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Fixed-time consensus disturbance rejection for high-order nonlinear multi-agent systems with input saturation

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Abstract

This study is devoted to investigating the fixed-time consensus disturbance rejection problem for a class of high-order nonlinear multi-agent systems (MASs) with input saturation. A distributed fixed-time state observer (FTSO) is first proposed to reconstruct the leader's states for each follower. Based on the estimated values, a fixed-time consensus protocol is designed via backstepping, where a fixed-time disturbance observer (FTDO) is used to provide the robustness against the external disturbances. The violation of the input saturation is prevented by introducing an auxiliary variable and the problem of "explosion of terms" suffered by the conventional backstepping design is also successfully avoided by a fixed-time differentiator. Detailed convergence results are presented by leveraging Lyapunov stability theory. Finally, the effectiveness of the proposed approach is numerically validated through simulation results.

Keywords

Fixed-time consensus, multi-agent systems, disturbance rejection, input saturation

Introduction

In recent years, distributed coordination of multi-agent systems (MASs) has attracted considerable attention due to its widely practical applications, such as sensor networks (Wang et al., 2017b), mobile robotic teams (ElAshry et al., 2019), formation control of unmanned aerial vehicles (Ai and Yu, 2019b; Jia et al., 2019), and salvo attack of multiple missiles (Ai et al., 2019). Being one of the fundamental issues about this topic, the multi-agent consensus problem, focusing on designing distributed protocols based on local information to steer all agents to reach an agreement on a common value of interest, has been extensively investigated during the past few decades (Ai, 2018, 2020; Ai et al., 2017; Ai and Wang, 2021; Mao et al., 2019; Zou et al., 2019, 2020b, 2021a, 2021b).

As a factor of limiting the system performance, input saturation appears in most engineering systems. Therefore, the multi-agent consensus problem with input saturation is both theoretically and practically concerned (Fu et al., 2020; Lv et al., 2020, 2021). Along with this fact, Meng et al. (2013) investigated the saturated consensus-tracking problem for neutrally stable systems over fixed and switching graphs. The extension of the results in Meng et al. (2013) to output consensus problems of generic linear systems over jointly connected networks and directed switching proximity topologies were presented in Su et al. (2014) and Fan et al. (2015), respectively. In Wang et al. (2017a), a dynamic scheduling approach was employed to design low-gain algorithms to solve the robust global consensus problem for linear systems, where the considered agents are subject to input saturation and input-additive uncertainties. The event-triggered semi-global consensus problem was solved by Wang et al. (2017c) for saturated linear systems in the presence of updating delays. Ding et al. (2018) concerned with the practical set consensus problem for MASs subject to input saturation while considering network-induced delays, data quantization, and aperiodic sampling. Thereafter, an impulsive algorithm was proposed by Liu et al. (2019) to solve the dynamic consensus problem for time-delay nonlinear systems with input saturation. A common characteristic of these works is that the presented consensus protocols are of asymptotic convergence, leading to an unpredictable and slow settling time.

In practice, the convergence rate performs as an important and significant index for evaluating the system performance and a prompt convergence also contributes to achieving better disturbance rejection properties and robustness against uncertainties (Ai and Yu, 2019a; Zou et al., 2020a, 2020c). This triggers intensive research on achieving finite-time consensus for MASs subject to input saturation while obtaining bounded settling times. In Lu et al. (2013), the finite-time consensus-tracking problem was investigated for double-

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integrator systems with bounded input over fixed and switching jointly digraphs. Lyu et al. (2016) further proposed an output feedback-based protocol to solve the consensus problem of double-integrator systems while guaranteeing both the finite-time agreement and the input saturation requirement. The extension of the results in Lyu et al. (2016) to the output consensus problem of uncertain second-order MASs over general directed communication graphs could be found in Cai and Xiang (2017). Thereafter, adaptive finite-time protocols were designed by Fu et al. (2019) to solve the consensus problem over switching topologies, where the considered agents were of second-order dynamics subject to uncertainties and input saturation. Although the aforementioned results are effective to follow, the settling time of the above finite-time consensus protocols is determined by the initial conditions. However, the initial conditions are usually unprocurable in practical applications, and thereby the settling time cannot be predetermined under this situation.

Actually, many practical tasks are desired to be accomplished within a predetermined time regardless of the initial conditions. Fortunately, fixed-time stability theory, proposed by Polyakov (2012), contributes to achieving consensus for MASs with a guaranteed settling time regardless of the initial conditions, under which situation fast convergence can be still achieved without prior known initial conditions. Along with this fact, Zhang and Duan (2018) proposed an output feedback-based fixed-time protocol to solve the leader–following consensus problem for second-order systems subject to input saturation. Compared with second-order systems, it is known that high-order systems are more attractive as many real agents can be described as or transformed into high-order systems (Tian et al., 2018; Zuo et al., 2018a). However, existing works on the fixed-time consensus of MASs with high-order dynamics are relatively few. Tian et al. (2017) and Zuo et al. (2018b) were devoted to the fixed-time consensus of high-order integrators over undirected graphs. The extension of these results to the multi-agent consensus problem over directed graphs was given by Zuo et al. (2019). Thereafter, robust fixed-time protocols were proposed by Shi et al. (2018, 2021) to solve the consensus-tracking problem for high-order systems subject to parametric uncertainties. Nevertheless, these researches do not take the input saturation into consideration, leading to a relatively large energy cost.

Inspired by the aforementioned observations, this study makes a further effort to investigate the fixed-time consensus disturbance rejection problem for high-order nonlinear MASs with input saturation. To achieve this goal, a fixed-time state observer (FTSO) is developed to reconstruct the leader's states, based on which a fixed-time disturbance observer (FTDO)-based protocol is proposed via backstepping to achieve the leader–follower consensus for the considered MAS. With the aid of Lyapunov stability theory, it is theoretically proven that the fixed-time leader–follower consensus can be achieved in the presence of input saturation. The main contributions of this study are threefold:

1. This paper investigates the fixed-time consensus problem for high-order nonlinear systems with input saturation, and thereby the considered problem can cover the existing work (Zhang and Duan, 2018), concerning with second-order MASs subject to input saturation, as special cases. Moreover, the focused agents are subject to input saturation, which renders the existing fixed-time consensus protocols (Shi et al., 2018, 2021; Tian et al., 2017; Zuo et al., 2018b, 2019) designed for high-order MASs not applicable to the our problem.
2. Compared with the finite-time consensus protocols with input saturation (Cai and Xiang, 2017; Fu et al., 2019; Lu et al., 2013; Lyu et al., 2016), the settling time of the closed-loop system can be predetermined in this study regardless of the initial conditions. This contributes to achieving a fast convergence under the situation that the initial conditions of agents are not available in advance.
3. The proposed fixed-time consensus protocol is determined without the information exchange on neighbors' inputs, and thereby it requires less communication cost and can exclude the communication loop problem encountered by the existing algorithms (Shi et al., 2021; Tian et al., 2017).

The remainder of this study is summarized as follows. Some preliminaries and problem formulation are first presented. Then, the proposed consensus protocol together with the corresponding stability analyses are given in detail. Finally, some numerical examples and conclusion remarks are drawn in this paper.

Preliminaries and problem formulation

Notations

In this paper, $\mathbb{R}^{n\times m}$, $\mathbb{R}^n$, and $\mathbb{R}^+$ denote a set of $n\times m$-dimensional real matrices, a set of n-dimensional real column vectors, and a set of positive real numbers, respectively. For a symmetric matrix $X\in\mathbb{R}^{n\times n}$, $\lambda_{\max}(X)$ and $\lambda_{\min}(X)$ represent its maximum and minimum eigenvalues, respectively. Let $\mathrm{diag}\{a_1,\ldots,a_N\}$ be a diagonal matrix with a_i on its diagonal. For any non-negative real number α and any real number x, the function $\mathrm{sig}^\alpha(x)$ is defined as $\mathrm{sig}^\alpha(x)=|x|^\alpha\mathrm{sign}(x)$, where $\mathrm{sign}(\cdot)$ denotes the signum function. For any given vector $x=[x_1,\ldots,x_n]^T\in\mathbb{R}^n$, one has $\mathrm{sig}^\alpha(x)=[\mathrm{sig}^\alpha(x_1),\ldots,\mathrm{sig}^\alpha(x_n)]^T$, $\mathrm{sign}(x)=[\mathrm{sign}(x_1)\ldots,\mathrm{sig}(x_n)]^T$, and $|x|^\alpha=[|x_1|^\alpha,\ldots,|x_n|^\alpha]^T$.

