher-order system as described in (1). For such a strict-feedback system, we use the classical backstepping method [36], with additional special treatment in each step, as detailed in what follows:

Step 1: Let $\alpha_1 = x_2 - z_2$ and according to (6), we can convert $\dot{x}_1 = \phi_1^T\theta(t) + x_2$ into the following z_1-dynamics:

$$\begin{aligned}\dot{z}_1 &= \Pi\left(\alpha_1 + z_2 + \phi_1^T\theta(t) + \frac{\Psi}{\Pi}\right)\\ &= \Pi\left(\alpha_1 + z_2 + \phi_1^T\hat{\theta} + \phi_1^T(\theta(t) - \ell_\theta) + \frac{\Psi}{\Pi}\right.\\ &\quad \left.+\ \phi_1^T(\ell_\theta - \hat{\theta})\right)\end{aligned} \tag{19}$$

where $\Pi(x_1,t)$ and $\Psi(x_1,t)$ are given below equation (8), and $\ell_\theta \in \mathbb{R}^q$ is an unknown constant vector. By Hadamard's lemma, one can express the regressor ϕ_1 as $\phi_1(x_1) = \Phi_1(x_1)x_1$, where $\Phi_1(x_1) \in \mathbb{R}^q$ is a smooth mapping. The third line of (19) will be treated by the following tuning function:

$$\tau_1(x_1,\beta) = \Gamma z_1\Phi_1\Pi x_1 \tag{20}$$

where $\Gamma = \Gamma^T \in \mathbb{R}^{q\times q}$ is the positive adaptation gain. Consider the Lyapunov function candidate

$$V_1 = \frac{1}{2}z_1^2 + \frac{1}{2}(\ell_\theta - \hat{\theta})^T\Gamma^{-1}(\ell_\theta - \hat{\theta}) \tag{21}$$

then, by recalling Remark 4

$$\begin{aligned}\dot{V}_1 = &\ \Pi\left(z_1\alpha_1 + z_1z_2 + \frac{\Psi_{x_1}}{\Pi}z_1x_1 + z_1\phi_1^T\hat{\theta} + z_1\phi_1^T\Delta_\theta\right)\\ &+ (\ell_\theta - \hat{\theta})^T\Gamma^{-1}(\tau_1 - \dot{\hat{\theta}})\end{aligned} \tag{22}$$

where $\Psi_{x_1} = \Psi/x_1$ is positive and invertible for all $x_1 \in \mathcal{L}_\infty$. Invoking Young's inequality, yields

$$z_1\phi_1^T\Delta_\theta = z_1\Phi_1^T\Delta_\theta x_1 \leq \frac{\delta_{\Delta_\theta}}{2}\Phi_1^T\Phi_1W_1^2z_1^2 + \frac{\delta_{\Delta_\theta}}{2}z_1^2 \tag{23}$$

where W_1 is shown in Remark 4. The virtual control law α_1 is designed as

$$\alpha_1(x_1,\bar{\beta}^{(1)},\hat{\theta}) = -\frac{1}{\Pi}(k_1 + \zeta_1)z_1 - \frac{\Psi_{x_1}}{\Pi}x_1 - \phi_1^T\hat{\theta} \tag{24}$$

where $\bar{\beta}^{(1)} = [\beta,\dot{\beta}]^{\mathrm{T}}$, $k_1 > 0$, and

$$\zeta_1 = \frac{1}{2}\left(\frac{1}{\epsilon_\psi} + \delta_{\Delta_\theta}\Pi\Phi_1^T\Phi_1W_1^2 + \Pi\delta_{\Delta_\theta} + (n-1)\delta_{\Delta_\theta}\right)$$

is the nonlinear damping gain with $\epsilon_\psi > 0$ and δ_{Δ_θ} being the "radius" of the compact set of $\theta(t)$. $\Pi(x_1,t) \in \mathbb{R}_+$, $\Phi_1 \in \mathbb{R}^q$ and $W_1 \in \mathbb{R}$ are computable functions. The resulting $\dot{V}_1$ is

$$\begin{aligned}\dot{V}_1 \leq &-k_1z_1^2 + \Pi z_1z_2 - \frac{(n-1)}{2}\delta_{\Delta_\theta}z_1^2 - \frac{1}{2\epsilon_\psi}z_1^2\\ &+ (\ell_\theta - \hat{\theta})^T\Gamma^{-1}(\tau_1 - \dot{\hat{\theta}}).\end{aligned} \tag{25}$$

The second term Πz_1z_2 in the right hand side of (22) can be canceled at the next step.

Step 2: Recall $\dot{x}_2 = x_3 + \phi_2^T(\underline{x}_2)\theta(t)$ and let $\alpha_2 = x_3 - z_3$. We rewrite $\dot{z}_2 = \dot{x}_2 - \dot{\alpha}_1$ as

$$\begin{aligned}\dot{z}_2 = &\ \alpha_2 + z_3 - \frac{\partial\alpha_1}{\partial x_1}x_2 - \frac{\partial\alpha_1}{\partial\hat{\theta}}\dot{\hat{\theta}} - \frac{\partial\alpha_1}{\partial\beta}\dot{\beta} - \frac{\partial\alpha_1}{\partial\dot{\beta}}\ddot{\beta}\\ &+ \phi_2^T\theta(t) - \frac{\partial\alpha_1}{\partial x_1}\phi_1^T\theta(t).\end{aligned} \tag{26}$$

