Letter

Encoding-Decoding-Based Recursive Filtering for Fractional-Order Systems

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Dear editor,

This letter focuses on the encoding-decoding-based recursive filtering problem for a class of fractional-order systems. For the purpose of protecting security of the wireless communication network, a dynamic-quantization-based encoding-decoding mechanism is introduced to encrypt the transmitted measurement. Specifically, the measurement outputs are first encoded by the encoder into codewords which are then transmitted over the wireless communication network. After received by the decoder, the codewords are decoded and then sent to the filter. In light of the mathematical properties of truncated Gaussian distribution, the variance of the encoding-decoding-induced error between the decoded measurement and the real measurement is derived. An upper bound on the filtering error covariance is first obtained based on the variance of the encoding-decoding-induced error. Then, the minimal upper bound is derived by choosing proper filter gain. Finally, the efficiency and superiority of the proposed algorithm are verified through a simulation example.

The fractional calculus is a generalization of the integral calculus, which was first mentioned by Leibniz and Laplace hospital in 1695 [1]. In recent decades, the fractional calculus has attracted great attention in the field of dynamic modeling and control engineering. In contrast to the classical integer calculus, the fractional calculus is able to model the complex systems more accurately. At the same time, some research results further demonstrate that the fractional-order system has its unique advantages. For example, in [2], the dynamics of packet channels in TCP/IP networks has been described by fractional calculus, which provides a new mathematical idea for describing the characteristics of packet traffic behavior. The fractional-order model of ultracapacitors has been constructed to estimate the charge state more accurately in [3]. It is worth mentioning that the Kalman filter is regarded as an optimal estimator for solving the filtering problem of fractional-order system on account of the concise recursive form and excellent real-time online computing capability [4], [5].

On the other hand, with the rapid development of the wireless communication networks, the network security problem becomes a really hot research topic [6], [7]. To secure data from leakage and theft during transmission, the digital communication strategy has been widely used because of the high anti-interference and data encryption ability [8]. In recent years, the encoding-decoding mechanism (EDM) has received considerable research attention in the state estimation problems [9]−[12]. For example, in [13], a distributed filtering method for smart grid with concatenated coding is proposed to alleviate the limitation of communication resources. A multiple description video coding scheme is applied to heterogeneous communication chain to improve the robustness of video diversity [14]. Under the EDM, the encoder first maps the measurement into codewords according to specific rules, and the codewords are then transmitted to the decoder. The decoder restores the received codewords and sends the decoded information to the filter. Finally, the filter estimates the system state based on the decoded information. Due to the encoding-decoding process, the security of the measurement is improved to prevent attackers from stealing useful information.

It is worth noting that, although the EDM has unique advantages in improving the network security, there are also some noteworthy problems. Due to the introduction of the EDM, the decoded measurement might deviate from the real measurement. As such, the main challenges are identified as: 1) how to characterize the encoding-decoding-induced error between the decoded measurement and real measurement and 2) how to mitigate the filtering performance deterioration from the encoding-decoding-induced error should be seriously taken into consideration in the filtering problem under the EDM. In this letter, we aim to solve these two problems by considering the recursive filtering problem for the fractional-order system under the EDM. The main contributions of this letter are identified as: 1) the filtering problem is, for the first time, considered for the fractional-order system under the EDM; 2) a random variable is introduced to represent the encoding-decoding-induced error, and the exact variance of the introduced random variable is calculated; and 3) a minimized upper bound is ensured on the filtering error covariance (FEC).

