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Velocity and input constrained distributed nash equilibrium seeking for multi-agent integrated game and control via event-triggered communication

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Abstract This paper investigates velocity and input constrained integrated game and control (IGC) problems for integrator-type multi-agent systems via event-triggered communication mechanism. To be specific, a distributed event-triggered scheme is firstly developed to seek the Nash equilibrium (NE) for single-integrator without considering the boundedness of control inputs, followed by which an adaptive NE seeking algorithm is also provided with no need of the smallest eigenvalue of the graph's Laplacian matrix as well as the number of agents. Then, by adapting the hyperbolic tangent function into the event-triggered NE seeking controller, the IGC problem of double-integrator is addressed while guaranteeing the velocity and input constraints. The convergence results are given through Lyapunov stability analyses, and the Zeno behavior is proven to be excluded by the MASs. Finally, numerical simulations are included to verify the effectiveness of the proposed method.

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Keywords Multi-agent systems · Integrated game and control · Velocity constraint · Input saturation · Event-triggered communication

Abbreviations

MASs	Multi-agent systems
NE	Nash equilibrium
IGC	Integrated game and control
ISS	Input-to-state stable

1 Introduction

With the penetration of game theory into more and more engineering applications, including wireless networks [1], sensor networks [2], electric vehicle charging [3–5], optical networks [6] and competition among energy resources [7], noncooperative games of multi-agent systems (MASs) on graphs have been of considerable interest in recent years. The goal is to design distributed learning algorithms via local information exchange based on which each agent optimizes its individual but coupled cost function to reach the corresponding Nash equilibrium (NE). Great progress has been made in this field recently; however, most NE seeking schemes are emphasized with attention being given to MASs without explicit dynamics (see [8–12] and references therein).

From the practical perspective, agents are commonly governed by inherent dynamics. With this in mind, researchers have shown an increased interest in the problem of integrated game and control (IGC) for MASs with various dynamics, where controllers were developed in the literature to steer the agents to seek the NE in a distributed way. In [13] and [14], distributed NE seeking algorithms were presented for single-integrator systems under fixed or switching communication topologies, where leader-follower consensus protocols were integrated for information sharing. Along with this line, the authors of [15] further extended the results in [13] to accommodate hybrid games, played by both continuous-time players and discrete-time players, and an extended state observer-based distributed NE seeking scheme was developed in [16] to achieve the IGC design for MASs subject to unmodeled and disturbance terms. Additionally, aggregative games were also considered in [17] and [18] for disturbed linear and uncertain perturbed nonlinear systems, respectively. Besides first-order systems concerned above, more recent attentions have focused on the IGC problem for second- or high-order MASs. The motivation behind this focus is that many force-actuated systems are with second-order dynamics and some underactuated systems can be transformed into high-order fully-actuated ones [19–23]. In [24], saturated gradient-based schemes were given to accommodate games played by double-integrator with bounded control input. The authors in [25] investigated the aggregative game for heterogeneous Euler-Lagrange systems. Distributed NE seeking strategies, augmented with a dynamic internal-model based component, were developed in [26] for multi-integrator systems subject to external disturbances. Moreover, our previous works mainly focused on robust NE seeking for high-order MASs with the consideration of input delay, communication delay, nonlinear dynamics or unmeasured inherent states [27–29].

The aforementioned works contribute to the field of multi-agent IGC problems by designing distributed NE seeking algorithms with continuous communication between neighbors. Nevertheless, from the practical perspective, each agent usually equips with on board communication modules with limited energy resources [30]. With this context in mind, the event-triggered mechanism has been considered as an effective option to design energy-saving controllers to reduce the communication burden. Along with this line, much effort has been made to design event-triggered protocols for multi-agent consensus with integral-type [31,32] or general linear [33,34] or nonlinear [35–37] systems, and more recent works on event-triggered control of MASs have been included in the review articles [38,39]. Despite the recent progress on event-triggered consensus, there have been relatively few works on event-triggered NE seeking for multi-agent IGC problems yet. This lies in the primary goal of this research.

When applying the controllers on practical distributed MASs, the ubiquity of constraints on the agents' dynamics should be considered. One of the most commonly encountered constraints is input saturation, caused by the finite actuation power of physical plants, and recent advances have been made on input constrained distributed coordination of MASs with integrator-type [40,41] or general linear [42–44] dynamics. Apart from the input saturation, velocity constraint is another commonly existing restriction in practical engineering systems and has motivated the study of MASs with bounded velocity in the literature [45,46]. Examples are fixed-wing UAVs, whose minimum speed should be maintained to provide necessary lift, or ground vehicles, whose maximum speed needs to be limited for safety concerns. On these aspects, a few recent attempts have been made on distributed coordination of MASs with both input and velocity constraints. In [47,48], distributed consensus protocols were presented for second-order discrete- and continuous-time MASs, respectively, with bounded velocity and input. Further studies for containment control and formation control of double-integrator were given in [49,50], respectively, under the challenging situation where the velocity and acceleration of each agent are both restricted within their desired ranges. Although the aforementioned results are effective to follow, very little attention has been paid to distributed NE seeking of multi-agent IGC problems under the case where the input and velocity constraints coexist. This lies the second goal of this research.

Motivated by the above observations, the objective of this study is to design distributed NE seeking algo-

rithms based on event-triggered neighboring communication for the IGC problems played by MASs, where the agents' dynamics are subject to velocity and input constraints. The considered problems are challenging as the velocity constraint and the input saturation would introduce high nonlinearity into the closed-loop system and the event-triggered communication mechanism would render the closed-loop system non-smooth. These raise difficulties on the algorithm design as well as the establishment of the convergence analyses and render the existing algorithms not applicable to tackle our problems. In comparison with the existing works concerning with distributed NE seeking problems [13–18,24–29], the main contributions of this study is threefold:

(1) A distributed event-triggered algorithm with static triggering condition and a distributed adaptive event-triggered scheme with dynamic triggering condition are developed to achieve NE seeking for single-integrator while excluding the unexpected Zeno behavior. With these proposed algorithms, continuous communication encountered in the existing literature are successfully excluded.
(2) Considering the velocity and input constraints commonly coexist in practical physical systems, the results concerning with single-integrator are further extended to accommodate the IGC problems for constrained double-integrator MASs by adapting the hyperbolic tangent function into the controllers. This renders the proposed NE seeking algorithms more suitable for practical applications than the existing ones.
(3) Benefiting from the adaptive strategy, the global information on the communication graph and the number of agents are also excluded from the proposed controllers. Therefore, the algorithms given in this study are designed in a more distributed way and can be used in the situation where there exist agents that leave or join the networked game.

The rest of the paper is organized as follows. Preliminary results and problem formulation are presented in Sect. 2. The main results, showing the proposed NE seeking strategies as well as the corresponding convergence analyses, are, respectively, given for single- and double-integrator MASs in Sects. 3 and 4. Numerical simulations are presented in Sect. 5. Conclusions are drawn in Sect. 6.

2 Preliminaries and problem formulation

2.1 Notations

Throughout the paper, $\mathbb{R}$ and $\mathbb{R}^+$ stand for the set of real and positive real numbers, respectively. 1_N (0_N) denotes the all ones (zeros) column vector. Given scalars (matrices) $A_1, \ldots, A_N$, $\operatorname{diag}\{A_1, \ldots, A_N\}$ is the diagonal matrix with A_i on the diagonal. Notations $\|\cdot\|$ and $\otimes$ denote the Euclidean norm and the Kronecker product. Moreover, the smallest (largest) eigenvalue of a symmetric matrix H is denoted as $\lambda_{\min}(H)$ ($\lambda_{\max}(H)$).

2.2 Graph theory

This paper considers the NE seeking of multi-agent IGC problems with sharing information with each other though networks. The corresponding communication topology is commonly described by an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, \ldots, N\}$ is the node set and $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ denotes the edge set. In the undirected graph $\mathcal{G}$, the communication between any two agents are bidirectional, and thereby we have $(i, j) \in \mathcal{E} \Leftrightarrow (j, i) \in \mathcal{E}$. The associated adjacency matrix $\mathcal{A} = \left[a_{i,j}\right] \in \mathbb{R}^{N\times N}$ of the graph $\mathcal{G}$ is defined such that $a_{i,j} = 1$ if $(j, i) \in \mathcal{E}$ and $a_{i,j} = 0$ otherwise. It should be noted that self-loop is prohibitive in this paper, that is $a_{i,i} = 0, \forall i \in \mathcal{V}$. Furthermore, the Laplacian matrix $\mathcal{L} = \left[l_{i,j}\right] \in \mathbb{R}^{N\times N}$ is commonly defined as $l_{i,i} = \sum_{j\neq i} a_{i,j} = |\mathcal{N}_i|$ with $\mathcal{N}_i \subseteq \mathcal{V}$ being the set of agent i's neighbors and $l_{i,j} = -a_{i,j}, i \neq j$. An undirected graph is said to be connected if there exists a path connecting any two agents. Notably, the Laplacian matrix $\mathcal{L}$ of an undirected and connected graph is symmetric positive semidefinite and has a simple zero eigenvalue [51,52].

2.3 Problem formulation

Consider the NE seeking problem for a noncooperative game played by a class of MASs with N agents (players) sharing information over a graph $\mathcal{G}$. Each agent $i \in \mathcal{V} = \{1, \ldots, N\}$ intends to solve the following IGC problem:

$$\begin{aligned} &\min_{x_i} \quad J_i(x_i, x_{-i}) \\ &\text{s.t.} \quad x_i^{(n)} = u_i \end{aligned} \tag{1}$$

where $x_i \in \mathbb{R}$ and $u_i \in \mathbb{R}$ are the action and the control input of agent i, respectively, the stacked vector $x_{-i} = \left[x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N\right]^T \in \mathbb{R}^{N-1}$ represents the actions of the MASs except agent i, and $J_i : \mathbb{R}^N \to \mathbb{R}$ is agent i's individual cost function. Moreover, the variable n stands for the system order, and $n = 1$ and $n = 2$ are, respectively, considered in this study. For simplicity of notation, let $x = [x_1, \ldots, x_N]^T \in \mathbb{R}^N$ and $J_i(x_i, x_{-i})$ can be rewritten as $J_i(x)$. According to [53], an interesting solution to problem (1) is the NE, defined as:

Definition 1 [53] Consider the game problem (1), an action profile $x^* = \left[x_i^*, x_{-i}^*\right] \in \mathbb{R}^N$ is an NE if

$$J_i\left(x_i^*, x_{-i}^*\right) \leq J_i\left(x_i, x_{-i}^*\right), \forall i \in \mathcal{V} \tag{2}$$

The objective of this paper is to design the control input u_i for each agent i via event-triggered communication mechanism such that the action profile x asymptotically converge to the NE x^*. Specifically, we will firstly present distributed NE seeking strategies in Sect. 3 for single-integrator MASs with ideal input to provide better insight on the event-triggered communication mechanism, and then, the IGC problem will be further considered in Sect. 4 for double-integrator MASs subject to velocity and input constraints.