Define $w_2(\underline{x}_2,\hat{\theta},\bar{\beta}^{(1)}) = \phi_2 - \frac{\partial\alpha_1}{\partial x_1}\phi_1$, then the second line of (26) can be rewritten as

$$w_2^T\theta(t) = w_2^T\hat{\theta} + w_2^T(\theta(t) - \ell_\theta) + w_2^T(\ell_\theta - \hat{\theta}). \tag{27}$$

Denote $\theta(t) - \ell_\theta$ by Δ_θ, and according to Assumption 1, there exist a known constant δ_{Δ_θ} such that $\delta_{\Delta_\theta} \geq |\Delta_\theta|$. Also note that z_1, and $\alpha_1(x_1,\bar{\beta}^{(1)},\hat{\theta})$ are smooth and $\alpha_1(0,\bar{\beta}^{(1)},\hat{\theta}) = 0$. The $\hat{\theta}$- and $\bar{\beta}^{(1)}$-dependent change of coordinates between $\underline{z}_2$ and $\underline{x}_2$ is smooth, invertible, and $\underline{x}_2 = 0 \Leftrightarrow \underline{z}_2 = 0$. Using Hadamard's lemma, one can directly express w_2 as $w_2 = W_2^T(\underline{x}_2,\bar{\beta}^{(1)},\hat{\theta})\underline{z}_2$, where $W_2(\underline{x}_2,\bar{\beta}^{(1)},\hat{\theta}) \in \mathbb{R}^{2\times q}$ is a smooth mapping. Therefore, one can calculate that

$$\begin{aligned}z_2w_2^T(\theta(t) - \ell_\theta) &= z_2\Delta_\theta^Tw_2 = z_2\Delta_\theta^TW_2^T\underline{z}_2\\ &\leq \frac{1}{2}\delta_{\Delta_\theta}|W_2|_{\mathsf{F}}^2z_2^2 + \frac{\delta_{\Delta_\theta}}{2}\underline{z}_2^T\underline{z}_2\\ &= \frac{\delta_{\Delta_\theta}}{2}\left(|W_2|_{\mathsf{F}}^2 + 1\right)z_2^2 + \frac{\delta_{\Delta_\theta}}{2}z_1^2\end{aligned} \tag{28}$$

where $\underline{z}_2^T\underline{z}_2 = z_1^2 + z_2^2$ is used and $|W_2|_{\mathsf{F}} = \sqrt{\sum_{i=1}^2\sum_{j=1}^q(W_{2ij})^2}$ denotes the Frobenius norm. Choosing the Lyapunov function candidate $V_2 = V_1 + \frac{1}{2}z_2^2$, its derivative along the trajectories of (1) is

$$\begin{aligned}\dot{V}_2 \leq &-k_1z_1^2 + \Pi z_1z_2 - \frac{(n-1)}{2}\delta_{\Delta_\theta}z_1^2 - \frac{1}{2\epsilon_\psi}z_1^2\\ &+ z_2\alpha_2 + z_2\left(-\frac{\partial\alpha_1}{\partial x_1}x_2 - \frac{\partial\alpha_1}{\partial\hat{\theta}}\dot{\hat{\theta}} - \frac{\partial\alpha_1}{\partial\beta}\dot{\beta} - \frac{\partial\alpha_1}{\partial\dot{\beta}}\ddot{\beta}\right)\\ &+ \frac{\delta_{\Delta_\theta}}{2}\left(|W_2|_{\mathsf{F}}^2 + 1\right)z_2^2 + \frac{\delta_{\Delta_\theta}}{2}z_1^2 + z_2w_2^T\hat{\theta} + z_2z_3\\ &+ (\ell_\theta - \hat{\theta})^T\Gamma^{-1}(\Gamma w_2z_2 + \tau_1 - \dot{\hat{\theta}}).\end{aligned} \tag{29}$$

According to (29), we design the tuning function as

$$\tau_2(\underline{x}_2,\bar{\beta}^{(1)},\hat{\theta}) = \tau_1 + \Gamma w_2 z_2. \tag{30}$$

In addition, the virtual control law α_2 is constructed as

$$\begin{aligned}\alpha_2(\underline{x}_2,\bar{\beta}^{(2)},\hat{\theta}) = & -\Pi z_1 - (k_2+\zeta_2)z_2 - w_2^T\hat{\theta} \\ & + \frac{\partial\alpha_1}{\partial x_1}x_2 + \frac{\partial\alpha_1}{\partial\beta}\dot{\beta} + \frac{\partial\alpha_1}{\partial\dot{\beta}}\ddot{\beta} + \frac{\partial\alpha_1}{\partial\hat{\theta}}\tau_2\end{aligned} \tag{31}$$

where $\bar{\beta}^{(2)} = [\beta,\dot{\beta},\ddot{\beta}]^T$, $k_2 > 0$, and $\zeta_2(\underline{x}_2,\bar{\beta}^{(1)},\hat{\theta})$ is the nonlinear damping gain, as follows:

$$\zeta_2 = \frac{1}{2}\left(\delta_{\Delta_\theta}|W_2|_{\mathsf{F}}^2 + (n-1)\delta_{\Delta_\theta} + \frac{1}{\epsilon_\psi}\right). \tag{32}$$

After some simplifications and using (30) and (31), we express (29) as

$$\begin{aligned}\dot{V}_2 \le & -k_1 z_1^2 - k_2 z_2^2 - \frac{1}{2}\left((n-2)\delta_{\Delta_\theta} + \frac{1}{\epsilon_\psi}\right)\underline{z}_2^T\underline{z}_2 \\ & + z_2 z_3 + \left(z_2\frac{\partial\alpha_1}{\partial\hat{\theta}} + (\ell_\theta - \hat{\theta})^T\Gamma^{-1}\right)(\tau_2 - \dot{\hat{\theta}})\end{aligned} \tag{33}$$

where $z_2 z_3$ can be canceled at the next step.

