Adaptive Uniform Performance Control of Strict-Feedback Nonlinear Systems With Time-Varying Control Gain

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***Abstract*—In this paper, we present a novel adaptive performance control approach for strict-feedback nonparametric systems with unknown time-varying control coefficients, which mainly includes the following steps. Firstly, by introducing several key transformation functions and selecting the initial value of the time-varying scaling function, the symmetric prescribed performance with global and semi-global properties can be handled uniformly, without the need for control re-design. Secondly, to handle the problem of unknown time-varying control coefficient with an unknown sign, we propose an enhanced Nussbaum function (ENF) bearing some unique properties and characteristics, with which the complex stability analysis based on specific Nussbaum functions as commonly used is no longer required. Thirdly, by utilizing the core-function information technique, the nonparametric uncertainties in the system are gracefully handled so that no approximator is required. Furthermore, simulation results verify the effectiveness and benefits of the approach.**

***Index Terms*—Adaptive control, enhanced Nussbaum function (ENF), strict-feedback systems, unified prescribed performance.**

I. Introduction

IN practice, it is not difficult to design a proper control scheme such that all signals in the closed-loop systems are bounded in the presence of parametric/nonparametric uncertainties [1]−[4]. However, the control problem becomes rather challenging if the sign of control coefficient is unknown. The first result was proposed in [5], where an adaptive control law using the so-called Nussbaum-type gain was designed. Motivated by such a technique, remarkable progresses were achieved by designing various adaptive control schemes so that the problem of unknown control direction with constant coefficient is solved ([6]−[8], to just name a few).

To deal with the case of unknown sign of control coefficient with time-varying yet unknown magnitude, a developed Nussbaum-based lemma was presented in [9]. Its fundamental idea is to establish a Nussbaum function-based inequality such that the Lyapunov-like function is upper bounded by a Nussbaum function based manner. Due to the great success of such an approach, many results have been developed for handling unknown control directions with time-varying control coefficients [10]−[12]. However, the stability proofs given in the above papers critically rely on some specific form of Nussbaum functions, which dramatically increases the complexity of stability analysis. To address such kind of issues, the work in [13] presented a general Nussbaum-gain-based lemma by developing some additional properties of Nussbaum functions and revealing its fundamental characteristics. This allows more types of Nussbaum functions to be employed to handle the problem of unknown control directions [14]−[17] and thus reduce the complexity of involved. Even so, it is still difficult to use such a lemma for strict-feedback nonlinear systems in the presence of unknown control directions. To our best knowledge, the main challenge is that there is no constructive guidance for us to design an adaptive law for Nussbaum functions so that it satisfies the precondition of the lemma in [13]. Therefore, motivated by the above discussion, how to design an adaptive law for Nussbaum functions and then use this lemma to solve the control direction problem of strict-feedback nonlinear systems denotes an interesting issue.

Furthermore, tracking a known reference with pre-specified performance is of great importance in the practical applications [18]. Up to now, some effective control methods on improving tracking performance have been proposed, see for example, prescribed performance bound (PPB)-based control [19] and funnel control (FC) [20]. By introducing an error transformation, the PPB controls ensure that the error/system state converges to a pre-given set within a pre-specified convergence rate. However, the corresponding results are essentially semi-global since they require that the bound informa-

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tion on the initial condition of the system must be available for control design, otherwise it is impossible to ensure the desired performance specifications [11], [12], [21], [22]. Fortunately, by constructing a class of performance funnels, the funnel control proposed in [20] was able to relax the limit on initial conditions of PPB control. Inspired by such an idea, there are some results developed for various kinds of nonlinear systems [23], [24]. Recently, by constructing a global performance function, an adaptive backstepping control developed in [25] was able to ensure the asymptotic tracking with transient performance. However, it is difficult to extend the method in [25] to non-parametric systems with time-varying control coefficients. The main difficulties/challenges come from the following three aspects. The first is that the tuning-function-based adaptive backstepping control cannot handle the problem of non-parametric uncertainty; The second challenge is the original lemma in [25] is invalid if the control gain is time-varying and unknown; The last but not the least is that the proposed method in [25] cannot unify the global and semi-global results without changing the control framework (see the detailed discussion in Remark 4). Thus, it is meaningful to develop new techniques to tackle the issue of uniform prescribed performance for non-parametric nonlinear systems with time-varying control coefficients.

In this paper, based on the parameter estimation technique, we aim to solve the uniform prescribed performance tracking problem for strict-feedback nonlinear systems with nonparametric uncertainties and unknown sign of control gain having time-varying magnitude. The main contributions of this article can be summarized as:

1) Different from the PPB-based controls [11], [12], [19], [21], [22], by constructing a unified function and a unique scaling function, the proposed control is flexible to handle the global or symmetric semi-global performance cases uniformly just by selecting the initial value of the time-varying scaling function properly, making the controller re-design and stability re-analysis not required;

2) Different from the specific form of Nussbaum functions in the existing literature [9]–[12], in this paper, by defining an enhanced Nussbaum function (ENF) and imposing a condition on the update law of Nussbaum argument, the developed control relaxes the complicated calculation and proof in the existing results; and

3) By extracting the core function information from the non-parametric uncertainty, no approximator (such as neural networks and fuzzy logic systems) is required, despite unknown control directions.

The remainder of the paper is organized as follows. Section II is the problem formulation. Some important functions for converting the required performance and the corresponding system transformations are introduced in Sections III-A and III-B, respectively. Section III-D presents the control design as well as the stability analysis. Simulation studies are shown in Section IV to verify the theoretical result. The paper is concluded in Section V.

II. Problem Formulation

In this paper, we consider the following nonparametric strict-feedback nonlinear system with arbitrarily given relative degree n:

$$\begin{cases} \dot{x}_j = f_j\left(\overline{x}_j, p_j\right) + g_j(t)\, x_{j+1},\ j = 1,\ldots,n-1 \\ \dot{x}_n = f_n(\overline{x}_n, p_n) + g_n(t)\, u(t) \\ y = x_1 \end{cases} \tag{1}$$

where $x_i \in \mathbb{R}$, $i = 1,\ldots,n$, is the system state with $\overline{x}_i = [x_1,\ldots,x_i]^T \in \mathbb{R}^i$, $y \in \mathbb{R}$ is the system output, and $u \in \mathbb{R}$ is a control signal, $p_i \in \mathbb{R}^{q_i}$ represents an unknown parameter vector, $f_i : \mathbb{R}^i \times \mathbb{R}^{q_i} \to \mathbb{R}$ denotes a smooth function, which contains the nonparametric uncertainty, $g_i(t) = b_i\omega_i(t)$ is a time-varying control gain/coefficient, where $b_i = 1$ or -1 represents the control direction and $\omega_i(t) > 0 : [0,\infty) \to \mathbb{R}$ denotes the magnitude, both of which are unavailable for control design[1].

Let $e(t) = y(t) - y_d(t)$ be the tracking error with $y_d(t)$ denoting the desired signal. The control objective of this paper is to develop an adaptive control algorithm for (1) so that:

O_1. All the closed-loop signals are bounded; and

O_2. The uniform prescribed performance of tracking error can be ensured for $t \geq 0$.

Assumption 1: The desired signal and its derivatives up to n-th are bounded, known, and piecewise continuous.

Assumption 2: There exist some unknown positive constants $\underline{\omega}_i$ and $\overline{\omega}_i$ so that $0 < \underline{\omega}_i \leq \omega_i(\cdot) < \overline{\omega}_i < \infty$. In addition, the sign of b_i is fixed, but unknown.

Assumption 3: There exist an unknown positive constant a_i, $i = 1,\ldots,n$, and an available core function $\kappa_i(\overline{x}_i)$ so that $|f_i(\overline{x}_i, p_i)| \leq a_i\kappa_i(\overline{x}_i)$, $\forall t \geq 0$.

III. Main Results

A. Performance Transformation

With respect to the second goal on prescribed tracking performance, we introduce the concept of unified function.

- Unified Function (UF).

