Letter

Norm-Based Adaptive Coefficient ZNN for Solving the Time-Dependent Algebraic Riccati Equation

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Dear Editor,

The time-dependent algebraic Riccati equation (TDARE) problem is applied to many optimal control industrial applications. It is susceptible to interference from measurement noises in the virtual environment, which current methods cannot effectively address. A norm-based adaptive coefficient zeroing neural network (NACZNN) model to solve the TDARE problem is proposed, with an adaptive scale coefficient based on the residual error norm to accelerate convergence speed to the theoretical solution. Momentum enhancement terms enable NACZNN to effectively solve the TDARE problem in real time when perturbed by measurement noise. Simulation experiments were designed and executed, and results confirm the NACZNN model's superior robustness and accuracy when solving the TDARE problem disturbed by noises in real time.

Introduction: The algebraic Riccati equation (ARE) is an essential branch of optimal control theory. It must be accurately solved in many applications [1], [2], which often requires the implicit positive or negative definite solution. For instance, the Lyapunov function approach to solve the matrix form differential Riccati equation in linear-quadratic optimal control requires the negative definite solution [3]. Consequently, the solution of the ARE problem has received much attention, and many models are proposed [4]. For example Korayem *et al.* present a controller with enhanced dynamic load performance based on state-dependent ARE, and apply it to the cooperative manipulators task. Nonetheless, traditional methods usually discretize the TDARE problem and transform it to a static problem. While current methods are effective for conventional ARE problems, computational shortcomings make them challenging in some cases. Considering that traditional methods are prone with low solution accuracy or system collapse due to lagging error when addressing the time-dependent problem [5]–[7]. To overcome these shortcomings, the derivative information in the TDARE problem is fully utilized in the ZNN model to effectively overcome the lagging error and improve the solution accuracy [8]–[10]. The ZNN model can achieve high accuracy in the aim problem, but noise interference can lead to collapse. Therefore, Jin *et al.* [11] present a gradient-based differential neural-solution to address the above issues, which possess higher accuracy for eliminating the solution residual with superior convergence. Then, Li *et al.* [12] design a strictly predefined-time convergent neural network (PTCZNN) model to accelerate convergence. It is worth noting that the learning rate of the above methods must be carefully adjusted [13]. Current models may not fully use the residual error information in the solution system, leading to truncation errors or collapse [14]. Consequently, this paper proposes a NACZNN model that can make full use of residual information. The model can maintains superior robustness and solution accuracy under the interference of measurement noises while accelerating the con-

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vergence of the solution. The contributions of this paper are summarized as follows:

1) A norm-based adaptive coefficient design framework is proposed. Compared with state-of-the-art methods, this method can accelerate convergence and enhance the model robustness.

2) NACZNN is proposed to solve the TDARE problem perturbed by noises, presenting a new technique to combine adaptive control with ZNN models.

3) The theoretical analyses and conclusions are presented from the perspective of stability theory, investigating the global convergence and robustness of the NACZNN model.

Preliminaries and scheme formulation: The TDARE problem [8] can be described as

$$A^T(t)X(t) + X(t)A(t) - X(t)M(t)X(t) + H(t) = 0 \qquad (1)$$

where the superscript^T denotes the transpose of a matrix or vector; and time-dependent matrices $A(t)$, $X(t)$, $M(t)$, and $H(t)$ are in $\mathbb{R}^{n \times n}$ with $M(t) = M^T(t)$, $H(t) = H^T(t)$. Monitoring and solving for the time-dependent matrix $X(t)$ involves the error function $E(t) = A^T(t)X(t) + X(t)A(t) - X(t)M(t)X(t) + H(t)$. Then, according to the definition of the original ZNN (OZNN) solution framework [13], the evolution direction of $E(t)$ should follow $\dot{E}(t) = -\gamma\Psi(E(t))$, where the scale coefficient $\gamma > 0$. Therefore, error function $E(t)$ can be expanded and rearranged as

$$A^T(t)\dot{X}(t) + \dot{X}(t)A(t) - \dot{X}(t)M(t)X(t) - X(t)M(t)\dot{X}(t)$$
$$= X(t)\dot{M}(t)X(t) - \dot{A}^T(t)X(t) - X(t)\dot{A}(t) - \dot{H}(t)$$
$$- \gamma\Psi(A^T(t)X(t) + X(t)A(t) - X(t)M(t)X(t) + H(t))$$