Step 3: Introducing $\alpha_3 = x_4 - z_4$ and according to $z_3 = x_3 - \alpha_2$, we can transform $\dot{x}_3 = x_4 + \phi_3^T(\underline{x}_3)\theta(t)$ to the following z_3-dynamics:

$$\begin{aligned}\dot{z}_3 = & \alpha_3 + z_4 - \frac{\partial\alpha_2}{\partial x_1}x_2 - \frac{\partial\alpha_2}{\partial x_2}x_3 - \frac{\partial\alpha_2}{\partial\beta}\dot{\beta} - \frac{\partial\alpha_2}{\partial\dot{\beta}}\ddot{\beta} \\ & - \frac{\partial\alpha_2}{\partial\ddot{\beta}}\dddot{\beta} - \frac{\partial\alpha_2}{\partial\hat{\theta}}\dot{\hat{\theta}} + w_3^T\hat{\theta} + w_3^T(\ell_\theta - \hat{\theta}) \\ & + w_3^T(\theta(t) - \ell_\theta).\end{aligned} \tag{34}$$

Now we choose the Lyapunov function candidate $V_3 = V_2 + \frac{1}{2}z_3^2$, then

$$\begin{aligned}\dot{V}_3 \le & -k_1 z_1^2 - k_2 z_2^2 - \frac{1}{2}\left((n-2)\delta_{\Delta_\theta} + \frac{1}{\epsilon_\psi}\right)\underline{z}_2^T\underline{z}_2 \\ & + z_3 z_4 + \left(z_2\frac{\partial\alpha_1}{\partial\hat{\theta}} + (\ell_\theta - \hat{\theta})^T\Gamma^{-1}\right)(\tau_2 - \dot{\hat{\theta}}) \\ & - z_3\left(\sum_{j=1}^{2}\frac{\partial\alpha_2}{\partial x_j}x_{j+1} + \sum_{j=0}^{2}\frac{\partial\alpha_2}{\partial\beta^{(j)}}\beta^{(j+1)} + \frac{\partial\alpha_2}{\partial\hat{\theta}}\dot{\hat{\theta}}\right) \\ & + z_3\alpha_3 + z_2 z_3 + z_3 w_3^T\hat{\theta} + z_3 w_3^T(\ell_\theta - \hat{\theta}) \\ & + z_3 w_3^T(\theta(t) - \ell_\theta)\end{aligned} \tag{35}$$

where $w_3(\underline{x}_3,\bar{\beta}^{(2)},\hat{\theta}) = \phi_3 - \frac{\partial\alpha_2}{\partial x_1}\phi_1 - \frac{\partial\alpha_2}{\partial x_2}\phi_2 \in \mathbb{R}^q$ is the new regressor vector, and it can be verified that $w_3(0,\bar{\beta}^{(2)},\hat{\theta}) = 0$. Using the analysis similar to that used in (27) and (28), one can express w_3 as $w_3 = W_3^T(\underline{x}_3,\bar{\beta}^{(2)},\hat{\theta})\underline{z}_3$, where $W_3 \in \mathbb{R}^{3\times q}$ is a smooth mapping. Therefore, we obtain an upper bound of the last line of (35), as follows:

$$z_3 w_3^T(\theta(t) - \ell_\theta) \le \frac{\delta_{\Delta_\theta}}{2}\left(|W_3|_{\mathsf{F}}^2 + 1\right)z_3^2 + \frac{\delta_{\Delta_\theta}}{2}\underline{z}_2^T\underline{z}_2. \tag{36}$$

Then, we design the following tuning function and virtual control law, respectively:

$$\tau_3(\underline{x}_3,\bar{\beta}^{(2)},\hat{\theta}) = \tau_2 + \Gamma w_3 z_3 \tag{37}$$

$$\begin{aligned}\alpha_3(\underline{x}_3,\bar{\beta}^{(3)},\hat{\theta}) = & -z_2 - (k_3+\zeta_3)z_3 - w_3^T\hat{\theta} + \frac{\partial\alpha_2}{\partial\hat{\theta}}\tau_3 \\ & + \sum_{j=1}^{2}\frac{\partial\alpha_2}{\partial x_j}x_{j+1} + \sum_{j=0}^{2}\frac{\partial\alpha_2}{\partial\beta^{(j)}}\beta^{(j+1)} + \frac{\partial\alpha_1}{\partial\hat{\theta}}\Gamma z_2 w_3\end{aligned} \tag{38}$$

where $\bar{\beta}^{(3)} = [\beta, \dot{\beta}, \ddot{\beta}, \dddot{\beta}]^T$, $k_3 > 0$, and

$$\zeta_3 = \frac{1}{2}\left(\delta_{\Delta_\theta}|W_3|_{\mathsf{F}}^2 + (n-2)\delta_{\Delta_\theta} + \frac{1}{\epsilon_\psi}\right). \tag{39}$$

Now, in virtue of (37) and (38), we can rewrite $\dot{V}_3$ as

$$\begin{aligned}\dot{V}_3 \le & -\sum_{j=1}^{3}k_j z_j^2 + z_3 z_4 - \frac{1}{2}\left((n-3)\delta_{\Delta_\theta} + \frac{1}{\epsilon_\psi}\right)\underline{z}_3^T\underline{z}_3 \\ & + \left(z_2\frac{\partial\alpha_1}{\partial\hat{\theta}} + z_3\frac{\partial\alpha_2}{\partial\hat{\theta}} + (\ell_\theta - \hat{\theta})^T\Gamma^{-1}\right)(\tau_3 - \dot{\hat{\theta}}).\end{aligned} \tag{40}$$

where $z_3 z_4$ can be canceled at the next step.