Definition 1: A real composite function $\mathcal{F}(\phi) \in \mathbb{R}$ with $\phi \in [-1,1]$ is called a unified function if it satisfies:

1) $\mathcal{F}(1) = \infty$, $\mathcal{F}(0) = 0$, and $\mathcal{F}(-1) = -\infty$;

2) $\mathcal{F}(\phi) + \mathcal{F}(-\phi) = 0$;

3) $\mathcal{F}(\phi) : (-1,1) \to (-\infty,\infty)$ is continuously differentiable with respect to (w.r.t.) ϕ; and

4) $\mu = \frac{\partial \mathcal{F}}{\partial \phi}(\phi) : (-1,1) \to \left[\underline{\mu},\infty\right)$ with $\underline{\mu} > 0$ being a constant and $\mu \to \infty$ if and only if $\phi \to \pm 1$.

To ensure that the tracking error is within the pre-given region for $\forall t \geq 0$, we impose a unique time-varying scaling function φ to replace ϕ in UF $\mathcal{F}$, which has the following features:

- $\varphi^{(k)}$, $k = 0,1,\ldots,n$, is bounded, known, and piecewise continuous; and
- $\varphi(t)$ with $\varphi(0) = \varphi_0$ is strictly monotonically decreasing w.r.t. time and $\lim_{t\to\infty}\varphi(t) = \varphi_f$ with $0 < \varphi_f < \varphi_0 \leq 1$ being some constants, namely, $\varphi(t) : [0,\infty) \to (\varphi_f, \varphi_0] \subsetneq [-1,1]$.

[1] $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}_+$ is the set of positive real numbers, and $\mathbb{R}^n$ represents the set of n−dimensional real vectors. Let $|\bullet|$ be the absolute value of a real number $\bullet$.

According to the definition of $\mathcal{F}$ and the properties of φ, it is easy to get the conclusion that:

1) $\mathcal{F}(\varphi(t))$ is strictly decreasing w.r.t. time, namely, $\mathcal{F}(\varphi(0)) = \mathcal{F}(\varphi_0) = b_0 > 0$ and $\lim_{t\to\infty} \mathcal{F}(\varphi(t)) = \mathcal{F}(\varphi_f) = b_f > 0$ with $0 < b_f < b_0$ denoting some constants; and

2) $\mathcal{F}(\varphi) : (\varphi_f, \varphi_0) \to (b_f, b_0)$ is continuously differentiable w.r.t. φ.

Obviously, there are many forms of functions satisfying the properties of unified function, for example, $\mathcal{F}(\varphi) = \frac{\varphi}{\sqrt{1-\varphi^2}}$ and $\mathcal{F}(\varphi) = \tan(\frac{\pi}{2}\varphi)$ with $\varphi = (\varphi_0 - \varphi_f)\exp(-t) + \varphi_f$.

By utilizing the time-varying scaling function φ and the properties of UF $\mathcal{F}$ in Definition 1, the problem of prescribed tracking performance is stated mathematically equivalent to

$$\mathcal{F}(-\varphi) < e < \mathcal{F}(\varphi) \tag{2}$$

with the initial value $e(0) = e_0$ satisfying $\mathcal{F}(-\varphi_0) = -b_0 < e_0 < \mathcal{F}(\varphi_0) = b_0$. In other words, the objective O_2 on the prescribed performance is converted into guaranteeing (2).

Remark 1: It is worth mentioning that selecting different initial values of time-varying scaling function φ, φ_0, may lead to different results:

Case 1: If $\varphi_0 = 1$, with the property of $\mathcal{F}$ in Definition 1, it follows that $\mathcal{F}(1) = b_0 = \infty$, which implies that the constraint on the initial error is vacuous, then the result is global;

Case 2: If $\varphi_0 < 1$, it is seen that $\mathcal{F}(\varphi_0) = b_0 < \infty$ with b_0 being a positive yet bounded constant, then the initial value has to satisfy $|e(0)| < \mathcal{F}(\varphi_0)$ and so the result is of a semi-global nature, which is similar to those in [11], [12], [19], [22].

Therefore, the developed control without changing controller structure gives a unified framework to achieve the global and semi-global prescribed tracking performance uniformly by only choosing different value of φ_0.

- Normalized Function (NF).

Actually, by noting that the form of (2) belongs to the error constraint problem, it is difficult and challenging to handle such an issue directly by utilizing the adaptive algorithm. In the following, we give a normalized function η w.r.t. the tracking error e to simplify the design difficulty of maintaining error constraint, i.e.,

$$\eta = \mathcal{F}^{-1}(e) \tag{3}$$

which is equivalent to

$$e = \mathcal{F}(\eta). \tag{4}$$

It must be emphasized that, according to the properties of UF, it shows that:

- $\eta \in (-1, 1)$ for $\forall e \in (-\infty, \infty)$;
- $\eta \to \pm 1$ as $e \to \pm\infty$;
- $\beta(e) = \frac{\partial \eta}{\partial e} \in (0, \overline{\beta}]$ is well defined for $\forall e \in (-\infty, \infty)$, where $\overline{\beta} > 0$ is a constant;
- $\beta(e) \to 0$ as $e \to \pm\infty$; and
- $\lim_{\eta\to 0} \mathcal{F}(\eta) = e = 0$.

Therefore, it follows from (4) that the performance given in (2) can be rewritten as:

$$\mathcal{F}(-\varphi) < \mathcal{F}(\eta) < \mathcal{F}(\varphi). \tag{5}$$

Since $\mathcal{F}(\cdot)$ is monotonic, the problem of unified prescribed performance in (2) (or (5)) can be further converted into

$$-\varphi < \eta < \varphi. \tag{6}$$

- Auxiliary Function (AF).

Let

$$\zeta = \eta/\varphi \tag{7}$$

be the so-called auxiliary function. As $\varphi > 0$, (6) becomes

$$-1 < \zeta = \eta/\varphi < 1, \text{ or } |\zeta| < 1 \tag{8}$$

and the problem of prescribed performance in (2) (or (5) or (6)) is further converted into guaranteeing (8).

Let ζ_0 and η_0 be the initial values of ζ and η at $t = 0$, respectively. To achieve the prescribed tracking performance, the initial values of e and η must satisfy $-b_0 < e_0 < b_0$ and $-\varphi_0 < \eta_0 < \varphi_0$, which implies from (8) that $-1 < \zeta_0 < 1$. Therefore, the problem of maintaining (8) for $\forall t \geq 0$ is converted into designing an adaptive control law such that $|\zeta| < 1$ for $\forall t > 0$. Furthermore, it is seen from (8) that this inequality actually belongs to the problem of constant yet symmetric constraint. Motivated by output/state/error-constrained control schemes [27], [28], if we are able to find a new variable such that the stabilization of the new variable is the sufficient condition of ensuring (8), then the constant and symmetric constraint is naturally achieved. Therefore, to this end, we give a definition of barrier function (BF) in what follows.

- Barrier Function (BF).

Definition 2: A real composite function $s(\varsigma)$ is a barrier function if it satisfies:

1) The initial value of ς is within the interval $(-1, 1)$;

2) $s(\varsigma) : (-1, 1) \to (-\infty, \infty)$ is continuously differentiable w.r.t. ς;

3) $s = \pm\infty \Leftrightarrow \varsigma \to \pm 1$;

4) $s = 0 \Leftrightarrow \varsigma = 0$;

5) $s \in L_\infty \Rightarrow |\varsigma| < 1$; and

6) There exists a constant $\underline{\rho} > 0$ such that $|\rho = \frac{\partial s}{\partial \varsigma}| \in [\underline{\rho}, \infty)$ over the interval $\Omega = \{\varsigma \in \mathbb{R} \mid |\varsigma| < 1\}$ and $\rho \to \pm\infty$ if and only if $\varsigma \to \pm 1$.

From the above discussion it is seen that the initial value of ζ is within $(-1, 1)$. According to Definition 2, if we are able to utilize the composite function s w.r.t. ζ (i.e., $s(\zeta)$) and to develop an advanced control method such that $s(\zeta) \in L_\infty$ for $\forall t \geq 0$, together with the analysis in (2)–(8), the unified prescribed tracking performance can be guaranteed. Thus we will focus on handling the stabilization problem of BF s in the remaining part of this paper.