where the mapping function $\Psi(\cdot): \mathbb{R}^{n \times n} \to \mathbb{R}^{n \times n}$ represents the activation function, which is generally constructed as a monotonically increasing odd function. On top of that, these matrices are constructed to further formulate the model: $B(t) = I \otimes (A^T(t) - X(t)M(t)) \in \mathbb{R}^{n^2 \times n^2}$, $C(t) = (A(t) - M(t)X(t))^T \otimes I \in \mathbb{R}^{n^2 \times n^2}$, $D(t) = I \otimes (X(t)\dot{M}(t) - \dot{A}^T(t)) \in \mathbb{R}^{n^2 \times n^2}$, $G(t) = \dot{A}^T(t) \otimes I \in \mathbb{R}^{n^2 \times n^2}$, $J(t) = A^T(t) \otimes I \in \mathbb{R}^{n^2 \times n^2}$, where the symbol $\otimes$ denotes the Kronecker product. We define vectors $\boldsymbol{x}(t) = \text{vec}(X(t)) \in \mathbb{R}^{n^2}$ and $\boldsymbol{k}(t) = \text{vec}(H(t)) \in \mathbb{R}^{n^2}$, where $\text{vec}(\cdot)$ represents the vectorization of a matrix. Therefore, the OZNN model to solve the TDARE problem (1) can be written as

$$(B(t) + C(t))\dot{\boldsymbol{x}}(t) = (D(t) - G(t))\boldsymbol{x}(t) - \boldsymbol{k}(t)$$
$$- \gamma\Psi\big((B(t) + J(t))\boldsymbol{x}(t) + \boldsymbol{k}(t)\big). \qquad (2)$$

NACZNN model construction: The NACZNN model adds a momentum term to overcome the lack of OZNN to eliminate noise influence. Then, we propose an adaptive coefficient based on the residual error to accelerate convergence speed. The evolution function for implementing the NACZNN model is defined as $\dot{E}(t) = -\omega(E(t))E(t) - \mu\int_0^t E(\tau)d\tau$, where $\omega(E(t)) > 0$ denotes the adaptive scale coefficient, and $\mu > 0$. We design following method for realizing mapping function $\omega(E(t)) = \|E(t)\|_F^{\eta} + \zeta$, where $\eta > 0$, $\zeta > 1$, and $\|\cdot\|_F$ denotes the Frobenius norm. Therefore, the NACZNN model (3) to solve the TDARE problem (1) is defined as

$$(B(t) + C(t))\dot{\boldsymbol{x}}(t) = (D(t) - G(t))\boldsymbol{x}(t) - \boldsymbol{k}(t)$$
$$- \omega(E(t))\big((B(t) + J(t))\boldsymbol{x}(t) + \boldsymbol{k}(t)\big)$$
$$- \mu\int_0^t \big((B(\tau) + J(\tau))\boldsymbol{x}(\tau) + \boldsymbol{k}(\tau)\big)d\tau. \qquad (3)$$

The properties of the different methods are shown in Table 1. It can be concluded that NACZNN (3) uses a novel adaptive scale coefficient to accelerate convergence, and maintains superior robustness under noise interference due to the full use of the momentum information of the solution system. In addition, because NACZNN (3) takes into account the derivative term of the solution system, the TDARE problem (1) can be solved without being affected by lagging error.

Table 1. Comparisons Among Different Neural Network Models For Solving the TDARE Problem (1)

Model	Derivative information involved	Integral information involved	Applied adaptive parameter	Anti noises
OZNN [8]	√	×	×	×
RNINN [15]	×	√	×	√
NCZNN [16]	√	×	×	×
PTCZNN [12]	√	×	×	×
NACZNN (3)	√	√	√	√

Convergence of NACZNN model: The following theorem is formulated to verify the global convergence of the NACZNN model (3).

Theorem 1: The solution generated by the NACZNN model (3) globally converges to the theoretical solution of TDARE problem (1) from any random initialization state.