For notational convenience, let $g_i(a) = \left.\frac{\partial J_i(x)}{\partial x_i}\right|_{x=a} \in \mathbb{R}$ be the partial gradient, $g(a) = [g_1(a), \ldots, g_N(a)]^T \in \mathbb{R}^N$ denote the pseudogradient, and $G(b) = [g_1(b_1), \ldots, g_N(b_N)]^T \in \mathbb{R}^N$ represent the extended pseudogradient, where $a, b_i \in \mathbb{R}^N, i = 1, \ldots, N$ and $b = \left[b_1^T, \ldots, b_N^T\right]^T \in \mathbb{R}^{N^2}$. It is easy to obtain that $G(b) = G(1_N \otimes a) = g(a)$ if $b_i = a$ for all $i = 1, \ldots, N$. In addition, some commonly used assumptions are made in preparation for our main results [13–18,24–29].

Assumption 1 The undirected graph $\mathcal{G}$ considered in this paper is connected.

Assumption 2 The cost function $J_i(x)$ is continuously differentiable in x and convex in x_i for all $i \in \mathcal{V}$.

Assumption 3 The map $g : \mathbb{R}^N \to \mathbb{R}^N$ is strongly monotone as well as Lipschitz continuous and the map $G : \mathbb{R}^{N^2} \to \mathbb{R}^N$ is Lipschitz continuous, that is there exist two constants $\gamma_1, \gamma_2 \in \mathbb{R}^+$ such that for any $x, y \in \mathbb{R}^N$ and $a, b \in \mathbb{R}^{N^2}$, we have

$$\begin{cases} (x-y)^T (g(x) - g(y)) \geq \gamma_1 \|x-y\|^2 \\ \|g(x) - g(y)\| \leq \gamma_2 \|x-y\| \\ \|G(a) - G(b)\| \leq \gamma_2 \|a-b\| \end{cases} \tag{3}$$

Remark 1 As the agents' actions are coupled in the cost functions, information sharing is required for the MASs such that the IGC problem can be solved. With this in line, assuming that the communication graph in Assumption 1 is connected is reasonable. Assumptions 2 and 3 have been commonly used in the literature concerning the IGC problems of MASs. With Assumption 2, the IGC problem can be addressed though the gradient descent approach. Moreover, under Assumptions 2 and 3, the considered game admit a unique NE and $g(x) = G(1_N \otimes x) = 0_N$ is a sufficient and necessary condition for $x = x^*$ [54]. This contributes to obtaining a global convergence result in this study.

3 NE seeking for single integrator

To provide better insight on the event-triggered communication mechanism, this section considers the IGC problem for single-integrator MASs without input saturation:

$$\begin{aligned} &\min_{x_i} \quad J_i(x_i, x_{-i}) \\ &\text{s.t.} \quad \dot{x}_i = u_i \end{aligned} \tag{4}$$

where the considered MASs can be viewed as in the example of networked velocity-actuated robots, and each agent can only access local information over graph $\mathcal{G}$. For such an IGC problem, various NE seeking strategies have been made from multiple perspectives in the literature, intensively for continuous communication cases [13–18]. Differently, we aim to make a further step on developing distributed NE seeking algorithms via event-triggered communication mechanism herein.

3.1 Event-triggered NE seeking

To seek the NE solution via event-triggered communication, the following distributed algorithm is designed for each agent $i \in \mathcal{V}$:

$$\begin{aligned} u_i &= -\theta g_i(y_i) \\ \dot{y}_i &= -\delta \left(\sum_{j \in \mathcal{N}_i} \left(y_i\left(t_i^c\right) - y_j\left(t_i^c\right)\right) + A_i\left(y_i\left(t_i^c\right) \right.\right. \\ &\qquad \left.\left. - x\left(t_i^c\right)\right)\right) \end{aligned} \tag{5}$$

where $\theta, \delta \in \mathbb{R}^+$ denote the controller parameters, $y_i \in \mathbb{R}^N$ and $y_j \in \mathbb{R}^N$ are the estimates on the agents' action profile x for agents i and j, respectively,

$A_i = \text{diag}\left\{a_{i,1}, \ldots, a_{i,N}\right\}$, and t_i^c represents the most recent triggering time.

Remark 2 y_i in (5) can be viewed as an observer on the action profile x. It should be noted that although x is the global information, the observer is designed in a distributed way. To be specific, let $y_i = \left[y_{i,1}, \ldots, y_{i,N}\right]^T$, where $y_{i,k}, k \in \mathcal{V}$ denotes the estimate on x_k, and from (5) we have $\dot{y}_{i,k} = -\delta\left(\sum_{j\in\mathcal{N}_i}\left(y_{i,k}\left(t_i^c\right) - y_{j,k}\left(t_i^c\right)\right) + a_{i,k}\left(y_{i,k}\left(t_i^c\right) - x_k\left(t_i^c\right)\right)\right)$ with $a_{i,k}$ being the element of the adjacency matrix of the communication graph. Therefore, only local information exchange is required for the proposed algorithm. Moreover, if agent k can be directly accessed by agent i, then the update of $y_{i,k}$ can be excluded from the observer; however, we keep it for notational convenience.

For notational convenience, let $\alpha_i = \sum_{j\in\mathcal{N}_i}\left(y_i - y_j\right)$ and $\beta_i = A_i\left(y_i - x\right)$. Then, the measurement errors can be defined as $\bar{\alpha}_i = \alpha_i\left(t_i^c\right) - \alpha_i$ and $\bar{\beta}_i = \beta_i\left(t_i^c\right) - \beta_i$. Under these circumstances, the triggering condition is designed as $t_{c+1}^i = \inf\left\{t | t > t_c^i, f_i\left(\alpha_i\left(t_i^c\right), \beta_i\left(t_i^c\right), \bar{\alpha}_i, \bar{\beta}_i\right) \geq 0\right\}$ with the corresponding static triggering function given by

$$f_i\left(\alpha_i, \beta_i, \bar{\alpha}_i, \bar{\beta}_i\right) = \|\bar{\alpha}_i\| + \left\|\bar{\beta}_i\right\| - \mu e^{-\nu t} - \frac{\varrho_i}{2\sqrt{2\left(1+\varrho_i^2\right)}}\left(\left\|\alpha_i\left(t_i^c\right)\right\| + \left\|\beta_i\left(t_i^c\right)\right\|\right) \tag{6}$$

where $\varrho_i \in \mathbb{R}^+$ will be determined later and $\mu, \nu \in \mathbb{R}^+$ are design parameters.

Define the error variables $e_{x,i} = x_i - x_i^*$ and $\tilde{y}_i = y_i - x$ for agent $i \in \mathcal{V}$. Then, the error dynamics can be obtained from (5) that

$$\begin{aligned} \dot{e}_{x,i} &= -\theta g_i\left(y_i\right) \\ \dot{\tilde{y}}_i &= -\delta\left(\alpha_i + \beta_i\right) - \delta\left(\bar{\alpha}_i + \bar{\beta}_i\right) - \dot{x} \end{aligned} \tag{7}$$

Let $\alpha = \left[\alpha_1^T, \ldots, \alpha_N^T\right]^T \in \mathbb{R}^{N^2}$, $\beta = \left[\beta_1^T, \ldots, \beta_N^T\right]^T \in \mathbb{R}^{N^2}$, $\bar{\alpha} = \left[\bar{\alpha}_1^T, \ldots, \bar{\alpha}_N^T\right]^T \in \mathbb{R}^{N^2}$, $\bar{\beta} = \left[\bar{\beta}_1^T, \ldots, \bar{\beta}_N^T\right]^T \in \mathbb{R}^{N^2}$, $e_x = x - x^* = [e_1, \ldots, e_N]^T \in \mathbb{R}^N$ and $\tilde{y} = y - 1_N \otimes x = \left[\tilde{y}_1^T, \ldots, \tilde{y}_N^T\right]^T \in \mathbb{R}^{N^2}$ with $y = \left[y_1^T, \ldots, y_N^T\right]^T \in \mathbb{R}^{N^2}$, and then the augmented systems of (7) can be expressed as:

$$\begin{aligned} \dot{e}_x &= -\theta G\left(y\right) \\ \dot{\tilde{y}} &= -\delta\left(\alpha + \beta\right) - \delta\left(\bar{\alpha} + \bar{\beta}\right) - 1_N \otimes \dot{x} \\ &= -\delta H \tilde{y} - \delta\left(\bar{\alpha} + \bar{\beta}\right) + \theta\left(1_N \otimes G\left(y\right)\right) \end{aligned} \tag{8}$$

where $H = \mathcal{L} \otimes I_N + \text{diag}\left\{A_1, \ldots, A_N\right\}$ is positive definite if Assumption 1 holds. The following theorem demonstrates the convergence to the NE.

Theorem 1 *Consider the IGC problem (4) of MASs with controller (5) and triggering function (6). Under Assumptions 1–3, selecting* $\theta > 0$, $\delta > \frac{\theta\gamma_2\left(\gamma_2\left(1+\sqrt{N}\|H\|\right)^2 + 4\gamma_1\sqrt{N}\|H\|\right)}{2(1-\sigma)\gamma_1\underline{\lambda}^2}$ *and* $\varrho_i^2 = \frac{\sigma\underline{\lambda}^2}{2\left(4|\mathcal{N}_i|^2+1\right)}$ *with* $0 < \sigma < 1$ *and* $\underline{\lambda} = \lambda_{\min}\left(H\right)$, *yields that (1) the unique NE* x^* *is globally asymptotically stable for the considered MAS; (2) the Zeno behavior can be excluded.*

Proof Proof of Theorem 1, Part (1) Consider the following Lyapunov function:

$$V_1 = e_x^T e_x + \tilde{y}^T H \tilde{y} \tag{9}$$

Taking the time derivative of V_1 along the trajectory (8), we have

$$\begin{aligned} \dot{V}_1 =& -2\theta e_x^T G\left(y\right) - 2\delta\tilde{y}^T H^2 \tilde{y} \\ &+ 2\theta\tilde{y}^T H\left(1_N \otimes G\left(y\right)\right) \\ &- 2\delta\tilde{y}^T H\left(\bar{\alpha} + \bar{\beta}\right) \\ =& -2\theta e_x^T\left(g\left(x\right) - g\left(x^*\right)\right) \\ &- 2\theta e_x^T\left(G\left(y\right) - G\left(1_N \otimes x\right)\right) \\ &+ 2\theta\tilde{y}^T H\left(1_N \otimes \left(G\left(y\right) - G\left(1_N \otimes x\right)\right)\right) \\ &+ 2\theta\tilde{y}^T H\left(1_N \otimes \left(g\left(x\right) - g\left(x^*\right)\right)\right) \\ &- 2\delta\tilde{y}^T H^2\tilde{y} - 2\delta\tilde{y}^T H\left(\bar{\alpha} + \bar{\beta}\right) \\ \leq& -2\gamma_1\theta\|e_x\|^2 - \left(\underline{\lambda}^2\delta - 2\gamma_2\theta\sqrt{N}\|H\|\right)\|\tilde{y}\|^2 \\ &+ 2\gamma_2\theta\left(1 + \|H\|\sqrt{N}\right)\|e_x\|\|\tilde{y}\| \\ &+ 2\delta\left(\|\bar{\alpha}\|^2 + \left\|\bar{\beta}\right\|^2\right) \end{aligned} \tag{10}$$

where Assumption 3 and the Young's inequality are used to obtain the above equations.