Remark 2: According to the definition of s, it is not difficult to see that $0 < \underline{\rho} \leq \rho < \infty$ or $-\infty < \rho \leq -\underline{\rho} < 0$, which implies that s is strictly monotonic w.r.t. ζ. As s is smooth w.r.t. ζ over the interval $(-1, 1)$, then if $s \in L_\infty$, it is easy to check that there exist some positive constants λ, $\underline{\beta}$, and $\overline{\rho}$ such that $|\zeta| \leq \lambda < 1$, $\beta \in [\underline{\beta}, \overline{\beta}]$, and $|\rho| \in [\underline{\rho}, \overline{\rho}]$.

B. System Transformation

Upon using the expression of ζ as shown in (7), with the definitions of η and φ, it is seen that

$$\frac{d\zeta}{dt} = \frac{\partial \zeta}{\partial \eta}\frac{\partial \eta}{\partial e}\dot{e} + \frac{\partial \zeta}{\partial \varphi}\dot{\varphi}. \tag{9}$$

Noting that $\frac{\partial \zeta}{\partial \eta} = \frac{1}{\varphi} > 0$, $\frac{\partial \eta}{\partial e} = \beta > 0$, and $\frac{\partial \zeta}{\partial \varphi} = -\frac{\eta}{\varphi^2}$, then (9) becomes

$$\frac{d\zeta}{dt} = \frac{1}{\varphi}\beta(\dot{x}_1 - \dot{y}_d) - \frac{\eta}{\varphi^2}\dot{\varphi}. \tag{10}$$

As $\frac{\partial s}{\partial \zeta} = \rho$, the derivative of s w.r.t. time is

$$\dot{s} = \frac{\partial s}{\partial \zeta}\dot{\zeta} = \rho\left(\frac{1}{\varphi}\beta(\dot{x}_1 - \dot{y}_d) - \frac{\eta}{\varphi^2}\dot{\varphi}\right) = r_1(\dot{x}_1 - \dot{y}_d) + r_2 \tag{11}$$

where $r_1 = \frac{\rho\beta}{\varphi}$ and $r_2 = -\frac{\rho\eta\dot{\varphi}}{\varphi^2}$. According to the properties of UF and BF, it can be checked that $r_1 > 0$ is well defined.

By replacing the equation of $\dot{x}_1$ with $\dot{s}$, (1) can be transformed to the following form:

$$\begin{cases} \dot{s} = r_1(g_1 x_2 + f_1 - \dot{y}_d) + r_2 \\ \dot{x}_k = g_k x_{k+1} + f_k,\ k = 2,\ldots,n-1 \\ \dot{x}_n = g_n u + f_n \\ y = x_1. \end{cases} \tag{12}$$

Therefore, it is obviously seen that if an adaptive control law is designed such that the barrier function s is bounded over $[0,\infty)$, then the objective O_2 on tracking performance is achieved.

C. Enhanced Nussbaum Function

For a smooth function $N(\chi): \mathbb{R} \to \mathbb{R}$, denote its positive and negative truncated functions by $N^+(\chi)$ and $N^-(\chi)$, respectively, i.e.,

$$N^+(\chi) = \max\{0, N(\chi)\},\ N^-(\chi) = \max\{0, -N(\chi)\}.$$

Obviously, the truncated functions satisfy the following properties: $N^+(\chi) \geq 0$, $N^-(\chi) \geq 0$ and $N(\chi) = N^+(\chi) - N^-(\chi)$. If a continuously differentiable function $N(\chi)$ satisfies

$$\limsup_{v\to\infty} \frac{1}{v}\left[\int_0^v N^+(\chi)d\chi - \int_0^v N^-(\chi)d\chi\right] = \infty \tag{13}$$

$$\limsup_{v\to\infty} \frac{1}{v}\left[\int_0^v N^-(\chi)d\chi - \int_0^v N^+(\chi)d\chi\right] = \infty \tag{14}$$

then it is called a Nussbaum function, which has been widely employed for coping with the problem of unknown control directions. In this paper, to handle the unknown time-varying control coefficient with unknown sign and to reduce the difficulty of stability analysis, the concept of *enhanced Nussbaum function (ENF)*, inspired by [13], is presented.

Definition 3: A continuously differentiable function $N(\chi)$ is called an ENF, if, for a constant $L > 1$, it satisfies

$$\lim_{v\to\infty} \frac{1}{v}\int_0^v N^+(\chi)d\chi = \infty; \limsup_{v\to\infty} \frac{\int_0^v N^+(\chi)d\chi}{\int_0^v N^-(\chi)d\chi} \geq L \tag{15}$$

$$\lim_{v\to\infty} \frac{1}{v}\int_0^v N^-(\chi)d\chi = \infty; \limsup_{v\to\infty} \frac{\int_0^v N^-(\chi)d\chi}{\int_0^v N^+(\chi)d\chi} \geq L. \tag{16}$$

According to the properties of Nussbaum function shown in (13) and (14), it is easy to check that the ENF also belongs to the Nussbaum function. To state the Nussbaum-based lemma later and to simplify the stability proof, we present the following lemma.

Lemma 1: Suppose a function $N(\chi)$ with $N(\chi) = N^+(\chi) - N^-(\chi)$ is an ENF, and let λ_1 and λ_2 be two real constants satisfying $\lambda_1\lambda_2 > 0$. If $\hat{L} = \min\left\{\frac{\lambda_1}{\lambda_2}, \frac{\lambda_2}{\lambda_1}\right\}L > 1$, then the function $\hat{N}(\chi) = \lambda_1 N^+(\chi) - \lambda_2 N^-(\chi)$ is also an ENF.

Proof: We only prove the case with $\lambda_1 > 0$ and $\lambda_2 > 0$, as the result for $\lambda_1, \lambda_2 < 0$ can be similarly derived. We denote $\hat{N}(\chi) = \hat{N}^+(\chi) - \hat{N}^-(\chi)$, where $\hat{N}^+(\chi) = \lambda_1 N^+(\chi)$ and $\hat{N}^-(\chi) = \lambda_2 N^-(\chi)$. Since $N(\chi)$ is an ENF, then it satisfies (15) and (16). It can be verified that $\lim_{v\to\infty} \frac{1}{v}\int_0^v \hat{N}^+(\chi)d\chi = \infty$ and $\lim_{v\to\infty} \frac{1}{v}\int_0^v \hat{N}^-(\chi)d\chi = \infty$. Furthermore, from the second property in (15), one is also able to prove that $\limsup_{v\to\infty} \frac{\int_0^v \hat{N}^+(\chi)d\chi}{\int_0^v \hat{N}^-(\chi)d\chi} = \limsup_{v\to\infty} \frac{\lambda_1}{\lambda_2}\frac{\int_0^v N^+(\chi)d\chi}{\int_0^v N^-(\chi)d\chi} \geq \frac{\lambda_1}{\lambda_2}L \geq \hat{L} > 1$. Similarly, we have $\limsup_{v\to\infty} \frac{\int_0^v \hat{N}^-(\chi)d\chi}{\int_0^v \hat{N}^+(\chi)d\chi} \geq \hat{L} > 1$. Therefore, $\hat{N}(\chi)$ is an ENF for $\lambda_1, \lambda_2 > 0$. ■

Lemma 2 [13]: Consider the continuously differentiable functions $V(t) \in \mathbb{R}_+$ and $\chi(t) \in \mathbb{R}$. Let $g(t): [0,\infty) \to \left[\underline{g}, \overline{g}\right]$ for two constants $\underline{g}$ and $\overline{g}$ satisfying $\underline{g}\overline{g} > 0$. If

$$V(t) \leq \int_0^t [g(\tau)N(\chi(\tau)) + w]\dot{\chi}(\tau)d\tau + \Lambda \tag{17}$$

$$\dot{\chi}(t) \geq 0,\ \forall t \geq 0 \tag{18}$$

for some constants w, Λ, and an ENF $N(\chi)$ with $L > \max\left\{\frac{\overline{g}}{\underline{g}}, \frac{\underline{g}}{\overline{g}}\right\}$, then $V(t)$ and $\chi(t)$ are bounded over $[0,\infty)$.