Proof: The ijth subsystem of NACZNN model (3) is described as $\dot{E}_{ij}(t) = -\omega(E(t))E_{ij}(t) - \mu\int_0^t E_{ij}(\tau)d\tau$ where $i \in 1, 2, ..., n$, $j \in 1, 2, ..., n$, and $E_{ij}(t)$ represents the ijth element of $E(t)$. The following Lyapunov candidate function is formulated to further investigate the convergence:

$$l(t) = \left(E_{ij}^2(t) + \mu\left(\int_0^t E_{ij}(\tau)d\tau\right)^2\right)/2 \geq 0 \qquad (4)$$

which indicates the Lyapunov function candidate $l(t) > 0$ when $E_{ij}(t) \neq 0$ or $\int_0^t E_{ij}(\tau)d\tau \neq 0$. If and only if $E_{ij}(t) = \int_0^t E_{ij}(\tau)d\tau = 0$ for $l(t) = 0$. Thus, the Lyapunov function candidate $l(t)$ is positive definite. Taking the time derivative of function (4)

$$\frac{dl(t)}{dt} = E_{ij}(t)\dot{E}_{ij}(t) + \mu E_{ij}(t)\int_0^t E_{ij}(\tau)d\tau$$

$$= E_{ij}(t)\left(\dot{E}_{ij}(t) + \mu\int_0^t E_{ij}(\tau)d\tau\right) = -\omega(E(t))E_{ij}^2(t) \leq 0.$$

That is, the Lyapunov function candidate $\dot{l}(t)$ is negative definite. Thus, according to Lyapunov theory, $E_{ij}(t)$ will ultimately converge to zero. It can be concluded that $E_{ij}(t)$ globally converges to zero. In summary, the error function $E(t)$ globally converges to zero over time. In other words, NACZNN (3) globally converges to the theoretical solution of the TDARE problem (1). ∎

Robustness of NACZNN model under different noises: We analyze the robustness of NACZNN (3) under different kinds of noise perturbation. The following two theorems are used to analyze the robustness of NACZNN (3) with conditions perturbed by constant and time-varying noise.

Theorem 2: The NACZNN model (3) perturbed by constant noise $\Theta(t) = \Theta \in \mathbb{R}^{n \times n}$ will converge to the solution of TDARE problem (1). That is to say, the solution generated by NACZNN model (3) globally converges to the theoretical solution of problem (1).

Proof: With constant noise, the noise-perturbed term can be written as $\Theta(t) = \Theta \in \mathbb{R}^{n \times n}$ and it ijth subelement is $\Theta_{ij} \in \mathbb{R}$. According to the definition of the Laplace transformation [15], the ijth subelement of NACZNN model (3) can be written as

$$sE_{ij}(s) - E_{ij}(0) = -\omega(E(s))E_{ij}(s) - \frac{\mu}{s}E_{ij}(s) + \Theta_{ij}. \qquad (5)$$

We rearrange (5) to obtain

$$E_{ij}(s) = \frac{s(E_{ij}(0) + \Theta_{ij})}{s^2 + s\omega(E(s)) + \mu}. \qquad (6)$$

Considering that $\lim_{t\to\infty}\omega(E(t)) = \lim_{s\to 0}\omega(E(s)) = \varrho > 0$, when $t \to \infty$, the poles of (6) are $s_1 = (-\varrho + \sqrt{\varrho^2 - 4\mu})/2$ and $s_2 = (-\varrho - \sqrt{\varrho^2 - 4\mu})/2$. Because $\varrho > 0$ and $\mu > 0$, it can be concluded that the two poles of function (6) locate on the left half-plane. Therefore, this system is stable, and the final value theorem applies to it, gives

$$\lim_{t\to\infty}E_{ij}(t) = \lim_{s\to 0}sE_{ij}(s) = \lim_{s\to 0}\frac{s(E_{ij}(0) + \Theta_{ij}/s)}{s^2 + s\varrho + \mu} = 0.$$

We can conclude that $\lim_{t\to\infty}\|E(t)\|_F = 0$. ∎

Theorem 3: The residual error of the NACZNN model (3) perturbed by time-varying noise $\Theta(t) = \Theta t \in \mathbb{R}^{n \times n}$ will eventually converge to a certain range. More precisely, the upper bound of residual error is $\|\Theta\|_F/\mu$. It is noteworthy that $\lim_{t\to\infty}\|E(t)\|_F = 0$ as $\mu \to +\infty$.

