

## Letter

# Secure Bipartite Tracking Control for Linear Leader-Following Multiagent Systems Under Denial-of-Service Attacks

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Dear editor,

This letter puts forward a secure feedback control scheme to bipartite tracking consensus for a set of generic linear autonomous agents subject to aperiodic and unknown denial-of-service (DoS) attacks. In contrast to the DoS attack model that disables all transmission channels simultaneously, we are concerned with a general DoS attack model with independent attacks over each transmission channel. Such malicious attacks not only destroy the connectivity of underlying network, but also induce the dynamic transmission of reachable information. A time-varying error system is built upon the proposed distributed controller and the designed feedback matrix. A sufficient condition in terms of the frequency and duration of DoS attacks is developed with the assistance of some techniques from graph theory and non-negative matrix theory such that the state error is guaranteed to asymptotically approach zero. In particular, our results are proved to be applicable for a class of multiagent systems (MASs) with the strictly unstable system matrix.

The resource advantages and huge developmental potentials of MASs have been demonstrated in industrial applications, especially in performing distributed tasks including traffic management, search and rescue, multi-unmanned aerial vehicle (UAV) patrol, while these tasks are usually difficult to be completed by an agent alone. There is a growing body of research devoted to the collaboration of multiple autonomous agents [1]–[4].

The network topology is considered to be one of the key factors that dominate the final behavior of the participating agents. Compared with the cooperative networks, signed networks can be seen everywhere, like social networks [5] and neural networks. The recent years have witnessed an increasing trend towards investigating the bipartite tracking consensus emerging from signed networks with single leader [6]–[9].

MASs in real-world networks are vulnerable to different cyber attacks that threaten their performance and stability. DoS attack, aiming at blocking and interrupting the information transmission by jamming the communication channels on MASs, is the current way of an attack widely adopted. Most of the existing papers involving MASs under DoS attacks [10]–[12] focus on the fully cooperative networks and/or the case of paralyzing all transmission channels

simultaneously. There is a lack of concerns on the MASs with antagonistic interactions and independent attacks over multiple transmission channels.

The above motives inspire us to investigate the problem of secure bipartite tracking control of generic linear agents in the presence of DoS attacks. The extensions from the bipartite tracking consensus of integrator agents to that of generic linear agents are technically hard. Moreover, the existing secure control approaches [10]–[13] are not applicable to this study. We first propose an effective secure feedback control scheme for the considered general DoS attack model. Then we establish time-varying error system based on the dynamic information flows under attacks. Finally, we present the mild sufficient condition associated with the frequency and duration of attacks by means of graph theory and nonnegative matrix theory. The system matrix is relaxed to be strictly unstable and we give an explicit expression of the upper bound of its spectral radius.

**Notations:**  $\mathbb{N}$  and  $\mathbb{R}^s$  define the sets of natural numbers and  $s$ -dimensional real column vectors. The Kronecker product is denoted by  $\otimes$ . A zero matrix with compatible dimensions is denoted by 0. An identity matrix with compatible dimensions is defined as  $I$ . For a  $s \times s$  real matrix  $W = [w_{ij}]$  with elements  $w_{ij}$ ,  $i, j = 1, \dots, s$ ,  $\Lambda_i[W]$  is the sum of the elements in the  $i$ th row of matrix  $W$ ;  $|W| = [|w_{ij}|]$  is a matrix with  $|w_{ij}|$  being the absolute value of  $w_{ij}$ ;  $\|W\|_\infty = \max_i \{\Lambda_i[|W|]\}$  defines the infinite norm of matrix  $W$ ;  $\text{diag}(W)$  defines a diagonal matrix and its diagonal elements are  $w_{11}, w_{22}, \dots, w_{ss}$ . Let  $W = [w_{ij}]$  be a  $s \times s$  nonnegative real matrix, it is row-stochastic if  $\Lambda_i[W] = 1$  for  $i = 1, 2, \dots, s$ ; it is sub-stochastic if  $\Lambda_i[W] \leq 1$ ,  $i = 1, 2, \dots, s$ .