Considering the triggering function (6), we have

$$\begin{aligned}
&\|\bar{\alpha}_i\|^2 + \|\bar{\beta}_i\|^2 \le \left(\|\bar{\alpha}_i\| + \|\bar{\beta}_i\|\right)^2 \\
&\le \frac{\varrho_i^2}{4\left(1+\varrho_i^2\right)}\left(\left\|\alpha_i\left(t_i^c\right)\right\| + \left\|\beta_i\left(t_i^c\right)\right\|\right)^2 + 2\mu^2 e^{-2\nu t} \\
&\le \frac{\varrho_i^2}{4\left(1+\varrho_i^2\right)}\left(\|\alpha_i\| + \|\beta_i\| + \|\bar{\alpha}_i\| + \|\bar{\beta}_i\|\right)^2 \\
&\quad + 2\mu^2 e^{-2\nu t} \\
&\le \frac{\varrho_i^2}{1+\varrho_i^2}\left(\|\alpha_i\|^2 + \|\beta_i\|^2 + \|\bar{\alpha}_i\|^2 + \|\bar{\beta}_i\|^2\right) \\
&\quad + 2\mu^2 e^{-2\nu t}
\end{aligned} \tag{11}$$

which yields that

$$\begin{aligned}
&\|\bar{\alpha}_i\|^2 + \|\bar{\beta}_i\|^2 \le \varrho_i^2\left(\|\alpha_i\|^2 + \|\beta_i\|^2\right) \\
&\quad + 2\mu^2\left(1+\varrho_i^2\right)e^{-2\nu t}
\end{aligned} \tag{12}$$

With the condition given in Theorem 1 and Eq. (12), the last term in (10) can be determined by

$$\begin{aligned}
&2\delta\left(\|\bar{\alpha}\|^2 + \|\bar{\beta}\|^2\right) = 2\delta\sum_{i=1}^{N}\left(\|\bar{\alpha}_i\|^2 + \|\bar{\beta}_i\|^2\right) \\
&\le 2\delta\sum_{i=1}^{N}\left(\varrho_i^2\left(\|\alpha_i\|^2 + \|\beta_i\|^2\right) + 2\mu^2\left(1+\varrho_i^2\right)e^{-2\nu t}\right) \\
&= 2\delta\sum_{i=1}^{N}\varrho_i^2\left(\left\|\sum_{j\in\mathcal{N}_i}\left(y_i - y_j\right)\right\|^2 + \|A_i\left(y_i - x\right)\|^2\right) \\
&\quad + 4\delta\mu^2\sum_{i=1}^{N}\left(1+\varrho_i^2\right)e^{-2\nu t} \\
&\le 2\delta\sum_{i=1}^{N}\varrho_i^2\left(\left\|\sum_{j\in\mathcal{N}_i}\left(\tilde{y}_i - \tilde{y}_j\right)\right\|^2 + \|\tilde{y}_i\|^2\right) \\
&\quad + 4\delta\mu^2\sum_{i=1}^{N}\left(1+\varrho_i^2\right)e^{-2\nu t} \\
&\le 2\delta\sum_{i=1}^{N}\varrho_i^2\left(|\mathcal{N}_i|\sum_{j\in\mathcal{N}_i}\|\tilde{y}_i - \tilde{y}_j\|^2 + \|\tilde{y}_i\|^2\right) \\
&\quad + 4\delta\mu^2\sum_{i=1}^{N}\left(1+\varrho_i^2\right)e^{-2\nu t} \\
&\le 2\delta\sum_{i=1}^{N}\varrho_i^2\left(4|\mathcal{N}_i|^2 + 1\right)\|\tilde{y}_i\|^2 + 4\delta\mu^2\sum_{i=1}^{N}\left(1+\varrho_i^2\right)e^{-2\nu t} \\
&= \sigma\underline{\lambda}^2\delta\|\tilde{y}\|^2 + \frac{2\delta N\mu^2\left(10+\sigma\underline{\lambda}^2\right)}{5}e^{-2\nu t}
\end{aligned} \tag{13}$$

Submitting (13) into (10) yields that

$$\begin{aligned}
\dot{V}_1 &\le -2\gamma_1\theta\|e_x\|^2 + 2\theta\gamma_2\left(1+\|H\|\sqrt{N}\right)\|e_x\|\|\tilde{y}\| \\
&\quad - \left((1-\sigma)\underline{\lambda}^2\delta - 2\theta\gamma_2\sqrt{N}\|H\|\right)\|\tilde{y}\|^2 \\
&\quad + \frac{2\delta N\mu^2\left(10+\sigma\underline{\lambda}^2\right)}{5}e^{-2\nu t} \\
&= -\left[\|e_x\|, \|\tilde{y}\|\right]P_1\left[\|e_x\|, \|\tilde{y}\|\right]^T \\
&\quad + \frac{2\delta N\mu^2\left(10+\sigma\underline{\lambda}^2\right)}{5}e^{-2\nu t}
\end{aligned} \tag{14}$$

with

$$P_1 = \begin{bmatrix} 2\gamma_1\theta & -\theta\gamma_2\left(1+\|H\|\sqrt{N}\right) \\ -\theta\gamma_2\left(1+\|H\|\sqrt{N}\right) & (1-\sigma)\underline{\lambda}^2\delta - 2\theta\gamma_2\sqrt{N}\|H\| \end{bmatrix} \tag{15}$$

To achieve convergence to the NE, the matrix P_1 should be positive definite, which is equivalent to $\theta > 0$ and $\delta > \frac{\theta\gamma_2\left(\gamma_2\left(1+\sqrt{N}\|H\|\right)^2 + 4\gamma_1\sqrt{N}\|H\|\right)}{2(1-\sigma)\gamma_1\underline{\lambda}^2}$. This holds as the conditions given in Theorem 1, and thereby we have

$$\begin{aligned}
\dot{V}_1 &\le -\lambda_{\min}\left(P_1\right)\left(\|e_x\|^2 + \|\tilde{y}\|^2\right) \\
&\quad + \frac{2\delta N\mu^2\left(10+\sigma\underline{\lambda}^2\right)}{5}e^{-2\nu t} \\
&\le -\kappa_1 V_1 + \kappa_2 e^{-2\nu t}
\end{aligned} \tag{16}$$

with $\kappa_1 = \lambda_{\min}\left(P_1\right)\min\{1, 1/\bar{\lambda}\}$, $\kappa_2 = 2\delta N\mu^2\left(10+\sigma\underline{\lambda}^2\right)/5$ and $\bar{\lambda} = \lambda_{\max}\left(H\right)$.

Solving (16) yields that

$$V_1\left(t\right) \le e^{-\kappa_1 t}V_1\left(0\right) + \kappa_2\psi\left(t\right) \tag{17}$$

where $\psi\left(t\right) = te^{-2\nu t}$ if $\kappa_1 = 2\nu$, and $\psi\left(t\right) = e^{-2\nu t}/\left(\kappa_1 - 2\nu\right)$ otherwise. Since $\lim_{t\to+\infty}\psi\left(t\right) = 0$, it follows from (17) that $\lim_{t\to+\infty}V_1\left(t\right) = 0$, which demonstrates the globally asymptotic convergence to the NE of the considered MASs governed by (5). This completes the proof of Theorem 1, Part (1). □

Proof of Theorem 1, Part (2) The time derivative of $\|\bar{\alpha}_i\|$ during $t \in \left[t_i^c, t_i^{c+1}\right)$ can be calculated as

$$\begin{aligned}
\frac{\mathrm{d}\|\bar{\alpha}_i\|}{\mathrm{d}t} &= \frac{\bar{\alpha}_i^T\dot{\bar{\alpha}}_i}{\|\bar{\alpha}_i\|} \le \|\dot{\bar{\alpha}}_i\| = \|\dot{\alpha}_i\| = \left\|\sum_{j\in\mathcal{N}_i}\left(\dot{y}_i - \dot{y}_j\right)\right\| \\
&= \delta\left\|\sum_{j\in\mathcal{N}_i}\left(\alpha_i\left(t_i^c\right) + \beta_i\left(t_i^c\right)\right.\right. \\
&\quad \left.\left. - \alpha_j\left(t_j^c\right) - \beta_j\left(t_j^c\right)\right)\right\|
\end{aligned} \tag{18}$$

The time derivative of $\left\|\bar{\beta}_i\right\|$ during $t \in \left[t_i^c, t_i^{c+1}\right)$ can be calculated as

$$\begin{aligned}\frac{\mathrm{d}\left\|\bar{\beta}_i\right\|}{\mathrm{d}t} &\leq \left\|\dot{\beta}_i\right\| = \|\dot{y}_i - \dot{x}\| \leq \|\dot{y}_i\| + \|\dot{x}\| \\ &= \theta \|G(y) - G(1_N \otimes x)\| \\ &\quad + \theta \left\|g(x) - g\left(x^*\right)\right\| \\ &\quad + \delta \left\|\alpha_i\left(t_i^c\right) + \beta_i\left(t_i^c\right)\right\| \\ &\leq \theta\gamma_2 \left(\|\tilde{y}\| + \|e_x\|\right) + \delta \left\|\alpha_i\left(t_i^c\right) + \beta_i\left(t_i^c\right)\right\| \\ &\leq \theta\gamma_2\sqrt{2}\sqrt{\|\tilde{y}\|^2 + \|e_x\|^2} \\ &\quad + \delta \left\|\alpha_i\left(t_i^c\right) + \beta_i\left(t_i^c\right)\right\| \\ &\leq \theta\kappa\gamma_2\sqrt{2V_1(0)} + \delta \left\|\alpha_i\left(t_i^c\right) + \beta_i\left(t_i^c\right)\right\|\end{aligned} \tag{19}$$

with $\kappa = \max\left\{1, 1/\underline{\lambda}\right\}$.