Problem formulation: Consider the following discrete fractional-order system:

$$\Delta^q x_{\epsilon+1} = A_\epsilon x_\epsilon + B_\epsilon \omega_\epsilon \tag{1}$$

$$x_{\epsilon+1} = \Delta^q x_{\epsilon+1} - \sum_{i=1}^{\epsilon+1} (-1)^i \Lambda_i x_{\epsilon+1-i} \tag{2}$$

$$y_\epsilon = C_\epsilon x_\epsilon + \nu_\epsilon \tag{3}$$

where

$$\Delta^q x_{\epsilon+1} = [\Delta^{q_1} x_{1,\epsilon+1}, \Delta^{q_2} x_{2,\epsilon+1}, \ldots, \Delta^{q_{n_x}} x_{n_x,\epsilon+1}]^T$$

$$\Lambda_i = \mathrm{diag}\left\{\begin{pmatrix} q_1 \\ i \end{pmatrix}, \begin{pmatrix} q_2 \\ i \end{pmatrix}, \ldots, \begin{pmatrix} q_{n_x} \\ i \end{pmatrix}\right\}$$

$$\begin{pmatrix} q_j \\ i \end{pmatrix} = \begin{cases} 1, & \text{if } i = 0 \\ \dfrac{q_j(q_j-1)\cdots(q_j-i+1)}{i!}, & i > 0 \end{cases}$$

and $x_\epsilon \in \mathbb{R}^{n_x}$ is the state vector with the initial value x_0 whose mean is $\bar{x}_0$ and covariance is $P_{0|0}$. $y_\epsilon \in \mathbb{R}^{n_y}$ denotes the measurement output, Δ is the fractional difference operator, $q_j (j = 1, 2, \ldots, n_x)$ are the corresponding fractional orders of state components $x_{j,\epsilon}$. ω_ϵ and ν_ϵ are mutually uncorrelated zero-mean white Gaussian noises with covariances $Q_\epsilon > 0$ and $Z_\epsilon > 0$, respectively.

In this letter, a dynamic-quantization-based EDM is applied to encrypt the data, where the encoder and the decoder are presented as follows:

Encoder:

$$\rho_\epsilon = d\left(\frac{1}{g_\epsilon} y_\epsilon\right) \tag{4}$$

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where ρ_ϵ is the codeword transmitted over the wireless communication network, g_ϵ is the scaling function, and the quantizer $d(\cdot)$ with quantization level $2L$+1 is chosen as follows:

$$d(\gamma) = \begin{cases} n, & \frac{2n-1}{2}\xi \le \gamma < \frac{2n+1}{2}\xi \\ L, & \gamma \ge \frac{2L-1}{2}\xi \\ -L, & \gamma < -\frac{2L-1}{2}\xi \end{cases}$$

where $n \in \{0, \pm1, \pm2, \ldots, \pm(L-1)\}$ and ξ denotes the quantization interval. For the uniform quantizer $d(\cdot)$ it is obtained that

$$\frac{1}{g_\epsilon} y_\epsilon = \xi\rho_\epsilon + \Theta_\epsilon \tag{5}$$

where $\Theta_\epsilon = [\theta_{1,\epsilon}, \theta_{2,\epsilon}, \ldots, \theta_{n_y,\epsilon}]^T$ with $\theta_{l,\epsilon} (l \in \{1, 2, \ldots, n_y\})$ being the quantization error.

Decoder:

$$\begin{cases} \tilde{y}_0 = 0 \\ \tilde{y}_\epsilon = \xi g_\epsilon \rho_\epsilon \end{cases} \tag{6}$$

where $\tilde{y}_\epsilon$ is the output of the decoder.

In this letter, the recursive filter is designed as follows:

$$\hat{x}_{\epsilon+1|\epsilon} = (A_\epsilon + \Lambda_1)\hat{x}_{\epsilon|\epsilon} - \sum_{i=2}^{\epsilon+1} (-1)^i \Lambda_i \hat{x}_{\epsilon+1-i|\epsilon+1-i} \tag{7}$$

$$\hat{x}_{\epsilon+1|\epsilon+1} = \hat{x}_{\epsilon+1|\epsilon} + G_{\epsilon+1}(\tilde{y}_{\epsilon+1} - C_{\epsilon+1}\hat{x}_{\epsilon+1|\epsilon}) \tag{8}$$