Proof: As $g(\cdot)$ has unknown but certain sign, without losing generality, we consider the case $g(\cdot) > 0$. Then, $g(\tau)N(\chi(\tau))$ can be rewritten as

$$\begin{aligned} g(\tau)N(\chi(\tau)) &= g(\tau)N^+(\chi(\tau)) - g(\tau)N^-(\chi(\tau)) \\ &\leq \overline{g}N^+(\chi(\tau)) - \underline{g}N^-(\chi(\tau)) \\ &= \hat{N}^+(\chi(\tau)) - \hat{N}^-(\chi(\tau)) = \hat{N}(\chi(\tau)) \end{aligned} \tag{19}$$

where $\hat{N}^+(\chi(\tau)) = \overline{g}N^+(\chi(\tau))$ and $\hat{N}^-(\chi(\tau)) = \underline{g}N^-(\chi(\tau))$. With the aid of Lemma 1, it is not difficult to check that $\hat{N}(\chi(\tau))$ is an ENF. Inequality (17) can also be expressed as

$$\begin{aligned} V(t) &\leq \int_0^t g(\tau)N(\chi(\tau))\dot{\chi}(\tau)d\tau + \int_0^t w\dot{\chi}(\tau)d\tau + \Lambda \\ &\leq \int_{\chi(0)}^{\chi(t)} \hat{N}(s)ds + w\chi(t) - w\chi(0) + \Lambda \\ &= \int_0^{\chi(t)} \hat{N}(s)ds - \int_0^{\chi(0)} \hat{N}(s)ds + w(\chi(t) - \chi(0)) + \Lambda. \end{aligned}$$

Denote a constant $c_0 = \int_0^{\chi(0)} \hat{N}(s)ds + w\chi(0) - \Lambda$, then one has

$$\int_0^{\chi(t)} \hat{N}(s)ds + w\chi(t) \geq c_0. \tag{20}$$

As $\hat{N}(s)$ is a Nussbaum function, together with the property of Nussbaum function and $\dot{\chi} \geq 0$, there exists $\chi^* > 1$ such that $\frac{1}{\chi^*}\int_0^{\chi^*} \hat{N}(s)ds < -|c_0| - a$. If $\chi(t)$ is not bounded over $[0,\infty)$, there exists $t^* > 0$ such that $\chi(t^*) = \chi^*$ and hence

$$\frac{1}{\chi^*}\int_0^{\chi^*}\hat{N}(s)ds < -|c_0| - a < \frac{c_0}{\chi(t^*)} - a. \tag{21}$$

As a result, $\int_0^{\chi^*}\hat{N}(s)ds + a\chi(t^*) < c_0$, this contradicts (20), which implies that $\chi(t)$ is bounded over $[0,\infty)$, so is $V(t)$. The same result can also be derived for $g < 0$. ■

Remark 3: The problem of constant control gain with unknown control directions is solved in the existing works [6], [7], [8], [25], whereas the corresponding Nussbaum lemma is no longer applicable to the case in this work as the control coefficient is time-varying, which may affect the stability analysis. Moreover, although some efforts have been made to handle the problem of time-varying case [10], [11], [12], the stability proofs must critically rely on the explicit calculation for the particularly chosen Nussbaum functions. If other forms of Nussbaum functions are employed, the corresponding complicated stability must be re-proved and re-analyzed. In this paper, as long as the employed Nussbaum function and its argument satisfy Definition 3 and (18), for the case of time-varying control coefficient, the onerous calculation process is not required, which reduces the difficulty of stability analysis.

D. Control Design

Here we present an ENF-based control for strict-feedback system (1) with unknown sign and magnitude of control coefficients. To this end, we utilize the coordinate transformation in what follows:

$$z_1 = s \tag{22}$$

$$z_j = z_j - \alpha_{j-1},\ j = 2,\ldots,n \tag{23}$$

where α_{j-1} is the virtual control law.

Before the control design, let

$$\theta_1 = \max\{1, a_1\} \tag{24}$$

$$\theta_j = \max\{1, a_1,\ldots,a_j,\overline{g}_1,\ldots,\overline{g}_{j-1}\}, j = 2,\ldots,n-1 \tag{25}$$

$$\theta_n = \max\{1, a_1,\ldots,a_n,\overline{g}_1,\ldots,\overline{g}_{n-1}\}. \tag{26}$$

Step 1: Differentiating $\frac{1}{2}z_1^2$ w.r.t. time and invoking the first equation of (12), one has

$$z_1\dot{z}_1 = z_1 r_1(f_1 + g_1 x_2 - \dot{y}_d) + z_1 r_2. \tag{27}$$

Note that $x_2 = z_2 + \alpha_1$, then (27) can be arranged as

$$z_1\dot{z}_1 = z_1 r_1 g_1 \alpha_1 + \Xi_1 \tag{28}$$

where $\Xi_1(\cdot) = z_1 r_1(f_1 - \dot{y}_d) + z_1 r_2 + z_1 r_1 g_1 z_2$ is an uncertain function.

Since the nonlinear function f_i does not satisfy the parametric composition condition, then the tuning-function based adaptive backstepping control [1] is no longer available. To solve this issue, we impose Assumption 3 in this paper for the nonparametric uncertainty so that we can obtain the deep-rooted information, making the approximator not required. Therefore, we have

$$z_1 r_1 f_1 \le |z_1||r_1|a_1\kappa_1 \tag{29}$$

$$-r_1 z_1 \dot{y}_d \le |z_1||r_1||\dot{y}_d| \tag{30}$$

$$z_1 r_2 \le |z_1||r_2| \tag{31}$$

$$g_1 r_1 z_1 z_2 \le r_1^2 z_1^2 + \frac{1}{4}\overline{g}_1^2 z_2^2. \tag{32}$$

Hence, Ξ_1 can also be upper bounded by the following form:

$$\Xi_1 \le \theta_1|z_1|\Phi_1 + r_1^2 z_1^2 + \frac{1}{4}\overline{g}_1^2 z_2^2 \tag{33}$$

where θ_1 is an unknown virtual constant as defined in (24), and

$$\Phi_1 = r_1^2\kappa_1^2 + r_1^2\dot{y}_d^2 + r_2^2 + \frac{3}{4} \tag{34}$$

is an available function.

For any positive function $\varepsilon(t): [0,\infty) \to \mathbb{R}_+$ and variable $\mathcal{Z} \in \mathbb{R}$, it holds that: $|\mathcal{Z}| \le \varepsilon(t) + \frac{|\mathcal{Z}|^2}{\sqrt{|\mathcal{Z}|^2 + \varepsilon^2(t)}}$ [14], [26], then we have

$$\theta_1|z_1|\Phi_1 \le \theta_1\varepsilon(t) + \frac{\theta_1 z_1^2\Phi_1^2}{\sqrt{z_1^2\Phi_1^2 + \varepsilon^2(t)}} \tag{35}$$

where $\varepsilon(t) \in \mathbb{R}_+$ is chosen to have the following properties: 1) $\varepsilon^{(k)}(t)$, $k = 0,1,\ldots,n-1$, is bounded, known, and piece-wise continuous; and 2) $\int_0^t \varepsilon(\tau)d\tau \le \delta < \infty$ with $\delta > 0$ being a constant. Thus, (33) becomes

$$\Xi_1 \le \theta_1\varepsilon(t) + \frac{\theta_1 z_1^2\Phi_1^2}{\sqrt{z_1^2\Phi_1^2 + \varepsilon^2(t)}} + r_1^2 z_1^2 + \frac{1}{4}\overline{g}_1^2 z_2^2 \tag{36}$$

which leads to

$$z_1\dot{z}_1 \le g_1 r_1 z_1 \alpha_1 + \theta_1\varepsilon(t) + \frac{\theta_1 z_1^2\Phi_1^2}{\sqrt{z_1^2\Phi_1^2 + \varepsilon^2(t)}} + r_1^2 z_1^2 + \frac{1}{4}\overline{g}_1^2 z_2^2. \tag{37}$$