**Preliminaries:** The underlying interactions of single leader and  $N$  followers are modeled as a signed digraph  $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ , where  $\mathcal{V} = \{u_0, u_1, \dots, u_N\}$  and  $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$  are the sets of vertexes and directed edges respectively. The subgraph composed of all followers is denoted by  $\mathcal{G} = (\mathcal{V}, \mathcal{E})$  with vertex subset  $\mathcal{V} = \{u_1, \dots, u_N\}$  and edge subset  $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ . A directed edge from  $u_i$  to  $u_j$  is denoted by  $(u_i, u_j)$ .  $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N \times N}$  defines a weighted adjacency matrix of the subgraph  $\mathcal{G}$ .  $a_{ij} \neq 0$  if and only if  $(u_j, u_i) \in \mathcal{E}$ . Denote  $\Xi_i = \{u_z : (u_z, u_i) \in \mathcal{E}\}$ , which is a set of neighbors of vertex  $u_i$ .  $\mathcal{B} = \text{diag}\{\beta_1, \beta_2, \dots, \beta_N\}$  is a diagonal matrix that evaluates the degree of interactions between single leader and followers, where  $\beta_i \neq 0$  if there is the directed edge  $(u_0, u_i)$ , otherwise  $\beta_i = 0$ . A directed path from  $u_i$  to  $u_j$ , denoted by  $\mathcal{P}_{u_i \rightarrow u_j}$ , is a sequence of edges  $(u_i, u_{z_1}), (u_{z_1}, u_{z_2}), \dots, (u_{z_s}, u_j)$  with the vertexes  $u_{z_1}, \dots, u_{z_s}$  being distinct. The length of a directed path  $\mathcal{P}_{u_i \rightarrow u_j}$  is  $\eta_{u_i \rightarrow u_j}$ .

**Definition 1** [14]: Consider the relations  $\mathcal{E}_a$  and  $\mathcal{E}_b$  associated with the set  $\mathcal{V}$ . The composition of  $\mathcal{E}_a$  and  $\mathcal{E}_b$ , denoted by  $\mathcal{E}_a \circ \mathcal{E}_b$ , is the relation that is made of ordered pair  $(u_i, u_j)$  with  $u_i, u_j \in \mathcal{V}$ , and for which there exists  $u_h \in \mathcal{V}$  satisfying  $(u_i, u_h) \in \mathcal{E}_a$  and  $(u_h, u_j) \in \mathcal{E}_b$ .

**Lemma 1** [15]: Let  $\zeta$  be the spectral radius of matrix  $W \in \mathbb{R}^{s \times s}$ , then there  $\exists r > 0$  such that  $\|W^l\|_\infty \leq r l^{s-1} \zeta^l$  for  $l \geq s$ .

**DoS attack model:** In our work, it is assumed that in each attack, only the transmission channels are attacked and the relevant information exchanges are interrupted. We consider a general attack model where the DoS attacks over multiple transmission channels are independent. A schematic of DoS attacks is shown in Fig. 1(a).

In view of the resource limitation, it is reasonable to allow that the adversary move to an asleep period to accumulate energy for the next action. As shown in Fig. 1(b), the whole time series consists of two parts: One is the active period subject to malicious attacks; the other is the asleep period for the next action. Here, we define  $M_l = [\bar{t}_l, \bar{t}_l + \bar{\Delta}_l]$ ,  $l \in \mathbb{N}$  as the active interval of the  $l$ th DoS attack, where  $\bar{\Delta}_l$  is the duration of the  $l$ th attack. Let  $\bar{\Delta}_l < \mathfrak{N}_l = \bar{t}_{l+1} - \bar{t}_l$  and  $\mathfrak{N}_l$ ,  $l \in \mathbb{N}$  is uniformly bounded. Define  $T^d = \cup_l M_l$  as the total time interval subject to DoS attacks and  $T^n = T \setminus T^d$  as a set of instants

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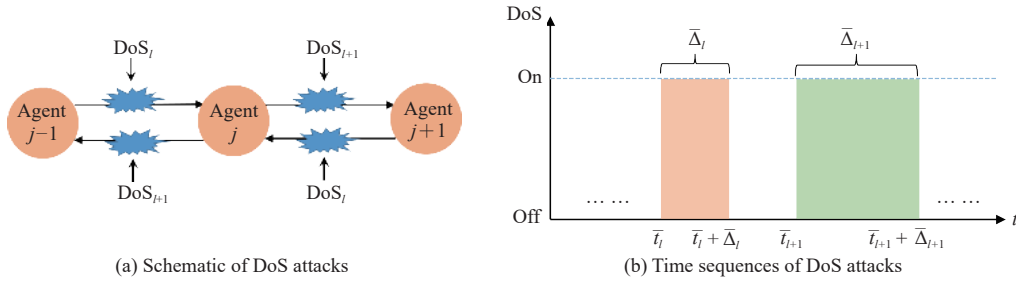


Fig. 1. Illustration of DoS attacks.

with normal information exchanges.

