Letter

A Semi-Looped-Functional for Stability Analysis of Sampled-Data Systems

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Dear Editor,

In recent decades, owing to the significance of sampled-data control way on reducing the burden of communication transmission and improving the control capability of networked control systems, the research of sampled-data systems has become increasingly important (see [1]–[5]).

For the basic stability problem of linear sampled-data systems, there are two popular methods recently, including the looped-functional method and the discontinuous Lyapunov functional method. On one hand, the looped-functional method was firstly introduced by [4]. Then, [6] combined Wirtinger's inequality to further improve the results. Afterwards, Zeng *et al*. [5] provided a two-sided looped-functional, which efficiently improved the previous functionals. Shao *et al*. [7], [8] obtained novel stability criteria based on new looped-functionals. And Park and Park [9] extended the looped-functionals without requiring them to be always continuous. On the other hand, the discontinuous Lyapunov functional method was introduced by [10] for the stability of sampled-data systems. Later, Lee and Park [11] provided free-matrix-based discontinuous Lyapunov functional. And our latest study [12] also dedicated to extend the two methods.

The above two methods played important roles in the stability analysis of sampled-data systems, and they were also combined to further solve more complex problems (see [13]–[15]). But, after observation, the two well-used methods are both with strict formal constraints, in details, the looped-functional terms of $\mathcal{V}(t)$ are required to be continuous and $\mathcal{V}(t_k)=0$ at sampling instants t_k or equivalent conditions, and the discontinuous Lyapunov functional terms of $V_d(t)$ have to satisfy $V_d(t_k)=0$ and $V_d(t)\geq 0$ when $t\neq t_k$. And the methods are based on the functional $V(t)$ being $\dot{V}(t)<0$ to ensure $V(t)$ decreasing. These strict conditions not only bring conservatism to the obtained results, but also hinder the further improvements of the methods. Recently, although Park *et al*. [9], [12] tried to extend the two methods through requiring the new terms $V_1(t)$ and $V_2(t)$ to satisfy $V_1(t_{k+1}^-)-V_2(t_k^+)>0$ and $V_1(t_{k+1}^+)=V_2(t_k^-)=0$, they still did not deeply extend and improve the previous methods, and did not relax the condition $\dot{V}(t)<0$ through considering the discontinuity of $V(t)$. Therefore, it is difficult but necessary to propose a new method to relax the above critical conditions of ensuring functionals decreasing, and then obtain improved results.

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Based on the above discussion, for the stability of sampled-data systems, the letter presents a semi-looped-functional method with relaxed constrain, which extends and improves the recent two common methods. Especially, the study does not require the functional to satisfy $\dot{V}(t)<0$ through considering its discontinuity. Thus, the new method flexibly leads to less conservative results. Numerical examples verify the effectiveness and superiority of the new method and results.

Notations: In the letter, $\mathbb{R}^n$ represents the n-dimensional Euclidean space, $\mathbb{R}^{n\times m}$ denotes the set of all $n\times m$-dimensional real matrices. "$*$" denotes symmetric term in matrix, "ε" means a sufficiently small positive scalar. For vectors μ_1 and μ_2, $\mathrm{col}\{\mu_1,\mu_2\}=[\mu_1^T,\mu_2^T]^T$. For square matrices X and Y, $\mathrm{Sym}(X)=X+X^T$, and $\mathrm{diag}\{X,Y\}=\begin{bmatrix}X & 0\\ 0 & Y\end{bmatrix}$. For functional $V(t)$, $V(t_k^-)=\lim_{t\to t_k^-}V(t)$ and $V(t_k^+)=\lim_{t\to t_k^+}V(t)$.