With the results shown in (18) and (19), we have

$$\begin{aligned}&\frac{d}{\mathrm{d}t}\left(\|\bar{\alpha}_i\| + \left\|\bar{\beta}_i\right\|\right) \\ &\leq \theta\kappa\gamma_2\sqrt{2V_1(0)} + \delta \left\|\alpha_i\left(t_i^c\right) + \beta_i\left(t_i^c\right)\right\| \\ &\quad + \delta \left\|\sum_{j\in\mathcal{N}_i}\left(\alpha_i\left(t_i^c\right) + \beta_i\left(t_i^c\right)\right.\right. \\ &\quad \left.\left. - \alpha_j\left(t_j^c\right) - \beta_j\left(t_j^c\right)\right)\right\| \\ &\leq \left(\theta\kappa\gamma_2\sqrt{2} + (|\mathcal{N}_i| + 1)\,\delta\sqrt{\bar{\lambda}}\right)\sqrt{V_1(0)}\end{aligned} \tag{20}$$

Since $\left\|\bar{\alpha}_i\left(t_i^c\right)\right\| + \left\|\bar{\beta}_i\left(t_i^c\right)\right\| = 0$, by leveraging Newton-Leibniz formula and (20) we have for all $t \in \left[t_i^c, t_i^{c+1}\right)$

$$\begin{aligned}&\|\bar{\alpha}_i(t)\| + \left\|\bar{\beta}_i(t)\right\| \\ &= \int_{t_i^c}^{t} \frac{d}{\mathrm{d}t}\left(\|\bar{\alpha}_i(\tau)\| + \left\|\bar{\beta}_i(\tau)\right\|\right)\mathrm{d}\tau \\ &\quad + \left(\left\|\bar{\alpha}_i\left(t_i^c\right)\right\| + \left\|\bar{\beta}_i\left(t_i^c\right)\right\|\right) \\ &= \int_{t_i^c}^{t} \frac{d}{\mathrm{d}t}\left(\|\bar{\alpha}_i(\tau)\| + \left\|\bar{\beta}_i(\tau)\right\|\right)\mathrm{d}\tau \\ &\leq \left(t - t_i^c\right)\left(\theta\kappa\gamma_2\sqrt{2} + (|\mathcal{N}_i| + 1)\,\delta\sqrt{\bar{\lambda}}\right)\sqrt{V_1(0)}\end{aligned} \tag{21}$$

As the next event will not be triggered before $f_i\left(\alpha_i, \beta_i, \bar{\alpha}_i, \bar{\beta}_i\right) = 0$, and thereby the next triggering time satisfies $\left\|\bar{\alpha}_i\left(t_i^{c+1}\right)\right\| + \left\|\bar{\beta}_i\left(t_i^{c+1}\right)\right\| = \frac{\varrho_i}{2\sqrt{2(1+\varrho_i^2)}}\left(\left\|\alpha_i\left(t_i^c\right)\right\| + \left\|\beta_i\left(t_i^c\right)\right\|\right) + \mu e^{-\nu t_i^{c+1}}$. Consequently, from (21), we have

$$\begin{aligned}&t_i^{c+1} - t_i^c \\ &\geq \frac{\varrho_i\left(\left\|\alpha_i\left(t_i^c\right)\right\| + \left\|\beta_i\left(t_i^c\right)\right\|\right) + 2\mu\sqrt{2\left(1+\varrho_i^2\right)}e^{-\nu t_i^{c+1}}}{2\left(\theta\kappa\gamma_2\sqrt{2} + (|\mathcal{N}_i| + 1)\,\delta\sqrt{\bar{\lambda}}\right)\sqrt{2\left(1+\varrho_i^2\right)V_1(0)}} \\ &> 0\end{aligned} \tag{22}$$

which demonstrates the exclusion of the Zeno behavior. This completes the proof of Theorem 1, Part 2).

It should be noting that the triggering condition (6) still requires continuous communication between neighboring agents. To accommodate this issue, we propose a scheme to monitor the triggering condition (6). With (20), the monitoring condition is designed as

$$\begin{aligned}E_i &= \left(\theta\kappa\gamma_2\sqrt{2V_1(0)} + (1 + |\mathcal{N}_i|)\,\delta \left\|\alpha_i\left(t_i^c\right)\right.\right. \\ &\quad \left.\left. + \beta_i\left(t_i^c\right)\right\|\right)\left(t - t_i^c\right) \\ &\quad + \int_{t_i^c}^{t}\left\|\sum_{j\in\mathcal{N}_i}\left(\alpha_j\left(t_j^c\right) + \beta_j\left(t_j^c\right)\right)\right\|\mathrm{d}\tau - \mu e^{-\nu t} \\ &\quad - \frac{\varrho_i}{2\sqrt{2\left(1+\varrho_i^2\right)}}\left(\left\|\alpha_i\left(t_i^c\right)\right\| + \left\|\beta_i\left(t_i^c\right)\right\|\right)\end{aligned} \tag{23}$$

Notably, Eq. (23) is a sufficient condition of the original triggering condition (6), and with (23), the next triggering time t_i^{c+1} can be determined without continuous communication. This completes the overall design of the event-triggered NE seeking algorithm.

3.2 Adaptive event-triggered NE seeking

As indicated in Theorem 1, the NE seeking scheme presented in Sect. 3.1 depends on the smallest eigenvalue $\underline{\lambda}$ of matrix H and the agents' number N. However, the matrix H is determined by the entire communication graph $\mathcal{G}$, which is global information. Moreover, the number of agents can hardly be obtained under distributed communication networks especially when there exist agents that join or leave the game.

An alternative and effective way to accommodate the aforementioned issues is to adjust the controller parameters adaptively. Along with this line, an adaptive event-triggered NE seeking algorithm is designed in this section as follows:

$$
\begin{aligned}
u_i &= -\theta g_i\left(y_i\right)\\
\dot{y}_i &= -\Delta_i\left(\sum_{j\in\mathcal{N}_i}\left(y_i\left(t_i^c\right)-y_j\left(t_i^c\right)\right)+A_i\left(y_i\left(t_i^c\right)\right.\right.\\
&\quad\left.\left.-x\left(t_i^c\right)\right)\right)\\
\dot{\Delta}_i &= \rho_i\left\|\sum_{j\in\mathcal{N}_i}\left(y_i\left(t_i^c\right)-y_j\left(t_i^c\right)\right)+A_i\left(y_i\left(t_i^c\right)\right.\right.\\
&\quad\left.\left.-x\left(t_i^c\right)\right)\right\|^2
\end{aligned}
\tag{24}
$$

where $\rho_i \in \mathbb{R}^+$ is a design parameter and Δ_i denotes the dynamic gain with its initial condition satisfying $\Delta_i(0) \in \mathbb{R}^+$.

The triggering function is designed as follows:

$$
\begin{aligned}
f_i\left(\alpha_i,\beta_i,\bar{\alpha}_i,\bar{\beta}_i\right) &= \|\bar{\alpha}_i\|+\|\bar{\beta}_i\|-\frac{\mu e^{-\nu t}}{\sqrt{1+\varrho\Delta_i}}\\
&\quad-\frac{1}{\sqrt{2(1+\varrho\Delta_i)}}\left\|\alpha_i\left(t_i^c\right)+\beta_i\left(t_i^c\right)\right\|
\end{aligned}
\tag{25}
$$

where $\varrho \in \mathbb{R}^+$. It is worth noting that $\Delta_i(t)$ is monotone increasing with (24) and thereby $\Delta_i(t) > 0$ holds as long as $\Delta_i(0) > 0$. Along with this line, the triggering function (25) is valid.

Similar to (8), the closed-loop system can be determined by

$$
\begin{aligned}
\dot{e}_x &= -\theta G(y)\\
\dot{\tilde{y}} &= -\Delta H\tilde{y}-\Delta\left(\bar{\alpha}+\bar{\beta}\right)+\theta\left(1_N\otimes G(y)\right)\\
\dot{\Delta}_i &= \rho_i\left\|\alpha_i\left(t_i^c\right)+\beta_i\left(t_i^c\right)\right\|^2
\end{aligned}
\tag{26}
$$

with $\Delta = \mathrm{diag}\{\Delta_1,\ldots,\Delta_N\}$. The following theorem demonstrates the convergence to the NE.

Theorem 2 *Consider the IGC problem (4) of MASs with controller (24) and triggering function (25). Under Assumptions 1–3, selecting $\delta > 0$ and $\rho_i > 0$ yields that (1) the unique NE x^* is globally asymptotically stable for the considered MAS; (2) the Zeno behavior can be excluded.*

Proof Proof of Theorem 2, Part (1) Consider the following Lyapunov function:

$$
V_2 = V_1+\sum_{i=1}^{N}\frac{1}{2\rho_i}\left(\Delta_i-\bar{\Delta}\right)^2
\tag{27}
$$

with $\bar{\Delta} \in \mathbb{R}^+$.

Similar to the proof of Theorem 1, the time derivate of V_2 along the trajectory (26) is given by

$$
\begin{aligned}
\dot{V}_2 &= -2\theta e_x^T G(y)+2\theta\tilde{y}^T H\left(1_N\otimes G(y)\right)\\
&\quad-2\sum_{i=1}^{N}\Delta_i\left(\alpha_i+\beta_i\right)^T\left(\alpha_i\left(t_i^c\right)+\beta_i\left(t_i^c\right)\right)\\
&\quad+\sum_{i=1}^{N}\left(\Delta_i-\bar{\Delta}\right)\left\|\alpha_i\left(t_i^c\right)+\beta_i\left(t_i^c\right)\right\|^2\\
&\le -2\gamma_1\theta\|e_x\|^2+2\gamma_2\theta\sqrt{N}\|H\|\|\tilde{y}\|^2\\
&\quad+2\gamma_2\theta\left(1+\|H\|\sqrt{N}\right)\|e_x\|\|\tilde{y}\|\\
&\quad-2\sum_{i=1}^{N}\Delta_i\left(\alpha_i+\beta_i\right)^T\left(\alpha_i\left(t_i^c\right)+\beta_i\left(t_i^c\right)\right)\\
&\quad+\sum_{i=1}^{N}\left(\Delta_i-\bar{\Delta}\right)\left\|\alpha_i\left(t_i^c\right)+\beta_i\left(t_i^c\right)\right\|^2
\end{aligned}
\tag{28}
$$