Step i (i = 3,…,n−1): We are now in the position to summarize the expression of the input signals by previous design steps.

$$\begin{cases} z_i = x_i - \alpha_{i-1} \\ w_i(\underline{x}_i,\bar{\beta}^{(i-1)},\hat{\theta}) = \phi_i - \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}\phi_j \\ \tau_i(\underline{x}_i,\bar{\beta}^{(i-1)},\hat{\theta}) = \tau_{i-1} + \gamma_\theta w_i z_i \\ \alpha_i(\underline{x}_i,\bar{\beta}^{(i)},\hat{\theta}) = -z_{i-1} - (k_i+\zeta_i)z_i - w_i^T\hat{\theta} \\ \qquad + \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}x_{j+1} + \sum_{j=0}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\beta^{(j)}}\beta^{(j+1)} \\ \qquad + \sum_{j=2}^{i-1}\frac{\partial\alpha_{j-1}}{\partial\hat{\theta}}\Gamma z_j w_i + \frac{\partial\alpha_{i-1}}{\partial\hat{\theta}}\tau_i \\ \zeta_i = \frac{1}{2}\left(\delta_{\Delta_\theta}|W_i|_{\mathsf{F}}^2 + (n+1-i)\delta_{\Delta_\theta} + \frac{1}{\epsilon_\psi}\right) \end{cases} \tag{41}$$

where $k_i > 0$, $\epsilon_\psi > 0$, $\bar{\beta}^{(i)} = [\beta,\dot{\beta},\ldots,\beta^{(i)}]^T \in \mathbb{R}^{i+1}$, and $W_i \in \mathbb{R}^{i\times q}$ is a smooth mapping. Based upon (41), the derivative of $V_i = V_{i-1} + \frac{1}{2}z_i^2$ can be computed as

$$\begin{aligned}\dot{V}_i \le & -\sum_{j=1}^{i}k_j z_j^2 + z_i z_{i+1} - \frac{1}{2}\left((n-i)\delta_{\Delta_\theta} + \frac{1}{\epsilon_\psi}\right)\underline{z}_i^T\underline{z}_i \\ & + \Big(\sum_{j=1}^{i-1}\frac{\partial\alpha_j}{\partial\hat{\theta}}z_{j+1} + (\ell_\theta - \hat{\theta})^T\Gamma^{-1}\Big)(\tau_i - \dot{\hat{\theta}}).\end{aligned} \tag{42}$$

Step n: This step is different from the previous steps. On one hand, the actual control law and update law of $\hat{\theta}$ should be designed in this step. On the other hand, we need to extend the congelation of variables for time-varying parameters in the feedback path to the scenario where time-varying parameters are in the input path.

To proceed, we rewrite $\dot{x}_n = \phi_n^T\theta(t) + b(t)u$ as

$$\begin{aligned}\dot{z}_n = & w_n^T\theta(t) + b(t)u - \frac{\partial\alpha_{n-1}}{\partial\hat{\theta}}\dot{\hat{\theta}} - \sum_{j=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial x_j}x_{j+1} \\ & - \sum_{j=0}^{n-1}\frac{\partial\alpha_{n-1}}{\partial\beta^{(j)}}\beta^{(j+1)}\end{aligned} \tag{43}$$

where $w_n = \phi_n - \sum_{j=1}^{n-1} \frac{\partial \alpha_{i-1}}{\partial x_j}\phi_j$. The main difference will start from the following design. For the next developments we need the following intermediate result by means of $u = \hat{\rho}\bar{u}$:

$$\begin{aligned} z_n\dot{z}_n &= z_n w_n^T\hat{\theta} + z_n w_n^T(\theta(t) - \ell_\theta) + z_n w_n^T(\ell_\theta - \hat{\theta}) \\ &\quad + z_n\bar{u} + z_n(b(t) - \ell_b)\hat{\rho}\bar{u} + z_n\ell_b\left(\frac{1}{\ell_b} - \hat{\rho}\right)\bar{u} \\ &\quad - z_n\frac{\partial\alpha_{n-1}}{\partial\hat{\theta}}\dot{\hat{\theta}} - z_n\sum_{j=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial x_j}x_{j+1} \\ &\quad - z_n\sum_{j=0}^{n-1}\frac{\partial\alpha_{n-1}}{\partial\beta^{(j)}}\beta^{(j+1)} \end{aligned} \tag{44}$$

where ℓ_b is an unknown constant which can be regard as the average of $b(t)$, and $\hat{\rho}$ is an "estimate" of $1/\ell_b$. Note that we need δ_{Δ_θ} to construct the nonlinear damping gain to cancel the effect of the unknown $\theta(t)$, as our previous steps do. However, the same method cannot be used directly for dealing with $b(t)$ since the perturbation term $z_1(b(t) - \ell_b)\hat{\rho}\bar{u}$ is coupled with the control input. Here we apply a special way to cope with the unknown time-varying quantities, i.e., designing $\bar{u}$ skillfully to ensure the perturbation term $z_n(b(t) - \ell_b)\hat{\rho}\bar{u}$ in the second of (44) is always negative.