Problem formulation and main method: Consider linear sampled-data system

$$\dot{x}(t)=Ax(t)+A_s x(t_k),\quad t\in[t_k,t_{k+1}) \tag{1}$$

where $x(t)\in\mathbb{R}^n$ is the state, A and $A_s\in\mathbb{R}^{n\times n}$ are known matrices, and sampling instants $t_k(k=0,1,\ldots)$ satisfy $t_{k+1}-t_k=h_k\in[h_l,h_u]$, where known scalars h_l and h_u satisfy $0\leq h_l\leq h_u$, and represent the lower and upper bounds of the sampling intervals $h_k(k=0,1,\ldots)$, respectively. Therefore, the system (1) can be with constant or aperiodic sampling.

The following Lemma 1 equivalently transforms the asymptotic stability of the system (1).

Lemma 1: For the system (1), the following two situations 1) and 2) are equivalent.

1) The system (1) is asymptotically stable;

2) The sampling state $x(t_k)$ asymptotically tends to 0.

Proof: From the situation 1), it is easy to get 2). Considering the uniform boundedness of the transfer function of the system (1) over $[t_k,t_{k+1})$, it is obtained that $x(t)$ asymptotically tends to 0 if sampled-data state $x(t_k)$ asymptotically tends to 0 as in [4]. Thus, 1) is obtained from 2). ■

The following Lemma 2 provides a theoretical basis for the improved functional compared with [5]–[7] and [9].

Lemma 2: For non-zero state $x(t)$, define differentiable and continuous function $V_0(t)$ for $t\in(t_k,t_{k+1})$. The following two statements 1) and 2) are equivalent.

1) The function $V_0(t)$ is decreasing between the adjacent sampling instants t_k and t_{k+1}, i.e.,

$$\triangle V_0^k := V_0(t_{k+1})-V_0(t_k)<0.$$

2) There exists functional $V(t)$ which is continuous and differentiable when $t\in(t_k,t_{k+1})$ such that

$$h_k\dot{V}(t)<(V(t_{k+1}^-)-V(t_k^+))-(V_0(t_{k+1})-V_0(t_k)). \tag{2}$$

Moreover, if 1) or 2) is satisfied and for $V_0(t)$ there are positive scalars ς_1, ς_2 and q such that

$$\varsigma_1|x(t)|^q\leq V_0(t)\leq\varsigma_2|x(t)|^q,\ \forall x(t)\in\mathbb{R}^n,\ t\geq t_0 \tag{3}$$

then the system (1) is asymptotically stable.

Proof: Assume 1) being satisfied. Then, we set $V(t)=-\frac{t}{h_k}\triangle V_0^k$ for $t\in(t_k,t_{k+1})$, and get $\dot{V}(t)=-\frac{1}{h_k}\triangle V_0^k$ and $(V(t_{k+1}^-)-V(t_k^+))-(V_0(t_{k+1})-V_0(t_k))=-2\triangle V_0^k$. Consider $\triangle V_0^k<0$, so 2) is satisfied.

Assume 2) being satisfied. Integrate both sides of (2) with respect to t over (t_k,t_{k+1}), then we get $V_0(t_{k+1})-V_0(t_k)<0$, that is, 1) being satisfied. Thus, 1) and 2) are equivalent.

For system (1), 1) is a sufficient condition for the situation 2) of Lemma 1 under (3). According to Lemma 1, considering the equivalence between 1) and 2) in Lemma 2, we know that if 1) or 2) is satisfied, the system (1) is asymptotically stable. ■

For simplicity, the following concise expressions are used:

$$d_1(t)=t-t_k,\ \ d_2(t)=t_{k+1}-t$$
$$\chi_1(t)=\frac{1}{d_1(t)}\int_{t_k}^{t}x(s)ds,\ \chi_2(t)=\frac{1}{d_2(t)}\int_{t}^{t_{k+1}}x(s)ds$$
$$\chi_0=\chi_1(t_{k+1})=\chi_2(t_k)$$
$$\chi_3(t)=x(t)-x(t_k),\ \ \chi_4(t)=x(t_{k+1})-x(t)$$
$$\zeta_1(t)=\mathrm{col}\{x(t),x(t_k),x(t_{k+1})\},\ \ \zeta_2(t)=\mathrm{col}\{x(t_k),x(t_{k+1})\}$$
$$\eta(t)=\mathrm{col}\{x(t),x(t_k),\chi_1(t),x(t_{k+1}),\chi_2(t),\chi_0\}$$
$$e_\kappa=[0_{n\times(\kappa-1)n},I_n,0_{n\times(6-\kappa)n}],\ \ \kappa=1,2,...,6$$
$$e_s=Ae_1+A_se_2,\ \ \mathrm{E}_1^T=\left[e_2^T,e_4^T\right],\ \ \mathrm{E}_2=e_1-e_2$$
$$\mathrm{E}_3=e_4-e_1,\ \ \mathrm{E}_4^T=\left[e_1^T,e_2^T,e_3^T\right],\ \ \mathrm{E}_5=e_1+e_2-2e_3$$
$$\mathrm{E}_6^T=\left[e_4^T,e_1^T,e_5^T\right],\ \ \mathrm{E}_7=e_4+e_1-2e_5,\ \ \mathrm{E}_8=Ae_3+A_se_2$$
$$\mathrm{E}_9=Ae_5+A_se_2,\ \ \mathrm{E}_{10}^T=\left[e_1^T,e_2^T,e_3^T,e_4^T,e_5^T\right].$$

Lemma 3: For symmetric matrices P, R_1, R_2, Y_1, Y_3, Y_4, S_1, S_2, S_3, S_4, H_1, X, matrices Y_2, H_2, H_3, Z, define the following semi-looped-functional $V(t)$ for the system (1):

$$V(t)=\begin{cases}V_0(t), & t=t_k\\ V_0(t)+V_1(t)+V_2(t)+V_3(t), & t\neq t_k\end{cases}\tag{4}$$

where

$$V_0(t)=x^T(t)Px(t)$$
$$V_1(t)=d_1(t)x^T(t)[Y_1,2Y_2]\zeta_1(t)+d_1(t)\chi_1^T(t)Y_3d_1(t)\chi_1(t)$$
$$+d_1(t)\chi_1^T(t)Y_4\chi_1(t)+\chi_3^T(t)(h_uS_1+h_lS_3)\chi_3(t)$$
$$V_2(t)=-d_2(t)x^T(t)[H_1,2H_2]\zeta_1(t)-2x^T(t)H_3d_2(t)\chi_2(t)$$
$$-\chi_4^T(t)(h_uS_2+h_lS_4)\chi_4(t)$$
$$V_3(t)=2\chi_3^T(t)Z\chi_4(t)+d_1(t)d_2(t)\zeta_2^T(t)X\zeta_2(t)$$
$$+d_2(t)\int_{t_k}^{t}\dot{x}^T(s)R_1\dot{x}(s)ds-d_1(t)\int_{t}^{t_{k+1}}\dot{x}^T(s)R_2\dot{x}(s)ds.$$

If there is

$$h_k\dot{V}(t)<V_1(t_{k+1}^-)-V_2(t_k^+)\tag{5}$$

when $t\in(t_k,t_{k+1})$, then $V_0(t_k)>V_0(t_{k+1})$.

Proof: From $V_1(t_k^+)=V_3(t_k^+)=V_2(t_{k+1}^-)=V_3(t_{k+1}^-)=0$ and $V_0(t)$ being continuous, there is $(V(t_{k+1}^-)-V_0(t_{k+1}))-(V(t_k^+)-V_0(t_k))=V_1(t_{k+1}^-)-V_2(t_k^+)$. So if (5) holds, then,

$$h_k\dot{V}(t)<(V(t_{k+1}^-)-V(t_k^+))-(V_0(t_{k+1})-V_0(t_k)).\tag{6}$$

From (6) and Lemma 2, we deduce $V_0(t_k)>V_0(t_{k+1})$.