where $\hat{x}_{\epsilon+1|\epsilon}$ and $\hat{x}_{\epsilon|\epsilon}$ are the one-step prediction and the estimate of x_ϵ at time instant ϵ, respectively. $G_{\epsilon+1}$ is the filter gain to be determined. Denote $e_{\epsilon+1|\epsilon} = x_{\epsilon+1} - \hat{x}_{\epsilon+1|\epsilon}$ and $e_{\epsilon+1|\epsilon+1} = x_{\epsilon+1} - \hat{x}_{\epsilon+1|\epsilon+1}$, respectively. Our aim is to design a recursive filter (7) and (8) such that the designed filter gain $G_{\epsilon+1}$ can ensure that the FEC has a minimal upper bound.

Main results:

Lemma 1: For the uniform quantizer $d(\cdot)$, one has

$$\mathrm{Tr}\left\{E\left\{\Theta_\epsilon\Theta_\epsilon^T\right\}\right\} = \sum_{l=1}^{n_y} M_{l,\epsilon}$$

where

$$M_{l,\epsilon} \triangleq \begin{cases} \frac{1}{g_\epsilon^2}\aleph_{l,\epsilon|1} - \frac{2n\xi}{g_\epsilon}\varsigma_{l,\epsilon|1} + (n\xi)^2, & \text{for } \tilde{y}_{l,\epsilon} = n\xi g_\epsilon \\ \frac{1}{g_\epsilon^2}\aleph_{l,\epsilon|2} - \frac{2L\xi}{g_\epsilon}\varsigma_{l,\epsilon|2} + (L\xi)^2, & \text{for } \tilde{y}_{l,\epsilon} = L\xi g_\epsilon \\ \frac{1}{g_\epsilon^2}\aleph_{l,\epsilon|3} + \frac{2L\xi}{g_\epsilon}\varsigma_{l,\epsilon|3} + (L\xi)^2, & \text{for } \tilde{y}_{l,\epsilon} = -L\xi g_\epsilon \end{cases}$$

$$\varsigma_{l,\epsilon|1} \triangleq \mu_{l,\epsilon} - \sigma_{l,\epsilon}^2 \frac{h(\bar{\hbar}_{\epsilon,n}) - h(\underline{\hbar}_{\epsilon,n})}{H(\bar{\hbar}_{\epsilon,n}) - H(\underline{\hbar}_{\epsilon,n})}$$

$$\varsigma_{l,\epsilon|2} \triangleq \mu_{l,\epsilon} + \sigma_{l,\epsilon}^2 \frac{h(\vartheta_\epsilon)}{1 - H(\vartheta_\epsilon)}$$

$$\varsigma_{l,\epsilon|3} \triangleq \mu_{l,\epsilon} - \sigma_{l,\epsilon}^2 \frac{h(-\vartheta_\epsilon)}{H(-\vartheta_\epsilon)}$$

$$\aleph_{l,\epsilon|1} \triangleq \sigma_{l,\epsilon}^2 \left[1 - \frac{\bar{\hbar}_{\epsilon,n} h(\bar{\hbar}_{\epsilon,n}) - \underline{\hbar}_{\epsilon,n} h(\underline{\hbar}_{\epsilon,n})}{H(\bar{\hbar}_{\epsilon,n}) - H(\underline{\hbar}_{\epsilon,n})}\right] + \mu_{l,\epsilon}\varsigma_{l,\epsilon|1}$$

$$\aleph_{l,\epsilon|2} \triangleq \sigma_{l,\epsilon}^2 \left[1 + \frac{\vartheta_\epsilon h(\vartheta_\epsilon)}{1 - H(\vartheta_\epsilon)}\right] + \mu_{l,\epsilon}\varsigma_{l,\epsilon|2}$$

$$\aleph_{l,\epsilon|3} \triangleq \sigma_{l,\epsilon}^2 \left[1 + \frac{\vartheta_\epsilon h(-\vartheta_\epsilon)}{H(-\vartheta_\epsilon)}\right] + \mu_{l,\epsilon}\varsigma_{l,\epsilon|3}$$