Consider the Lyapunov function candidate

$$V_n = V_{n-1} + \frac{|\ell_b|}{2\gamma_\rho}\left(\frac{1}{\ell_b} - \hat{\rho}\right)^2 + \frac{1}{2}z_n^2 \tag{45}$$

where $\gamma_\rho > 0$, then

$$\begin{aligned} \dot{V}_n &= \dot{V}_{n-1} - \frac{|\ell_b|}{\gamma_\rho}\left(\frac{1}{\ell_b} - \hat{\rho}\right)\dot{\hat{\rho}} + z_n\dot{z}_n \\ &\le -\sum_{j=1}^{n-1}k_j z_j^2 - \frac{1}{2}\left(\delta_{\Delta_\theta} + \frac{1}{\epsilon_\psi}\right)\underline{z}_{n-1}^T\underline{z}_{n-1} \\ &\quad + \left(\sum_{j=1}^{n-1}\frac{\partial\alpha_j}{\partial\hat{\theta}}z_{j+1} + (\ell_\theta - \hat{\theta})^T\Gamma^{-1}\right)(\tau_{n-1} - \dot{\hat{\theta}}) \\ &\quad + z_{n-1}z_n + z_n w_n^T\hat{\theta} + z_n w_n^T\Delta_\theta + z_n w_n^T(\ell_\theta - \hat{\theta}) \\ &\quad + z_n\bar{u} + z_n\Delta_b\hat{\rho}\bar{u} - z_n\sum_{j=0}^{n-1}\frac{\partial\alpha_{n-1}}{\partial\beta^{(j)}}\beta^{(j+1)} \\ &\quad - z_n\sum_{j=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial x_j}x_{j+1} - z_n\frac{\partial\alpha_{n-1}}{\partial\hat{\theta}}\dot{\hat{\theta}} \\ &\quad + z_n\ell_b\left(\frac{1}{\ell_b} - \hat{\rho}\right)\bar{u} - \frac{|\ell_b|}{\gamma_\rho}\left(\frac{1}{\ell_b} - \hat{\rho}\right)\dot{\hat{\rho}} \end{aligned} \tag{46}$$

where $\Delta_b = b(t) - \ell_b$ and $w_n = \phi_n - \sum_{j=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial x_j}\phi_j$. Now, to cancel the third and last lines of (46), we design the update laws for the parameters $\hat{\theta}$ and $\hat{\rho}$, as follows:

$$\begin{aligned} \dot{\hat{\theta}} &= \tau_n = \tau_{n-1} + \Gamma w_n z_n \\ &= \Gamma\Big(z_1\Phi_1\Pi x_1 + \sum_{j=2}^{n}w_j z_j\Big) \end{aligned} \tag{47}$$

$$\dot{\hat{\rho}} = -\gamma_\rho \operatorname{sgn}(\ell_b)z_n\bar{u}. \tag{48}$$

Remark 5: Define $\Omega(\underline{x}_n, \bar{\beta}^{(n)}, \hat{\theta}) = z_{n-1} + w_n^T\hat{\theta} - \frac{\partial\alpha_{n-1}}{\partial\hat{\theta}}\tau_n - \sum_{j=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial x_j}x_{j+1} - \sum_{j=0}^{n-1}\frac{\partial\alpha_{n-1}}{\partial\beta^{(j)}}\beta^{(j+1)} - \sum_{j=2}^{n-1}\frac{\partial\alpha_{j-1}}{\partial\hat{\theta}}\Gamma z_j w_n$. It can be further verified, for $i = 1,\ldots,n-1$, that α_i, w_i, τ_i and Ω are smooth, and $\alpha_i = w_i = \tau_i = \Omega = 0$ if $\underline{x}_i = 0$. Note also that the coordinate transformation

$$z_1 = \frac{\beta(t)\psi(x_1)}{\beta^2(t) - \psi^2(x_1)} \tag{49}$$

and $z_i = x_i - \alpha_{i-1}$ $(i > 1)$ is also smooth, invertible and $x_i = 0 \Leftrightarrow z_i = 0$. According to Hadamard's lemma, $w_n(x, \bar{\beta}^{(n-1)}, \hat{\theta})$ and Ω can be expressed as $w_n = W_n^T\boldsymbol{v}z$ and $\Omega = \bar{\Omega}^T\boldsymbol{z}$, respectively, with $\boldsymbol{z} = [z_1,\ldots,z_n]^T$, $W_n \in \mathbb{R}^{n\times q}$ and $\bar{\Omega} \in \mathbb{R}^n$ being smooth mappings.

Applying Young's inequality with $\epsilon_\psi > 0$, yields

$$z_n\Omega = z_n\bar{\Omega}^T\boldsymbol{z} \le \frac{1}{2}\left(\epsilon_\Omega|\bar{\Omega}|^2 + \frac{1}{\epsilon_\Omega}\right)z_n^2 + \frac{1}{2\epsilon_\Omega}\underline{z}_{n-1}^T\underline{z}_{n-1}$$

$$z_n w_n^T\Delta_\theta \le \frac{\delta_{\Delta_\theta}}{2}\left(|W_n|_{\mathsf{F}}^2 + 1\right)z_n^2 + \frac{\delta_{\Delta_\theta}}{2}\underline{z}_{n-1}^T\underline{z}_{n-1}.$$