By leveraging the Young's inequality, we have

$$
\begin{aligned}
&-2\sum_{i=1}^{N}\Delta_i\left(\alpha_i+\beta_i\right)^T\left(\alpha_i\left(t_i^c\right)+\beta_i\left(t_i^c\right)\right)\\
&=2\sum_{i=1}^{N}\Delta_i\left(\bar{\alpha}_i+\bar{\beta}_i\right)\left(\alpha_i\left(t_i^c\right)+\beta_i\left(t_i^c\right)\right)\\
&\quad-2\sum_{i=1}^{N}\Delta_i\left\|\alpha_i\left(t_i^c\right)+\beta_i\left(t_i^c\right)\right\|^2\\
&\le -\sum_{i=1}^{N}\Delta_i\left\|\alpha_i\left(t_i^c\right)+\beta_i\left(t_i^c\right)\right\|^2+\sum_{i=1}^{N}\Delta_i\left\|\bar{\alpha}_i+\bar{\beta}_i\right\|^2
\end{aligned}
\tag{29}
$$

and

$$
\begin{aligned}
&-\bar{\Delta}\sum_{i=1}^{N}\left\|\alpha_i\left(t_i^c\right)+\beta_i\left(t_i^c\right)\right\|^2\\
&=-\frac{\bar{\Delta}}{2}\sum_{i=1}^{N}\left\|\alpha_i\left(t_i^c\right)+\beta_i\left(t_i^c\right)\right\|^2-\frac{\bar{\Delta}}{2}\sum_{i=1}^{N}\|\alpha_i+\beta_i\|^2\\
&\quad-\frac{\bar{\Delta}}{2}\sum_{i=1}^{N}\left\|\bar{\alpha}_i+\bar{\beta}_i\right\|^2-\bar{\Delta}\sum_{i=1}^{N}\left(\alpha_i+\beta_i\right)^T\left(\bar{\alpha}_i+\bar{\beta}_i\right)\\
&\le -\frac{\bar{\Delta}}{2}\sum_{i=1}^{N}\left\|\alpha_i\left(t_i^c\right)+\beta_i\left(t_i^c\right)\right\|^2-\frac{\bar{\Delta}}{4}\sum_{i=1}^{N}\|\alpha_i+\beta_i\|^2\\
&\quad+\frac{\bar{\Delta}}{2}\sum_{i=1}^{N}\left\|\bar{\alpha}_i+\bar{\beta}_i\right\|^2
\end{aligned}
\tag{30}
$$

Submitting (29) and (30) into (28) yields that

$$\begin{aligned}\dot{V}_2 \leq & \frac{\bar{\Delta}}{2}\sum_{i=1}^{N}\left(\left(1+\frac{2}{\bar{\Delta}\varrho}\cdot\varrho\Delta_i\right)\|\bar{\alpha}_i+\bar{\beta}_i\|^2-\|\alpha_i\left(t_i^c\right)\right.\\ & \left.+\beta_i\left(t_i^c\right)\|^2\right)\\ & -2\gamma_1\theta\|e_x\|^2-\left(\frac{\bar{\Delta}\underline{\lambda}^2}{4}-2\gamma_2\theta\sqrt{N}\|H\|\right)\|\tilde{y}\|^2\\ & +2\gamma_2\theta\left(1+\|H\|\sqrt{N}\right)\|e_x\|\|\tilde{y}\|\\ = & \frac{\bar{\Delta}}{2}\sum_{i=1}^{N}\left(\left(1+\frac{2}{\bar{\Delta}\varrho}\cdot\varrho\Delta_i\right)\|\bar{\alpha}_i+\bar{\beta}_i\|^2-\|\alpha_i\left(t_i^c\right)\right.\\ & \left.+\beta_i\left(t_i^c\right)\|^2\right)\\ & -\left[\|e_x\|,\|\tilde{y}\|\right]P_2\left[\|e_x\|,\|\tilde{y}\|\right]^T \end{aligned} \tag{31}$$

where

$$P_2=\begin{bmatrix}2\gamma_1\theta & -\theta\gamma_2\left(1+\|H\|\sqrt{N}\right)\\ -\theta\gamma_2\left(1+\|H\|\sqrt{N}\right) & \frac{\underline{\lambda}^2\bar{\Delta}}{4}-2\theta\gamma_2\sqrt{N}\|H\|\end{bmatrix} \tag{32}$$

It is easy to verify that the matrix P_2 is positive definite if $\delta>0$ and $\bar{\Delta}>\frac{2\theta\gamma_2\left(\gamma_2\left(1+\sqrt{N}\|H\|\right)^2+4\gamma_1\sqrt{N}\|H\|\right)}{\gamma_1\underline{\lambda}^2}$. Choosing $\bar{\Delta}$ large enough such that $\bar{\Delta}>\max\left\{\frac{2\theta\gamma_2\left(\gamma_2\left(1+\sqrt{N}\|H\|\right)^2+4\gamma_1\sqrt{N}\|H\|\right)}{\gamma_1\underline{\lambda}^2},\frac{2}{\varrho}\right\}$ and considering the triggering condition (25), it follows from (31) that

$$\begin{aligned}\dot{V}_2 \leq & \frac{\bar{\Delta}}{2}\sum_{i=1}^{N}\left((1+\varrho\Delta_i)\left(\|\bar{\alpha}_i\|+\|\bar{\beta}_i\|\right)^2-\|\alpha_i\left(t_i^c\right)\right.\\ & \left.+\beta_i\left(t_i^c\right)\|^2\right)\\ & -\lambda_{\min}\left(P_2\right)\left(\|e_x\|^2+\|\tilde{y}\|^2\right)\\ \leq & \frac{\bar{\Delta}}{2}\sum_{i=1}^{N}\left((1+\varrho\Delta_i)\left(\frac{\mu e^{-\nu t}}{\sqrt{1+\varrho\Delta_i}}\right.\right.\\ & \left.+\frac{1}{\sqrt{2(1+\varrho\Delta_i)}}\|\alpha_i\left(t_i^c\right)+\beta_i\left(t_i^c\right)\|\right)^2\\ & \left.-\|\alpha_i\left(t_i^c\right)+\beta_i\left(t_i^c\right)\|^2\right)\\ & -\lambda_{\min}\left(P_2\right)\left(\|e_x\|^2+\|\tilde{y}\|^2\right)\\ \leq & \frac{\bar{\Delta}}{2}\sum_{i=1}^{N}\left((1+\varrho\Delta_i)\left(\frac{2\mu^2e^{-2\nu t}}{1+\varrho\Delta_i}\right.\right.\\ & \left.+\frac{1}{1+\varrho\Delta_i}\|\alpha_i\left(t_i^c\right)+\beta_i\left(t_i^c\right)\|^2\right)\\ & -\|\alpha_i\left(t_i^c\right)+\beta_i\left(t_i^c\right)\|^2\right)\\ & -\lambda_{\min}\left(P_2\right)\left(\|e_x\|^2+\|\tilde{y}\|^2\right)\\ \leq & -\lambda_{\min}\left(P_2\right)\left(\|e_x\|^2+\|\tilde{y}\|^2\right)+\mu^2\bar{\Delta}Ne^{-2\nu t}\end{aligned} \tag{33}$$

which demonstrates the globally asymptotic convergence to the NE for the considered MASs governed by (24). This completes the proof of Theorem 2, Part (1). □

Proof of Theorem 2, Part 2) The proof is similar to that of Theorem 1, Part 2), and thereby is omitted here. □

To exclude the requirement of continuous communication, the following condition can be used to monitor the triggering function (25):

$$\begin{aligned}E_i = & \left(\theta\kappa\gamma_2\sqrt{2V_1(0)}+(1+|\mathcal{N}_i|)\delta\|\alpha_i\left(t_i^c\right)\right.\\ & \left.+\beta_i\left(t_i^c\right)\|\right)\left(t-t_i^c\right)\\ & +\int_{t_i^c}^{t}\left\|\sum_{j\in\mathcal{N}_i}\left(\alpha_j\left(t_j^c\right)+\beta_j\left(t_j^c\right)\right)\right\|\mathrm{d}\tau\\ & -\frac{\mu e^{-\nu t}}{\sqrt{1+\varrho\Delta_i}}\\ & -\frac{1}{\sqrt{2(1+\varrho\Delta_i)}}\|\alpha_i\left(t_i^c\right)+\beta_i\left(t_i^c\right)\|\end{aligned} \tag{34}$$

Notably, Eq. (34) is a sufficient condition of the origin triggering condition (25) and continuous communication is successfully excluded herein. This completes the overall design of the event-triggered NE seeking algorithm.

Remark 3 From Theorem 2, it is clear that the controller gains are determined with no need of the global information on the communication graph and the number of agents, and thereby compared with (5) the algorithm in (24) is designed in a fully distributed manner. In addition, compared with the static triggering function (6) the dynamic triggering condition (25) provides more freedom to regulate the triggering frequency by introducing the dynamic gain Δ_i, which contributes to further reducing the communication cost.

Remark 4 Despite of the superiorities of the adaptive event-triggered algorithm (24) compared with the

event-triggered one (5) as mentioned in Remark 3, it should be noted that the NE seeking algorithm (24) is more complicated than (5) and the transient-state performance of (24) will not be as good as that of (5) owing to the introduction of the adaptation law. Therefore, one can choose the appropriate algorithm depending on whether the communication network and the number of players will change or not in a game. To be specific, if there exist agents that leave or join the networked game, the fully distributed algorithm (24) is the priority, otherwise it prefers (5).

Remark 5 The adaptive NE seeking algorithm herein is partly motivated by our previous work [29], where similar scheme was presented to seek the NE for high-order nonlinear MASs. Differently, the algorithm developed in [29] still requires the number of agents as well as continuous communication between neighbors while this study overcomes these drawbacks.

4 NE seeking for double-integrator with constrained velocity and input

This section considers the event-triggered IGC problem for double-integrator MASs with the consideration of velocity constraint and input saturation. The motivation is twofold: (1) many real agents are acceleration-actuated systems that can be described by double-integrator; (2) velocity constraint and input saturation are commonly encountered constraints on the agents' dynamics in practice and might lead to serious performance degradation if not properly handled.

Consider the IGC problem for double-integrator as follows:

$$\begin{aligned} &\min_{x_i} \quad J_i\left(x_i, x_{-i}\right) \\ &\text{s.t.} \quad \begin{cases} \dot{x}_i = v_i \\ \dot{v}_i = u_i \end{cases} \end{aligned} \tag{35}$$

where $v_i \in \mathbb{R}$ is the velocity of agent i. In addition, we have the velocity constraint $|v_i(t)| \leq \bar{v}$ and the input constraint $|u_i(t)| \leq \bar{u}$ here where $\bar{v} \in \mathbb{R}^+$ and $\bar{u} \in \mathbb{R}^+$ are known upper bounds.