Considering the quadratic function as $V_1 = \frac{1}{2}z_1^2 + \frac{1}{2\gamma_1}\tilde{\theta}_1^2$, where $\hat{\theta}_1$ is the estimate of θ_1, $\tilde{\theta}_1 = \theta_1 - \hat{\theta}_1$ is the estimate error, and $\gamma_1 > 0$, then the derivative of V_1 along (37) is

$$\dot{V}_1 \le g_1 r_1 z_1 \alpha_1 + \theta_1\varepsilon(t) + \frac{\theta_1 z_1^2\Phi_1^2}{\sqrt{z_1^2\Phi_1^2 + \varepsilon^2(t)}} + r_1^2 z_1^2 + \frac{1}{4}\overline{g}_1^2 z_2^2 - \frac{1}{\gamma_1}\tilde{\theta}_1\dot{\hat{\theta}}_1. \tag{38}$$

To handle the unknown sign of control gain with time-varying yet unknown magnitude, we employ the ENF to design the virtual control law α_1,

$$\alpha_1 = \frac{1}{r_1}N_1(\chi_1)\left(c_1 z_1 + r_1^2 z_1 + \frac{\hat{\theta}_1 z_1\Phi_1^2}{\sqrt{z_1^2\Phi_1^2 + \varepsilon^2(t)}}\right) \tag{39}$$

$$\dot{\chi}_1 = c_1 z_1^2 + \frac{\hat{\theta}_1 z_1^2\Phi_1^2}{\sqrt{z_1^2\Phi_1^2 + \varepsilon^2(t)}} + r_1^2 z_1^2 \tag{40}$$

$$\dot{\hat{\theta}}_1 = \gamma_1\frac{z_1^2\Phi_1^2}{\sqrt{z_1^2\Phi_1^2 + \varepsilon^2(t)}},\ \hat{\theta}_1(0) \ge 0 \tag{41}$$

where $c_1 > 0$ is a design parameter, $\hat{\theta}_1(0) \geq 0$ and $\chi_1(0)$ are the arbitrarily chosen initial values of $\hat{\theta}_1(t)$ and $\chi_1(t)$, respectively.

Substituting the first virtual control and adaptive law as shown in (39) and (40) into (38) and adding and subtracting $\dot{\chi}_1$ in the right-hand side of (38), it is deduced that

$$\dot{V}_1 \leq [g_1(t)N_1(\chi_1)+1]\dot{\chi}_1 - c_1 z_1^2 + \theta_1\varepsilon(t) + \frac{1}{4}\bar{g}_1^2 z_2^2. \tag{42}$$

Integrating (42) over $[0,t]$, we have

$$V_1(t) + c_1\int_0^t z_1^2(\tau)d\tau \leq \int_0^t [g_1(\tau)N_1(\chi_1(\tau))+1]\dot{\chi}_1(\tau)d\tau + \Pi_1 + \frac{1}{4}\bar{g}_1^2\int_0^t z_2^2(\tau)d\tau \tag{43}$$

where $\Pi_1 = V_1(0)+\theta_1\delta$. If the developed control guarantees that $\int_0^t z_2^2(\tau)d\tau$ is bounded, i.e., $\frac{1}{4}\bar{g}_1^2\int_0^t z_2^2(\tau)d\tau \leq \Omega_1$ with Ω_1 being a positive constant, then (43) can be checked as

$$V_1 + c_1\int_0^t z_1^2(\tau)d\tau \leq \Pi_1' + \int_0^t [g_1(\tau)N_1(\chi_1(\tau))+1]\dot{\chi}_1(\tau)d\tau$$

where $\Pi_1' = \Pi_1 + \Omega_1$. In this case, Lemma 2 can be applied to the above inequality. The problem on the boundedness of $\int_0^t z_2^2(\tau)d\tau$ will be coped with in the final step (see the analysis in (67) and (68)).

Step i $(i = 2,\ldots,n-1)$*:* The derivative of z_i, by considering $x_{i+1} = z_{i+1}+\alpha_i$, is computed as

$$\dot{z}_i = f_i + g_i z_{i+1} + g_i\alpha_i - \sum_{k=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_k}(f_k + g_k x_{k+1}) + \Delta_i \tag{44}$$

with

$$\dot{\alpha}_{i-1} = \sum_{k=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_k}(f_k + g_k x_{k+1}) - \Delta_i \tag{45}$$

$$\Delta_i = -\sum_{k=0}^{i-1}\frac{\partial\alpha_{i-1}}{\partial y_d^{(k)}}y_d^{(k+1)} - \sum_{k=0}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\varphi^{(k)}}\varphi^{(k+1)} - \sum_{k=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\chi_k}\dot{\chi}_k - \sum_{k=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\hat{\theta}_k}\dot{\hat{\theta}}_k - \sum_{k=0}^{i-2}\frac{\partial\alpha_{i-1}}{\partial\varepsilon^{(k)}(t)}\varepsilon^{(k+1)}(t). \tag{46}$$

Then the derivative of $\frac{1}{2}z_i^2$ along (44) is

$$z_i\dot{z}_i = g_i z_i\alpha_i + \Xi_i \tag{47}$$

where $\Xi_i = g_i z_i z_{i+1} + z_i f_i - z_i\sum_{k=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_k}(f_k+g_k x_{k+1}) + z_i\Delta_i$.

Similar to (29)–(36), Ξ_i can be upper bounded by

$$\Xi_i \leq \theta_i\varepsilon(t) + \theta_i\psi_i(\cdot) + z_i^2 + \frac{1}{4}\bar{g}_i^2 z_{i+1}^2 \tag{48}$$

$$\psi_i(\cdot) = \frac{z_i^2\Phi_i^2}{\sqrt{z_i^2\Phi_i^2+\varepsilon^2(t)}} \tag{49}$$

where θ_i, $i = 2,\ldots,n-1$, is defined in (25), and

$$\Phi_i = \kappa_i^2 + \sum_{k=1}^{i-1}\left[\left(\frac{\partial\alpha_{i-1}}{\partial x_k}\right)^2 + \frac{1}{4}\right]\kappa_k + \Delta_i^2 + \frac{1}{4} + \sum_{k=1}^{i-1}\left(\left(\frac{\partial\alpha_{i-1}}{\partial x_k}x_{k+1}\right)^2 + \frac{1}{4}\right) \tag{50}$$

is an available function.

Substituting (48) into (47), we have

$$z_i\dot{z}_i \leq g_i z_i\alpha_i + z_i^2 + \theta_i\varepsilon(t) + \theta_i\psi_i + \frac{1}{4}\bar{g}_i^2 z_{i+1}^2. \tag{51}$$

By utilizing the characteristics of ENF, the following Nussbaum-gain technique based virtual control law is given as:

$$\alpha_i = N_i(\chi_i)\left(c_i z_i + z_i + \frac{\hat{\theta}_i z_i\Phi_i^2}{\sqrt{z_i^2\Phi_i^2+\varepsilon^2(t)}}\right) \tag{52}$$

$$\dot{\chi}_i = c_i z_i^2 + z_i^2 + \hat{\theta}_i\psi_i \tag{53}$$

$$\dot{\hat{\theta}}_i = \gamma_i\psi_i,\ \hat{\theta}_i(0) \geq 0 \tag{54}$$

where c_i and γ_i are positive parameters, $\hat{\theta}_i$ is the estimated value of θ_i, and $\hat{\theta}_i(0) \geq 0$ and $\chi_i(0)$ are the initial values of $\hat{\theta}_i(t)$ and $\chi_i(t)$, respectively.

Choosing the quadratic function as $V_i = \frac{1}{2}z_i^2 + \frac{1}{2\gamma_i}\tilde{\theta}_i^2$, where $\tilde{\theta}_i = \theta_i - \hat{\theta}_i$, the derivative of V_i w.r.t. time is

$$\dot{V}_i \leq g_i z_i\alpha_i + z_i^2 + \theta_i\varepsilon(t) + \theta_i\psi_i + \frac{1}{4}\bar{g}_i^2 z_{i+1}^2 - \frac{1}{\gamma_i}\tilde{\theta}_i\dot{\hat{\theta}}_i. \tag{55}$$

Substituting the virtual control and adaptive law as defined in (52)–(54) into (55), we have

$$\dot{V}_i \leq [g_i N_i(\chi_i)+1]\dot{\chi}_i - c_i z_i^2 + \theta_i\varepsilon(t) + \frac{1}{4}\bar{g}_i^2 z_{i+1}^2. \tag{56}$$

Solving the above inequality yields

$$V_i(t) + c_i\int_0^t z_i^2(\tau)d\tau \leq \Pi_i + \int_0^t [g_i(\tau)N_i(\chi_i(\tau))+1]\dot{\chi}_i(\tau)d\tau + \frac{1}{4}\bar{g}_i^2\int_0^t z_{i+1}^2(\tau)d\tau \tag{57}$$

where $\Pi_i = V_i(0)+\theta_i\delta$.

