Letter

Predefined-Time Backstepping Stabilization of Autonomous Nonlinear Systems

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Dear Editor,

In this letter, the backstepping technique is applied to solve the predefined-time stabilization problem for autonomous nonlinear systems, where nonlinear terms are unknown but satisfy a linear growth condition. A linear time-varying control law is designed to drive all states of an autonomous nonlinear system to the origin for an *a priori* preassigned time. Numerical examples are included to validate the obtained theoretical results and demonstrate effectiveness of the proposed control law. The conducted simulations show that the designed control input provides a lesser magnitude and is easier computable than some existing control laws.

There are different approaches to solving the stabilization problem for autonomous systems. The most researched one is asymptotic stabilization, where the control objective is to ensure asymptotic convergence of the system states to an equilibrium as time goes to infinity [1]. In many cases, the asymptotic convergence is provided by a linear autonomous feedback, which is easily computable and implementable. However, the asymptotic convergence has a considerable disadvantage: since the system state actually never reaches an equilibrium, the separation principle must be rigorously substantiated. To overcome this difficulty, much attention has recently been paid to designing finite-time and fixed-time convergent control laws and estimating their convergence (settling) times, in particular, for uncertain systems [2]−[7]. However, for the finite-time convergent strategies, their convergence times still depend on state initial values and may diverge to infinity as the state initial values grow. The convergence times are uniformly bounded for fixed-time convergent control laws, but the calculated upper convergence time estimate may be unreasonably larger than the real convergence time. In addition, in most fixed-time strategies, the settling time cannot be assigned *a priori*. This presents a challenge to design control laws where the fixed settling time can be assigned arbitrarily.

An innovative idea has been proposed to include the convergence time in the designed control law explicitly as a parameter, called predefined-time or prescribed-time convergence [8]−[10]. This enables the control designer to assign the fixed settling time arbitrarily or at will. Note that only time-varying (non-autonomous) control laws are able to realize predefined-time convergence, if the control law is required to be smooth. There have recently been a number of papers presenting predefined-time convergent algorithms for various classes of nonlinear systems [11]−[16].

This letter considers a class of autonomous nonlinear systems, where nonlinear terms are unknown but satisfy a linear growth condition, similarly to [15]. As known, the linear growth condition essentially means that the system state cannot diverge to infinity in finite time. Therefore, it is a conventional condition for nonlinear uncertainties in a practical system. The predefined-time convergent linear time-varying control law, stabilizing the system state at the origin in presence of unknown nonlinearities, is designed based on the conventional backstepping technique, which makes it straightforwardly obtainable and intuitively consistent. The conducted numerical simulations show that the designed control input provides a lesser magnitude and, in some cases, faster real convergence times and is easier computable than some existing control laws, such as those proposed in [15].

Predefined-time convergence: Consider an autonomous system

$$\begin{aligned} \dot{x}(t) &= f(x,u) \\ x(t_0) &= x_0, \quad t \geq t_0 \end{aligned} \tag{1}$$

where $x(t)$ is the system state, $u(t)$ is the control input, and t_0 is the initial time moment.

Definition 1 [10]: The system (1) is called predefined-time convergent to the origin, if

1) It is fixed-time convergent to the origin, i.e., for any initial state x_0, there exists a positive constant $T_{\max} > 0$ independent of x_0, such that $x(t) = 0$ for all $t \geq T_{\max}$;

2) $T_{\max}$ is independent of initial conditions and disturbances and can be explicitly assigned as a parameter of the control input; and

3) $T_{\max} \geq T_f$, where T_f is the true convergence time.

Problem statement: Consider the following autonomous nonlinear system [15]:

$$\begin{cases} \dot{x}_1 = x_2 + \phi_1(x,u) \\ \dot{x}_2 = x_3 + \phi_2(x,u) \\ \quad \vdots \\ \dot{x}_{n-1} = x_n + \phi_{n-1}(x,u) \\ \dot{x}_n = u + \phi_n(x,u) \end{cases} \tag{2}$$

where $x = [x_1, x_2, \ldots, x_n]^T \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}$ is the control input, and $\phi_i(x,u)$, $i = 1,2,\ldots,n$, are unknown continuous functions (disturbances) satisfying a linear growth condition, i.e., there exist some known constants $c_{ij} \geq 0$, $i, j = 1,2,\ldots,n$, such that

$$|\phi_i(t,x,u)| \leq c_{i1}|x_1| + c_{i2}|x_2| + \cdots + c_{in}|x_n|. \tag{3}$$

The control problem is to design a smooth control law driving the state of the system (2) at the origin in predefined time in the sense of Definition 1.