Useful lemmas and definitions

Definition 1 (Polyakov, 2012). Consider the following system

$$\dot{x}=f(x),\quad x(0)=x_0 \tag{1}$$

where $x\in\mathbb{R}^n$ and $f(\cdot):\mathbb{R}^n\to\mathbb{R}^n$ is a smooth function. Suppose that the origin is an equilibrium point of (1). The origin is said to be globally finite-time stable if it is globally asymptotically stable and any solution $x(x_0, t)$ reaches the origin within a given settling time $T(x_0)$ with $T(\cdot):\mathbb{R}^n \to \mathbb{R}^+$, that is, $x(x_0, t) = 0, \forall t \geqslant T(x_0)$:

Definition 2 (Polyakov, 2012). The origin of the system (1) is said to be globally fixed-time stable if it is globally finite-time stable and any solution $x(x_0, t)$ reaches the origin within a bounded settling time $T \in \mathbb{R}^+$ regardless of the initial conditions x_0.

Lemma 1 (Polyakov, 2012). Consider the system (1), if there exists a positive definite function $V(x)$ such that

$$\dot{V}(x) \leqslant -\alpha V^p(x) - \beta V^q(x) \tag{2}$$

holds, where $\alpha > 0$, $\beta > 0$, $p > 1$, and $0 < q < 1$, then the origin of (1) is globally fixed-timely stable within the settling time bounded by

$$T \leqslant \frac{1}{\alpha(p-1)} + \frac{1}{\beta(1-q)} \tag{3}$$

Lemma 2 (Zuo et al., 2018b). For any non-negative real constants $x_i, i = 1, ..., n$, there exists a constant $p \in \mathbb{R}^+$ such that

$$\begin{cases} \sum_{i=1}^{n} x_i^p \geqslant \left(\sum_{i=1}^{n} x_i\right)^p & 0 < p \leqslant 1 \\ \sum_{i=1}^{n} x_i^p \geqslant n^{1-p}\left(\sum_{i=1}^{n} x_i\right)^p & p > 1 \end{cases} \tag{4}$$

Problem formulation

The MAS considered in this study consists of one leader indexed by 0 and N followers indexed by $1, ..., N$, respectively. The dynamics of the leader are given as follows

$$\begin{cases} \dot{x}_{0,1} = x_{0,2} \\ \vdots \\ \dot{x}_{0,n-1} = x_{0,n} \\ \dot{x}_{0,n} = f(x_{0,n}) + u_0 \end{cases} \tag{5}$$

and the dynamics of follower $i (i = 1, \ldots, N)$ are described by

$$\begin{cases} \dot{x}_{i,1} = x_{i,2} \\ \vdots \\ \dot{x}_{i,n-1} = x_{i,n} \\ \dot{x}_{i,n} = f(x_{i,n}) + \mathrm{sat}(u_i) + d_i \end{cases} \tag{6}$$

where $x_i = [x_{i,1}, \ldots, x_{i,n}]^T \in \mathbb{R}^n$ and $u_i \in \mathbb{R}$ denote the state vector and control input, respectively; $d_i \in \mathbb{R}$ represents the external disturbance; $f(\cdot):\mathbb{R} \to \mathbb{R}$ is a smooth nonlinear function, and $\mathrm{sat}(\cdot)$ stands for the saturation nonlinearity given by

$$\mathrm{sat}(u_i) = \begin{cases} u_i & |u_i| < u^i_{\max} \\ u^i_{\max}\mathrm{sign}(u_i) & |u_i| \geqslant u^i_{\max} \end{cases} \tag{7}$$

with $u^i_{\max} \in \mathbb{R}^+$ being the known upper bound.

Assumption 1. The nonlinear function $f(\cdot)$ is Lipschitz continuous, and thereby there exists a constant $\mu \in \mathbb{R}^+$ such that

$$|f(a) - f(b)| \leqslant \mu|a - b|, \ \forall a, b \in \mathbb{R} \tag{8}$$

Assumption 2. The leader's input u_0 is not accessible to any followers but bounded by a known constant $u^0_{\max} \in \mathbb{R}^+$, that is, $|u_0| \leqslant u^0_{\max}$.

Assumption 3. The disturbance d_i is first-order differentiable, and there exists a constant $\mathcal{D} \in \mathbb{R}^+$ such that $|\dot{d}_i| \leqslant \mathcal{D}$ holds.

Remark 1. The considered MAS is governed by high-order nonlinear dynamics with input saturation, which can be viewed as extensions of the existing works (Tian et al., 2017; Zuo et al., 2018b, 2019) by adding input saturation and nonlinear terms and also cover the results given by Zhang and Duan (2018) concerning with second-order systems. This raises the challenging difficulty of this study.

Remark 2. As indicated in Ding (2015), many practical disturbances can be approximated by sinusoidal functions with various amplitudes, frequencies, and phases, and thereby assumption 3 is moderate and has been commonly used for disturbance observer design (Ai and Yu, 2019a; Zhang et al., 2018).

The communication connections among followers can be described by a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, in which $\mathcal{V} = \{1, ..., N\}$ and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ denote the node set and the edge set, respectively. An edge $\varepsilon_{i,j} \in \mathcal{E}$ means that agent i can receive information from agent j. Self-loops are not allowed in this paper, that is, $\varepsilon_{i,i} \notin \mathcal{E}, \forall i \in \mathcal{V}$. Let $\mathcal{A} \in \mathbb{R}^{N \times N}$ be an adjacency matrix, with entries $a_{i,j} = 1$ if $\varepsilon_{i,j} \in \mathcal{E}$ and $a_{i,j} = 0$ otherwise. The Laplacian matrix $\mathcal{L} = [\mathcal{L}_{i,j}]_{N \times N}$ can be defined as follows

$$\mathcal{L}_{i,j} = \begin{cases} -a_{i,j} & i \neq j \\ \sum_{k=1, k \neq i}^{N} a_{i,k} & i = j \end{cases} \tag{9}$$

The information interconnection between the leader and the followers is given by $\mathcal{H} = \mathrm{diag}\{h_1, \ldots, h_N\}$, where $h_i = 1$ if the ith follower can access the leader, otherwise $h_i = 0$:

Assumption 4. The graph $\mathcal{G}$ among followers is undirected and connected. The leader cannot receive information from any followers and at least one follower can access the leader.

Lemma 3 (Zuo et al., 2018b). The matrix $L = [l_{i,j}]_{N \times N} = \mathcal{L} + \mathcal{H}$ is positive definite if assumption 4 holds.

The following definition is introduced to describe the fixed-time leader–follower consensus problem, which also represents the control objective of this study:

Definition 3. The fixed-time leader–follower consensus is achieved by a distributed protocol u_i if there exists a settling time $T_s \in \mathbb{R}^+$ independent of the initial conditions such that

$$\begin{cases} \lim_{t \to T_s} \|x_i(t) - x_0(t)\| = 0 \\ x_i(t) = x_0(t),\ \forall t > T_s \end{cases} \tag{10}$$

holds for all $i \in \mathcal{V}$.

Main results

FTSO design

If the leader's states cannot be accessed by all followers, an FTSO is designed in a distributed manner to reconstruct the leader's states for each follower.

Let $\hat{x}^0_{i,k} \in \mathbb{R}$ be the estimated value of the leader's state $x_{0,k}, k = 1, ..., n$, made by each follower $i \in \mathcal{V}$. The distributed FTSO is given as follows

$$\begin{cases} \dot{\hat{x}}^0_{i,k} = -\alpha_{1,k}\mathrm{sig}^{\frac{p_1}{q_1}}\left(\hat{e}^0_{i,k}\right) - \alpha_{2,k}\mathrm{sig}^{\frac{q_1}{p_1}}\left(\hat{e}^0_{i,k}\right) \\ \qquad + \hat{x}^0_{i,k+1}(k = 1, ..., n-1) \\ \dot{\hat{x}}^0_{i,n} = -\alpha_{1,n}\mathrm{sig}^{\frac{p_1}{q_1}}\left(\hat{e}^0_{i,n}\right) - \alpha_{2,n}\mathrm{sig}^{\frac{q_1}{p_1}}\left(\hat{e}^0_{i,n}\right) \\ \qquad -\alpha_3\hat{e}^0_{i,n} - \alpha_4\mathrm{sign}\left(\hat{e}^0_{i,n}\right) + f\left(\hat{x}^0_{i,n}\right) \end{cases} \tag{11}$$

where $\alpha_{1,j}, \alpha_{2,j}, \alpha_3, \alpha_4 \in \mathbb{R}^+, j = 1, \ldots, n,$ denote the design parameters, p_1 and q_1 are positive odd integers satisfying $p_1 < q_1$, and $\hat{e}^0_{i,k} = \sum_{j=1}^N a_{i,j}(\hat{x}^0_{i,k} - \hat{x}^0_{j,k}) + h_i(\hat{x}^0_{i,k} - x_{0,k})$.