Step n: The derivative of z_n is

$$\dot{z}_n = f_n + g_n u - \sum_{k=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial x_k}(f_k + g_k x_{k+1}) + \Delta_n \tag{58}$$

where $\Delta_n = -\sum_{k=0}^{n-1}\frac{\partial\alpha_{n-1}}{\partial y_d^{(k)}}y_d^{(k+1)} - \sum_{k=0}^{n-1}\frac{\partial\alpha_{n-1}}{\partial\varphi^{(k)}}\varphi^{(k+1)} - \sum_{k=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial\chi_k}\dot{\chi}_k - \sum_{k=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial\hat{\theta}_k}\dot{\hat{\theta}}_k - \sum_{k=0}^{n-2}\frac{\partial\alpha_{n-1}}{\partial\varepsilon^{(k)}(t)}\varepsilon^{(k+1)}(t)$. Then we further have

$$z_n\dot{z}_n = g_n z_n u + \Xi_n \leq g_n z_n u + \theta_n\psi_n + \theta_n\varepsilon(t) \tag{59}$$

where $\Xi_n = z_n f_n - z_n\sum_{k=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial x_k}(f_k+g_k x_{k+1}) + z_n\Delta_n$, θ_n is defined in (26), $\psi_n(\cdot) = \frac{z_n^2\Phi_n^2}{\sqrt{z_n^2\Phi_n^2+\varepsilon^2(t)}}$, and $\Phi_n = \kappa_n^2 + \sum_{k=1}^{n-1}\left(\left(\frac{\partial\alpha_{n-1}}{\partial x_k}\right)^2+\frac{1}{4}\right)\kappa_k + \sum_{k=1}^{n-1}\left(\left(\frac{\partial\alpha_{n-1}}{\partial x_k}x_{k+1}\right)^2+\frac{1}{4}\right) + \Delta_n^2 + \frac{5}{4}$.

The actual control law and adaptive law are given by

$$u = N_n(\chi_n)\left(c_n z_n + \frac{\hat{\theta}_n z_n\Phi_n^2}{\sqrt{z_n^2\Phi_n^2+\varepsilon^2(t)}}\right) \tag{60}$$

$$\dot{\chi}_n = c_n z_n^2 + \hat{\theta}_n\psi_n \tag{61}$$

$$\dot{\hat{\theta}}_n = \gamma_n \psi_n,\ \hat{\theta}_n(0) \geq 0 \tag{62}$$

where $c_n > 0$, $\gamma_n > 0$ are design parameters, $\hat{\theta}_n$ is the estimate value of θ_n, and $\hat{\theta}_n(0) \geq 0$ and $\chi_n(0)$ are the initial values of $\hat{\theta}_n(t)$ and $\chi_n(t)$, respectively.

According to the developed control algorithm, we state the following theorem.

Theorem 1: For the strict-feedback nonlinear system (1) with the time-varying control coefficients of an unknown sign. Under Assumptions 1–3, if the control law (60)–(62) is applied, the control objectives $O_1 - O_2$ are ensured.

Proof: Define the Lyapunov function candidate as

$$V_n = \frac{1}{2}z_n^2 + \frac{1}{2\gamma_n}\tilde{\theta}_n^2 \tag{63}$$

where $\tilde{\theta}_n = \theta_n - \hat{\theta}_n$. Taking the derivative of V_n w.r.t. time along (58) yields

$$\dot{V}_n \leq g_n z_n u + \theta_n \varepsilon(t) + \theta_n \psi_n - \frac{1}{\gamma_n}\tilde{\theta}_n \dot{\hat{\theta}}_n. \tag{64}$$

Substituting the true control law (60) and adaptive laws (61) and (62) into (64), it is deduced that

$$\dot{V}_n \leq [g_n N_n(\chi_n) + 1]\dot{\chi}_n - c_n z_n^2 + \theta_n \varepsilon(t) \tag{65}$$

then it follows that:

$$V_n + c_n \int_0^t z_n^2(\tau)d\tau \leq \Pi_n + \int_0^t [g_n(\tau)N_n(\chi_n(\tau)) + 1]\dot{\chi}_n(\tau)d\tau \tag{66}$$

where $\Pi_n = V_n(0) + \theta_n \delta$.

Thus, together with (61), (66), and Lemma 2, it is ensured that $V_n(t)$ and χ_n are bounded for $\forall t \geq 0$, then from the definition of V_n, it follows that $z_n \in L_\infty$ and $\tilde{\theta}_n \in L_\infty$, then it is further shown that $\hat{\theta}_n$ is bounded. As $\chi_n(t) \in L_\infty$ and $\dot{\chi}_n = c_n z_n^2 + \hat{\theta}_n \psi_n$, it can be rewritten in the integral form as

$$c_n \int_0^t z_n^2(\tau)d\tau \leq \chi_n(t) - \chi_n(0), \tag{67}$$

in which $\hat{\theta}_n \psi_n \geq 0$ is utilized, then it is seen from (67) that the boundedness of $\int_0^t z_n^2(\tau)d\tau$ is ensured. Therefore, according to the analysis in Step 1 and applying (17) $(n-1)$ times, we can conclude that

$$V_i, z_i, \hat{\theta}_i, \chi_i, \int_0^t z_i^2(\tau)d\tau \tag{68}$$

for $i = 1, \ldots, n-1$, are bounded. As $z_1 = s$, then it is shown from Remark 2 that, there exist positive constants λ, $\underline{\rho}$, $\bar{\rho}$, $\underline{\beta}$, and $\bar{\beta}$ such that $|\zeta| \leq \lambda < 1$, $\underline{\rho} \leq |\rho| \leq \bar{\rho}$, and $\underline{\beta} \leq \beta \leq \bar{\beta}$. Thus, according to the properties of $\mathcal{F}$ and η, it is not difficult to prove that e, r_1, and r_2 are bounded as y_d, $\varphi \in (\varphi_f, \varphi_0]$, and $\dot{\varphi}$ are bounded, which further implies that there exist positive constants $\underline{r}_1$ and $\bar{r}_1$ so that $0 < \underline{r}_1 \leq r_1 \leq \bar{r}_1 < \infty$. Since $z_1 = x_1 - y_d$ and $y_d \in L_\infty$, then $x_1 \in L_\infty$ and $\kappa_1 \in L_\infty$, then it implies from (34) that $\Phi_1 \in L_\infty$, which further follows from the definitions of $\dot{\hat{\theta}}_1$ and $\dot{\chi}_1$ as given in (41) and (40) that $\dot{\chi}_1 \in L_\infty$ and $\dot{\hat{\theta}}_1 \in L_\infty$. For the ENF, as $\chi_1 \in L_\infty$, it is easily verified that $N_1(\chi_1)$ is bounded, then it follows that the virtual control law α_1 and $\dot{z}_1$ are bounded. Similarly, using an induction argument, we can get the conclusion that x_i, κ_i, Φ_i, $\dot{\chi}_i$, $\dot{\hat{\theta}}_i$, $\dot{z}_i$, $(i = 2, \ldots, n)$, the virtual control law α_j $(j = 2, \ldots, n-1)$, and the true control law u are bounded. Hence, all signals in the closed-loop systems are bounded.

In addition, note that $s(t) \in L_\infty$ holds for $\forall t \geq 0$, then upon utilizing the discussion in Section III-A the prescribed tracking performance is guaranteed. Finally, note that $z_1 \in L_2 \bigcap L_\infty$ and $\dot{z}_1 \in L_\infty$, $i = 1, \ldots, n$, then by using the Barbalat's Lemma it is ensured that $\lim_{t\to\infty} z_1(t) \to 0$, namely, $\lim_{t\to\infty} s(t) \to 0$, then it follows from the definition of s that $\lim_{t\to\infty} \zeta(t) \to 0$. As $\zeta = \frac{\eta(t)}{\varphi(t)}$ and $0 < b_f \leq \varphi(t) \leq 1$, then it is shown that $\lim_{t\to\infty} \eta(t) \to 0$; then by utilizing the property of $\eta(t)$, it is ensured that $e(t) \to 0$ as $t \to \infty$. ■

To facilitate the understanding of our design procedure, a detailed block diagram of the proposed control is presented in Fig. 1.