This problem was originally investigated in [15]. Here, we present an alternative solution to this problem and demonstrate its advantages with respect to some performance indices, such as a control input magnitude. Note that the linear growth condition (3) does not have to be in a triangular form, as it was assumed in [15]. In contrast to [11], the system (2) contains unmatched nonlinear disturbances appearing in each system equation.

Control design:

Second-order system: Consider the autonomous nonlinear two-dimensional system (2) as

$$\begin{cases} \dot{x}_1 = x_2 + \phi_1(x,u) \\ \dot{x}_2 = u + \phi_2(x,u) \end{cases} \tag{4}$$

where the nonlinear disturbances satisfy the linear growth conditions

$$\begin{aligned} \phi_1(x,u) &\leq c_{11}|x_1| + c_{12}|x_2| \\ \phi_2(x,u) &\leq c_{21}|x_1| + c_{22}|x_2| \end{aligned} \tag{5}$$

with some known constants $c_{ij} > 0$, $i, j = 1,2$, and the control input is

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given as

$$u(t)=\begin{cases}\frac{1}{1+\rho_{12}}\left(-x_1-\frac{\partial\psi_1}{\partial t}-x_2\frac{\partial\psi_1}{\partial x_1}\right)\\ -\psi_2-\rho_{21}x_1, & t_0\le t<t_f\\ 0, & t\ge t_f\end{cases} \quad (6)$$

where $\psi_1=\eta_1x_1/(t_f-t)$, $\psi_2=\eta_2z_2/(t_f-t)$, and the control gains $\eta_1,\eta_2,\rho_{12},\rho_{21}>0$ are defined in the following theorem.

Theorem 1: The control law (6) drives the state $x(t)$ of the system (4) at the origin for an *a priori* pre-assigned time t_f and stays there afterwards for any $t\ge t_f$, or, in other words, the closed-loop system (4), (6) is predefined-time convergent to the origin, if the following conditions hold: $\eta_1>c_{11}(t_f-t)+1$, $\eta_2>c_{22}(t_f-t)+1$, $\rho_{12}>c_{12}$, $\rho_{21}>c_{21}$.

Proof: Using the backstepping technique, assign the desired value x_{2d} as

$$x_{2d}=\frac{-\eta_1x_1}{(t_f-t)}-\rho_{12}x_2=-\psi_1-\rho_{12}x_2. \quad (7)$$

Applying the change of variables

$$z_2=x_2-x_{2d}=x_2+\psi_1+\rho_{12}x_2 \quad (8)$$

and taking the time derivative, one obtains

$$\dot z_2=\dot x_2+\frac{\partial\psi_1}{\partial x_1}\dot x_1+\frac{\partial\psi_1}{\partial t}+\rho_{12}\dot x_2. \quad (9)$$

Therefore, the transformed system takes the form

$$\dot x_1=z_2-\psi_1-\rho_{12}x_2+\phi_1(x,u)$$
$$\dot z_2=(1+\rho_{12})(u(t)+\phi_2(x,u))+\frac{\partial\psi_1}{\partial x_1}\dot x_1+\frac{\partial\psi_1}{\partial t}.$$

Choose the Lyapunov function as

$$V_2=V_1+\frac{1}{2}z_2^2 \quad (10)$$

where $V_1=(1/2)x_1^2$ is the Lyapunov function for the first-order system. Then, taking the time derivative of V_2 yields

$$\dot V_2=\frac{\partial\psi_1}{\partial x_1}\dot x_1+z_2\dot z_2=x_1\dot x_1+z_2\dot z_2. \quad (11)$$

Substituting $\dot x_1$ and $\dot z_2$ into the last equation results in

$$\dot V_2=x_1(z_2-\psi_1-\rho_{12}x_2+\phi_1)+z_2\left[(1+\rho_{12})(u+\phi_2)+\frac{\partial\psi_1}{\partial x_1}\dot x_1+\frac{\partial\psi_1}{\partial t}\right]. \quad (12)$$

Then, the control law is defined as

$$u(t)=\begin{cases}\frac{1}{1+\rho_{12}}\left(-x_1-\frac{\partial\psi_1}{\partial t}-x_2\frac{\partial\psi_1}{\partial x_1}\right)\\ -\psi_2-\rho_{21}x_1, & t_0\le t<t_f\\ 0, & t\ge t_f\end{cases} \quad (13)$$

where $\psi_2=\eta_2z_2/(t_f-t)$.