$$\underline{\hbar}_{\epsilon,n} \triangleq \frac{(2n-1)\xi g_\epsilon}{2}, \ \bar{\hbar}_{\epsilon,n} \triangleq \frac{(2n+1)\xi g_\epsilon}{2}, \ \vartheta_\epsilon \triangleq \frac{2L-1}{2}\xi g_\epsilon$$

$$h(y_{l,\epsilon}) = \frac{1}{\sqrt{2\pi}\sigma_{l,\epsilon}} e^{-\frac{(y_{l,\epsilon} - \mu_{l,\epsilon})^2}{2\sigma_{l,\epsilon}^2}}$$

$$H(y_{l,\epsilon}) = \int_{-\infty}^{y_{l,\epsilon}} \frac{1}{\sqrt{2\pi}\sigma_{l,\epsilon}} e^{-\frac{(t - \mu_{l,\epsilon})^2}{2\sigma_{l,\epsilon}^2}} dt$$

$$\mu_{l,\epsilon} = (C_\epsilon \hat{x}_{\epsilon|\epsilon-1})_l, \ \sigma_{l,\epsilon}^2 = Z_{l,\epsilon}.$$

Here, $h(y_{l,\epsilon})$ and $H(y_{l,\epsilon})$ are the probability density function and the cumulative distribution function of $y_{l,\epsilon}$, respectively. $(C_\epsilon \hat{x}_{\epsilon|\epsilon-1})_l$ is the lth component of $C_\epsilon \hat{x}_{\epsilon|\epsilon-1}$ and $Z_{l,\epsilon}$ means the lth pivot diagonal element of Z_ϵ.

Proof: When $y_{l,\epsilon} \in [\underline{\hbar}_{\epsilon,n}, \bar{\hbar}_{\epsilon,n})$, the expectation of $y_{l,\epsilon}$ under this condition is

$$E\left\{y_{l,\epsilon} | \underline{\hbar}_{\epsilon,n} \le y_{l,\epsilon} < \bar{\hbar}_{\epsilon,n}, x_\epsilon\right\} = \int_{\underline{\hbar}_{\epsilon,n}}^{\bar{\hbar}_{\epsilon,n}} \frac{y_{l,\epsilon} h(y_{l,\epsilon})}{H(\bar{\hbar}_{\epsilon,n}) - H(\underline{\hbar}_{\epsilon,n})} dy_{l,\epsilon} = \varsigma_{l,\epsilon|1}.$$

Moreover, we have

$$E\left\{y_{l,\epsilon}^2 | \underline{\hbar}_{\epsilon,n} \le y_{l,\epsilon} < \bar{\hbar}_{\epsilon,n}, x_\epsilon\right\} = \int_{\underline{\hbar}_{\epsilon,n}}^{\bar{\hbar}_{\epsilon,n}} \frac{y_{l,\epsilon}^2 h(y_{l,\epsilon})}{H(\bar{\hbar}_{\epsilon,n}) - H(\underline{\hbar}_{\epsilon,n})} dy_{l,\epsilon} = \aleph_{l,\epsilon|1}.$$

Then, it is obtained that

$$\begin{aligned} &E\left\{\theta_{l,\epsilon}^2 | \underline{\hbar}_{\epsilon,n} \le y_{l,\epsilon} < \bar{\hbar}_{\epsilon,n}, x_\epsilon\right\} \\ &= E\left\{\left(\frac{1}{g_\epsilon} y_{l,\epsilon} - n\xi\right)^2 | \underline{\hbar}_{\epsilon,n} \le y_{l,\epsilon} < \bar{\hbar}_{\epsilon,n}, x_\epsilon\right\} = M_{l,\epsilon|1}. \end{aligned}$$

Similarly, when $y_{l,\epsilon} \ge \vartheta_\epsilon$ and $y_{l,\epsilon} < -\vartheta_\epsilon$, then, $E\left\{\theta_{l,\epsilon}^2 | y_{l,\epsilon} \ge \vartheta_\epsilon, x_\epsilon\right\} = M_{l,\epsilon|2}$ and $E\left\{\theta_{l,\epsilon}^2 | y_{l,\epsilon} < -\vartheta_\epsilon, x_\epsilon\right\} = M_{l,\epsilon|3}$. ■

Assumption 1 [15]: In this letter, we assume that $E\{e_{i|i} e_{j|j}\} = 0$ for $i \ne j$.