As Fig. 1, $V(t)$ satisfies $V(t_k)>V(t_{k+1})$ (i.e., $V_0(t_k)>V_0(t_{k+1})$) and is not required $\dot{V}(t)<0$. In $V(t)$, new term $V_1(t)$ or $V_2(t)$ satisfies $V_1(t_k^+)=0$, $V_1(t_{k+1}^-)\neq 0$ or $V_2(t_k^+)\neq 0$, $V_2(t_{k+1}^-)=0$, so $V(t)$ is called semi-looped-functional. ■

Remark 1: The extended looped-functionals in [9] satisfy $(V(t_{k+1}^-)-V(t_k^+))-(V_0(t_{k+1})-V_0(t_k))>0$, and the terms in [12] is with $V_1(t_k^+)=0$, $V_1(t_{k+1}^-)\geq 0$ or $V_2(t_k^+)\leq 0$, $V_2(t_{k+1}^-)=0$. These terms extend the common looped-functional terms $\mathcal{V}(t)$ (being continuous and $\mathcal{V}(t_k)=0$) in [4]–[7] and discontinuous functional terms $V_d(t)$ (satisfying $V_d(t_k)=0$ and $V_d(t)\geq 0$ when $t\neq t_k$) in [10], [11]. Further, the semi-looped-functional $V(t)$ merely require $V_1(t_{k+1}^-)-V_2(t_k^+)>h_k\dot{V}(t)$ (i.e., $(V(t_{k+1}^-)-V(t_k^+))-(V_0(t_{k+1})-V_0(t_k))>h_k\dot{V}(t)$) to ensure $V(t_k)>V(t_{k+1})$. And if set $V_1(t_{k+1}^-)=0$ or $V_2(t_k^+)=0$, then $V_1(t)$ or $V_2(t)$ degenerates into looped-functional term; if set $V_1(t_{k+1}^-)-V_2(t_k^+)>0$, then $V_1(t)$ and $V_2(t)$ degenerate into extended looped-functional terms. Thus, the new method constructs a flexible functional, and it further extends and improves the previous methods.

Remark 2: The previous methods usually require functionals to be derivative negative definite as in [4]–[12], while we do not require such a condition through considering the discontinuities of $V(t)$. As in [9] and [12], the terms $V_1(t)$ and $V_2(t)$ with $V_1(t_{k+1}^-)-V_2(t_k^+)>0$ (equivalent to $(V(t_{k+1}^-)-V(t_k^+))-(V_0(t_{k+1})-V_0(t_k))>0$) already lead to improved results, then (5) is more relaxed than $\dot{V}(t)<0$, and we even do not require $V_1(t_{k+1}^-)-V_2(t_k^+)>0$. Therefore, we improve the results through relaxing the requirement $\dot{V}(t)<0$ in view of the discontinuity of $V(t)$.

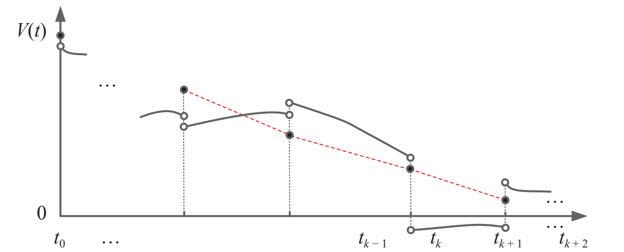

Fig. 1. Schematic illustration of semi-looped-functional $V(t)$.

Remark 3: Both terms $V_1(t)$ and $V_2(t)$ are novelly selected as asymmetric, so $V_1(t_{k+1}^-)$ and $V_2(t_k^+)$ are asymmetric. Thus, our previous method in [12] can not directly ensure $V_1(t_{k+1}^-)\geq 0$ and $V_2(t_k^+)\leq 0$, while the new method ensures $h_k\dot{V}(t)<V_1(t_{k+1}^-)-V_2(t_k^+)$ by the cooperation of $V_1(t)$, $V_2(t)$ and $\dot{V}(t)$. And setting $V_1(t_{k+1}^-)\geq 0$, $V_2(t_k^+)\leq 0$, $\dot{V}(t)<0$, the new method is reduced to being similar to the method in [12].