Thus, $\dot V_2$ can be rewritten as

$$\dot V_2=x_1(-\psi_1-\rho_{12}x_2+\phi_1)+z_2(1+\rho_{12})(-\rho_{21}x_1-\psi_2+\phi_2). \quad (14)$$

Taking the upper bounds of ϕ_1 and ϕ_2 in (5) yields

$$\dot V_2<x_1\left(-\eta_1\frac{x_1}{(t_f-t)}-\rho_{12}x_2+c_{11}x_1+c_{12}x_2\right)+z_2(1+\rho_{12})\left(-\rho_{21}x_1-\eta_2\frac{z_2}{(t_f-t)}+c_{21}x_1+c_{22}x_2\right). \quad (15)$$

Using $z_2=x_2-x_{2d}$,

$$z_2=x_2+\frac{\eta_1x_1}{(t_f-t)}+\rho_{12}x_2=(1+\rho_{12})x_2+\frac{\eta_1x_1}{(t_f-t)}$$

and it follows that:

$$x_2=\frac{z_2}{(1+\rho_{21})}-\frac{\eta_1x_1}{(t_f-t)(1-\rho_{12})}. \quad (16)$$

Substituting (16) into (15) implies that if the control gains are assigned as

$$\eta_1>c_{11}(t_f-t)+1$$
$$\eta_2>c_{22}(t_f-t)+1>\frac{1}{1+\rho_{12}}\left(c_{22}(t_f-t)+1\right)$$
$$\rho_{12}>c_{12}$$
$$\rho_{21}>c_{21}>c_{21}-\left(\frac{1}{1+\rho_{12}}\right)\left(\frac{c_{22}\eta_1}{(t_f-t)}\right)$$

taking into account $0<1/(1+\rho_{12})<1$, then

$$\dot V_2<-\frac{x_1^2}{(t_f-t)}-\frac{z_2^2}{(t_f-t)}=-\frac{1}{(t_f-t)}(x_1^2+z_2^2). \quad (17)$$

In view of (10), the preceding equation can be rewritten as

$$\dot V_2<\frac{-2V_2}{(t_f-t)} \quad (18)$$

which implies that

$$\frac{\dot V}{V}<\frac{-2}{t_f-t}. \quad (19)$$

Therefore, the solution of the differential inequality (18) satisfies

$$V(t)<K(t_f-t)^2,\quad t_0\le t<t_f \quad (20)$$

where $K=V(x_0)/(t_f-t_0)^2$. Therefore, following (20), if $t=t_f$, then $V_2(t_f)=0$, which yields $x_1(t_f)=x_2(t_f)=0$, in view of (10). Given that $u(t)=0$ for $t\ge t_f$ and the disturbances vanish at $t=t_f$, in view of the condition (5), the system states remain equal to zero, $x_1(t)=x_2(t)=0$, for $\forall t\ge t_f$. ■

Remark 1: The system state $x(t)$ remains at the origin after reaching it at $t=t_f$, since the disturbances vanish at $t=t_f$, in view of the condition (3). If the disturbances are not vanishing, an additional control input is required to maintain the system state at the equilibrium (see, e.g., [10], [16]).

Simulations: Consider the following autonomous nonlinear two-dimensional system [15]:

$$\begin{cases}\dot x_1=x_2+x_1\sin(x_2^2)\\ \dot x_2=u+x_1^{\frac{2}{3}}x_2^{\frac{1}{3}}.\end{cases} \quad (21)$$

In this case, $|\phi_1(x,u)|=|x_1\sin(x_2^2)|\le|x_1|$, then $c_{11}=1$ and $c_{12}=0$. On the other hand, $|\phi_2(x,u)|=|x_1x_1x_2|^{\frac{1}{3}}\le(|x_1|+|x_1|+|x_2|)/3=(2/3)|x_1|+(1/3)|x_2|$, thus, $c_{21}=2/3$ and $c_{22}=1/3$. The control input is defined by (6). The convergence time is set to $t_f=1$ and the control gains are assigned as $\eta_1=5$, $\eta_2=3$, $\rho_{12}=0$, and $\rho_{21}=0.7$. Following [15], the initial condition is selected as $x_0=[5,50]$. Fig. 1 shows the time histories of the system states (21) for the control law (6), whereas Fig. 2 shows the time histories of the system states (21) for the control law proposed in Theorem 1 of [15] with the same t_f and x_0. It can be observed that the magnitude of the control input (13) is less than that of the control input proposed in [15]. Note that the control law (6) is smooth at $t=t_f$. In addition, the computation of the gains of the control law (6) is more straightforward than those of the control law proposed in [15].