Definite the error signal $\tilde{x}^0_{i,k} = \hat{x}^0_{i,k} - x_{0,k}, k = 1, \ldots, n,$ whose dynamics can be obtained based on equation (11) as follows

$$\begin{cases} \dot{\tilde{x}}^0_{i,k} = -\alpha_{1,k}\mathrm{sig}^{\frac{p_1}{q_1}}\left(\tilde{e}^0_{i,k}\right) - \alpha_{2,k}\mathrm{sig}^{\frac{q_1}{p_1}}\left(\tilde{e}^0_{i,k}\right) \\ \qquad + \tilde{x}^0_{i,k+1}(k = 1, ..., n-1) \\ \dot{\tilde{x}}^0_{i,n} = -\alpha_{1,n}\mathrm{sig}^{\frac{p_1}{q_1}}\left(\tilde{e}^0_{i,n}\right) - \alpha_{2,n}\mathrm{sig}^{\frac{q_1}{p_1}}\left(\tilde{e}^0_{i,n}\right) \\ \qquad -\alpha_3\tilde{e}^0_{i,n} - \alpha_4\mathrm{sign}\left(\tilde{e}^0_{i,n}\right) + f\left(\hat{x}^0_{i,n}\right) \\ \qquad -f(x_{0,n}) - u_0 \end{cases} \tag{12}$$

where $\tilde{e}^0_{i,k} = \sum_{j=1}^N a_{i,j}(\tilde{x}^0_{i,k} - \tilde{x}^0_{j,k}) + h_i\tilde{x}^0_{i,k}$.

Let $\tilde{x}^0_k = [\tilde{x}^0_{1,k}, \ldots, \tilde{x}^0_{N,k}]^T \in \mathbb{R}^N$ be a compact vector, and thereby equation (12) can be rewritten as

$$\begin{cases} \dot{\tilde{x}}^0_k = -\alpha_{1,k}\mathrm{sig}^{\frac{p_1}{q_1}}\left(L\tilde{x}^0_k\right) - \alpha_{2,k}\mathrm{sig}^{\frac{q_1}{p_1}}\left(L\tilde{x}^0_k\right) \\ \qquad + \tilde{x}^0_{k+1}(k = 1, ..., n-1) \\ \dot{\tilde{x}}^0_n = -\alpha_{1,n}\mathrm{sig}^{\frac{p_1}{q_1}}\left(L\tilde{x}^0_n\right) - \alpha_{2,n}\mathrm{sig}^{\frac{q_1}{p_1}}\left(L\tilde{x}^0_n\right) \\ \qquad -\alpha_3 L\tilde{x}^0_n - \alpha_4\mathrm{sign}\left(L\tilde{x}^0_n\right) + \hat{f}^0 - f^0 \\ \qquad -U_0 \end{cases} \tag{13}$$

where $U_0 = 1_N \otimes u_0$, $\hat{f}^0 = [f(\hat{x}^0_{1,n}), \ldots, f(\hat{x}^0_{N,n})]^T$, and $f^0 = 1_N \otimes f(x_{0,n})$. The following theorem shows the fixed-time convergence of the error signals $\tilde{x}^0_k, k = 1, \ldots, n$:

Theorem 1. Suppose that assumptions 1, 2, and 4 hold. If the observer parameters are chosen appropriately such that

$$\begin{array}{ll} \alpha_{1,k} \geqslant \dfrac{\sigma_1}{(2\lambda_{\min}(L))^{\frac{p_1+q_1}{2q_1}}} & \alpha_{2,k} \geqslant \dfrac{\sigma_2}{N^{\frac{p_1-q_1}{2p_1}}(2\lambda_{\min}(L))^{\frac{p_1+q_1}{2p_1}}} \\ \alpha_3 \geqslant \dfrac{\mu^2 + \lambda^2_{\min}(L)}{2\lambda^2_{\min}(L)} & \alpha_4 \geqslant u^0_{\max} \end{array} \tag{14}$$

for all $k = 1, \ldots, n$, where $\sigma_1, \sigma_2 \in \mathbb{R}^+$, then the error signals $\tilde{x}^0_k, k = 1, \ldots, n$ can be stabilized at the origin within a bounded settling time regardless of the initial conditions.

Proof. Consider the following Lyapunov candidate function

$$V_1 = \frac{1}{2}\sum_{k=1}^n \left(\tilde{x}^0_k\right)^T L\tilde{x}^0_k \tag{15}$$

Differentiating equation (15) along with the trajectory equation (13) yields that

$$\begin{aligned} \dot{V}_1 = & -\sum_{k=1}^n \alpha_{1,k}\left(\tilde{x}^0_k\right)^T L\mathrm{sig}^{\frac{p_1}{q_1}}\left(L\tilde{x}^0_k\right) - \alpha_3\left(\tilde{x}^0_n\right)^T L^2\tilde{x}^0_n \\ & -\sum_{k=1}^n \alpha_{2,k}\left(\tilde{x}^0_k\right)^T L\mathrm{sig}^{\frac{q_1}{p_1}}\left(L\tilde{x}^0_k\right) + \sum_{k=1}^{n-1}\left(\tilde{x}^0_k\right)^T L\tilde{x}^0_{k+1} \\ & -\alpha_4\left(\tilde{x}^0_n\right)^T L\mathrm{sign}\left(L\tilde{x}^0_n\right) + \left(\tilde{x}^0_n\right)^T L\left(\hat{f}^0 - f^0 - U_0\right) \\ \leqslant & -\sum_{k=1}^n \alpha_{1,k}\sum_{i=1}^N\left|\sum_{j=1}^N l_{i,j}\tilde{x}^0_{j,k}\right|^{\frac{p_1+q_1}{q_1}} + \frac{\left(\hat{f}^0 - f^0\right)^T\left(\hat{f}^0 - f^0\right)}{2} \\ & -\sum_{k=1}^n \alpha_{2,k}\sum_{i=1}^N\left|\sum_{j=1}^N l_{i,j}\tilde{x}^0_{j,k}\right|^{\frac{p_1+q_1}{p_1}} - \frac{2\alpha_3 - 1}{2}\left(\tilde{x}^0_n\right)^T L^2\tilde{x}^0_n \\ & -\left(\alpha_4 - u^0_{\max}\right)\sum_{i=1}^N\left|\sum_{j=1}^N l_{i,j}\tilde{x}^0_{j,n}\right| + \sum_{k=1}^{n-1}\left(\tilde{x}^0_k\right)^T L\tilde{x}^0_{k+1} \\ \leqslant & -\frac{2\alpha_3 - 1}{2}\left(\tilde{x}^0_n\right)^T L^2\tilde{x}^0_n + \frac{\mu^2}{2}\left(\tilde{x}^0_n\right)^T\tilde{x}^0_n + \sum_{k=1}^n\left(\tilde{x}^0_k\right)^T L\tilde{x}^0_{k+1} \\ \leqslant & 2V_1 - \frac{2\lambda^2_{\min}(L)\alpha_3 - \lambda^2_{\min}(L) - \mu^2}{2\lambda^2_{\min}}\left(\tilde{x}^0_n\right)^T L^2\tilde{x}^0_n \\ \leqslant & 2V_1 \end{aligned} \tag{16}$$

where $\alpha_3 \geqslant \mu^2 + \lambda^2_{\min}(L)/2\lambda^2_{\min}(L)$, $\alpha_4 \geqslant u^0_{\max}$, $\lambda^2_{\min}(L)^2 I_N \leqslant L^2$, and the Lipschitz condition in assumption 1 are used to obtain the above formula.

As indicated in equation (16), V_1 is bounded during all time, meaning that the error signal $\tilde{x}^0_k, k = 1, \ldots, n,$ will not escape to infinity in any finite-time interval [0, t]. To show the fixed-time convergence of the error signals, consider the following Lyapunov candidate function