Main results: Now we provide the main results based on the new method.

Theorem 1: Given positive scalars h_l and h_u, the system (1) is asymptotically stable if there exist positive definite symmetric matrices $P,R_1,R_2\in\mathbb{R}^{n\times n}$, symmetric matrices $Y_1,Y_3,Y_4,H_1,S_1,S_2,S_3,S_4\in\mathbb{R}^{n\times n}$, $X\in\mathbb{R}^{2n\times 2n}$, matrices $Y_2,H_2\in\mathbb{R}^{n\times 2n}$, $H_3,Z\in\mathbb{R}^{n\times n}$, $N_\iota\in\mathbb{R}^{3n\times n}(\iota=1,2,3,4)$, $W_1,W_2\in\mathbb{R}^{5n\times n}$, such that

$$S_1+S_2>0,S_3+S_4<0\tag{7}$$

$$\begin{bmatrix}\Sigma(t_k)-\Lambda & \sqrt{h_k}\Omega_2\\ * & -\mathcal{R}_2\end{bmatrix}<0,\ \begin{bmatrix}\Sigma(t_{k+1})-\Lambda & \sqrt{h_k}\Omega_1\\ * & -\mathcal{R}_1\end{bmatrix}<0\tag{8}$$

where

$$\Sigma(\theta)=\Phi_0+\Phi_1(\theta)+\Phi_2(\theta)+\Phi_3(\theta)+\Phi_4(\theta)$$
$$\Phi_0=\mathrm{Sym}\{e_1^TPe_s\}$$
$$\Phi_1(\theta)=\mathrm{Sym}\{e_1^TY_2\mathrm{E}_1+e_1^TY_4e_3+e_s^T(h_uS_1+h_lS_3)\mathrm{E}_2\}$$
$$+d_1(\theta)\mathrm{Sym}\{e_s^TY_1e_1+e_s^TY_2\mathrm{E}_1+e_1^TY_3e_3\}$$
$$+e_1^TY_1e_1-e_3^TY_4e_3$$
$$\Phi_2(\theta)=\mathrm{Sym}\{e_1^TH_2\mathrm{E}_1+e_1^TH_3e_1+e_s^T(h_uS_2+h_lS_4)\mathrm{E}_3\}$$
$$-d_2(\theta)\mathrm{Sym}\{e_s^TH_1e_1+e_s^TH_2\mathrm{E}_1+e_s^TH_3e_5\}+e_1^TH_1e_1$$
$$\Phi_3(\theta)=\mathrm{Sym}\{e_s^TZ\mathrm{E}_3-\mathrm{E}_2^TZe_s+\mathrm{E}_4^TN_1\mathrm{E}_2+\mathrm{E}_4^TN_2\mathrm{E}_5$$
$$+\mathrm{E}_6^TN_3\mathrm{E}_3+\mathrm{E}_6^TN_4\mathrm{E}_7\}+[d_2(\theta)-d_1(\theta)]\mathrm{E}_1^TX\mathrm{E}_1$$
$$+d_2(\theta)e_s^TR_1e_s+d_1(\theta)e_s^TR_2e_s$$
$$\Phi_4(\theta)=\mathrm{Sym}\{d_1(\theta)\mathrm{E}_{10}^TW_1\mathrm{E}_8-\mathrm{E}_{10}^TW_1\mathrm{E}_2+d_2(\theta)\mathrm{E}_{10}^TW_2\mathrm{E}_9$$
$$-\mathrm{E}_{10}^TW_2\mathrm{E}_3\}$$
$$\Lambda=\mathrm{Sym}\{e_4^TY_2\mathrm{E}_1+e_2^TH_2\mathrm{E}_1+e_2^TH_3e_6\}$$
$$+e_4^TY_1e_4+h_ke_6^TY_3e_6+e_6^TY_4e_6+e_2^TH_1e_2$$
$$+(e_4-e_2)^T(S_1+S_2+S_3+S_4)(e_4-e_2)$$
$$\Omega_2=[\mathrm{E}_6^TN_3,\mathrm{E}_6^TN_4],\ \ \mathcal{R}_2=\mathrm{diag}\{R_2,3R_2\}$$
$$\Omega_1=[\mathrm{E}_4^TN_1,\mathrm{E}_4^TN_2],\ \ \mathcal{R}_1=\mathrm{diag}\{R_1,3R_1\}$$