4.1 Useful definitions and lemmas

Before moving on, some useful definitions and lemmas are introduced here. Consider the following system

$$\dot{x} = f(x, u) \tag{36}$$

Definition 2 [55] System (36) is input-to-state stable (ISS) if there exist functions $\Upsilon_1 \in \mathcal{KL}$ and $\Upsilon_2 \in \mathcal{K}$ such that for any initial condition $x(0)$ and any bounded input $u(t)$, the solution $x(t)$ satisfies

$$\|x(t)\| \leq \Upsilon_1\left(\|x(0)\|, t\right) + \Upsilon_2\left(\sup_{0 \leq \tau \leq t} \|u(\tau)\|\right) \tag{37}$$

Lemma 1 [55] *If there exists a continuously differentiable function $V(x)$ such that*

$$\Upsilon_3(\|x\|) \leq V(x) \leq \Upsilon_4(\|x\|) \tag{38}$$

$$\frac{\partial V}{\partial x} f(x, u) \leq -\Upsilon_5(x), \forall \|x\| \geq \Upsilon_6(\|u\|) > 0 \tag{39}$$

hold, where $\Upsilon_6 \in \mathcal{K}$, $\Upsilon_3, \Upsilon_4 \in \mathcal{K}_\infty$ and $\Upsilon_5(x)$ is positive definite, then system (36) is ISS.

Lemma 2 [55] *Consider the following cascade system:*

$$\dot{x}_1 = f_1(x_1, x_2) \tag{40}$$

$$\dot{x}_2 = f_2(x_2) \tag{41}$$

where $x_1 \in \mathbb{R}^n$ and $x_2 \in \mathbb{R}^m$ denote the states and $f_1 : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $f_2 : \mathbb{R}^m \to \mathbb{R}^m$ are locally Lipschitz functions. If (40) is ISS with x_2 as an input and the origin of system (41) is globally uniformly asymptotically stable, then the origin of the cascade system (40)–(41) is globally uniformly asymptotically stable.

Lemma 3 [45] *Consider the following system:*

$$\dot{v}_i = -K\left(v_i - v_i^r\right) \tag{42}$$

with $v_i, v_i^r \in \mathbb{R}$ and $K \in \mathbb{R}^+$. If $|v_i(0)| \leq \bar{v}$ and $\left|v_i^r(t)\right| \leq \bar{v}$ are always satisfied, then the velocity $v_i(t)$ is always bounded by $|v_i(t)| \leq \bar{v}$, where $\bar{v} \in \mathbb{R}^+$ is the upper bound.

4.2 Adaptive event-triggered NE seeking for double-integrator

The following NE seeking scheme is proposed for each agent $i \in \mathcal{V}$:

$$\begin{aligned} \dot{z}_i &= -\theta g_i(y_i) \\ \dot{y}_i &= -\Delta_i\left(\sum_{j \in \mathcal{N}_i}\left(y_i\left(t_i^c\right) - y_j\left(t_i^c\right)\right) + A_i\left(y_i\left(t_i^c\right) - z\left(t_i^c\right)\right)\right) \\ \dot{\Delta}_i &= \rho_i\left\|\sum_{j \in \mathcal{N}_i}\left(y_i\left(t_i^c\right) - y_j\left(t_i^c\right)\right) + A_i\left(y_i\left(t_i^c\right) - z\left(t_i^c\right)\right)\right\|^2 \\ u_i &= -k_1 v_i - k_2 \tanh\left(x_i + v_i/k_1 - z_i\right) \end{aligned} \tag{43}$$

where $z_i \in \mathbb{R}$ is considered as a auxiliary system, $y_i \in \mathbb{R}^N$ denotes the estimate on $z = [z_1, \ldots, z_N]^T \in \mathbb{R}^N$, and $k_1, k_2 \in \mathbb{R}^+$ stand for design parameters.

Similar to (25), the triggering function can be designed as

$$f_i\left(\alpha_i, \beta_i, \bar{\alpha}_i, \bar{\beta}_i\right) = \|\bar{\alpha}_i\| + \|\bar{\beta}_i\| - \frac{\mu e^{-\nu t}}{\sqrt{1+\varrho\Delta_i}} - \frac{1}{\sqrt{2(1+\varrho\Delta_i)}}\left\|\alpha_i\left(t_i^c\right) + \beta_i\left(t_i^c\right)\right\| \tag{44}$$

where $\beta_i = A_i\left(y_i\left(t_i^c\right) - z\left(t_i^c\right)\right)$.

Let $e_z = z - x^*$, $\tilde{y} = y - 1_N \otimes z$, $\xi = [\xi_1 \ldots, \xi_N]^T = x + v/k_1 - z$ and $v = [v_1 \ldots, v_N]^T \in \mathbb{R}^N$. The overall closed-loop systems can be determined as

$$\begin{aligned}
\dot{e}_z &= -\theta G(y)\\
\dot{\tilde{y}} &= -\Delta H\tilde{y} - \Delta\left(\bar{\alpha} + \bar{\beta}\right) + \theta\left(1_N \otimes G(y)\right)\\
\dot{\Delta}_i &= \rho_i\left\|\alpha_i\left(t_i^c\right) + \beta_i\left(t_i^c\right)\right\|^2\\
\dot{\xi} &= -k_2\tanh(\xi) + \theta G(y)\\
\dot{v} &= -k_1 v - k_2\tanh(\xi)
\end{aligned} \tag{45}$$

where $\tanh(\xi) = [\tanh(\xi_1), \ldots, \tanh(\xi_N)]^T$. The following theorem demonstrates the convergence to the NE.

Theorem 3 *Consider the IGC problem (35) of MASs with controller (43) and triggering function (44). Under Assumptions 1–3, if the initial velocity satisfies $|v_i(0)| \le \bar{v}, \forall i \in \mathcal{V}$, then selecting $\delta > 0$, $\rho_i > 0$, $0 < k_1 < \bar{u}/(2\bar{v})$ and $0 < k_2 < k_1 \min\{\bar{v}, 4\}$ yields that (1) the unique NE x^* is globally asymptotically stable for the considered MAS; (2) the Zeno behavior can be excluded; (3) the velocity and input constraints are always satisfied, that is $|v_i(t)| \le \bar{v}$ and $|u_i(t)| \le \bar{u}$ hold for all $i \in \mathcal{V}$.*

Proof Proof of Theorem 3, Part (1) Note that the whole system (45) is in cascade form with subsystem $(e_z, \tilde{y})$ generating external input for subsystem (ξ, v). Therefore, the idea of the proof is to use the ISS arguments to show the convergence to the NE.

Consider the following Lyapunov function for subsystem $(e_z, \tilde{y})$:

$$V_3 = e_z^T e_z + \tilde{y}^T H\tilde{y} + \sum_{i=1}^{N}\frac{1}{2\rho_i}\left(\Delta_i - \bar{\Delta}\right)^2 \tag{46}$$

with $\bar{V}_3 = e_z^T e_z + \tilde{y}^T H\tilde{y}$.

Similar to the proof of Theorem 2, Part (1), the time derivate of V_3 can be obtained by

$$\dot{V}_3 \le -\lambda_{\min}(P_2)\left(\|e_z\|^2 + \|\tilde{y}\|^2\right) + \mu^2\bar{\Delta}Ne^{-2\nu t} \tag{47}$$

It is easy to obtain that $\lim_{t\to+\infty}\|e_z\| = 0$ and $\lim_{t\to+\infty}\|\tilde{y}\| = 0$ hold, which demonstrate that subsystem $(e_z, \tilde{y})$ is globally asymptotically stable.

Next, we will show subsystem (ξ, v) is ISS with respect to $(e_z, \tilde{y})$. Consider the following Lyapunov function:

$$V_4 = \log(\cosh\xi) + \frac{1}{2}v^T v \tag{48}$$

whose time derivate can be obtained by

$$\begin{aligned}
\dot{V}_4 &= \tanh(\xi)\dot{\xi} + v^T\dot{v}\\
&= -k_2\|\tanh(\xi)\|^2 - k_1\|v\|^2 + \theta\tanh(\xi)^T G(y)\\
&\quad - k_2 v^T\tanh(\xi)\\
&\le -k_2\|\tanh(\xi)\|^2 - k_1\|v\|^2 + k_2\|v\|\|\tanh(\xi)\|\\
&\quad + \theta\|\tanh(\xi)\|\|(G(y) - G(1_N\otimes z))\|\\
&\quad + \theta\|\tanh(\xi)\|\left\|\left(g(z) - g\left(x^*\right)\right)\right\|\\
&\le -[\|\tanh(\xi)\|, \|v\|]\begin{bmatrix} k_2 & -k_2/2\\ -k_2/2 & k_1\end{bmatrix}\\
&\quad [\|\tanh(\xi)\|, \|v\|]^T + \gamma_2\theta\|\tanh(\xi)\|\left(\|e_z\| + \|\tilde{y}\|\right)\\
&\le -\frac{k_1 + k_2 - \sqrt{(k_1-k_2)^2 + k_2^2}}{2}\left(\|\tanh(\xi)\|^2\right.\\
&\quad \left. + \|v\|^2\right)\\
&\quad + \gamma_2\theta\sqrt{2\left(\|\tanh(\xi)\|^2 + \|v\|^2\right)\left(\|e_z\|^2 + \|\tilde{y}\|^2\right)}
\end{aligned} \tag{49}$$

From (49), if $0 < k_2 < 4k_1$ we have $\varsigma = k_1 + k_2 - \sqrt{(k_1-k_2)^2 + k_2^2} > 0$. Let $\chi_1 = \left[\tanh(\xi)^T, v^T\right]^T$ and $\chi_2 = \left[e_z^T, \tilde{y}^T\right]^T$, it follows from (49) that

$$\begin{aligned}
\dot{V}_4 &\le -\frac{\varsigma}{2}\|\chi_1\|^2 + \gamma_2\theta\sqrt{2}\|\chi_1\|\|\chi_2\|\\
&= -\frac{\varsigma(1-\varepsilon)}{2}\|\chi_1\|^2 - \|\chi_1\|\left(\frac{\varsigma\varepsilon}{2}\|\chi_1\| - \gamma_2\theta\sqrt{2}\|\chi_2\|\right)
\end{aligned} \tag{50}$$

Therefore, for any $0 < \varepsilon < 1$, we have

$$\dot{V}_4 \le -\frac{\varsigma(1-\varepsilon)}{2}\|\chi_1\|^2 \le 0, \forall\|\chi_1\| \ge \frac{2\sqrt{2}\gamma_2\theta}{\varepsilon\varsigma}\|\chi_2\| \tag{51}$$

This indicates that subsystem (ξ, v) is ISS with $(e_z, \tilde{y})$ as the input by Lemma 1, followed by which subsystem (ξ, v) is asymptotically stable by Lemma 2.

Consequently, from the aforementioned analyses, we have $\lim_{t\to+\infty}\|e_z\| = \lim_{t\to+\infty}\|z - x^*\| =$

Fig. 1 Communication graph

0, $\lim_{t\to+\infty}\|\xi\| = \lim_{t\to+\infty}\|x+v/k_1-z\| = 0$ and $\lim_{t\to+\infty}\|v\| = 0$, which indicates that $\lim_{t\to+\infty}\|x-x^*\| = 0$. This demonstrates the convergence to the NE and completes the proof. □

Proof of Theorem 3, Part (2) The proof is similar to that of Theorem 2, Part 2), and thereby is omitted here.