Finally, we choose the actual control law $u = \hat{\rho}\bar{u}$ such that the time-varying perturbed term $z_n\Delta_b\hat{\rho}\bar{u}$ is nonpositive

$$\begin{cases} \bar{u} = -\kappa\left(x, \beta, \ldots, \beta^{(n)}, \hat{\theta}\right)z_n \\ \kappa = k_n + \frac{1}{2}\left(\delta_{\Delta_\theta}|W_n|_{\mathsf{F}}^2 + \delta_{\Delta_\theta} + \frac{1}{\epsilon_\Omega} + \epsilon_\Omega|\bar{\Omega}|^2\right) \end{cases} \tag{50}$$

where $k_n > 0$. Inserting (47)−(50) into (46), yields

$$\dot{V}_n \le -\sum_{j=1}^{n}k_j z_j^2 - \kappa\Delta_b\hat{\rho}(t)z_n^2. \tag{51}$$

V. Stability Analysis

Firstly, it can be shown that $\hat{\rho}(t)$ in the right hand side of (51) is a monotonic increasing (or decreasing) function by calculating equation (48) as $\dot{\hat{\rho}} = \gamma_\rho \operatorname{sgn}(\ell_b)\kappa z_n^2$. In addition, one can select $\hat{\rho}(0) > 0$ when $0 < \ell_b \le b(t)$ (in this case, $\Delta_b > 0$) to make sure that $\hat{\rho}(t) > 0$, thereby obtaining $-\kappa\Delta_b\hat{\rho}z_n^2 < 0$. Similarly, one can select $\hat{\rho}(0) < 0$ when $b(t) \le \ell_b < 0$ (in this case, $\Delta_b < 0$) to make sure that $\hat{\rho}(t) < 0$, thereby obtaining $z_n\Delta_b\hat{\rho}\bar{u} = -\kappa\Delta_b\hat{\rho}z_n^2 \le 0$ again. Therefore, (51) can be simplified as $\dot{V}_n \le -\sum_{j=1}^{n}k_j z_j^2 \le 0$, which guarantees that z, $\hat{\theta}$, and $\hat{\rho}$ are bounded for all $t \ge 0$.

Next, in view of Remark 4 and the boundedness of z, it follows that W_1, $1/\Pi$ and Π are bounded, and therefore τ_1 and α_1 are bounded, which further proves the boundedness of x_2 along with the coordinate transformation $x_2 = z_2 + \alpha_1$ and the boundedness of w_2 due to (41). Hence W_2, τ_2 and α_2 are also bounded. Following this line of argument, the boundedness of state x_i, virtual control α_i $(i = 3,\ldots,n-1)$, and the actual control input u are ensured. In addition, it is seen from (47) and (48) that $\dot{\hat{\theta}} \in \mathcal{L}_\infty$ and $\dot{\hat{\rho}} \in \mathcal{L}_\infty$. To show the asymptotic constancy of $\hat{\theta}$ and $\hat{\rho}$, it follows from $\dot{V}_n \le -\sum_{j=1}^{n}k_j z_j^2$ that $z_j \in \mathcal{L}_2$; then, from (47) and (48) we get $\dot{\hat{\theta}} \in \mathcal{L}_1$ and $\dot{\hat{\rho}} \in \mathcal{L}_1$; by using the argument similar to Theorem 3.1 in [35], it is concluded that $\hat{\theta}$ and $\hat{\rho}$ have a limit as $t \to \infty$.

Finally, it follows from (19), (26), (34), (43) and (51) that $\dot{\boldsymbol{z}} \in \mathcal{L}_\infty$ and $\boldsymbol{z} \in \mathcal{L}_2 \cap \mathcal{L}_\infty$. Then, using Barbalat's lemma yields $\lim_{t\to\infty}z(t) = 0$, which further indicates that $\lim_{t\to\infty}x(t) = 0$.

Therefore, the closed-loop system is asymptotically stable. By virtue of Lemma 1, we get that output $x_1(t)$ is always constrained within the prescribed performance funnel $\digamma_\beta = \{(t,x_1) \in [R_{\geq 0}) \times \mathbb{R} \| x_1(t)|/\psi^{-1}(\beta(t)) < 1\}$.

The above facts prove the following result.

Theorem 2: Suppose that the design procedure is applied to the nonlinear system (1) with fast time-varying parameters in feedback and input paths, then, the closed-loop system is asymptotically stable and the system output $x_1(t)$ is always confined within the prescribed performance funnel $\digamma_\beta = \{(t,x_1) \in \mathbb{R}_{\geq 0} \times \mathbb{R} \| x_1(t)|/\psi^{-1}(\beta(t)) < 1\}$ and ultimately decays to zero. Furthermore, $\lim_{t\to\infty}\hat{\theta}$ and $\lim_{t\to\infty}\hat{\rho}$ exist but they are not necessarily equal to ℓ_θ and $1/\ell_b$. In addition, the control input and update laws remain uniformly bounded over $[0,\infty)$. ■

Remark 6: The proposed controller primarily consists of three units: the robust unit, the $\theta(t)$-adaptive unit and the $b(t)$-adaptive unit. Note that the $\theta(t)$-adaptive unit is completely equivalent to the design of update laws in classical adaptive control since we use the unknown constant ℓ_θ to replace $\theta(t)$. The time-varying perturbation term $\Delta_\theta(t)$ caused by $\theta(t) - \ell_\theta$ is allocated to the robust unit for processing. This is an easy-to-understand and easy-to-implement solution, in other words, the proposed controller is simple in structure and user-friendly in design. In addition, the $b(t)$-adaptive unit is deliberately designed for unknown and time-varying control gain, whose main purpose is to ensure the perturbation term $z_n\Delta_b\hat{\rho}\bar{u} \leq 0$, thereby avoiding the adaptive parameter drifting caused by unknown gains.