Remark 1: Assumption 1 is a general assumption of fractional Kalman filtering for the main reason that the correlation between cross terms $e_{i|i}$ and $e_{j|j}$ for $i \ne j$ is difficult to calculate. In addition, this assumption has little effect on filtering performance.

Theorem 1: The recursions of the one-step prediction error covariance $P_{\epsilon+1|\epsilon} \triangleq E\{e_{\epsilon+1|\epsilon} e_{\epsilon+1|\epsilon}^T\}$ and the FEC $P_{\epsilon+1|\epsilon+1} \triangleq E\{e_{\epsilon+1|\epsilon+1} e_{\epsilon+1|\epsilon+1}^T\}$ are given as follows:

$$\begin{aligned} P_{\epsilon+1|\epsilon} &= (A_\epsilon + \Lambda_1) P_{\epsilon|\epsilon} (A_\epsilon + \Lambda_1)^T + B_\epsilon Q_\epsilon B_\epsilon^T \\ &\quad + \sum_{i=2}^{\epsilon+1} \Lambda_i P_{\epsilon+1-i|\epsilon+1-i} \Lambda_i^T \end{aligned} \tag{9}$$

$$\begin{aligned} P_{\epsilon+1|\epsilon+1} &= E\left\{\Omega_{1,\epsilon+1} + \Omega_{1,\epsilon+1}^T - \Omega_{2,\epsilon+1} - \Omega_{2,\epsilon+1}^T \right. \\ &\quad \left. + g_{\epsilon+1}^2 G_{\epsilon+1} \Theta_{\epsilon+1} \Theta_{\epsilon+1}^T G_{\epsilon+1}^T\right\} \\ &\quad + (I - G_{\epsilon+1} C_{\epsilon+1}) P_{\epsilon+1|\epsilon} (I - G_{\epsilon+1} C_{\epsilon+1})^T \\ &\quad + G_{\epsilon+1} Z_{\epsilon+1} G_{\epsilon+1}^T \end{aligned} \tag{10}$$

where

$$\Omega_{1,\epsilon+1} \overset{\Delta}{=} g_{\epsilon+1}(I-G_{\epsilon+1}C_{\epsilon+1})e_{\epsilon+1|\epsilon}\Theta_{\epsilon+1}^T G_{\epsilon+1}^T$$

$$\Omega_{2,\epsilon+1} \overset{\Delta}{=} g_{\epsilon+1}G_{\epsilon+1}\nu_{\epsilon+1}\Theta_{\epsilon+1}^T G_{\epsilon+1}^T.$$

Proof: Taking (1)–(8) into consideration, one has

$$e_{\epsilon+1|\epsilon} = (A_\epsilon+\Lambda_1)e_{\epsilon|\epsilon}+B_\epsilon\omega_\epsilon-\sum_{i=2}^{\epsilon+1}(-1)^i\Lambda_i e_{\epsilon+1-i|\epsilon+1-i} \quad (11)$$

$$e_{\epsilon+1|\epsilon+1} = (I-G_{\epsilon+1}C_{\epsilon+1})e_{\epsilon+1|\epsilon}-G_{\epsilon+1}\nu_{\epsilon+1}+g_{\epsilon+1}G_{\epsilon+1}\Theta_{\epsilon+1}. \quad (12)$$

According to Assumption 1, $P_{\epsilon+1|\epsilon}$ and $P_{\epsilon+1|\epsilon+1}$ are directly derived by (11) and (12), respectively. ■