and h_k in (8) is replaced by h_l and h_u successively.

Proof: Choose the semi-looped-functional $V(t)$ in (4). Taking the derivative of $V(t)$ for $t\in(t_k,t_{k+1})$ yields

$$\dot{V}(t)=\dot{V}_0(t)+\dot{V}_1(t)+\dot{V}_2(t)+\dot{V}_3(t).\tag{9}$$

Through Wirtinger's inequality as in [6], we know in $\dot{V}_3(t)$

$$-\int_{t_k}^{t}\dot{x}^T(s)R_1\dot{x}(s)ds\leq d_1(t)\eta^T(t)\Omega_1\mathcal{R}_1^{-1}\Omega_1^T\eta(t)$$
$$+\ \eta^T(t)\mathrm{Sym}\{\mathrm{E}_4^TN_1\mathrm{E}_2+\mathrm{E}_4^TN_2\mathrm{E}_5\}\eta(t)\tag{10}$$

$$-\int_{t}^{t_{k+1}} \dot{x}^T(s)R_2\dot{x}(s)ds \le d_2(t)\eta^T(t)\Omega_2\mathcal{R}_2^{-1}\Omega_2^T\eta(t) +\eta^T(t)\mathrm{Sym}\{\mathrm{E}_6^T N_3\mathrm{E}_3+\mathrm{E}_6^T N_4\mathrm{E}_7\}\eta(t). \quad (11)$$

From integrating (1) from t_k to t and t to t_{k+1}, there is

$$0 = 2\eta^T(t)\mathrm{E}_{10}^T W_1(d_1(t)\mathrm{E}_8-\mathrm{E}_2)\eta(t) \quad (12)$$

$$0 = 2\eta^T(t)\mathrm{E}_{10}^T W_2(d_2(t)\mathrm{E}_9-\mathrm{E}_3)\eta(t). \quad (13)$$

Combining (9)−(13), we deduce

$$\dot{V}(t) \le \eta^T(t)\Psi(t)\eta(t) \quad (14)$$

where $\Psi(t)=\Sigma(t)+d_1(t)\Omega_1\mathcal{R}_1^{-1}\Omega_1^T+d_2(t)\Omega_2\mathcal{R}_2^{-1}\Omega_2^T$.

Besides, considering (7), from the definition of Λ, we know

$$V_1(t_{k+1}^-)-V_2(t_k^+) \ge h_k\eta^T(t)\Lambda\eta(t). \quad (15)$$

Based on (14) and (15), $\Psi(t)-\Lambda<0$ ensures

$$h_k\dot{V}(t) < V_1(t_{k+1}^-)-V_2(t_k^+). \quad (16)$$

Through convex combination, $\Psi(t)-\Lambda<0$ holds for $t\in(t_k,t_{k+1})$ when $\Psi(t_k)-\Lambda<0$ and $\Psi(t_{k+1})-\Lambda<0$, which are guaranteed by (8) based on Schur complement. And through convex combination, (8) is ensured for $h_k\in[h_l,h_u]$ by replacing h_k with h_l and h_u successively.