Remark 7: The control scheme involves the selection of $\{k_i\}_{i=1}^n > 0$, $\hat{\theta}(0) \geq 0$, $\hat{\rho}(0) > 0$, $\delta_{\Delta_\theta} > 0$, $\epsilon_\psi > 0$, $\gamma_\theta > 0$, $\gamma_\rho > 0$ and $\Gamma > 0$, which theoretically can be chosen quite arbitrarily by users. Certain compromises between convergence rate and control effort needs to be made when making the selection for those parameters for a given system. For example, the parameters k_i and δ_{Δ_θ} are proportional to convergence rate and control effort in this paper. Thus, reducing the input effort will cause the convergence rate to slow down. However, it is worth noting that the prescribed constraint rule will not be violated no matter how the parameters are selected.

Remark 8: In [37], an exponential function $\beta(t)$ that increases monotonically with time is included in the control law and the adaptive law, resulting in zero-error exponential regulation of the closed-loop system. However, the adaptation scaling gain $\beta(t)$ has grown so large to hardly even be computable as $t \to \infty$. Therefore, it needs to add a saturation to $\beta(t)$ for implementation, which obviously results in some loss of control accuracy. In contrast, the proposed method provides a simpler solution, and without loss of final control accuracy, completely eliminates the necessity for the control gain to grow with time.

Remark 9: Different from traditional prescribed performance control (see, for instance, [25]–[27]) that can only achieve bounded regulation, the size of the regulation residual set is reversely proportional to the control gain, such that higher final control precision is essentially the price of large control gain, and the proposed control method is able to steer each system state to zero asymptotically without the need for prohibitively large controller gain. Furthermore, no matter how small the control gain is, $-\psi^{-1}(\beta) < x_1(t) < \psi^{-1}(\beta)$ always holds.

Remark 10: Our control scheme benefits from the Chen & Astolfi's method [1]–[3] in dealing with unknown time-varying parameters. Furthermore, by introducing the performance function and employing a novel coordinate transformation, our control scheme is able to explicitly address the global transient behavior of system output, together with its steady-state performance.

VI. Simulation

To verify the effectiveness of the proposed control method, we consider the following system[3]:

$$\begin{aligned} \dot{x}_1 &= \theta(t)x_1 + x_2 \\ \dot{x}_2 &= b(t)u \\ y(t) &= x_1 \end{aligned} \tag{52}$$

with fast time-varying parameters[4]

$$b(t) = 2 + 0.1\cos(x_1) + \mathrm{sgn}(x_1x_2) \tag{53}$$

$$\theta(t) = 2 + 0.8\sin(t) + \sin(x_1x_2) + 0.2\sin(x_1t) + \mathrm{sgn}(\sin(t)). \tag{54}$$

It is not difficult to verify that Assumptions 1 and 2 are satisfied. The control objective is to make the state x_1 move back to zero at a prescribed rate no slower than exponentially and ultimately converges to zero. Now we consider three controllers: Controller 1 is the adaptive controller proposed by Chen & Astolfi in [1]; Controller 2 is the semi-global adaptive prescribed performance controller which can be obtained by combining Controller 1 and the controller proposed in [25]; Controller 3 is the global controller proposed in Theorem 2. In fact, Controller 2 can be viewed as a special case of Controller 3. For fair comparison, we set $[x_1(0); x_2(0)] = [1; -1]$, $k_1 = k_2 = \gamma_\rho = 0.1$, $\delta_{\Delta_\theta} = 1$, $\Gamma = 0.1I$, $\hat{\theta}(0) = 0$ and $\hat{\rho}(0) = 0.25$ for all controllers. In addition, we select $\beta(t) = 4e^{-0.4t} + 0.1$ and $z_1 = \tan(\pi x_1/(2\beta))$ for Controller 2, and select $\beta_1(t) = 0.9e^{-0.4t} + 0.1$ for Controller 3.

The responses of the state signals are shown in Figs. 1 and 2, and the responses of control input signals are shown in Fig 3. The evolution of adaptive parameters $\hat{\theta}$ and $\hat{\rho}$ are shown in Figs. 4 and 5, respectively. In addition, we also illustrate the time-varying parameters $\theta(t)$ and $b(t)$ in Fig 6, which shows that the state-dependent parameters are fast time-varying and nondifferentiable. From these simulation results, we know that the proposed controllers outperforms the adaptive controller in [1], since the transient behavior of the system can be confined to a prescribed performance boundary. In particular, comparing Controller 1 with Controllers 2 and 3, one can find

[3] Note that when $\theta(t)$ is an unknown constant and $b(t) = 1$, this model is a simplified version of the one studied by [37], where the exponential regulation is proposed for a class of strict-feedback systems with known control gain and unknown constants θ.

[4] Here $b(t)$ and $\theta(t)$ are fast time-varying parameters and they are only piecewise continuous yet $b(t)$ may undergo sudden changes. Therefore, some classical adaptive schemes [15], [19] are not available because those methods require the parameters be slow time-varying (i.e., here exists a parameter ϵ such that $|\dot{\theta}(t)| < \epsilon$ and $|\dot{b}(t)| < \epsilon$).

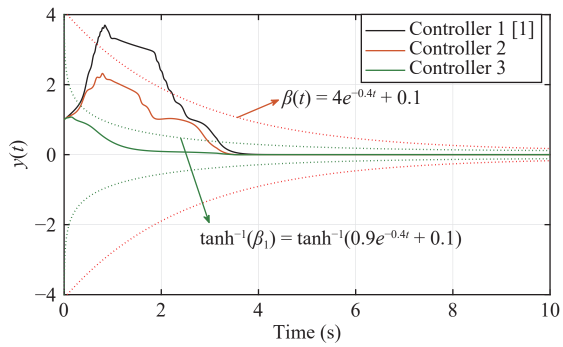

Fig. 1. The evolution of $y(t)$.