Theorem 2: Given the positive scalars α and β. The solutions $\Psi_{\epsilon+1|\epsilon+1}$ to the following two recursive matrix equations:

$$\Psi_{\epsilon+1|\epsilon} = (A_\epsilon+\Lambda_1)\Psi_{\epsilon|\epsilon}(A_\epsilon+\Lambda_1)^T+B_\epsilon Q_\epsilon B_\epsilon^T + \sum_{i=2}^{\epsilon+1}\Lambda_i\Psi_{\epsilon+1-i|\epsilon+1-i}\Lambda_i^T \quad (13)$$

$$\begin{aligned}\Psi_{\epsilon+1|\epsilon+1} =& (1+\alpha)(I-G_{\epsilon+1}C_{\epsilon+1})\Psi_{s+1|\epsilon}(I-G_{\epsilon+1}C_{\epsilon+1})^T \\ &+(1+\alpha^{-1}+\beta^{-1})\times g_{\epsilon+1}^2\sum_{l=1}^{n_y}M_{l,\epsilon+1}G_{\epsilon+1}G_{\epsilon+1}^T \\ &+(1+\beta)G_{\epsilon+1}Z_{\epsilon+1}G_{\epsilon+1}^T \end{aligned}\quad (14)$$

with initial condition $0<P_{0|0}\le\Psi_{0|0}$ is an upper bound of $P_{\epsilon+1|\epsilon+1}$. In addition, $\Psi_{\epsilon+1|\epsilon+1}$ is minimized by the filter gain

$$G_{\epsilon+1}=(1+\alpha)\Psi_{\epsilon+1|\epsilon}C_{\epsilon+1}^T\Xi_{\epsilon+1}^{-1} \quad (15)$$

where

$$\begin{aligned}\Xi_{\epsilon+1} =& (1+\alpha)C_{\epsilon+1}\Psi_{\epsilon+1|\epsilon}C_{\epsilon+1}^T+(1+\beta)Z_{\epsilon+1}+(1+\alpha^{-1} \\ &+\beta^{-1})g_{\epsilon+1}^2\sum_{l=1}^{n_y}M_{l,\epsilon+1}I.\end{aligned}$$

Proof: By using $ab^T+ba^T\le\delta aa^T+\delta^{-1}bb^T$, we have

$$\begin{aligned}&E\left\{\Omega_{1,\epsilon+1}+\Omega_{1,\epsilon+1}^T-\Omega_{2,\epsilon+1}-\Omega_{2,\epsilon+1}^T\right\} \\ &\le \alpha(I-G_{\epsilon+1}C_{\epsilon+1})P_{\epsilon+1|\epsilon}(I-G_{\epsilon+1}C_{\epsilon+1})^T \\ &\quad+(\alpha^{-1}+\beta^{-1})g_{\epsilon+1}^2G_{\epsilon+1}E\{\Theta_{\epsilon+1}\Theta_{\epsilon+1}^T\}G_{\epsilon+1}^T \\ &\quad+\beta G_{\epsilon+1}Z_{\epsilon+1}G_{\epsilon+1}^T.\end{aligned}\quad (16)$$

Applying Lemma 1 and (16) to (10) yields

$$\begin{aligned}P_{\epsilon+1|\epsilon+1} \le& (1+\alpha)(I-G_{\epsilon+1}C_{\epsilon+1})P_{\epsilon+1|\epsilon}(I-G_{\epsilon+1}C_{\epsilon+1})^T \\ &+(1+\alpha^{-1}+\beta^{-1})\times g_{\epsilon+1}^2\sum_{l=1}^{n_y}M_{l,\epsilon+1}G_{\epsilon+1}G_{\epsilon+1}^T \\ &+(1+\beta)G_{\epsilon+1}Z_{\epsilon+1}G_{\epsilon+1}^T.\end{aligned}\quad (17)$$