From (16) and Lemma 3, there is $V_0(t_k)>V_0(t_{k+1})$. And because of $V_0(t_k)=x^T(t_k)Px(t_k)>0$, according to Lemma 2, the system (1) is asymptotically stable. ■

Remark 4: Theorem 1 is with less conservativeness. In Theorem 1, let $S_3=S_4=-\varepsilon I_n$, $Y_3=Y_4=H_3=0$, and remove positive definite $(e_4-e_2)^T(S_1+S_2)(e_4-e_2)$ in Λ to tighten (8), then Theorem 1 degenerates into being equivalent to the simplified Theorem 1 of [9] with Wirtinger's inequality. This shows that Theorem 1 is with relaxed LMIs to more effectively describe the stability of the system (1). In details, after the above operations, setting $h_uS_1=\tilde{T}_1$, $h_uS_2=\tilde{T}_3$, $Y_1=\mathrm{Sym}(\tilde{Q}_1^3+\tilde{Q}_1^4)$, $H_1=\mathrm{Sym}(\tilde{Q}_1^1+\tilde{Q}_1^2)$, $Y_2=\begin{bmatrix}\tilde{Q}_2^3-\tilde{Q}_1^3\\ \tilde{Q}_2^4-\tilde{Q}_1^3-2\tilde{Q}_1^4\end{bmatrix}$, $H_2=\begin{bmatrix}\tilde{Q}_2^1-2\tilde{Q}_1^1-\tilde{Q}_1^2\\ \tilde{Q}_2^2-\tilde{Q}_1^2\end{bmatrix}$, and $Z=\tilde{Z}+2\tilde{T}_2$, where $\tilde{T}_2,\tilde{Q}_1^v,\tilde{Q}_2^v,\tilde{Z}\in\mathbb{R}^{n\times n}(v=1,2,3,4)$ are any matrices and $\tilde{T}_1,\tilde{T}_3\in\mathbb{R}^{n\times n}$ are any symmetric matrices, Theorem 1 is reduced to that of [9]. Thus, compared with [9] based on the extended looped-functional, Theorem 1 based on the semi-looped-functional reduces conservatism and provides a simplified result. And considering that [4]–[7], [12] and [16] also use the looped-functional method, the new method is also able to be combined with these results to lead to improvements.

We also provide Corollary 1 to demonstrate the new method.

Corollary 1: Given positive scalars h_l and h_u, the system (1) is asymptotically stable if there exist positive definite symmetric matrices P, R_1, R_2, symmetric matrices Y_1, Y_3, Y_4, H_1, X, matrices Y_2, H_2, H_3, Z, $N_\iota(\iota=1,2,3,4)$, W_1, W_2, such that

$$\begin{bmatrix}\hat{\Sigma}(t_k)-\hat{\Lambda} & \sqrt{h_k}\Omega_2\\ * & -\mathcal{R}_2\end{bmatrix}<0,\quad \begin{bmatrix}\hat{\Sigma}(t_{k+1})-\hat{\Lambda} & \sqrt{h_k}\Omega_1\\ * & -\mathcal{R}_1\end{bmatrix}<0 \quad (17)$$

where $\hat{\Sigma}(\theta)=\Phi_0+\hat{\Phi}_1(\theta)+\hat{\Phi}_2(\theta)+\Phi_3(\theta)+\Phi_4(\theta)$, and Φ_0, Φ_3, Φ_4, Ω_2, Ω_1, $\mathcal{R}_2$, $\mathcal{R}_1$ are as in Theorem 1, $\hat{\Phi}_1$, $\hat{\Phi}_2$, $\hat{\Lambda}$ are as Φ_1, Φ_2, Λ in Theorem 1 without the terms related to S_1, S_2, S_3, S_4, and h_k in (17) is replaced by h_l and h_u successively.

Proof: Letting $S_1=S_2=\varepsilon I_n$ and $S_3=S_4=-\varepsilon I_n$ in Theorem 1, Corollary 1 is obtained. ■

Remark 5: Corollary 1 is also with less conservatism. Letting $Y_3=Y_4=H_3=0$, Corollary 1 degenerates into being equivalent to the simplified Theorem 1 of [5] with Wirtinger's inequality. In details, after the above operations, setting Y_1, H_1, Y_2, H_2 as in Remark 4, Corollary 1 is reduced to that of [5]. Thus, compared with [5] based on the two-sided looped-functional, Corollary 1 reduces conservativeness and simplifies result.