Proof: According to the definition of the Laplace transformation, the ith subelement of the NACZNN model (3) can be written as

$$sE_{ij}(s) - E_{ij}(0) = -\omega(E(s))E_{ij}(s) - \frac{\mu}{s}E_{ij}(s) + \frac{\Theta_{ij}}{s^2} \qquad (7)$$

where Θ_{ij}/s^2 denotes the Laplace transformation of the time-varying noise $\Theta_{ij}t$ term. Considering that $\lim_{t\to\infty}\omega(E(t)) = \lim_{s\to 0}\omega(E(s)) = \varrho > 0$, (7) is written as $E_{ij}(s) = s(E_{ij}(0) + \Theta_{ij})/(s^2 + s\varrho + \mu)$. Considering the definition of the Laplace final value theorem and Theorem 2, we have $\lim_{t\to\infty}E_{ij}(t) = |\Theta_{ij}|/\mu$. Furthermore, it is concluded when $\mu \to +\infty$ for $\lim_{t\to\infty}\|E(t)\|_F = 0$. ∎

Simulations: We performed simulations to evaluate the effectiveness of NACZNN (3) to solve the TDARE problem (1), using MATLAB R2018a with an Intel Core i5-8300H CPU at 2.30 GHz, and Windows 10 operating system. We test our method on the following TDARE problem (1) example which is constructed as follows:

$$A(t) = \begin{bmatrix} 5 + \sin(t) & \cos(t) \\ -\cos(t) & 5 + \sin(t) \end{bmatrix}$$

$$M(t) = \begin{bmatrix} \delta_1(t) & 0 \\ 0 & \delta_2(t) \end{bmatrix}, \quad H(t) = \begin{bmatrix} \theta_1(t) & 0 \\ 0 & \theta_2(t) \end{bmatrix}$$

where time-dependent parameters are provided as

$$\delta_1(t) = (4 + e^{-t} - \cos(t))^2, \quad \delta_2(t) = \left(2 + \frac{1}{t+1} + \sin(t)\right)^2$$

$$\theta_1(t) = 3 + \left(\frac{1}{t+1}\right)^2 + \sin(t), \quad \theta_2(t) = 6 + (2 + e^{-t})^2 - \cos(t).$$

The NACZNN model (3) is compared with other competitive methods including OZNN [8], NCZNN [16], and PTCZNN [12] models. For comparison, the adaptive scale coefficient for realizing the NACZNN model (3) is designed as $\omega(E(t)) = \|E(t)\|_F^5 + 5$. Besides, the MATLAB routine ode45 is adopted for implementing these models.

In this experiment, the NACZNN model (3) and compared methods are adopted to solve the TDARE problem (1) in an ideal environment. Specifically, the corresponding visual comparative results are reported in Fig. 1. It can be found that the NACZNN model (3) achieves the highest solution accuracy, which reaches order 10^{-6} and has no significant increase in convergence time. Unfortunately, OZNN [8], NCZNN [16], and PTCZNN [12] models are shown to converge to only order 10^{-4}. Because NACZNN model (3) employed the norm-based adaptive coefficient $\omega(\cdot)$, there is no need to repeatedly adjust the convergence coefficients to achieve the desired experimental results, as with other models. It is noted that t (s) in the figures denotes time in seconds. In addition, Fig. 2 shows the residual error $\|E(t)\|_F$ and its logarithm generated by the NACZNN model (3) under different values of η and ζ, which is adopted for further investigating the influence of parameter selection. It can be observed from the Fig. 2 that the ratio of parameter η to parameter ζ is 1:2, the NACZNN model (3) converges to zero faster. From the Fig. 2, when the parameter η remains constant at the value of 5, the convergence speed of the NACZNN model (3) increases with the increase of ζ.