$$V_2 = \frac{1}{2}\left(\tilde{x}^0_n\right)^T L\tilde{x}^0_n \tag{17}$$

By leveraging lemma 2, the time derivative of V_2 can be calculated by

$$\begin{aligned}
\dot{V}_2 =& -\alpha_{1,n}\left(\tilde{x}_n^0\right)^T L\mathrm{sig}^{\frac{p_1}{q_1}}\left(L\tilde{x}_n^0\right) - \alpha_3\left(\tilde{x}_n^0\right)^T L^2\tilde{x}_n^0 \\
&- \alpha_{2,n}\left(\tilde{x}_n^0\right)^T L\mathrm{sig}^{\frac{q_1}{p_1}}\left(L\tilde{x}_n^0\right) - \alpha_4\left(\tilde{x}_n^0\right)^T L\mathrm{sign}\left(L\tilde{x}_n^0\right) \\
&+ \left(\tilde{x}_n^0\right)^T L\left(\hat{f}^0 - f^0 - U_0\right) \\
\leqslant& -\alpha_{1,n}\sum_{i=1}^{N}\left|\sum_{j=1}^{N} l_{i,j}\tilde{x}_{j,n}^0\right|^{\frac{p_1+q_1}{q_1}} \\
&- \alpha_{2,n}\sum_{i=1}^{N}\left|\sum_{j=1}^{N} l_{i,j}\tilde{x}_{j,n}^0\right|^{\frac{p_1+q_1}{p_1}} \\
&- \left(\alpha_3 - \frac{\mu^2 + \lambda_{\min}^2(L)}{2\lambda_{\min}^2(L)}\right)\left(\tilde{x}_n^0\right)^T L^2\tilde{x}_n^0 \\
&- \left(\alpha_4 - u_{\max}^0\right)\sum_{i=1}^{N}\left|\sum_{j=1}^{N} l_{i,j}\tilde{x}_{j,n}^0\right| \\
\leqslant& -\alpha_{1,n}\left(\sum_{i=1}^{N}\left|\sum_{j=1}^{N} l_{i,j}\tilde{x}_{j,n}^0\right|^2\right)^{\frac{p_1+q_1}{2q_1}} \\
&- \alpha_{2,n}N^{\frac{p_1-q_1}{2p_1}}\left(\sum_{i=1}^{N}\left|\sum_{j=1}^{N} l_{i,j}\tilde{x}_{j,n}^0\right|^2\right)^{\frac{p_1+q_1}{2p_1}} \\
=& -\alpha_{1,n}\left(\left(\tilde{x}_n^0\right)^T L^2\tilde{x}_n^0\right)^{\frac{p_1+q_1}{2q_1}} \\
&- \alpha_{2,n}N^{\frac{p_1-q_1}{2p_1}}\left(\left(\tilde{x}_n^0\right)^T L^2\tilde{x}_n^0\right)^{\frac{p_1+q_1}{2p_1}} \\
\leqslant& -\alpha_{1,n}(2\lambda_{\min}(L))^{\frac{p_1+q_1}{2q_1}}\left(\frac{1}{2}\left(\tilde{x}_n^0\right)^T L\tilde{x}_n^0\right)^{\frac{p_1+q_1}{2q_1}} \\
&- \alpha_{2,n}N^{\frac{p_1-q_1}{2p_1}}(2\lambda_{\min}(L))^{\frac{p_1+q_1}{2p_1}}\left(\frac{1}{2}\left(\tilde{x}_n^0\right)^T L\tilde{x}_n^0\right)^{\frac{p_1+q_1}{2p_1}} \\
\leqslant& -\sigma_1 V_2^{\frac{p_1+q_1}{2q_1}} - \sigma_2 V_2^{\frac{p_1+q_1}{2p_1}}
\end{aligned} \tag{18}$$

According to lemma 1, the error signal $\tilde{x}_n^0$ can be stabilized at the origin within a fixed settling time bounded by

$$T_{s,1} = \frac{2q_1}{\sigma_1(q_1 - p_1)} + \frac{2p_1}{\sigma_2(q_1 - p_1)} \tag{19}$$

Following the convergence of the error signal $\tilde{x}_n^0$, that is, $t \geqslant T_{s,1}$, the dynamics of $\tilde{x}_{n-1}^0$ reduce to

$$\dot{\tilde{x}}_{n-1}^0 = -\alpha_{1,n-1}\mathrm{sig}^{\frac{p_1}{q_1}}\left(L\tilde{x}_{n-1}^0\right) - \alpha_{2,n-1}\mathrm{sig}^{\frac{q_1}{p_1}}\left(L\tilde{x}_{n-1}^0\right) \tag{20}$$

Similarly, consider the following Lyapunov function

$$V_3 = \frac{1}{2}\left(\tilde{x}_{n-1}^0\right)^T L\tilde{x}_{n-1}^0 \tag{21}$$

whose time derivative can be directly calculated by

$$\begin{aligned}
\dot{V}_3 =& -\alpha_{1,n-1}\sum_{i=1}^{N}\left|\sum_{j=1}^{N} l_{i,j}\tilde{x}_{j,n-1}^0\right|^{\frac{p_1+q_1}{q_1}} \\
&-\alpha_{2,n-1}\sum_{i=1}^{N}\left|\sum_{j=1}^{N} l_{i,j}\tilde{x}_{j,n-1}^0\right|^{\frac{p_1+q_1}{p_1}} \\
\leqslant& -\alpha_{1,n-1}\left(\sum_{i=1}^{N}\left|\sum_{j=1}^{N} l_{i,j}\tilde{x}_{j,n-1}^0\right|\right)^{\frac{p_1+q_1}{q_1}} \\
&-\alpha_{2,n-1}N^{\frac{p_1-q_1}{2p_1}}\left(\sum_{i=1}^{N}\left|\sum_{j=1}^{N} l_{i,j}\tilde{x}_{j,n-1}^0\right|\right)^{\frac{p_1+q_1}{p_1}} \\
=& -\alpha_{1,n-1}\left(\left(\tilde{x}_{n-1}^0\right)^T LL\tilde{x}_{n-1}^0\right)^{\frac{p_1+q_1}{q_1}} \\
&-\alpha_{2,n-1}N^{\frac{p_1-q_1}{2p_1}}\left(\left(\tilde{x}_{n-1}^0\right)^T LL\tilde{x}_{n-1}^0\right)^{\frac{p_1+q_1}{p_1}} \\
\leqslant& -\alpha_{1,n-1}(2\lambda_{\min}(L))^{\frac{p_1+q_1}{2q_1}} V_3^{\frac{p_1+q_1}{2q_1}} \\
&-\alpha_{2,n-1}N^{\frac{p_1-q_1}{2p_1}}(2\lambda_{\min}(L))^{\frac{p_1+q_1}{2p_1}} V_3^{\frac{p_1+q_1}{2p_1}} \\
\leqslant& -\sigma_1 V_3^{\frac{p_1+q_1}{2q_1}} - \sigma_2 V_3^{\frac{p_1+q_1}{2p_1}}
\end{aligned} \tag{22}$$

Notably, the error signal $\tilde{x}_{n-1}^0$ can be stabilized at the origin within a fixed settling time bounded by $2T_{s,1}$. Recursively, it is easy to verify that the error signal $\tilde{x}_1^0$ will converge to zero within a fixed settling time bounded by $nT_{s,1}$. This concludes the proof of theorem 1:

Remark 3. The proof of theorem 1 indicates that the observer error $\tilde{x}_k^0$ is bounded during all time and $\hat{x}_k^0 = 1_N \otimes x_{0,k}$ holds for all $t \geqslant nT_{s,1}$. Therefore, the leader's state $x_{0,k}$ can be successfully reconstructed by $\hat{x}_{i,k}^0$ for each follower within a bounded settling time regardless of the initial conditions, which makes the design of the consensus protocol feasible and effective.

FTDO design

To accommodate the external disturbances suffered by the followers, an FTDO is introduced for each follower based on the results given by Basin et al. (2017) as follows

$$\begin{cases}
\dot{\hat{x}}_{i,1} = -\gamma_1\mathrm{sig}^{a_1}(\hat{x}_{i,1} - x_{i,1}) + \hat{x}_{i,2} \\
\qquad -\gamma_1\mathrm{sig}^{b_1}(\hat{x}_{i,1} - x_{i,1}) \\
\qquad \vdots \\
\dot{\hat{x}}_{i,n-1} = -\gamma_{n-1}\mathrm{sig}^{a_{n-1}}(\hat{x}_{i,1} - x_{i,1}) + \hat{x}_{i,n} \\
\qquad -\gamma_{n-1}\mathrm{sig}^{b_{n-1}}(\hat{x}_{i,1} - x_{i,1}) \\
\dot{\hat{x}}_{i,n} = -\gamma_n\mathrm{sig}^{a_n}(\hat{x}_{i,1} - x_{i,1}) + \mathrm{sat}(u_i) + f(x_{i,n}) + \hat{d}_i \\
\qquad -\gamma_n\mathrm{sig}^{b_n}(\hat{x}_{i,1} - x_{i,1}) \\
\dot{\hat{d}}_i = -\gamma_{n+1}\mathrm{sig}^{a_{n+1}}(\hat{x}_{i,1} - x_{i,1}) \\
\qquad -\gamma_{n+1}\mathrm{sig}^{b_{n+1}}(\hat{x}_{i,1} - x_{i,1})
\end{cases} \tag{23}$$

where $\hat{x}_{i,k}$ and $\hat{d}_i$ represent the estimated values of $x_{i,k}, k = 1, ..., n$ and $d_i, i = 1, ..., N$, respectively, and the observer parameters satisfy that $a_1 \in (1 - \iota, 1)$, $b_1 \in (1, 1 + \iota)$, $a_l = la_1 - (l - 1)$, and $b_l = lb_1 - (l - 1), l = 2, ..., n + 1$, with $\iota \in \mathbb{R}^+$ being a small constant. Furthermore, the observer gains $\gamma_l, l = 1, ..., n + 1$, are selected such that the following matrix

$$A = \begin{bmatrix} -\gamma_1 & 1 & 0 & \cdots & 0 \\ -\gamma_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\gamma_n & 0 & 0 & \cdots & 1 \\ -\gamma_{n+1} & 0 & 0 & \cdots & 0 \end{bmatrix} \tag{24}$$

is Hurwitz. The following theorem shows the fixed-time convergence of the error signal $\left|\hat{d}_i - d_i\right|, i \in \mathcal{V}$:

Theorem 2. With assumption 3, if there exist positive definite matrices $P \in \mathbb{R}^{(n+1)\times(n+1)}$ and $Q \in \mathbb{R}^{(n+1)\times(n+1)}$ such that the following equation

$$PA + A^T P = -Q \tag{25}$$

holds, and then the error signal $\left|\hat{d}_i - d_i\right|, i = 1, ..., N$, can be stabilized at the origin within a fixed settling time bounded by

$$T_{s,2} = \frac{\lambda_{\max}^{2-a_1}(P)}{(1 - a_1)\lambda_{\min}(Q)} + \frac{\lambda_{\min}(P)}{(b_1 - 1)\rho^{b_1 - 1}\lambda_{\max}(Q)} \tag{26}$$

where $\rho \leqslant \lambda_{\min}(P)$ denotes a positive constant.