**Problem description:** Consider a MAS with single leader and  $N$  followers, the dynamics of single leader and the  $i$ th follower are as follows:

$$\begin{aligned} p_0(k+1) &= A p_0(k) \\ p_i(k+1) &= A p_i(k) + B \mu_i(k) \end{aligned} \quad (1)$$

where  $p_i \in \mathbb{R}^m$  and  $\mu_i \in \mathbb{R}^n$  are the state vector and control input of the  $i$ th agent;  $A \in \mathbb{R}^{m \times m}$  and  $B \in \mathbb{R}^{m \times n}$  are, respectively, the system matrix and the input matrix. Denote  $\Xi_i(k) = \{u_j : u_j \in \Xi_i \text{ and } (u_j, u_i) \text{ is attacked at time } k\}$ . The control input  $\mu_i(k)$  subject to DoS attacks is designed as

$$\mu_i(k) = \begin{cases} K \sum_{u_j \in \Xi_i \setminus \Xi_i(k)} |a_{ij}| (\text{sgn}(a_{ij}) p_j(k) - p_i(k)) \\ \quad + K \sigma(k) |\beta_i| (\text{sgn}(\beta_i) p_0(k) - p_i(k)), & k \in T^d \\ K \sum_{u_j \in \Xi_i} |a_{ij}| (\text{sgn}(a_{ij}) p_j(k) - p_i(k)) \\ \quad + K |\beta_i| (\text{sgn}(\beta_i) p_0(k) - p_i(k)), & k \in T^n \end{cases} \quad (2)$$

where  $\sigma(k) = 0$  if  $(u_0, u_i)$  is attacked at time  $k$ , otherwise  $\sigma(k) = 1$ .

**Definition 2:** The secure bipartite tracking control of MAS with the generic linear dynamics (1) and the controller (2) is said to be reached if for any initial states

$$\lim_{k \rightarrow \infty} \|p_i(k) - p_0(k)\|_{\infty} = 0, \quad \forall u_i \in \mathcal{V}.$$

**Main results:** Without loss of generality, the signed digraph  $\mathcal{G}$  is structurally balanced with a bipartition  $\{\mathcal{V}_1, \mathcal{V}_2\}$  of node set  $\mathcal{V}$ , and  $\beta_i > 0$  ( $\beta_i < 0$ ) if  $(u_0, u_i) \in \bar{\mathcal{E}}$  and  $u_i \in \mathcal{V}_1$  ( $u_i \in \mathcal{V}_2$ ). Consider that there are  $q$  followers in  $\mathcal{V}_1$  and  $N-q$  followers in  $\mathcal{V}_2$ , i.e.,  $\mathcal{V}_1 = \{u_1, \dots, u_q\}$ ,  $\mathcal{V}_2 = \{u_{q+1}, u_{q+2}, \dots, u_N\}$ .

Denote

$$\begin{aligned} \tilde{e}_i(k) &= p_i(k) - p_0(k), \quad i = 1, \dots, q \\ \tilde{e}_i(k) &= -p_i(k) - p_0(k), \quad i = q+1, \dots, N. \end{aligned}$$

Let  $\tilde{e}(k) = [\tilde{e}_1^T(k), \tilde{e}_2^T(k), \dots, \tilde{e}_N^T(k)]^T$ , then, an error system is established below

$$\tilde{e}(k+1) = [I_N \otimes A - (\mathcal{D}(k) + \mathcal{B}(k) - |\mathcal{A}(k)|) \otimes BK] \tilde{e}(k) \quad (3)$$

where

$$\begin{aligned} |\mathcal{A}(k)|_{ij} &= a_{ij}, \text{ if } k \in T^n \text{ or } k \in T^d, \quad u_j \in \Xi \setminus \Xi_i(k) \\ |\mathcal{A}(k)|_{ij} &= 0, \text{ if } k \in T^d, \quad u_j \in \Xi_i(k). \end{aligned}$$

$\mathcal{B}(k) = \text{diag}\{|\beta_1(k)|, \dots, |\beta_N(k)|\}$  with

$$\begin{aligned} \beta_i(k) &= \beta_i, \text{ if } k \in T^n \text{ or } k \in T^d, \quad u_0 \in \Xi \setminus \Xi_i(k) \\ \beta_i(k) &= 0, \text{ if } k \in T^d, \quad u_j \in \Xi_i(k) \end{aligned}$$

$$\mathcal{D}(k) = \text{diag} \left\{ \sum_{j=1}^N |a_{1j}(k)|, \sum_{j=1}^N |a_{2j}(k)|, \dots, \sum_{j=1}^N |a_{Nj}(k)| \right\}.$$

For convenience, a time-varying signed digraph  $\tilde{\mathcal{G}}(k) = (\tilde{\mathcal{V}}, \tilde{\mathcal{E}}(k))$  is introduced to describe the successfully transmitted information flows

under attacks. The subgraph with only followers is denoted by  $\mathcal{G}(k) = (\mathcal{V}, \mathcal{E}(k))$  with the adjacency matrix  $\mathcal{A}(k)$ . It follows that  $\mathcal{G}(k) \subset \tilde{\mathcal{G}}$  and  $\mathcal{G}(k) \subset \mathcal{G}$ . Denote  $\Xi_i(k) = \{u_j : (u_j, u_i) \in \mathcal{E}(k)\}$ , then there holds that  $\Xi_i(k) = \Xi_i \setminus \Xi_i(k)$  if  $k \in T^d$ ;  $\Xi_i(k) = \Xi_i$  if  $k \in T^n$ .