Remark 6: Based on the semi-looped-functional method, Theorem 1 and Corollary 1 provide improved stability results for the system (1). To simply and clearly demonstrate the new method, the method is combined with well-used Wirtinger's inequality in [6] and its corresponding single integral (12) and (13). The method can also be combined with the latest technologies to be further improved, for example free-matrix-based inequality and its corresponding double integral equations in [5], N-order canonical Bessel-Legendre inequalities and their corresponding integral equations in [16], the relaxed integral inequalities in [17], more extended terms in [9], and other efficient technologies in [8], [11] and [12]. Besides, the functional can also be augmented to further lead to better results.

Numerical examples: We provide two examples to verify the method and results.

Example 1 [5]: $A=\begin{bmatrix}-2 & 0\\ 0 & -0.9\end{bmatrix}$, $A_s=\begin{bmatrix}-1 & 0\\ -1 & -1\end{bmatrix}$.

Consider system (1) with the above matrices, which is assumed to be with aperiodic sampling and $h_l=10^{-5}$. There is a fact that accurate sampling interval range of stable system reflects good effectiveness of stability condition. So under fixed h_l, larger maximum sampling upper bound h_{um} of stable system represents better effectiveness of methods and results.

Table 1 presents each h_{um} obtained by Corollary 1 and the literature (here we provide Corollary 1 with constrains in Remark 3 to represent [12], and represent [5] and [9] by their Theorems 1 with Wirtinger's inequality, and so is in Example 2). From Table 1, Corollary 1 achieves the most accurate h_{um}, which shows that our result is the least conservative.

Table 1. Maximum Aperiodic Sampling Upper Bounds h_{um} Under $h_l=10^{-5}$

Results	h_{um}	Numbers of decision variables
[4]	2.5156	$5n^2+2n$
[6]	2.5156	$12n^2+3n$
[7]	2.5199	$24n^2+3n$
[11]	2.8554	$36n^2+6n$
[12]	2.9765	$30.5n^2+3.5n$
[9]	3.0621	$36.5n^2+2.5n$
[5]	3.0621	$34.5n^2+1.5n$
Corollary 1	3.0735	$33.5n^2+3.5n$

Example 2 [12]: $A=\begin{bmatrix}-1 & -1\\ -1 & -2\end{bmatrix}$, $A_s=\begin{bmatrix}0 & 0\\ -1 & -2\end{bmatrix}$.

The system is with aperiodic sampling and $h_l=0$. Table 2 lists each h_{um} calculated by various latest results. Compared with the others, Theorem 1 and Corollary 1 are the least conservative, which still be effective for large sampling intervals.

Table 2. Maximum Aperiodic Sampling Upper Bounds h_{um} UNDER $h_l=0$

Results	h_{um}	Numbers of decision variables
[7]	2.09	$24n^2+3n$
[11]	2.18	$36n^2+6n$
[12]	3.04	$30.5n^2+3.5n$
[5]	3.99	$34.5n^2+1.5n$
[9]	3.99	$36.5n^2+2.5n$
Corollary 1	4.91	$33.5n^2+3.5n$
Theorem 1	5.21	$35.5n^2+5.5n$

The above examples are used to numerically verify the theoretical improvements of Theorem 1 and Corollary 1 compared with [5] and [9] as in Remarks 4 and 5. And compared with some other previous results, although the computational complexity increases, our results obtain obviously better effectiveness.

Conclusion: The letter provides a semi-looped-functional method for the stability of sampled-data systems, which extends and improves the previous methods. Importantly, the method does not require the functional to be derivative negative definite through its discontinuity. The new method therefore leads to less conservative stability results compared with the literature. Two examples clearly show the effectiveness and improvements of the provided method and results.

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