Proof. The proof of theorem 2 is similar to that of equation 37, theorem 2, and therefore is omitted here:

Remark 4. As indicated in Basin et al. (2017), the error signal $\left|\hat{d}_i - d_i\right|$ is bounded during all time, and $\hat{d}_i = d_i$ holds for all $t \geqslant T_{s,2}$ regardless of the initial conditions. Along with this fact, $\hat{d}_i$ can be used to provide the capacity of disturbance rejection for the followers.

Fixed-time consensus protocol design

As the system (6) is in the strict-feedback form, the backstepping technique is used to proceed with the consensus protocol design based on the FTSO (11) and the FTDO (23) for each follower $i \in \mathcal{V}$, step by step in the subsequent text:

Step 1: Let $\delta_{i,1} = x_{i,1} - \hat{x}_{i,1}^0$. Differentiating $\delta_{i,1}$ with respect to time gives that

$$\dot{\delta}_{i,1} = x_{i,2} - \hat{x}_{i,2}^0 + \alpha_{1,1}\mathrm{sig}^{\frac{p_1}{q_1}}\left(\tilde{e}_{i,1}^0\right) + \alpha_{2,1}\mathrm{sig}^{\frac{q_1}{p_1}}\left(\tilde{e}_{i,1}^0\right) \tag{27}$$

Select the intermediate control law $x_{i,2}^* \in \mathbb{R}$ for the system (27) as follows

$$x_{i,2}^* = -\beta_{1,1}\mathrm{sig}^{\frac{p_2}{q_2}}(\delta_{i,1}) - \beta_{2,1}\mathrm{sig}^{\frac{q_2}{p_2}}(\delta_{i,1}) + \hat{x}_{i,2}^0 \tag{28}$$

where $\beta_{1,1}, \beta_{2,1} \in \mathbb{R}^+$ denote the design parameters, p_2 and q_2 are positive odd integers satisfying $p_2 < q_2$.

Step $k (k = 2, \ldots, n-2)$: Define the error signal $\delta_{i,k} = x_{i,k} - x_{i,k}^*$, whose dynamics can be determined by

$$\dot{\delta}_{i,k} = x_{i,k+1} - \dot{x}_{i,k}^* \tag{29}$$

Choose the intermediate control law $x_{i,k+1}^* \in \mathbb{R}$ for the system (29) as follows

$$\begin{aligned} x_{i,k+1}^* = &-\beta_{1,k}\mathrm{sig}^{\frac{p_2}{q_2}}(\delta_{i,k}) - \beta_{2,k}\mathrm{sig}^{\frac{q_2}{p_2}}(\delta_{i,k}) \\ &+ \dot{x}_{i,k}^* - \delta_{i,k-1} \end{aligned} \tag{30}$$

where $\beta_{1,k}, \beta_{2,k} \in \mathbb{R}^+$ denote the design parameters

Step $n-1$: Let $\delta_{i,n-1} = x_{i,n-1} - x_{i,n-1}^*$. The time derivative of $\delta_{i,n-1}$ can be obtained by

$$\dot{\delta}_{i,n-1} = x_{i,n} - \dot{x}_{i,n-1}^* \tag{31}$$

Select the intermediate control law $x_{i,n}^* \in \mathbb{R}$ for the system (31) as follows

$$\begin{aligned} x_{i,n}^* = &-\beta_{1,n-1}\mathrm{sig}^{\frac{p_2}{q_2}}(\delta_{i,n-1}) - \beta_{2,n-1}\mathrm{sig}^{\frac{q_2}{p_2}}(\delta_{i,n-1}) \\ &-\delta_{i,n-2} - \omega_i + \dot{x}_{i,n-1}^* \end{aligned} \tag{32}$$

where $\beta_{1,n-1}, \beta_{2,n-1} \in \mathbb{R}^+$ denote the design parameters, and $\omega_i \in \mathbb{R}$ stands for an auxiliary variable to accommodate the input saturation and is updated by

$$\dot{\omega}_i = -\omega_i^{\frac{p_3}{q_3}} - \omega_i^{\frac{q_3}{p_3}} - u_i + \mathrm{sat}(u_i) \tag{33}$$

where p_3 and q_3 are positive odd integers satisfying $p_3 < q_3$.

Step n: Define the error signal $\delta_{i,n} = x_{i,n} - x_{i,n}^* - \omega_i$, whose dynamics can be determined by

$$\dot{\delta}_{i,n} = f(x_{i,n}) + u_i + d_i + \omega_i^{\frac{p_3}{q_3}} + \omega_i^{\frac{q_3}{p_3}} - \dot{x}_{i,n}^* \tag{34}$$

The final control input u_i can be chosen as follows

$$\begin{aligned} u_i = &-\beta_{1,n}\mathrm{sig}^{\frac{p_2}{q_2}}(\delta_{i,n}) - \beta_{2,n}\mathrm{sig}^{\frac{q_2}{p_2}}(\delta_{i,n}) - \delta_{i,n-1} \\ &-f(x_{i,n}) - \hat{d}_i - \omega_i^{\frac{p_3}{q_3}} - \omega_i^{\frac{q_3}{p_3}} + \dot{x}_{i,n}^* \end{aligned} \tag{35}$$

where $\beta_{1,n}, \beta_{2,n} \in \mathbb{R}^+$ denote the design parameters:

Remark 5. As shown in (34), the violation of the input saturation is prevented by introducing the first-order filter (33), which makes the design of the consensus protocol feasible and effective.

As indicated in (30), (32), and (35), the time derivative $\dot{x}_{i,k}^*, k = 2, \ldots, n$, is required in the intermediate control laws and the final control input. It is worth noting that $\dot{x}_{i,k}^*$ is difficult to be determined analytically with the increasing of the system order, leading to the problem of "explosion of terms" (Swaroop et al., 2000). To accommodate this problem, a fixed-time differentiator is constructed based on the FTDO to estimate the time derivatives of the intermediate control laws as follows

$$\begin{cases} \dot{x}_{i,k}^{d} = -\gamma_1 \mathrm{sig}^{a_1}\left(x_{i,k}^{d} - x_{i,k}^{*}\right) + v_{i,k}^{d} \\ \qquad -\gamma_1 \mathrm{sig}^{b_1}\left(x_{i,k}^{d} - x_{i,k}^{*}\right) \\ \dot{v}_{i,k}^{d} = -\gamma_2 \mathrm{sig}^{a_2}\left(x_{i,k}^{d} - x_{i,k}^{*}\right) \\ \qquad -\gamma_2 \mathrm{sig}^{b_2}\left(x_{i,k}^{d} - x_{i,k}^{*}\right) \end{cases} \tag{36}$$

where $x_{i,k}^{d} \in \mathbb{R}$ and $v_{i,k}^{d} \in \mathbb{R}$ denote the estimates on $x_{i,k}^{*}$ and $\dot{x}_{i,k}^{*}, k = 2, \ldots, n$. As indicated in theorem 2, $\dot{x}_{i,k}^{*}$ can be reconstructed by $v_{i,k}^{d}$ within a fixed settling time bounded by $T_{s,2}$:

Remark 6. It is known that the dynamic surface control (DSC) approach has been proven to be effective to accommodate the problem of "explosion of terms" (Swaroop et al., 2000). However, only asymptotic boundedness of the error signal $\dot{x}_{i,k}^{*} - v_{i,k}^{d}$ is guaranteed by the DSC technique. Different from the DSC, the fixed-time differentiator (36) is used in this paper to avoid the problem of "explosion of terms," which contributes to stabilizing the error signal $\dot{x}_{i,k}^{*} - v_{i,k}^{d}$ at the origin within a bounded settling time.