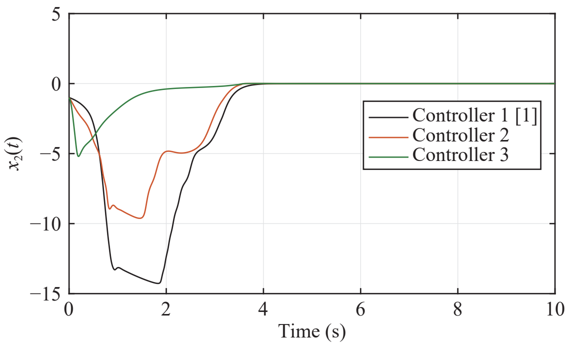

Fig. 2. The evolution of $x_2(t)$.

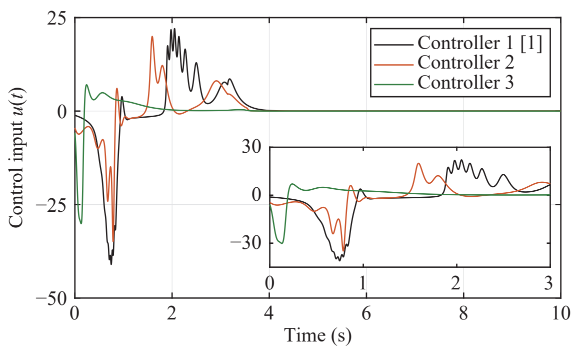

Fig. 3. The evolution of $u(t)$.

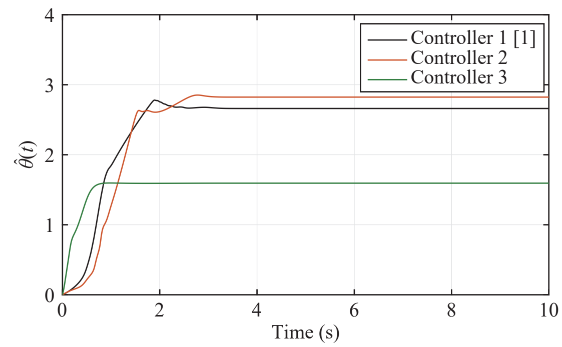

Fig. 4. The evolution of $\hat{\theta}(t)$.

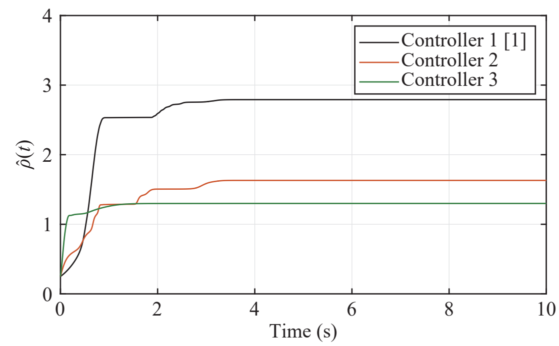

Fig. 5. The evolution of $\hat{\rho}(t)$.

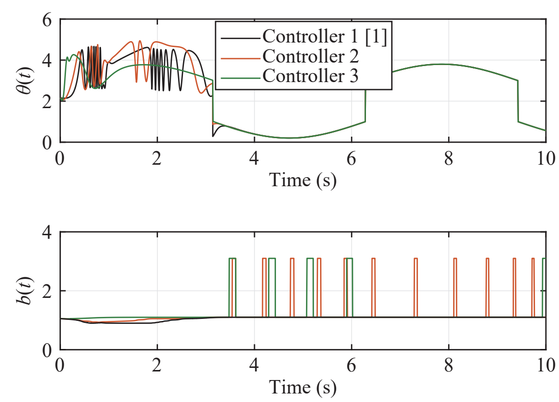

Fig. 6. System time-varying parameters $\theta(t)$ and $b(t)$.

a counterintuitive phenomenon, that is, a faster system response can be achieved without increasing the control effort for the same choice of controller parameters. In short, all results show that the proposed methods are powerful enough to stabilize the nonlinear system with fast time-varying parameters.

VII. Conclusion

This work presents an adaptive control strategy with guaranteed performance for strict-feedback nonlinear systems involving fast time-varying parameters. It is shown that with this strategy, not only each system state is regulated to zero asymptotically, but also the system output is strictly confined within an exponentially decaying boundary, making system output well behaved during the transient period and steady-state phase. We start with a simple scalar system with time-varying parameters in the feedback path and input path to illustrate our core idea in addressing time-varying parameters and output performance constraints simultaneously. By using classical backstepping technology and nonlinear damping, we then extend our method to a higher-order system and remove the need for overparametrization. Furthermore, the diversity of performance function selection and the diversity of normalized function selection together with the independence on initial conditions imply the universality of our controller, and simulation comparisons confirm the effectiveness and benefits of these methods.

Prior to this work, the prevailing wisdom in adaptive control in the context of exponential stability for time-varying systems was that certain persistent excitation conditions (sufficiently rich signals) must be present. Here in this work we develop a method that achieves exponential convergence, pointwise in time, without the need for PE conditions. Interesting future research topics include studying exponential/finite-/prescribed-time regulation of nonlinear systems with unknown time-varying parameters and control directions, studying robustness with respect to external disturbances via adaptive disturbance rejection, and borrowing the ideas in [38]–[40] to explore the distributed control of multi-agent/large-scale systems with time-varying parameters.

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