Control design:

General-order system: To design a smooth predefined-time stabilizing control law for an n-dimensional autonomous nonlinear system, there exist two ways: First, the control gains can be calculated recursively at each step (similarly to the considered two-dimensional case), using the Lyapunov stability approach, where the Lyapunov function at the i-th step is defined as $V_i=V_{i-1}+(1/2)z_i^2$.

Second, the Routh-Hurwitz stability criterion can be employed as follows. At the i-th step, the desired virtual control input can be represented as

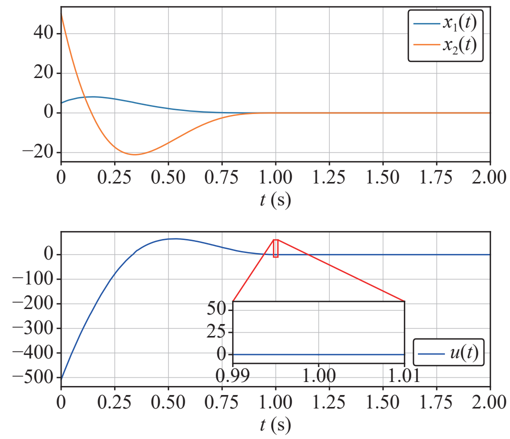

Fig. 1. Time histories of the states (21) and the control law (13) for the second-order system.

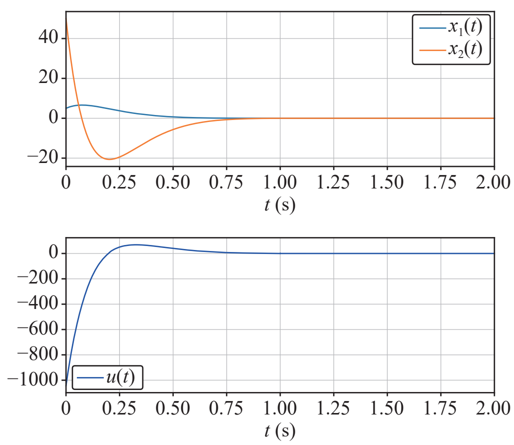

Fig. 2. Time histories of the state (21) and the control law proposed in [15] for the second-order system.

$$x_{id} = -\psi_i - \sum_{i,j=1}^{n} \rho_{ij} x_j + \zeta_i \tag{22}$$

for $i = 1,\ldots,n$, where ζ_i denotes all backstepping-induced terms accumulated from the previous steps and

$$\psi_i = \eta_i \frac{z_i(t)}{(t_f - t)}. \tag{23}$$

Assuming that the control gains satisfy the conditions

$$\eta_i > c_{ii}(t_f - t) + 1$$
$$\rho_{ij} > c_{ij}$$

for $i, j = 1,\ldots,n$, the predefined-time convergence of the closed-loop system to the origin would follow, if the dynamic matrix of the linear part of the system

$$A = \begin{pmatrix} -\eta_1 & -\rho_{12} & -\cdots & -\rho_{1n} \\ -\rho_{21} & -\eta_2 & \cdots & -\rho_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -\rho_{n1} & \cdots & \cdots & -\eta_n \end{pmatrix} \tag{24}$$

is Hurwitz. Note that the conditions obtained in the paragraph after (16) in the proof of Theorem 1 are more relaxed than the conditions given in the statement of Theorem 1. However, they are intentionally strengthened to unify them with the conditions induced by the fact that the matrix A in (24) is Hurwitz.

Conclusions: This letter has presented smooth predefined-time convergent backstepping-based control laws for autonomous nonlinear systems with linearly growing nonlinearities in two-dimensional and n-dimensional cases. In the two-dimensional case, the control input is obtained explicitly. The performance of the developed algorithm is verified via numerical simulations, which show the reliable predefined-time convergence to the origin. and demonstrate its advantages with respect to some existing control laws.

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