Proof Proof of Theorem 3, Part (3) From (43), the velocity subsystem v_i can be rewritten as

$$\dot{v}_i = u_i = -k_1\left(v_i + \frac{k_2}{k_1}\tanh\left(z_i - x_i - v_i/k_1\right)\right) \tag{52}$$

By leveraging Lemma 3, if selecting k_1 and k_2 appropriately such that $0 < k_2 \le k_1\bar{v}$ holds, then the velocity $v_i(t)$ is always bounded by $|v_i(t)| \le \bar{v}$. Moreover, from (52), we have the control input is bounded by $|u_i(t)| \le 2k_1\bar{v}$. Therefore, if k_1 is chosen such that $0 < k_1 \le \bar{u}/(2\bar{v})$, then $|u_i(t)| \le \bar{u}$ holds for all $i \in \mathcal{V}$. This completes the proof of Theorem 3, Part 3). □

Similar to (34), the following condition can be used to monitor the triggering function to exclude continuous communication (44):

$$\begin{aligned} E_i = &\Big(\theta\kappa\gamma_2\sqrt{2\bar{V}_3(0)} + (1+|\mathcal{N}_i|)\,\delta\,\|\alpha_i(t_i^c) \\ &+\beta_i(t_i^c)\|\Big)(t-t_i^c) \\ &+\int_{t_i^c}^{t}\left\|\sum_{j\in\mathcal{N}_i}\left(\alpha_j(t_j^c)+\beta_j(t_j^c)\right)\right\|\mathrm{d}\tau \\ &-\frac{\mu e^{-\nu t}}{\sqrt{1+\varrho\Delta_i}} \\ &-\frac{1}{\sqrt{2(1+\varrho\Delta_i)}}\|\alpha_i(t_i^c)+\beta_i(t_i^c)\| \end{aligned} \tag{53}$$

This completes the overall design of the event-triggered NE seeking algorithm.

Remark 6 As mentioned in Remark 2, y_i in (43) denotes an observer on the auxiliary state profile z. Therefore, it is suggested that the design parameters ρ_i and θ should be appropriately adjusted (e.g. $\rho_i > \theta$) such that the observer can achieve more faster convergence rate than the controller. This contributes to improving the performance of the closed-loop systems.

Remark 7 The results presented in this section is related to [24], where input constrained NE seeking problem was accommodated for double-integrator via saturated gradient-based algorithms. In contrary to [24], both input and velocity constraints are considered in the proposed NE seeking algorithm (43) by adapting the hyperbolic tangent function. Moreover, the developed scheme can exclude continuous communication and global information on the communication graph as well as the number of agents.

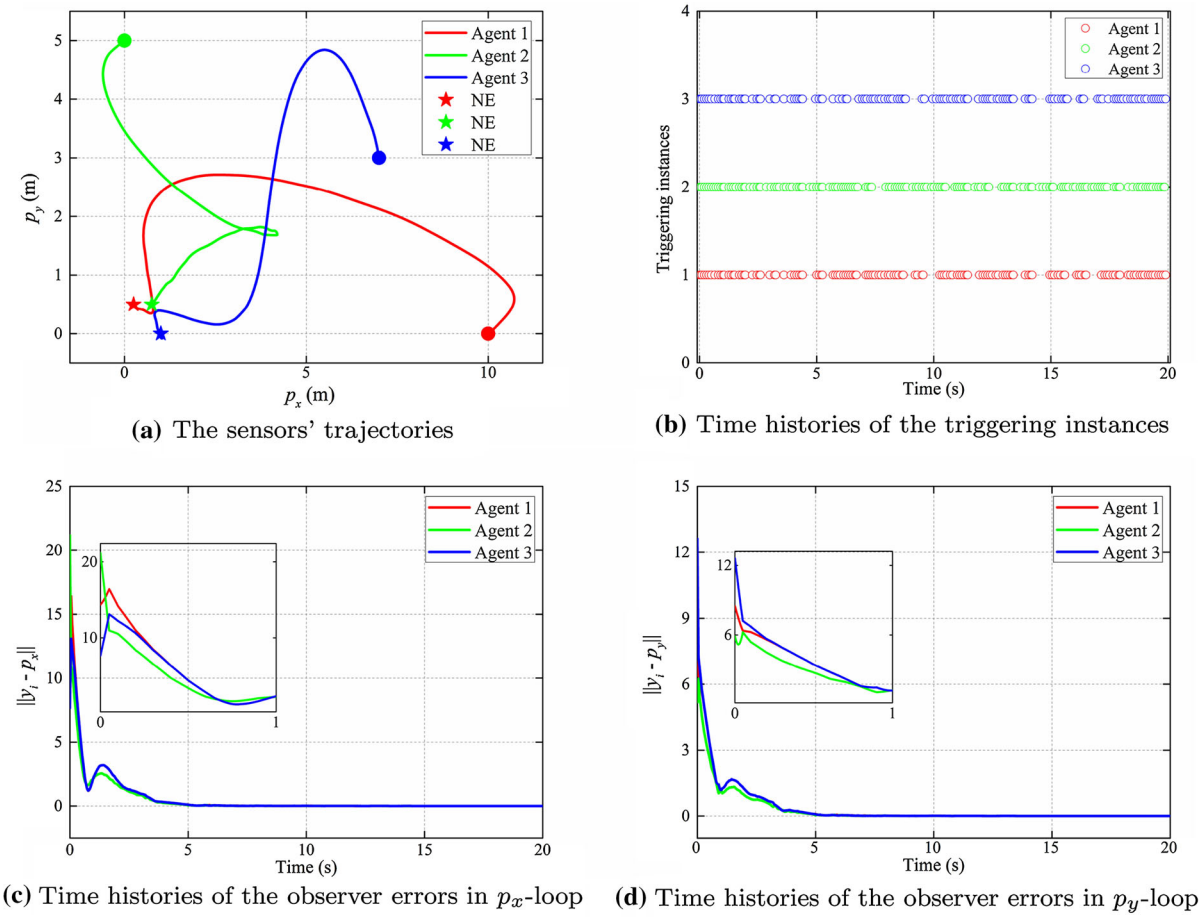
5 Simulation results

In this section, we consider the connectivity control problem for $N = 3$ two-dimensional mobile sensors, each of which is carried by a velocity- or acceleration-actuated vehicle. Let $p_x = [p_{x,1}, p_{x,2}, p_{x,3}]^T$ and $p_y = [p_{y,1}, p_{y,2}, p_{y,3}]^T$ denote the sensors' positions along p_x-axis and p_y-axis, respectively, and thereby the dynamics of sensors can be divided into two subsystems: p_x-subsystem and p_y-subsystem. The objective function of each sensor i can be specified as [2]

$$\begin{aligned} J_i(p_x, p_y) = &\sum_{j=1, j\neq i}^{3}\left((p_{x,i}-p_{x,j})^2 + (p_{y,i}-p_{y,j})^2\right) \\ &+ p_{x,i}^2 + p_{y,i}^2 + [p_{x,i}, p_{y,i}]\,r_{1,i} + r_{2,i} \end{aligned} \tag{54}$$

where $r_{1,i} \in \mathbb{R}^2$ and $r_{2,i} \in \mathbb{R}$ are fixed in the simulations as $r_{1,1} = [2, -2]^T$, $r_{1,2} = [-2, -2]^T$, $r_{1,3} = [-4, 2]^T$, $r_{2,1} = r_{2,2} = 3$ and $r_{2,3} = 6$. The first line of (54) can be considered as a global objective (e.g. connectivity preserving) while the second line of (54) can be viewed as a local objection (e.g. source seeking) [2]. Therefore, solving the game problem (54) intends to find a tradeoff between the local objective and the global objective for the MASs.

By directly calculation, the considered game admits a unique NE at $\left[\left(p_x^*\right)^T, \left(p_y^*\right)^T\right] = [0.25, 0.75, 1.0, 0.5, 0.5, 0.0]$ m. Notably, we can apply the proposed algorithms to solve the game problem for p_x-subsystem and p_y-subsystem separately. Moreover, we assume that the seeking algorithms are synchronously updated for p_x-subsystem and p_y-subsystem as long as there



(a) The sensors' trajectories

(b) Time histories of the triggering instances

(c) Time histories of the observer errors in p_x-loop

(d) Time histories of the observer errors in p_y-loop

Fig. 2 Simulation results generated by event-triggered algorithm (5)

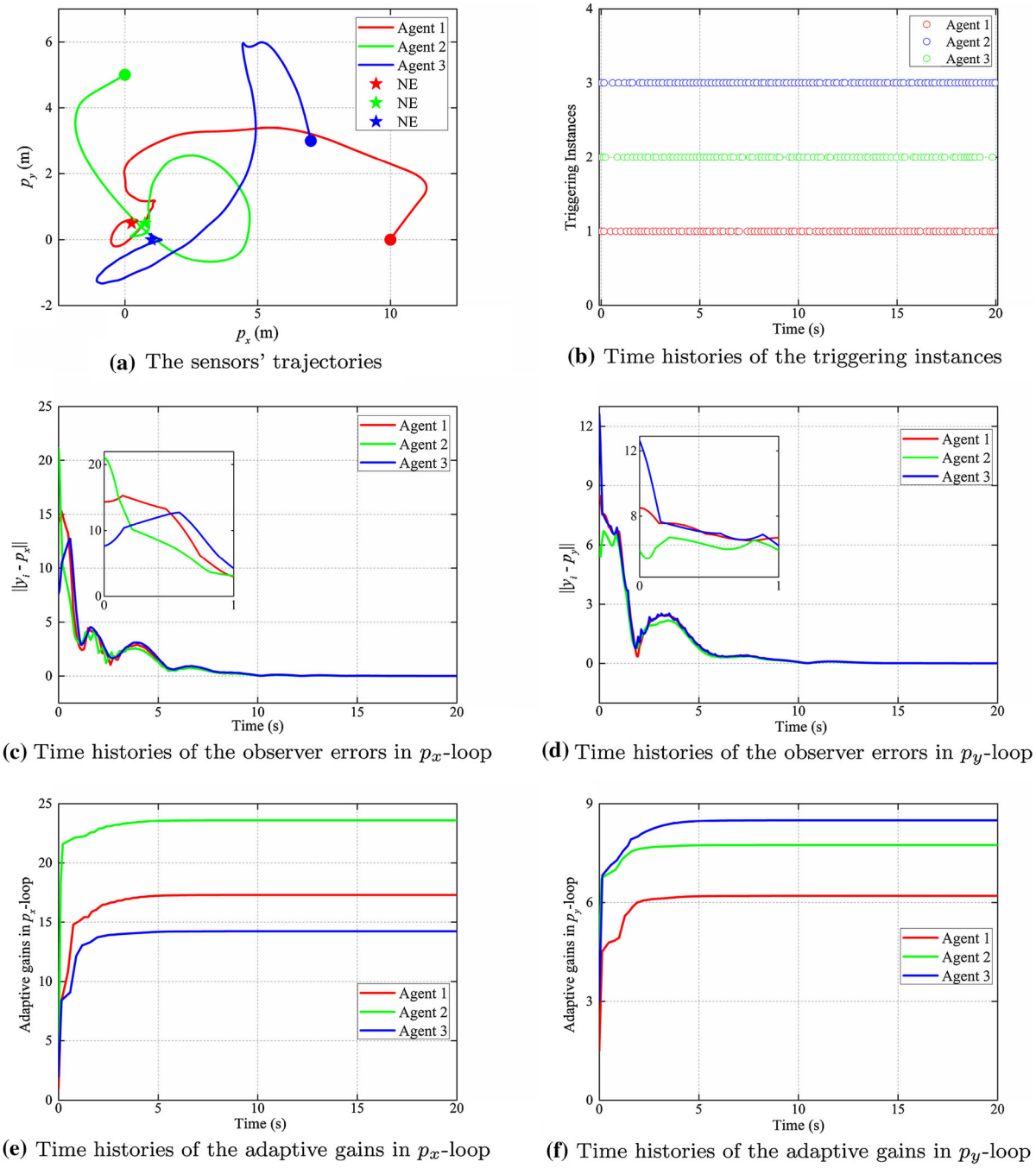
exists any triggering condition for the two subsystems that are satisfied. The communication graph of the considered sensor network is described by Fig. 1. The initial positions of the sensors are fixed as $p_x = [10, 0, 7]^T$ m and $p_y = [0, 5, 3]^T$ m. In the following, simulations are carried out for velocity- and acceleration-actuated vehicles successively.