Fig. 3(a) shows the comparative experimental results of the simulated global convergence and robustness among the different models when solving the TDARE problem under the interference environment of constant noise $\Theta(t) = \Theta = [2]^{4 \times 1}$. As shown in the Fig. 3(b), the residual error $\|E(t)\|_F$ of the NACZNN model (3) converges sharply to the order of 10^{-4} in 5 seconds. In contrast, the the residual errors $\|E(t)\|_F$ of PTCZNN, NCZNN and OZNN with parameter $\gamma = 5$ remain relatively high, on the order of 10^0. Then, the simulation experiments of the robustness of the NACZNN model (3) are carried out under the interference of time-varying noise $\Theta(t) = \Theta t = t \times [2]^{4 \times 1}$. As shown in Fig. 3(b), the NACZNN model (3) still maintains high accuracy disturbed by time-varying noise, on the order of 10^{-2} and 10^{-1}. However, the residual error $\|E(t)\|_F$ of compared
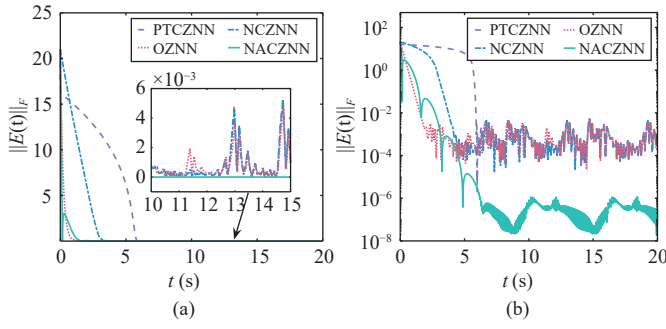
Fig. 1. Experimental results of simulations. (a) Comparsion of residual error $\|E(t)\|_F$ between different models with noise-free case; (b) The logarithm of residual error $\|E(t)\|_F$.
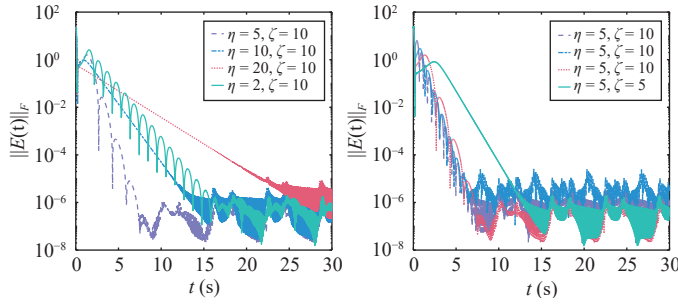

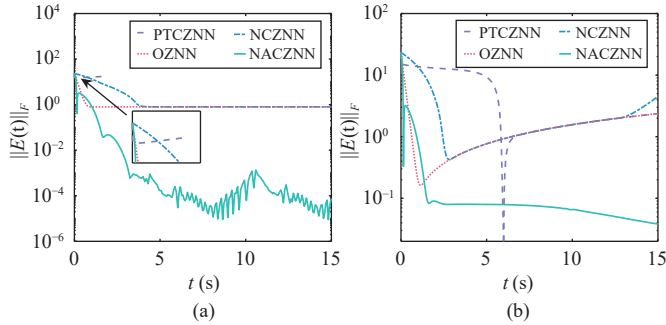Fig. 2. Comparison among different paramter adopted for the NACZNN model (3).

Fig. 3. Comparing the robustness of different models. (a) Residual error $\|E(t)\|_F$ of models perturbed by constant noises $\Theta = [2]^{4\times1}$; (b) Residual error $\|E(t)\|_F$ of models perturbed by time-varying noises $\Theta t = t \times [2]^{4\times1}$.

models are on the order of 10^{-1} and increase over time. Therefore, it is obvious that the NACZNN model (3) shown superior robustness in the presence of various noises.

Conclusion: We have proposed NACZNN (3) to solve the TDARE problem (1). NACZNN adds a integral term and a norm-based adaptive coefficient based on the original ZNN. It not only has significantly higher accuracy than other commonly used models, but maintains global convergence and robustness under constant and time-varying noise conditions. Simulations have showed the feasibility and superiority of the NACZNN model. It is worth mentioning that this is the first proposal of a model to solve the TDARE problem with stronger robustness to various kinds of noises. These are major breakthroughs in both the ZNN field and the research of dynamic problem solutions. Our future work will applied the NACZNN to the complex-valued problem and further investigate the potential of the proposed adaptive coefficients for achieving better convergence and robustness. Besides, we will also look for practical

applications with time-varying or complex-valued characteristics such as localization system, filter design, algorithmic platforms for UAVs, etc.

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