Owing to the fixed-time differentiator (36), the intermediate control laws and the final control input can be rewritten by

$$\begin{cases} x_{i,2}^{*} = & -\beta_{1,1}\mathrm{sig}^{\frac{p_2}{q_2}}(\delta_{i,1}) - \beta_{2,1}\mathrm{sig}^{\frac{q_2}{p_2}}(\delta_{i,1}) + \hat{x}_{i,2}^{0} \\ x_{i,k+1}^{*} = & -\beta_{1,k}\mathrm{sig}^{\frac{p_2}{q_2}}(\delta_{i,k}) - \beta_{2,k}\mathrm{sig}^{\frac{q_2}{p_2}}(\delta_{i,k}) - \delta_{i,k-1} \\ & + v_{i,k}^{d} (k = 2, \ldots, n-2) \\ x_{i,n}^{*} = & -\beta_{1,n-1}\mathrm{sig}^{\frac{p_2}{q_2}}(\delta_{i,n-1}) - \beta_{2,n-1}\mathrm{sig}^{\frac{q_2}{p_2}}(\delta_{i,n-1}) \\ & -\delta_{i,n-2} - \omega_i + v_{i,n-1}^{d} \\ u_i = & -\beta_{1,n}\mathrm{sig}^{\frac{p_2}{q_2}}(\delta_{i,n}) - \beta_{2,n}\mathrm{sig}^{\frac{q_2}{p_2}}(\delta_{i,n}) - \delta_{i,n-1} \\ & -f(x_{i,n}) - \hat{d}_i - \omega_i^{\frac{p_3}{q_3}} - \omega_i^{\frac{q_3}{p_3}} + v_{i,n}^{d} \end{cases} \tag{37}$$

Substituting (37) into (27), (29), (31), and (34) yields that

$$\begin{cases} \dot{\delta}_{i,1} = -\beta_{1,1}\mathrm{sig}^{\frac{p_2}{q_2}}(\delta_{i,1}) - \beta_{2,1}\mathrm{sig}^{\frac{q_2}{p_2}}(\delta_{i,1}) + \delta_{i,2} \\ \qquad + \alpha_{1,1}\mathrm{sig}^{\frac{p_1}{q_1}}\left(\tilde{e}_{i,1}^{0}\right) + \alpha_{2,1}\mathrm{sig}^{\frac{q_1}{p_1}}\left(\tilde{e}_{i,1}^{0}\right) \\ \dot{\delta}_{i,k} = -\beta_{1,k}\mathrm{sig}^{\frac{p_2}{q_2}}(\delta_{i,k}) - \beta_{2,k}\mathrm{sig}^{\frac{q_2}{p_2}}(\delta_{i,k}) - \delta_{i,k-1} \\ \qquad + \delta_{i,k+1} + v_{i,k}^{d} - \dot{x}_{i,k}^{*} (k = 2, \ldots, n-1) \\ \dot{\delta}_{i,n} = -\beta_{1,n}\mathrm{sig}^{\frac{p_2}{q_2}}(\delta_{i,n}) - \beta_{2,n}\mathrm{sig}^{\frac{q_2}{p_2}}(\delta_{i,n}) - \delta_{i,n-1} \\ \qquad + d_i - \hat{d}_i + v_{i,n}^{d} - \dot{x}_{i,n}^{*} \end{cases} \tag{38}$$

The following theorem shows the leader–follower consensus can be achieved by the protocol (35) for the MAS within a bounded settling time:

Theorem 3. Suppose that assumptions 1–4 hold. If the controller parameters are chosen appropriately such that

$$\beta_{1,k} \geqslant \frac{\sigma_3}{2^{\frac{p_2+q_2}{2q_2}}} \quad \beta_{2,k} \geqslant \frac{\sigma_4}{N^{\frac{p_2-q_2}{2p_2}} 2^{\frac{p_2+q_2}{2p_2}}} \tag{39}$$

for all $k = 1, \ldots, n$, where $\sigma_3, \sigma_4 \in \mathbb{R}^+$, then the fixed-time consensus of the considered MAS can be realized by the consensus protocol (35) associated with the FTSO (11) and the FTDO (23) regardless of the initial conditions.

Proof. Consider the following Lyapunov candidate function

$$V_4 = \frac{1}{2}\sum_{k=1}^{n} \delta_{i,k}^2 \tag{40}$$

whose time derivative along the trajectory (38) can be calculated by

$$\begin{aligned} \dot{V}_4 = & -\sum_{k=1}^{n} \beta_{1,k}\delta_{i,k}\mathrm{sig}^{\frac{p_2}{q_2}}(\delta_{i,k}) - \sum_{k=1}^{n} \beta_{2,k}\delta_{i,k}\mathrm{sig}^{\frac{q_2}{p_2}}(\delta_{i,k}) \\ & + \delta_{i,1}\left(\alpha_{1,1}\mathrm{sig}^{\frac{p_1}{q_1}}\left(\tilde{e}_{i,1}^{0}\right) + \alpha_{2,1}\mathrm{sig}^{\frac{q_1}{p_1}}\left(\tilde{e}_{i,1}^{0}\right)\right) \\ & + \sum_{k=2}^{n} \delta_{i,k}\left(v_{i,k}^{d} - \dot{x}_{i,k}^{*}\right) + \delta_{i,n}\left(d_i - \hat{d}_i\right) \\ \leqslant & -\sum_{k=1}^{n} \beta_{1,k}|\delta_{i,k}|^{\frac{p_2+q_2}{q_2}} - \beta_2 \sum_{k=1}^{n} \beta_{2,k}|\delta_{i,k}|^{\frac{p_2+q_2}{p_2}} \\ & + \frac{1}{2}\left(\alpha_{1,1}\mathrm{sig}^{\frac{p_1}{q_1}}\left(\tilde{e}_{i,1}^{0}\right) + \alpha_{2,1}\mathrm{sig}^{\frac{q_1}{p_1}}\left(\tilde{e}_{i,1}^{0}\right)\right)^2 \\ & + \frac{1}{2}\sum_{k=2}^{n}\left(v_{i,k}^{d} - \dot{x}_{i,k}^{*}\right)^2 + \frac{1}{2}\left(d_i - \hat{d}_i\right)^2 \\ & + \frac{1}{2}\left(\sum_{k=1}^{n} \delta_{i,k}^2 + \delta_{i,n}^2\right) \\ \leqslant & 2V_4 + C \end{aligned} \tag{41}$$

where $C = 1/2(\alpha_{1,1}\mathrm{sig}^{p_1/q_1}(\tilde{e}_{i,1}^{0}) + \alpha_{2,1}\mathrm{sig}^{q_1/p_1}(\tilde{e}_{i,1}^{0}))^2 + \frac{1}{2}(d_i - \hat{d}_i)^2 + 1/2\sum_{k=2}^{n}(v_{i,k}^{d} - \dot{x}_{i,k}^{*})^2$ is bounded by recalling theorems 1 and 2. Solving equation (41) during the time interval $t \in [0, nT_{s,1} + T_{s,2}]$ gives that

$$V_4(t) \leqslant \left(V_4(0) + \frac{C}{2}\right)e^{2(nT_{s,1} + T_{s,2})} - \frac{C}{2} \tag{42}$$

which explicitly indicates that $V_4(t)$ is bounded during the convergent phase of the FTSO and the FTDO.

Following the convergence of the FTSO and the FTDO, the error dynamics (38) can be rewritten as follows

$$\begin{cases} \dot{\delta}_{i,1} = -\beta_{1,1}\mathrm{sig}^{\frac{p_2}{q_2}}(\delta_{i,1}) - \beta_{2,1}\mathrm{sig}^{\frac{q_2}{p_2}}(\delta_{i,1}) + \delta_{i,2} \\ \dot{\delta}_{i,k} = -\beta_{1,k}\mathrm{sig}^{\frac{p_2}{q_2}}(\delta_{i,k}) - \beta_{1,k}\mathrm{sig}^{\frac{q_2}{p_2}}(\delta_{i,k}) - \delta_{i,k-1} \\ \qquad + \delta_{i,k+1} (k = 2, \ldots, n-1) \\ \dot{\delta}_{i,n} = -\beta_{1,n}\mathrm{sig}^{\frac{p_2}{q_2}}(\delta_{i,n}) - \beta_{2,n}\mathrm{sig}^{\frac{q_2}{p_2}}(\delta_{i,n}) - \delta_{i,n-1} \end{cases} \tag{43}$$