Then, on the basis of mathematical induction, we arrive at $P_{\epsilon+1|\epsilon+1}\le\Psi_{\epsilon+1|\epsilon+1}$. Finally, by using the completing-the-square method, one has

$$\begin{aligned}\Psi_{\epsilon+1|\epsilon+1} =& (G_{\epsilon+1}-(1+\alpha)\Psi_{\epsilon+1|\epsilon}C_{\epsilon+1}^T\Xi_{\epsilon+1}^{-1})\Xi_{\epsilon+1}(G_{\epsilon+1} \\ &-(1+\alpha)\Psi_{\epsilon+1|\epsilon}C_{\epsilon+1}^T\Xi_{\epsilon+1}^{-1})^T \\ &-(1+\alpha)^2\Psi_{\epsilon+1|\epsilon}C_{\epsilon+1}^T\Xi_{\epsilon+1}^{-1}C_{\epsilon+1}\Psi_{\epsilon+1|\epsilon}+(1+\alpha)\Psi_{\epsilon+1|\epsilon}.\end{aligned}\quad (18)$$

In view of (18), the optimal filter gain $G_{\epsilon+1}$ is calculated as (15). ■

Remark 2: When the dynamic-quantization-based EDM is used, the filtering performance may be affected by parameters g_ϵ (zoom function) and ξ (quantization interval). According to practical engineering experience, g_ϵ can be dynamically selected to ensure that the encoded data falls within the quantizer range as far as possible. In general, the more suitable g_ϵ selected, the better filtering performance is. Meanwhile, the smaller quantization interval is, the better the filtering performance gets.

Remark 3: Compared with the existing results, in this letter, a random variable subject to truncated Gaussian distribution is introduced to describe the relationship between the decoded output and the measurement output, and the variance of the encoding-decoding-induced error is calculated accurately to compensate for the influence of the EDM on the measurement output and improve the filtering performance.

Simulation examples: In this section, we aim to compare our proposed filtering algorithm with one that ignores quantization error, where the filtering algorithm without quantization error means that $y_\epsilon=\tilde{y}_\epsilon$. We consider a fractional discrete-time representation of ultra-capacitor [1] with parameters

$$A_\epsilon=\begin{bmatrix}0 & 1+0.02\sin(2\epsilon)\\ 0.035311 & 0.001815\end{bmatrix},\ B_\epsilon=\begin{bmatrix}0\\1\end{bmatrix}$$

$$C_\epsilon=\begin{bmatrix}-0.0186+0.001\cos(3\epsilon) & 0.18843\end{bmatrix},\ q=\begin{bmatrix}0.2 & 0.2\end{bmatrix}.$$

In the simulation, we set $x_{0|0}=[\ 0.2\quad 0.6\ ]^T$, $P_{0|0}=100I\in\mathbb{R}^{2\times 2}$, $Q_\epsilon=0.25$ and $Z_\epsilon=0.01$. For the EDM, the parameters are selected as $g_\epsilon=0.1$, $\xi=0.4$, and $L=10$. The mean square error (MSE) is defined as: $MSE=(1/N)\sum_{j=1}^{N}\sum_{i=1}^{2}(x_{i,\epsilon}-\hat{x}_{i,\epsilon|\epsilon})$ where $N=300$ is the number of independent experiments.

The simulation results are shown in Figs. 1−4. Figs. 1 and 2 plot the actual states and the estimated states with and without quantization error, respectively. The trace of $\Psi_{\epsilon|\epsilon}$ and the MSE are shown in Fig. 3. In addition, Fig. 4 depicts the filtering errors with and without quantization error. It is seen from Figs. 1, 2 and 4 that the proposed filtering algorithm performs well for the fractional-order systems under the EDM.