**Assumption 1:** Let the feedback matrix  $K = \tau B^T (BB^T)^{-1} A$  with  $\tau > 0$ , where the input matrix  $B$  is of row full rank.

In this way, we get an equivalent form of system (3)

$$\tilde{e}(k+1) = [\Theta(k) \otimes A] \tilde{e}(k) \quad (4)$$

where  $\Theta(k) = I_N - \tau \mathcal{D}(k) - \tau \mathcal{B}(k) + \tau |\mathcal{A}(k)|$ . Denote

$$\tilde{\mathcal{A}}(k) = \begin{bmatrix} 1 & 0_{1 \times N} \\ \alpha(k) & \Theta(k) \end{bmatrix}$$

with  $\alpha(k) = [\alpha_1(k), \alpha_2(k), \dots, \alpha_N(k)]^T$  and  $\alpha_i(k) = 1 - \Lambda_i[\Theta(k)]$ ,  $i = 1, \dots, N$ , where  $\Lambda_i[\Theta(k)]$  is the sum of the  $i$ th row of matrix  $\Theta(k)$ . A auxiliary digraph  $\tilde{\mathcal{G}}(k) = (\tilde{\mathcal{V}}, \tilde{\mathcal{E}}(k), \tilde{\mathcal{A}}(k))$  with  $\tilde{\mathcal{V}} = \{0, 1, \dots, N\}$  is introduced to assist in the subsequent analysis. Significantly, it is derived that  $(0, s) \in \tilde{\mathcal{G}}(k)$  only if  $(u_0, u_s) \in \mathcal{G}(k)$ ,  $s \in \{1, 2, \dots, N\}$ ;  $(s_1, s_2) \in \tilde{\mathcal{G}}(k)$  only if  $(u_{s_1}, u_{s_2}) \in \mathcal{G}(k)$ ,  $s_1, s_2 \in \{1, 2, \dots, N\}$ .

**Lemma 2:** Consider infinite time intervals  $[\tilde{t}_l, \tilde{t}_{(l+1)\theta})$ ,  $l \in \mathbb{N}$  with  $\theta = \max\{\eta_{u_0 \rightarrow u_i} : i = 1, \dots, N\}$ . Let  $\tau < 1/\max\{\Lambda_i[\mathcal{A}] + \beta_i : i = 1, \dots, N\}$ . With Assumption 1, there yields that  $(0, z) \in \tilde{\mathcal{E}}(\tilde{t}_l) \circ \tilde{\mathcal{E}}(\tilde{t}_l + 1) \circ \dots \circ \tilde{\mathcal{E}}(\tilde{t}_{(l+1)\theta} - 1)$  for  $z \in \{1, 2, \dots, N\}$  and  $l \in \mathbb{N}$  if there exists a directed spanning tree with the root  $u_0$  in  $\tilde{\mathcal{G}}$ .

**Proof:** With  $\tau < 1/\max\{\Lambda_i[\mathcal{A}] + \beta_i : i = 1, \dots, N\}$ ,  $\Theta(k)$ ,  $k \in \mathbb{N}$  is a sub-stochastic matrix and  $[\Theta(k)]_{ii} > 0$  for all  $i$ . It follows that  $\tilde{\mathcal{A}}(k)$ ,  $k \in \mathbb{N}$  is a row-stochastic matrix with  $[\tilde{\mathcal{A}}(k)]_{ii} > 0$ , which means that each node in  $\tilde{\mathcal{G}}(k)$  has self-loop. If there exists a directed spanning tree with the root  $u_0$  in  $\tilde{\mathcal{G}}$ , there exists a directed path  $\mathcal{P}_{u_0 \rightarrow u_i}$  composed of a sequence of edges  $(u_{g_0}, u_{g_1})$ ,  $(u_{g_1}, u_{g_2})$ , ...,  $(u_{g_{h-1}}, u_{g_h})$  for each follower  $u_i$ , where  $u_{g_0} = u_0$ ,  $u_{g_h} = u_i$  and  $g_1, \dots, g_h \in \{1, \dots, N\}$ .