The time derivative of $V_4(t), t > nT_{s,1} + T_{s,2}$, can be directly given based on (41) by

$$\begin{aligned} \dot{V}_4 = & -\sum_{k=1}^{n} \beta_{1,k}\delta_{i,k}\mathrm{sig}^{\frac{p_2}{q_2}}(\delta_{i,k}) - \sum_{k=1}^{n} \beta_{2,k}\delta_{i,k}\mathrm{sig}^{\frac{q_2}{p_2}}(\delta_{i,k}) \\ = & -\sum_{k=1}^{n} \beta_{1,k}|\delta_{i,k}|^{\frac{p_2+q_2}{q_2}} - \sum_{k=1}^{n} \beta_{2,k}|\delta_{i,k}|^{\frac{p_2+q_2}{p_2}} \\ \leqslant & -\frac{\sigma_3}{2^{\frac{p_2+q_2}{2q_2}}}\sum_{k=1}^{n}\left(|\delta_{i,k}|^2\right)^{\frac{p_2+q_2}{2q_2}} \\ & -\frac{\sigma_4}{N^{\frac{p_2-q_2}{2p_2}} 2^{\frac{p_2+q_2}{2p_2}}}\sum_{k=1}^{n}\left(|\delta_{i,k}|^2\right)^{\frac{p_2+q_2}{2p_2}} \\ = & -\sigma_3 V_4^{\frac{p_2+q_2}{2q_2}} - \sigma_4 V_4^{\frac{p_2+q_2}{2p_2}} \end{aligned} \tag{44}$$

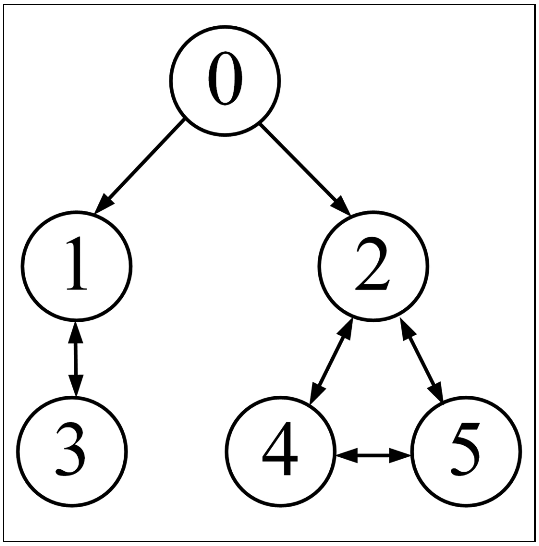

Figure 1. Communication topology.

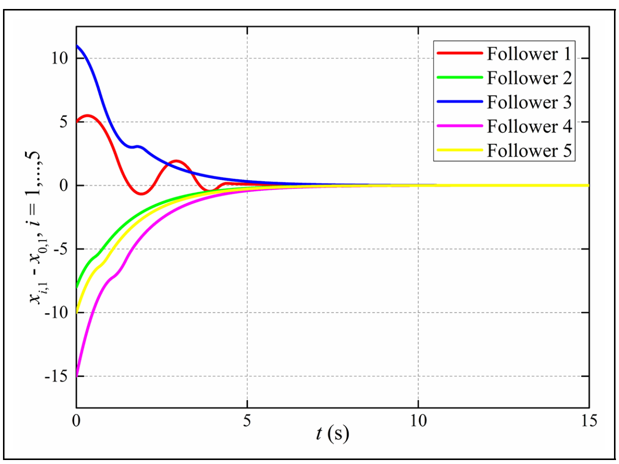

Figure 2. Time histories of the consensus errors $x_{i,1} - x_{0,1}, i = 1, 2, 3, 4, 5$.

According to lemma 1, the error signal $\delta_{i,k}, k = 1, \ldots, n, i = 1, \ldots, N$, can be stabilized at the origin within a fixed settling time bounded by $nT_{s,1} + T_{s,2} + T_{s,3}$ with

$$T_{s,3} = \frac{2q_2}{\sigma_3(q_2 - p_2)} + \frac{2p_2}{\sigma_4(q_2 - p_2)} \tag{45}$$

which explicitly demonstrates that the fixed-time leader–follower consensus is achieved by the considered MAS. This completes the proof of theorem 3:

Remark 7. It is worth noting that the settling time derived in theorem 3 can be predetermined independent of the initial conditions. To be specific, the convergence rate of the closed-loop system can be theoretically guaranteed even if the initial conditions are prior unknown. Therefore, the proposed consensus protocol is significantly different from the existing finite-time algorithms (Cai and Xiang, 2017; Fu et al., 2019; Lu et al., 2013; Lyu et al., 2016).

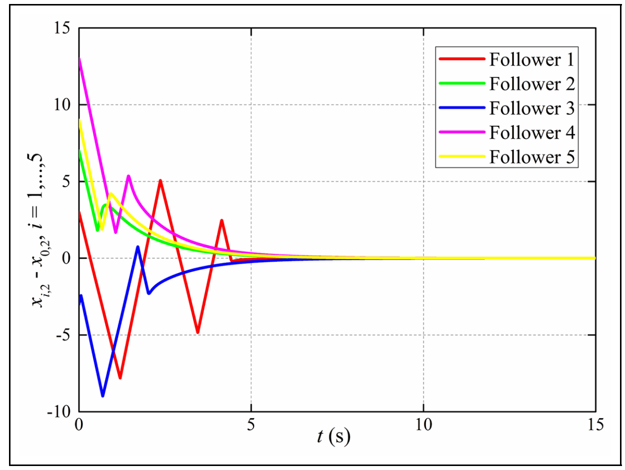

Figure 3. Time histories of the consensus errors $x_{i,2} - x_{0,1}, i = 1, 2, 3, 4, 5$.

Simulation results

This section provides simulation examples to demonstrate the effectiveness of the proposed consensus protocol. Consider an MAS with one leader and five followers ($N = 5$), among which the communication topology is described in Figure 1. It is easy to verify that assumption 4 holds.

The dynamics of the leader and the followers are given by

$$\begin{cases} \dot{x}_{0,1} = x_{0,2} \\ \dot{x}_{0,2} = f(x_{0,2}) + u_0 \end{cases} \tag{46}$$

and

$$\begin{cases} \dot{x}_{i,1} = x_{i,2} \\ \dot{x}_{i,2} = f(x_{i,2}) + \mathrm{sat}(u_i) + d_i \end{cases} \tag{47}$$

respectively, where $f(x_{i,3}) = 0.1\sin(x_{i,3}), i = 0, 1, ..., 5$, $d_j = \cos((j-1)t/3 + (j-1)\pi/5), j = 1, ..., 5$ and $u_0 = -\sin(0.5t)$. Therefore, it is easy to verify that $\mu = 0.1$ and $u^0_{\max} = 1$. Moreover, the upper bound of the control input is given by $u^i_{\max} = 10, i = 1 \ldots, 5$.

The initial values of the agents are fixed as $x_0(0) = [5, -3]^T$, $x_1(0) = [10, 0]^T$, $x_2(0) = [-3, 4]^T$, $x_3(0) = [16, -6]^T$, $x_4(0) = [-10, 10]^T$, and $x_5(0) = [-18, -5]^T$. The initial conditions of the FTSO and the FTDO are chosen as $\hat{x}^0_{i,k}(0) = x_{i,k}(0), k = 1, 2$, and $\hat{d}_i(0) = 0, i = 1, ..., 5$, respectively.

The design parameters of the FTSO are selected as $p_1 = 3$, $q_1 = 5$, $\alpha_{1,1} = \alpha_{1,2} = 0.8$, $\alpha_{2,1} = \alpha_{2,2} = 1.2$, $\alpha_3 = 5$, and $\alpha_4 = 1$. The design parameters of the FTDO are determined based on Basin et al. (2017) as $\gamma_1 = 24$, $\gamma_2 = 216$, $\gamma_4 = 1296$, $a_1 = 0.9$, $a_2 = 0.8$, $a_3 = 0.7$, $b_1 = 1.1$, $b_2 = 1.2$, and $b_3 = 1.3$. Moreover, the controller parameters are chosen as $p_2 = 5$, $q_2 = 7$, $\beta_{1,1} = \beta_{2,1} = 0.1$ and $\beta_{1,2} = \beta_{2,2} = 0.3$,.

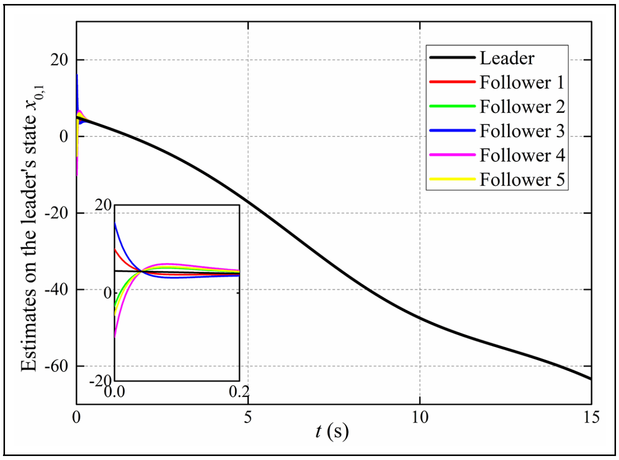

Figure 4. Time histories of the FTSO.