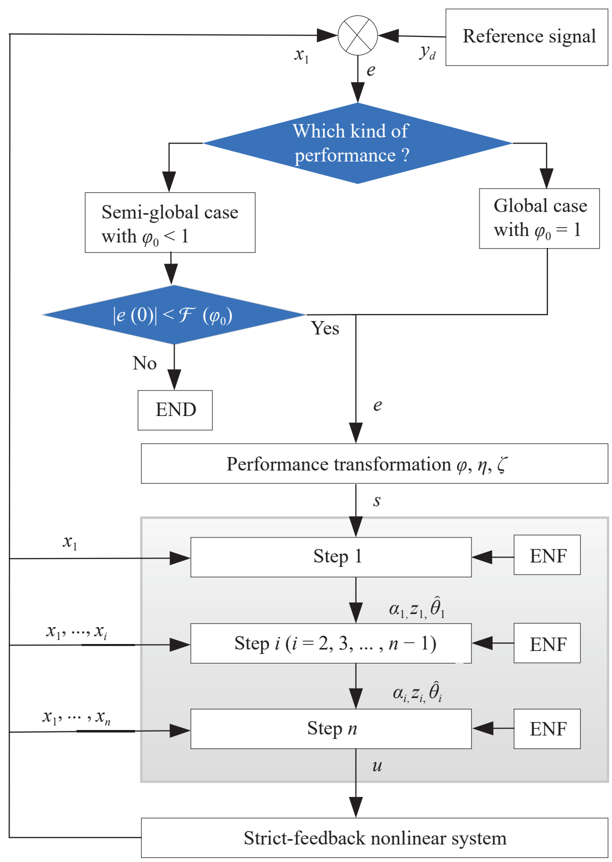

Fig. 1. The designed procedure diagram.

Remark 4: The differences between our previous result [25] and this work are mainly exhibited in the system model and the control goal: 1) the non-parametric uncertainty (rather than the parametric uncertainty) in the system model; 2) the time-varying (rather than constant) control coefficient with unknown control direction; and 3) the uniform prescribed performance (rather than global performance). It is quite difficult and challenging for solving the above issues as the approach in [25] is invalid, for example, for the non-parametric uncertainty, the parameter decomposition condition in [25] does not hold so that the tuning-function-based control cannot be adopted; For the time-varying control coefficient, the Nussbaum lemma in [25] is still ineffective. To solve the above problems, different yet more advanced technical development must be adopted in this paper, we utilize the "core information" technique in Assumption 3 to handle the non-parametric uncertainty, reveal some additional properties of Nussbaum functions in Definition 3 and Lemma 2 to cope with the time-varying control gain, and construct a unified function $\mathcal{F}(\varphi)$ in Definition 1 to solve the problem of uniform prescribed performance. Therefore, it is seen that, compared with [25], the proposed scheme has great contributions in control design.

Remark 5: To achieve the given objectives in this paper, two major difficulties are encountered, as discussed below.

1) The first is how to ensure the uniform performance for strict-feedback non-parametric systems, which cannot be solved with existing approaches. For example, the PPB control schemes [19], [22] for guaranteeing prescribed performance cannot be applied, because they always require a constraint on the initial error. In this paper, we construct a unified function $\mathcal{F}$ and a unique time-varying scaling function $\varphi(t)$, by selecting different initial values of $\varphi(t)$, the uniform prescribed performances can be handled. Furthermore, by employing the deep-rooted information, the nonparametric uncertainty in the system can be greatly handled without involving any approximator; and

2) The second is how to reduce the complex stability analysis caused by the problem of time-varying control gains with unknown control directions. For most of existing Nussbaum-gain results [11], [12], [14], only a special Nussbaum function can be proved to be effective based on the explicit calculation on the particular function used. However, the associated stability proof of closed-loop systems is quite complex. In this paper, an ENF is proposed. By establishing some important properties (i.e., (15) and (16)) and imposing some extra condition on the argument of Nussbaum function (i.e., (18)), the developed Nussbaum-gain-based Lemma 2 does not rely on the explicit calculation for the particularly chosen Nussbaum functions. This facilitates the stability analysis with great convenience and flexibility.

IV. Simulation Studies

A. Validity Verification of the Proposed Control

To illustrate the effectiveness of the developed control algorithm, the following strict-feedback nonlinear system is considered:

$$\begin{cases} \dot{x}_1 = f_1(x_1, p_1) + g_1(t)x_2 \\ \dot{x}_2 = f_2(\bar{x}_2, p_2) + g_2(t)u \\ y = x_1 \end{cases} \tag{69}$$

where $f_1(x_1) = 0.2\exp\left(-p_1 x_1^2\right)\sin(p_{12}x_1) + p_{13}x_1$, $f_2(\bar{x}_2) = p_{21}x_1x_2 + p_{22}\sin(p_{23}x_2)$, the system parameters are given as: $p_1 = [p_{11}, p_{12}, p_{13}]^T = [0.5, 1, 1]^T$ and $p_2 = [p_{21}, p_{22}, p_{23}]^T = [0.2, 0.1, 0.5]^T$. The control coefficients are chosen as: $g_1(t) = 3 + 0.1\sin(t)$, $g_2(t) = -2 + 0.2\cos(t)$, whereas the signs and the values of g_1 and g_2 are unavailable for controller design. As the nonlinear function f_i does not satisfy the parameter decomposition condition and the control coefficient g_i is not only unknown but also time-varying, then the approach in [25] is no longer effective.

Noting that if φ_0 is selected as 1, the corresponding control (60)−(62) is actually a global result, as it is independent of initial conditions of system states. To verify and illustrate this, the following three sets of initial conditions of system states are considered: $x(0) = [x_1(0), x_2(0)]^T = [0.9, -1]^T$, $[0.5, -0.5]^T$, and $[-1, 0]^T$. The other initial conditions are chosen as: $\hat{\theta}_1(0) = \hat{\theta}_2(0) = 0$, $\chi_1(0) = 1$, and $\chi_2(0) = 0$; the desired signal is chosen as $y_d(t) = 0.3\sin(t)$. According to the analysis in [13], it is shown that $N_1(\chi_1) = \cos(\pi\chi_1/2)\exp(\chi_1^2)$ and $N_2(\chi_2) = \cos(\pi\chi_2/2)\exp(\chi_2^2)$ are ENFs and thus they are applicable to the case of unknown control directions with time-varying control coefficients. To implement the control algorithm in the simulation, the scaling function is given as $\varphi(t) = (\varphi_0 - \varphi_f)\exp(-0.5t) + \varphi_f$, the design parameters are selected as: $c_1 = 5$, $c_2 = 10$, $\gamma_1 = 0.1$, $\gamma_2 = 0.01$, $\varphi_0 = 1$, and $\varphi_f = 0.08$. The integral function $\varepsilon(t)$ is given by $\varepsilon(t) = \exp(-0.4t)$. The simulation results are presented in Figs. 2−5. It is seen from Fig. 2 that, under the developed control scheme (60)−(62) for nonlinear systems with unknown control directions, the global prescribed performance of tracking error e is achieved. The evolutions of control input, the trajectories of parameters of Nussbaum functions (χ_1 and χ_2), and the responses of parameter estimates ($\hat{\theta}_1$ and $\hat{\theta}_2$) are plotted in Figs. 3−5, respectively, which show that the closed-loop signals are bounded for all time.

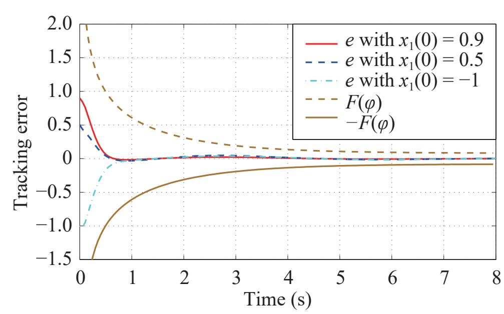

Fig. 2. The trajectories of tracking error with prescribed boundary under different initial values.