Divide the interval  $[\tilde{t}_l, \tilde{t}_{l+1})$  into two sub-intervals  $[\tilde{t}_l, \tilde{t}_l + \tilde{\Delta}_l)$  and  $[\tilde{t}_l + \tilde{\Delta}_l, \tilde{t}_{l+1})$ . In the sub-interval  $[\tilde{t}_l, \tilde{t}_l + \tilde{\Delta}_l)$ , the edge  $(u_{g_j}, u_{g_{j+1}})$  may be maliciously attacked such that  $(u_{g_j}, u_{g_{j+1}}) \notin \tilde{\mathcal{E}}(k)$ ,  $k \in [\tilde{t}_l, \tilde{t}_l + \tilde{\Delta}_l)$ . But, in the sub-interval  $[\tilde{t}_l + \tilde{\Delta}_l, \tilde{t}_{l+1})$ , the node  $u_{g_j}$  has access to the state vectors of its all neighbors. In other words,  $(u_{g_j}, u_{g_{j+1}}) \in \tilde{\mathcal{E}}(k)$ ,  $k \in [\tilde{t}_l + \tilde{\Delta}_l, \tilde{t}_{l+1})$ . Accordingly, one gets that  $(g_j, g_{j+1}) \in \tilde{\mathcal{E}}(k)$ ,  $k \in [\tilde{t}_l + \tilde{\Delta}_l, \tilde{t}_{l+1})$ . Since the nodes in  $\tilde{\mathcal{G}}(k)$ ,  $k \in \mathbb{N}$  have self-loops, we get that  $(g_j, g_{j+1}) \in \tilde{\mathcal{E}}(\tilde{t}_l) \circ \tilde{\mathcal{E}}(\tilde{t}_l + 1) \circ \dots \circ \tilde{\mathcal{E}}(\tilde{t}_{l+1} - 1)$ , where  $(g_j, g_j) \in \tilde{\mathcal{E}}(\tilde{t}_l)$ , ...,  $(g_j, g_j) \in \tilde{\mathcal{E}}(\tilde{t}_l + \tilde{\Delta}_l - 1)$ ,  $(g_j, g_{j+1}) \in \tilde{\mathcal{E}}(\tilde{t}_l + \tilde{\Delta}_l)$ ,  $(g_{j+1}, g_{j+1}) \in \tilde{\mathcal{E}}(\tilde{t}_l + \tilde{\Delta}_l + 1)$ , ...,  $(g_{j+1}, g_{j+1}) \in \tilde{\mathcal{E}}(\tilde{t}_{l+1} - 1)$ .

It follows that  $(g_j, g_{j+1}) \in \tilde{\mathcal{E}}(\tilde{t}_{l+j}) \circ \tilde{\mathcal{E}}(\tilde{t}_{l+j} + 1) \circ \dots \circ \tilde{\mathcal{E}}(\tilde{t}_{l+j+1} - 1)$  for all  $j = 0, \dots, h-1$ . It thus follows that  $(0, g_h) \in \tilde{\mathcal{E}}(\tilde{t}_l) \circ \tilde{\mathcal{E}}(\tilde{t}_l + 1) \circ \dots \circ \tilde{\mathcal{E}}(\tilde{t}_{l+h} - 1)$ . Since  $h$  is the length of the directed path  $\mathcal{P}_{u_0 \rightarrow u_i}$ , one gets that  $h \leq \theta$ . One thus infers that  $(0, z) \in \tilde{\mathcal{E}}(\tilde{t}_l) \circ \tilde{\mathcal{E}}(\tilde{t}_l + 1) \circ \dots \circ \tilde{\mathcal{E}}(\tilde{t}_{(l+1)\theta} - 1)$  for all  $z \in \{1, 2, \dots, N\}$  and  $k \in \mathbb{N}$ . ■

**Theorem 1:** Consider infinite time intervals  $[\tilde{t}_l, \tilde{t}_{(l+1)\theta})$ ,  $l \in \mathbb{N}$ , where  $\theta = \max\{\eta_{u_0 \rightarrow u_i} : i = 1, \dots, N\}$ . With Assumption 1, let  $\tau < 1/\max\{\Lambda_i[\mathcal{A}] + \beta_i : i = 1, \dots, N\}$ , then the secure bipartite tracking control associated with the MAS (1) and the controller (2) can be

guaranteed if: 1) The system matrix  $A$  is with the spectral radius  $\zeta < 1/\sqrt{\theta\varpi} \sqrt{1-(1-\hat{\mu})\bar{\phi}^{-\theta\varpi-1}}$ , where  $\hat{\mu} = \sup_{k \in \mathbb{N}} \{\Lambda_i[\Theta(k)] : \Lambda_i[\Theta(k)] < 1, i = 1, 2, \dots, N\}$ ,  $\bar{\phi} = \min_{k \in \mathbb{N}} \{[\Theta(k)]_{ij} \neq 0 : i, j = 1, 2, \dots, N\}$ ,  $\varpi = \max_{l \in \mathbb{N}} \{\bar{\omega}_{\bar{l}_l}^{\bar{l}_l+1} : \bar{\omega}_{\bar{l}_l}^{\bar{l}_l+1} \text{ defines the time number in } [\bar{l}_l, \bar{l}_{l+1}]\}$ ; 2) There exists a directed spanning tree with the root  $u_0$  in  $\bar{\mathcal{G}}$ .