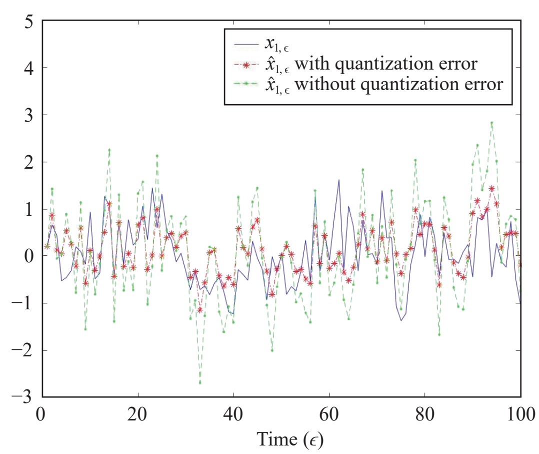

Fig. 1. State $x_{1,\epsilon}$ and its estimate.

Conclusions: In this letter, the encoding-decoding-based recursive filtering problem has been addressed for a class of fractional-order

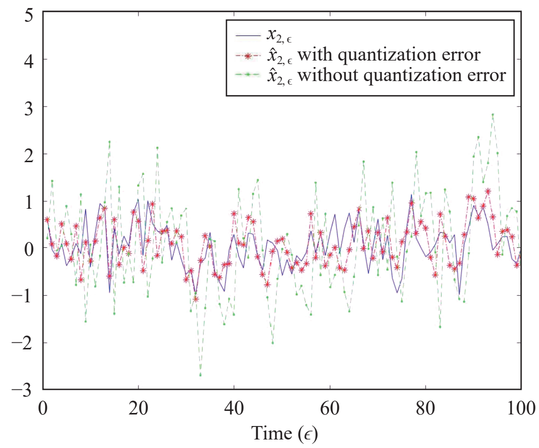

Fig. 2. State $x_{2,\epsilon}$ and its estimate.

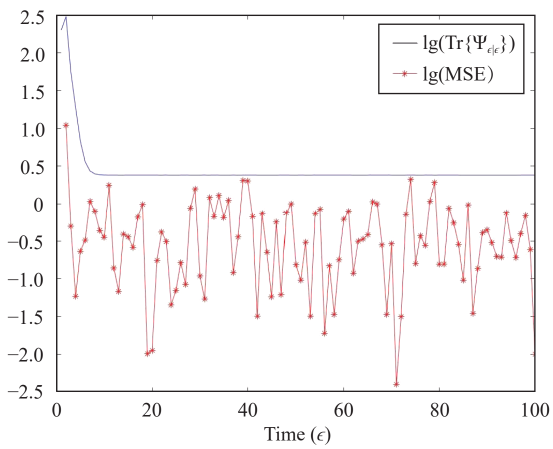

Fig. 3. The trace of $\Psi_{\epsilon|\epsilon}$ and the MSE.

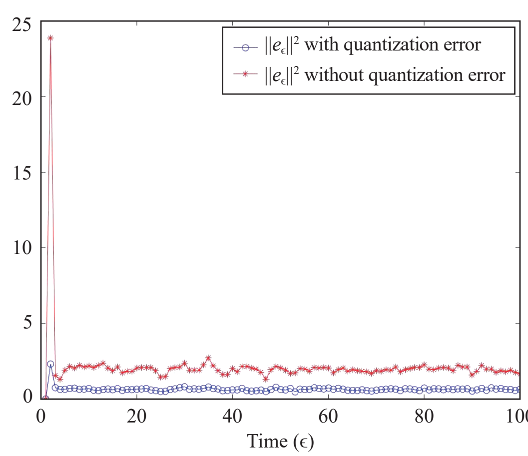

Fig. 4. Filtering error with and without quantization error.

systems. The dynamic-quantization-based EDM has been introduced into the wireless communication network to encrypt the measurement data. A quantization error has been introduced to characterize the relationship between the decoded output and the real measurement, and the exact variance of the introduced quantization error has been calculated. By solving two Riccati difference equations and appropriately designing the filter gain, a minimum upper bound on the FEC has been attained. Finally, the effectiveness of the proposed filtering algorithm has been verified by a simulation example.

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