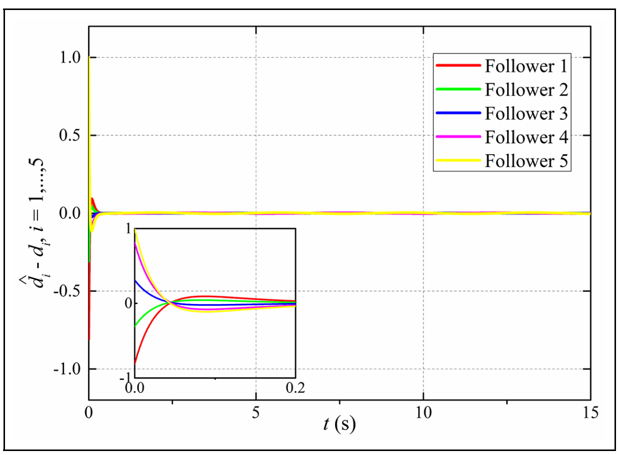

Figure 5. Time histories of the errors of the FTDO.

The simulation results are shown through Figures 2 to 7. Figures 2 and 3 present the time histories of the consensus errors, which explicitly show that the consensus can be achieved by the MASs governed by the proposed fixed-time protocol. The simulation results of the FTSO and the FTDO are respectively given in Figures 4 and 5, which demonstrates the effectiveness of the FTSO and the FTDO. Figures 6 and 7 give the time histories of the saturated control inputs $\mathrm{sat}(u_i)$ and the actual inputs u_i, respectively. It should be noted that due to the persistent external disturbances, the control inputs cannot be regulated to zeros with the consensus being achieved. Consequently, with the aid of the proposed fixed-time algorithm, the leader–follower consensus can be achieved successfully by the MAS with bounded control inputs.

To further provide better insights on the effect of the input saturation on the proposed approach, simulation studies with respect to various input bounds ($u^i_{max} = 10, 20, 30, 40, 50$) are illustrated in Figures 8 and 9, showing the time histories of the total consensus errors $E = \sum_{i=1}^{5} \sum_{j=1}^{2} |x_{i,j} - x_{0,j}|$ and the total control costs $\sum_{i=1}^{5} |\mathrm{sat}(u_i)|$. Notably, the simulation results explicitly indicate that the proposed consensus protocol can be used to tackle different input saturation.

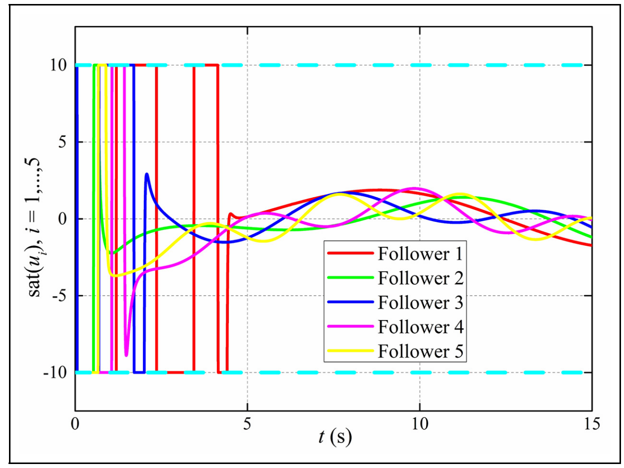

Figure 6. Time histories of the saturated inputs $\mathrm{sat}(u_i), i = 1, 2, 3, 4, 5$.

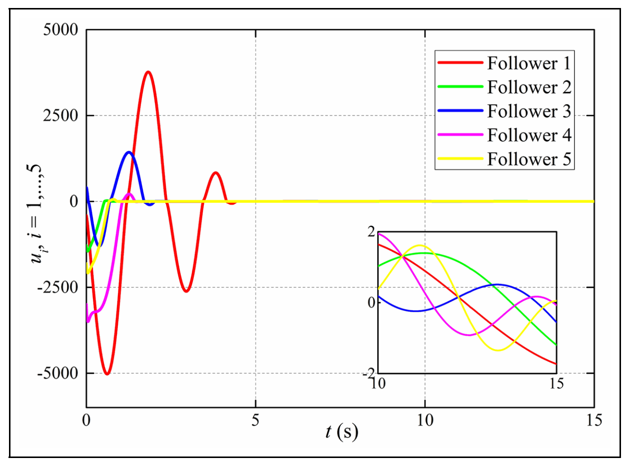

Figure 7. Time histories of the actual inputs $u_i, i = 1, 2, 3, 4, 5$.

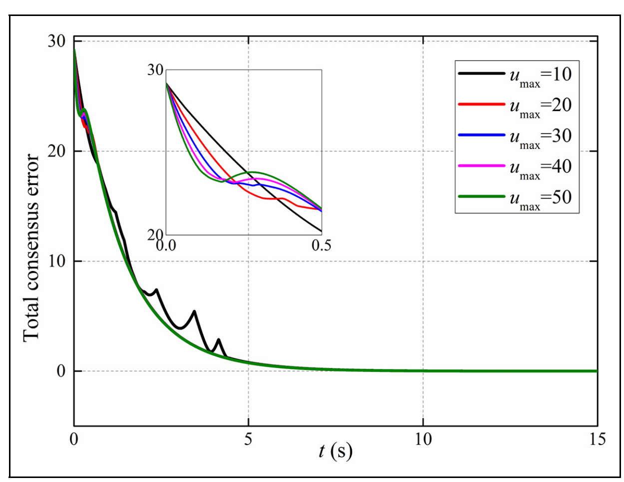

Figure 8. The total consensus errors with various input bounds.

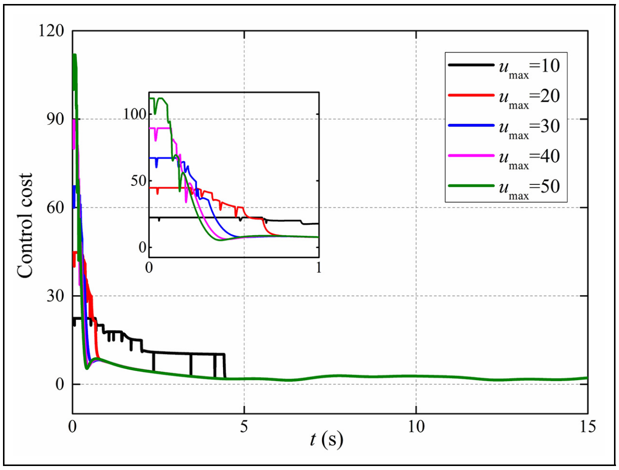

Figure 9. The total control costs with various input bounds.

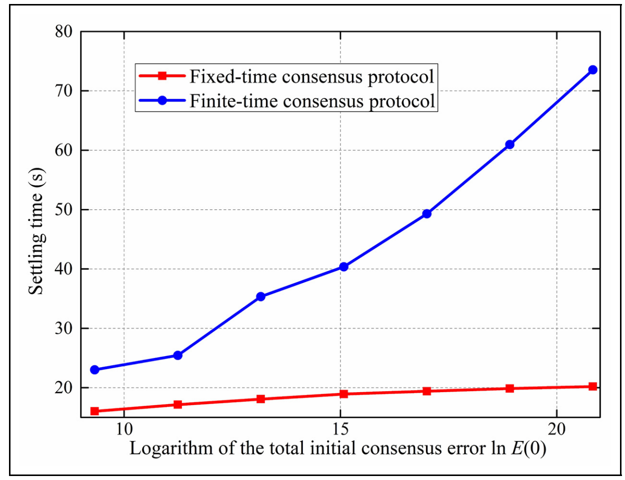

Figure 10. The settling times with various initial consensus errors.

Finally, to show the superior advantage of the proposed fixed-time consensus protocol compared with the existing finite-time algorithms, simulation results with respect to various initial conditions are presented in Figure 10, showing the settling times with respect to logarithm of the initial total consensus errors $\ln E(0)$. It should be noted that the settling time is obtained when the total consensus error reaches a threshold 10^{-2}. Evidently, fast convergence can be always achieved by the proposed fixed-time consensus algorithm while the settling time gained by the finite-time algorithm grows unboundedly with the increase of the initial consensus error.

Conclusion

In this study, the fixed-time consensus disturbance rejection problem for high-order nonlinear MASs was investigated considering input saturation. To achieve this goal, an FTSO and an FTDO were designed to estimate the leader's states and compensate for the external disturbances, respectively, for each follower. Based on the estimated values, backstepping was used to construct a distributed consensus protocol, where the problem of "explosion of terms" was avoided by a fixed-time differentiator. Rigorous stability analysis explicitly demonstrated that the leader–follower consensus could be achieved within a fixed-time settling time regardless of the initial conditions. Finally, the effectiveness of the proposed approach was verified through simulation results.

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