Proof: Lemma 2 indicates that  $(0, z) \in \bar{\mathcal{E}}(\bar{l}_\theta) \circ \bar{\mathcal{E}}(\bar{l}_\theta + 1) \circ \dots \circ \bar{\mathcal{E}}(\bar{l}_{(l+1)\theta} - 1)$  for all  $z \in \{1, 2, \dots, N\}$ ,  $l \in \mathbb{N}$ . As such, there  $\exists \varrho_0, \varrho_1, \varrho_2, \dots, \varrho_\lambda \in \{0, 1, \dots, N\}$  such that  $(\varrho_0, \varrho_1) \in \bar{\mathcal{E}}(\bar{l}_\theta)$ ,  $(\varrho_1, \varrho_2) \in \bar{\mathcal{E}}(\bar{l}_\theta + 1)$ ,  $\dots$ ,  $(\varrho_{\lambda-1}, \varrho_\lambda) \in \bar{\mathcal{E}}(\bar{l}_{(l+1)\theta} - 1)$  with  $\varrho_0 = 0, \varrho_\lambda = z$ . It is fact that  $\lambda \leq \theta\varpi$ . Accordingly, there  $\exists \varrho_\omega \neq 0$  with  $1 \leq \omega \leq \lambda$  such that  $\varrho_i = 0, i < \omega$ , which leads to that  $\Lambda_{\varrho_\omega}[\Theta(\bar{l}_\theta + \omega - 1)] \leq \hat{\mu}$ . One further yields that

$$\begin{aligned} & \Lambda_{\varrho_{\omega+1}}[\Theta(\bar{l}_\theta + \omega)\Theta(\bar{l}_\theta + \omega - 1)] \\ &= \sum_{c \neq \varrho_\omega}^N [\Theta(\bar{l}_\theta + \omega)]_{\varrho_{\omega+1}c} \Lambda_c[\Theta(\bar{l}_\theta + \omega - 1)] \\ & \quad + [\Theta(\bar{l}_\theta + \omega)]_{\varrho_{\omega+1}\varrho_\omega} \Lambda_{\varrho_\omega}[\Theta(\bar{l}_\theta + \omega - 1)] \\ & \leq 1 - [\Theta(\bar{l}_\theta + \omega)]_{\varrho_{\omega+1}\varrho_\omega} + [\Theta(\bar{l}_\theta + \omega)]_{\varrho_{\omega+1}\varrho_\omega} \hat{\mu} \\ & \leq 1 - (1 - \hat{\mu})\bar{\phi}. \end{aligned}$$

Denote  $\Psi_{\bar{l}_\theta}^{\bar{l}_{(l+1)\theta}} = \Theta(\bar{l}_{(l+1)\theta})\Theta(\bar{l}_{(l+1)\theta} - 1) \dots \Theta(\bar{l}_\theta)$ . Next, we prove that  $\Lambda_{\varrho_\lambda}[\Psi_{\bar{l}_\theta}^{\bar{l}_{(l+1)\theta}}] \leq 1 - (1 - \hat{\mu})\bar{\phi}^{\lambda-\omega}$  by using the mathematical induction. It is assumed that  $\Lambda_{\varrho_h}[\Psi_{\bar{l}_\theta}^{\bar{l}_{(l+1)\theta}+h-1}] \leq 1 - (1 - \hat{\mu})\bar{\phi}^{h-\omega}$  holds for all  $\omega < h \leq \lambda - 1$ , then one deduces that

$$\begin{aligned} \Lambda_{\varrho_{h+1}}[\Psi_{\bar{l}_\theta}^{\bar{l}_{(l+1)\theta}+h}] &= \sum_{c \neq \varrho_h}^N [\Theta(\bar{l}_\theta + h)]_{\varrho_{h+1}c} \Lambda_c[\Psi_{\bar{l}_\theta}^{\bar{l}_{(l+1)\theta}+h-1}] \\ & \quad + [\Theta(\bar{l}_\theta + h)]_{\varrho_{h+1}\varrho_h} \Lambda_{\varrho_h}[\Psi_{\bar{l}_\theta}^{\bar{l}_{(l+1)\theta}+h-1}] \\ & \leq 1 - (1 - \hat{\mu})\bar{\phi}^{h+1-\omega}. \end{aligned}$$

We get that  $\Lambda_{\varrho_\lambda}[\Psi_{\bar{l}_\theta}^{\bar{l}_{(l+1)\theta}}] \leq 1 - (1 - \hat{\mu})\bar{\phi}^{\lambda-\omega}$ . Therefore,