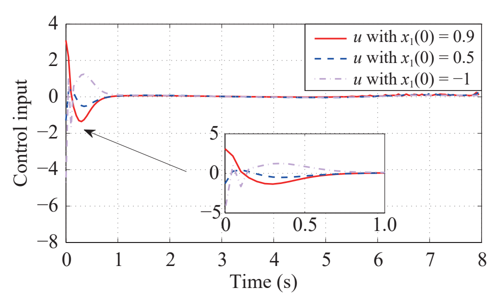

Fig. 3. The trajectories of control input under different initial values.

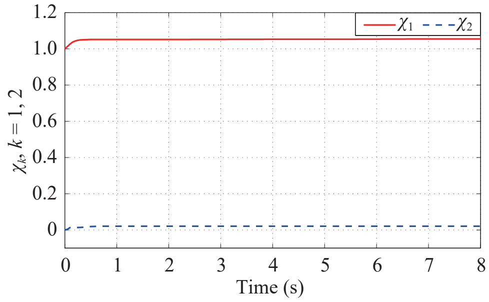

Fig. 4. The trajectories of parameters χ_1 and χ_2 under initial condition $x(0)=[-1,0]^T$.

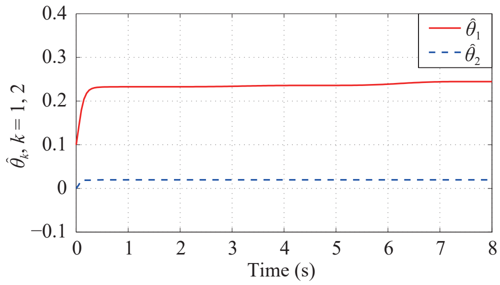

Fig. 5. The trajectories of parameter estimates $\hat{\theta}_1$ and $\hat{\theta}_2$ under initial condition $x(0)=[-1,0]^T$.

B. Comparison With the Existing Works

To verify the merits of the developed algorithm in Section III-D, we mainly give a comparison between the proposed control and the funnel control in [29]. To make a fair comparison, we use the following one-link robotic system in [30] for simulation:

$$\begin{cases}\dot{x}_1 = x_2\\ \dot{x}_2 = \dfrac{1}{J}u - \dfrac{1}{J}(Dx_2 + MG\check{L}\sin(x_1))\end{cases} \tag{70}$$

where x_1 and $x_2 = \dot{x}_1$ denote the angular positive and velocity, respectively. The detailed definitions of J, D, M, and $\check{L}$ can be found in [30] and the system parameters are given as $MG\check{L}=5$, $D=2$, and $J=1$. The initial states are chosen as $x(0)=[x_1(0),x_2(0)]^T=[1,-1]^T$ and the reference signal is $y_d(t)=0.5\sin(t)$.

Now we give the detailed formulations of the funnel control in [29] and the proposed control in this paper.

Funnel control:

$$\begin{cases} u = -\check{\mu}\left[(\cos(\chi_2)-\chi_2\sin(\chi_2))^2e^2 + N_2^2\right]\chi_2^4(1+\check{\omega}^2)\vartheta \\ \quad -\check{\mu}N_2e \\ N_2 = \chi_2\cos(\chi_2),\ \chi_2 = \dfrac{1}{1-\mathcal{F}^2e^2} \\ \vartheta = \check{\omega} - N_2e,\ \dot{\check{\omega}} = -\check{\mu}\check{\omega}+u,\ e = x_1 - y_d \end{cases} \tag{71}$$

where $\check{\mu}=10$ is a positive parameter, $\mathcal{F}$ denotes the prescribed performance boundary which is formulated as $\mathcal{F}=\frac{\sqrt{\ell}\varphi}{\sqrt{1-\varphi^2}}$ with $\varphi(t)=(1-0.08)\exp(-0.8t)+0.08$ and $\ell = 1-0.08^2$.

Proposed control:

$$\begin{cases} \alpha_1 = -\dfrac{1}{r_1}(c_1z_1+r_2)-r_1z_1+\dot{y}_d,\ z_1 = s \\ u = N_2\left(c_2z_2+\dfrac{\hat{\theta}_2z_2\Phi_2^2}{\sqrt{z_2^2\Phi_2^2+\varepsilon^2(t)}}\right),\ N_2=\chi_2\cos(\chi_2) \\ \dot{\chi}_2 = c_2z_2^2+\dfrac{\hat{\theta}_2z_2^2\Phi_2^2}{\sqrt{z_2^2\Phi_2^2+\varepsilon^2(t)}} \\ \dot{\hat{\theta}}_2 = \gamma_2\dfrac{z_2^2\Phi_2^2}{\sqrt{z_2^2\Phi_2^2+\varepsilon^2(t)}},\ \ \hat{\theta}_2(0)\geq 0 \\ z_2 = x_2-\alpha_1,\ \ \Phi_2=\kappa_2+|\Delta_2|,\ \kappa_2=|x_2|+1 \\ \Delta_2 = -\displaystyle\sum_{k=0}^{1}\left(\frac{\partial\alpha_1}{\partial y_d^{(k)}}y_d^{(k+1)}+\frac{\partial\alpha_1}{\partial\varphi^{(k)}}\varphi^{(k+1)}\right)-\frac{\partial\alpha_1}{\partial x_1}x_2 \end{cases} \tag{72}$$

where the design parameters and positive function are given as $c_1=3$, $c_2=6$, $\gamma_2=0.01$, and $\varepsilon(t)=\exp(-0.4t)$. The initial values of parameter estimates are chosen as $\hat{\theta}_2(0)=0$ and $\chi_2(0)=-0.9$. Under the funnel control (71) and the proposed control (72), the simulation results are shown in Figs. 6 and 7. It is easily seen that the proposed control has better transient performances than these in the funnel control.

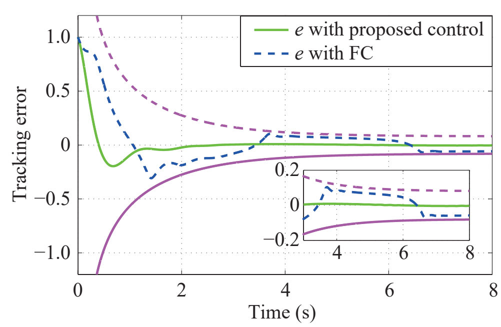

Fig. 6. The trajectories of tracking error under the proposed control and the funnel control.

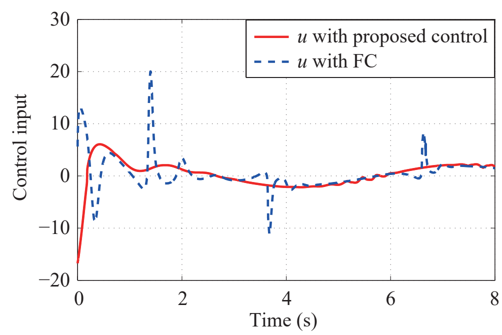

Fig. 7. The trajectories of control signal under the proposed control and the funnel control.

V. Conclusions

An adaptive uniform prescribed performance control strategy has been developed for strict-feedback nonlinear systems with nonparametric uncertainties and unknown control directions. By utilizing some function transformations and developing some additional features of Nussbaum functions, together with the adaptive backstepping technique, the proposed control exhibits the following features: 1) the boundedness of all signals in the closed-loop systems is ensured; 2) the uniform prescribed tracking performance can be ensured without changing the control structure; and 3) the complicated stability proof in the existing Nussbaum-based results is avoided. It is worth emphasizing that although the uniform performance can be ensured by utilizing the developed control, we only achieve the symmetric result, which sacrifices the overshoot of the tracking error in some degree. Moreover, only the single system is considered in this paper. Noting that network systems nowadays have received more and more attention due to its widely applications [31]–[33] (such as microgrids, distributed systems, and mobile robots), therefore, we will study the uniform prescribed performance problem of networked systems in the future work.

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