5.1 Velocity-actuated MASs

The velocity-actuated MASs can be described as

$$\begin{aligned} &p_x\text{−subsystem}: \dot{p}_x = u_x \\ &p_y\text{−subsystem}: \dot{p}_y = u_y \end{aligned} \tag{55}$$

The simulation results generated by the event-triggered algorithm (5) and the adaptive scheme (24) are given in Figs. 2 and 3, respectively, illustrating the sensors' trajectories, the time histories of the observer errors and the time histories of the adaptive gains. As illustrated in these figures, the sensors' positions are steered to converge to the NE with the observer errors being stabilized at the origin, which explicitly verify Theorems 1 and 2. It can be also seen from these figures that the observer achieves faster convergence rate than the controller. This contributes to obtaining better systems performance. The triggering instances of each sensor are shown in Figs. 2b and 3b, which show that



(a) The sensors' trajectories

(b) Time histories of the triggering instances

(c) Time histories of the observer errors in p_x-loop

(d) Time histories of the observer errors in p_y-loop

(e) Time histories of the adaptive gains in p_x-loop

(f) Time histories of the adaptive gains in p_y-loop

Fig. 3 Simulation results generated by adaptive event-triggered algorithm (24)

the Zeno behavior as well as continuous communication are successfully excluded, which helps to reducing the communication costs. Moreover, from Table 1, we can see that the triggering numbers are reduced by introducing the adaptive gains. This is due to that the triggering condition in (25) is adjusted by the adaptive gains. Finally, Fig. 3f–g gives the time histories of the adaptive gains for p_x-subsystem and p_y-subsystem, respectively, which implies that the adaptive gains converge to finite steady-state values.

5.2 Acceleration-actuated MASs

The acceleration-actuated MASs can be described as

$$\begin{aligned} p_x-\text{subsystem}: &\begin{cases} \dot{p}_{x,i} = v_{x,i} \\ \dot{v}_{x,i} = u_{x,i} \end{cases} \\ p_y-\text{subsystem}: &\begin{cases} \dot{p}_{y,i} = v_{y,i} \\ \dot{v}_{y,i} = u_{y,i} \end{cases} \end{aligned} \tag{56}$$

where the velocities and the accelerations are bounded by $|v_{x,i}| \leq \bar{v}$, $|v_{y,i}| \leq \bar{v}$, $|u_{x,i}| \leq \bar{u}$ and $|u_{y,i}| \leq \bar{u}$ with $\bar{v} = 2\text{m/s}$ and $\bar{u} = 4\text{m/s}^2$. From Theorem 3, we can select that $k_1 = 0.95$ and $k_2 = 1.8$.

To provide better insight on the effectiveness of the constrained algorithm (43), we compare it with the following unconstrained algorithm by excluding the hyperbolic tangent function:

$$u_i = -k_1 v_i - k_2 (x_i + v_i/k_1 - z_i) \tag{57}$$

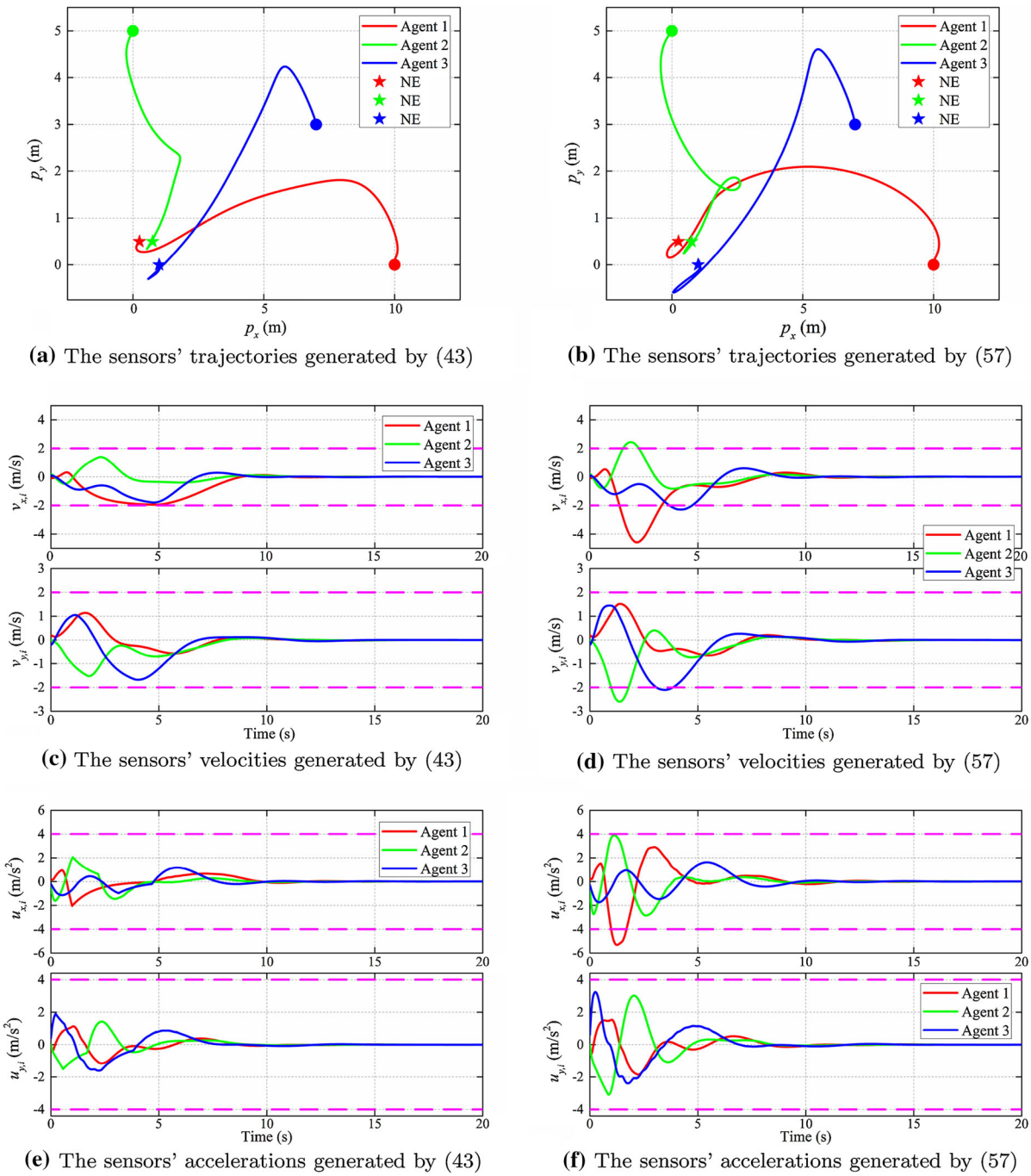
The simulation results generated by the constrained algorithm (43) and the unconstrained one (57) are presented in Fig. 4 including the sensors' trajectories, velocities and accelerations. As illustrated in the figure, the sensors' positions are steered to converge to the NE with the velocities and the accelerations being stabilized at the origin for both the two algorithms. However, the velocities and the accelerations generated by the proposed algorithm (43) are constrained within the given bounds while those generated by the unconstrained one (57) violate the constraints. Consequently, Theorem 3 is verified. Additionally, as illustrated in Fig. 4e, the accelerations given by the constrained algorithm (43) is much less than the given bounds, which lies in the conservatism of the proposed approach. How to design more feasible NE seeking algorithm under which the upper bounds of the inputs can be sufficiently exploited deserves future research.

Table 1 Triggering numbers under algorithms (5) and (24)

	Agent 1	Agent 2	Agent 3
Algorithm (5)	133	148	130
Algorithm (24)	107	92	101

6 Conclusion

The IGC problems of integrator-type MASs were studied in this paper, where the agents share information over communication networks. Specifically, a distributed NE seeking algorithm with static triggering condition and an adaptive distributed NE seeking scheme with dynamic triggering condition were developed for single-integrator to provide better insight on the event-triggered communication mechanism. It should be noted that with the adaptive seeking scheme, the agents can update their actions without knowing the entire communication graph as well as the number of agents. Thereafter, the proposed adaptive event-triggering algorithm was extended to accommodate the IGC problem of double-integrator, where the agents were subject to velocity constraint and input saturation, by integrating the hyperbolic tangent function into the controller. Based on the convergence analyses and numerical experiments, it is shown that the proposed algorithms would steer the agents' actions to reach the NE under certain given conditions. Future work will focus on the distributed finite-time NE seeking for multi-agent IGC problems with more general dynamics especially when the communication topology is directed and occasionally connected.



(a) The sensors' trajectories generated by (43)

(b) The sensors' trajectories generated by (57)

(c) The sensors' velocities generated by (43)

(d) The sensors' velocities generated by (57)

(e) The sensors' accelerations generated by (43)

(f) The sensors' accelerations generated by (57)

Fig. 4 Simulation results generated by the constrained algorithm (43) and the unconstrained algorithm (57)

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Data availability Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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