$$\begin{aligned} & \Lambda_{\varrho_\lambda}[\Psi_{\bar{l}_\theta}^{\bar{l}_{(l+1)\theta}}] \\ &= \sum_{c_1=1}^N \left[ \Psi_{\bar{l}_\theta}^{\bar{l}_{(l+1)\theta}-1} \right]_{\varrho_\lambda c_1} \Lambda_{c_1}[\Theta(\bar{l}_\theta + \omega - 2) \dots \Theta(\bar{l}_\theta)] \\ &= \sum_{c_1=1}^N \left[ \Psi_{\bar{l}_\theta}^{\bar{l}_{(l+1)\theta}-1} \right]_{\varrho_\lambda c_1} \sum_{c_2=1}^N [\Theta(\bar{l}_\theta + \omega - 2)]_{c_1 c_2} \\ & \quad \times \sum_{c_3=1}^N [\Theta(\bar{l}_\theta + \omega - 3)]_{c_2 c_3} \dots \sum_{c_\omega=1}^N [\Theta(\bar{l}_\theta)]_{c_{\omega-1} c_\omega} \\ & \leq \Lambda_{\varrho_\lambda}[\Psi_{\bar{l}_\theta}^{\bar{l}_{(l+1)\theta}-1}] \leq 1 - (1 - \hat{\mu})\bar{\phi}^{\lambda-\omega} \\ & \leq 1 - (1 - \hat{\mu})\bar{\phi}^{-\theta\varpi-1} \end{aligned}$$

where the last inequality holds based on the facts that  $\hat{\mu} < 1, \bar{\phi} < 1, \lambda < \theta\varpi$  and  $\omega > 1$ . It follows that  $\|\Psi_{\bar{l}_\theta}^{\bar{l}_{(l+1)\theta}-1}\|_\infty \leq 1 - (1 - \hat{\mu})\bar{\phi}^{-\theta\varpi-1}$ . It continues to follow that:

$$\begin{aligned} & \lim_{k \rightarrow \infty} \|\Theta(k)\Theta(k-1) \dots \Theta(0)\|_\infty \\ &= \lim_{\varphi \rightarrow \infty} \left\| \Psi_{\bar{l}_{(\varphi-1)\theta}}^{\bar{l}_{\varphi\theta}-1} \Psi_{\bar{l}_{(\varphi-2)\theta}}^{\bar{l}_{(\varphi-1)\theta}-1} \dots \Psi_{\bar{l}_0}^{\bar{l}_{\theta}-1} \right\|_\infty \\ &\leq \lim_{\varphi \rightarrow \infty} \left\| \Psi_{\bar{l}_{(\varphi-1)\theta}}^{\bar{l}_{\varphi\theta}-1} \right\|_\infty \left\| \Psi_{\bar{l}_{(\varphi-2)\theta}}^{\bar{l}_{(\varphi-1)\theta}-1} \right\|_\infty \dots \left\| \Psi_{\bar{l}_0}^{\bar{l}_{\theta}-1} \right\|_\infty \\ &\leq \lim_{\varphi \rightarrow \infty} \left[ 1 - (1 - \hat{\mu})\bar{\phi}^{\theta\varpi-1} \right]^\varphi. \end{aligned}$$

With Lemma 1, there  $\exists r > 0$  such that  $\|A^k\|_\infty \leq rk^{m-1}\zeta^k$  for  $k \geq m$ .

It yields that

$$\begin{aligned} & \lim_{k \rightarrow \infty} \|\Theta(k-1)\Theta(k-2) \dots \Theta(0) \otimes A^k\|_\infty \\ &= \lim_{k \rightarrow \infty} \|\Theta(k-1)\Theta(k-2) \dots \Theta(0)\|_\infty \cdot \|A^k\|_\infty \\ &\leq \lim_{\varphi \rightarrow \infty} [1 - (1 - \hat{\mu})\bar{\phi}^{\theta\varpi-1}]^\varphi r(\varphi\theta\varpi)^{m-1} \zeta^{\varphi\theta\varpi} = 0 \end{aligned}$$

where  $\lim_{\varphi \rightarrow \infty} [1 - (1 - \hat{\mu})\bar{\phi}^{\theta\varpi-1}]^\varphi r(\varphi\theta\varpi)^{m-1} \zeta^{\varphi\theta\varpi} = 0$  holds under the condition that  $\zeta < 1/\sqrt{\theta\varpi} \sqrt{1-(1-\hat{\mu})\bar{\phi}^{-\theta\varpi-1}}$ . There yields that

$$\begin{aligned} & \lim_{k \rightarrow \infty} \|\tilde{z}(k)\|_\infty \\ &= \lim_{k \rightarrow \infty} \|[\Theta(k-1) \otimes A]\tilde{z}(k-1)\|_\infty \\ &= \lim_{k \rightarrow \infty} \|\Theta(k-1)\Theta(k-2) \dots \Theta(0) \otimes A^k\|_\infty \cdot \|\tilde{z}(0)\|_\infty = 0 \end{aligned}$$

which indicates that the bipartite tracking consensus under DoS attacks is achieved. ■

**Remark 1:** Our result extends the secure consensus in [10]–[12] under DoS attacks from cooperative networks to signed networks, and extends the secure bipartite consensus subject to DoS attacks in [13] from integrator agents to generic linear agents. Moreover, in some previous works on generic linear MASSs, the pair  $(A, B)$  may be relaxed to be stabilizable, but it needs some additional conditions on the connectivity of the network topology or the stability of the system matrix. Compared to these works, the attempt of the feedback matrix  $K = \tau B^T (BB^T)^{-1} A$  in this letter is to build the weakest possible connectivity condition on the one hand, and on the other hand to make the system matrix in a strictly unstable mode and specify the upper bound of its spectral radius, which is considered to be more practical in real-world systems.

**Numerical example:** A MAS is made up of single leader with state vector  $p_0 \in \mathbb{R}^3$  and seven followers with state vectors  $p_i \in \mathbb{R}^3, i = 1, \dots, 7$ . These agents are connected via a communication topology in Fig. 2. From Fig. 2, the subgraph  $\bar{\mathcal{G}}$  of followers is structurally balanced. Specifically,  $\mathcal{V}^{(1)} = \{u_1, u_2, u_3\}$  and  $\mathcal{V}^{(2)} = \{u_4, \dots, u_7\}$ . Assume that at time  $4l, l \in \mathbb{N}$ , the attacker launches the  $l$ th DoS attack, and the duration of the attack is  $\bar{\Delta}_l = 2$ . The attack strategies of the attacker are as follows: 1) The communication channels  $(u_0, u_1), (u_1, u_3), (u_4, u_5), (u_5, u_3)$  are paralyzed at time  $4l, l \in \mathbb{N}$ ; 2) The communication channels  $(u_0, u_4), (u_3, u_2), (u_5, u_6), (u_7, u_4)$  are paralyzed at time  $4l+1, l \in \mathbb{N}$ ; 3) The communication channels  $(u_0, u_7), (u_2, u_1), (u_6, u_7), (u_1, u_4)$  are paralyzed at time  $4l+2, l \in \mathbb{N}$ . The parameters are  $\theta = 3, \tau = 1/4, \hat{\mu} = 3/4, \bar{\phi} = 1/4$  and  $\varpi = 4$ , which satisfies the condition 1) in Theorem 1. The system matrix  $A$  and the input matrix with row full rank are, respectively, given by

$$A = \begin{bmatrix} 0.8469 & -0.2989 & 0.1993 \\ 0.2291 & 0.8269 & -0.2989 \\ -0.3985 & 0.2989 & 0.7970 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

The state trajectories of agents are captured by Fig. 3.

**Conclusions:** The secure bipartite tracking control has been investigated for generic linear MASSs under the aperiodic and unknown DoS attacks in this letter. A general DoS attack model with independent attacks over multiple transmission channels has been

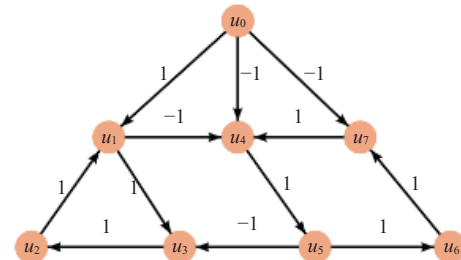


Fig. 2. The underlying communication topology  $\bar{\mathcal{G}}$ .

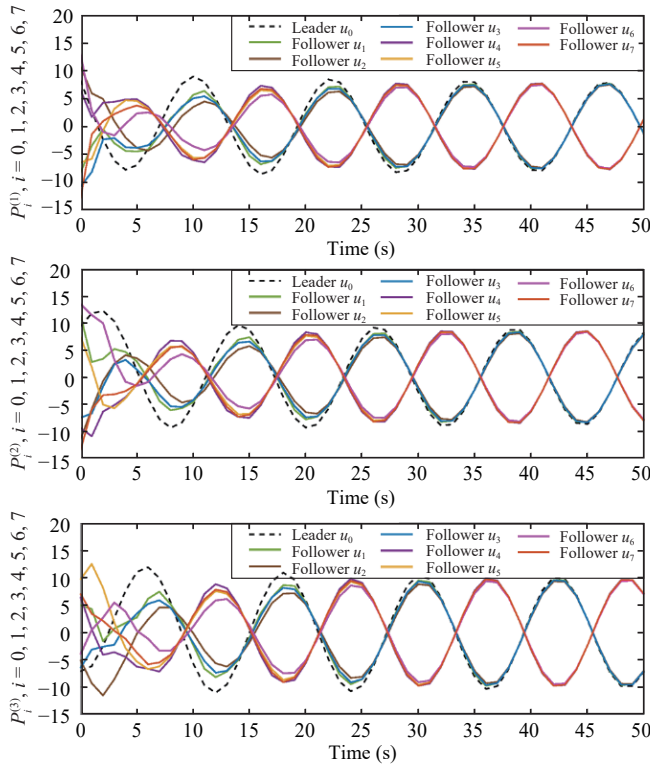


Fig. 3. The states tracking of single leader and multiple followers.

considered. A secure feedback control scheme and the effective parameter selection strategy have been proposed. Applying some properties of non-negative matrix theory and graph theory, a sufficient condition involving the frequency and duration of DoS attacks has been derived, and the system matrix has been proven to be strictly unstable.

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