Fully Distributed Nash Equilibrium Seeking for High-Order Players With Actuator Limitations

Maojiao Ye, *Senior Member, IEEE*, Qing-Long Han, *Fellow, IEEE*, Lei Ding, *Senior Member, IEEE*, and Shengyuan Xu

***Abstract*—This paper explores the problem of distributed Nash equilibrium seeking in games, where players have limited knowledge on other players' actions. In particular, the involved players are considered to be high-order integrators with their control inputs constrained within a pre-specified region. A linear transformation for players' dynamics is firstly utilized to facilitate the design of bounded control inputs incorporating multiple saturation functions. By introducing consensus protocols with adaptive and time-varying gains, the unknown actions for players are distributively estimated. Then, a fully distributed Nash equilibrium seeking strategy is exploited, showcasing its remarkable properties: 1) ensuring the boundedness of control inputs; 2) avoiding any global information/parameters; and 3) allowing the graph to be directed. Based on Lyapunov stability analysis, it is theoretically proved that the proposed distributed control strategy can lead all the players' actions to the Nash equilibrium. Finally, an illustrative example is given to validate effectiveness of the proposed method.**

***Index Terms*—Actuator limitation, directed networks, games, Nash equilibrium.**

I. Introduction

GAME theory has found ever-increasing potentials for applications in large-scale distributed systems, such as smart grids [1] and intelligent traffic systems [2]. As a fundamental and key issue to be addressed for game theoretical applications in such systems, Nash equilibrium seeking in neighboring-communication environments has attracted considerable attention in the past several years [3]–[5]. In practical control engineering implementation, communication issues [6] (e.g., undirected or directed), system dynamics [7] and actuator limitations [8] are all critical factors that may seriously influence control design and implementation. In this regard, to promote the penetration of game theoretical approaches for distributed control systems, it is essential to develop distributed Nash equilibrium seeking strategies taking these factors into full consideration. To broaden the applicable fields of distributed games, some efforts have been made to deal with high-order players [4]. However, for games with high-order players, there have been few results regarding actuator limitations and fully distributed designs under directed graphs.

It is well recognized that, due to hard physical constraints, it is inevitable for players to suffer from the limitation of control inputs in practical distributed game applications, which probably causes degradation or even damage of control performance. In order to address this issue, [9] and [10] made some trials to deal with distributed Nash equilibrium seeking for games in systems with actuator limitations. More specifically, bounded controls were constructed for first- and second-order integrator-type systems to find the Nash equilibrium [9]. Moreover, high-order players were considered [10]. Backstepping techniques were employed for the control design and "explosion of terms" induced by backstepping was addressed through a fixed-time sliding mode observer. However, the seeking strategies [9], [10] contain centralized control gains whose explicit quantification requires the knowledge on the network structure, the size of the game as well as the players' objective functions. These centralized information can hardly be known by each player or is at least computational costly to be calculated, thus limiting their practical applications.

As centralized information can hardly be obtained by every engaged player in practical situations, the tuning of control gains is in fact a matter of trial and error. In particular, when communication structures change or there is any player joining/leaving the game, the control gains for the designed strategies may need to be re-quantified, which implies the loss of plug and play property [11]. To address this problem, fully distributed control laws for games are developed [11]–[13] by proposing adaptive designs for control gains in neighboring communication environment. However, it should be pointed out that existing adaptive designs are only applicable for undirected graphs and cannot be directly extended to deal with directed graphs. To the best of the authors' knowledge, how to achieve fully distributed Nash equilibrium seeking under directed graphs is still an open and challenging issue. Furthermore, it is noted that, practical actuator limitations introduce

Manuscript received April 3, 2022; revised May 4, 2022; accepted May 18, 2022. This work was supported by the National Natural Science Foundation of China (62222308, 62173181, 62073171, 62221004), the Natural Science Foundation of Jiangsu Province (BK20220139, BK20200744), Jiangsu Specially-Appointed Professor (RK043STP19001), the Young Elite Scientists Sponsorship Program by China Association for Science and Technology (CAST) (2021QNRC001), 1311 Talent Plan of Nanjing University of Posts and Telecommunications, and the Fundamental Research Funds for the Central Universities (30920032203). Recommended by Associate Editor Tao Yang. *(Corresponding authors: Qing-Long Han and Lei Ding.)*

Citation: M. Ye, Q.-L. Han, L. Ding, and S. Xu, "Fully distributed Nash equilibrium seeking for high-order players with actuator limitations," *IEEE/CAA J. Autom. Sinica*, vol. 10, no. 6, pp. 1434–1444, Jun. 2023.

M. Ye and S. Xu are with the School of Automation, Nanjing University of Science and Technology, Nanjing 210094, China (e-mail: ye0003ao@e.ntu.edu.sg; syxu@njust.edu.cn).

Q.-L. Han is with the School of Science, Computing and Engineering Technologies, Swinburne University of Technology, Melbourne VIC 3122, Australia (e-mail: qhan@swin.edu.au).

L. Ding is with the Institute of Advanced Technology, Nanjing University of Posts and Telecommunications, Nanjing 210023, China (e-mail: dl522@163.com).

Color versions of one or more of the figures in this paper are available online at http://ieeexplore.ieee.org.

Digital Object Identifier 10.1109/JAS.2022.105983

high nonlinearity and bring some difficulties in distributed control design for games, but they are not taken into account in existing fully distributed schemes. Therefore, it is a nontrivial and challenging task to establish Nash equilibrium seeking strategies under bounded controls in a fully distributed manner, especially when communication topologies are directed.

Motivated by the observations above, this paper aims to develop fully distributed control laws for high-order players, which can accommodate actuator limitations and directed communication structures. Highlighting the improvements for existing algorithms, the contributions and novelties of this paper are stated as follows.

1) This paper solves games with high-order players whose control inputs are required to be bounded in a fully distributed fashion. By employing a linear transformation to convert the players' dynamics, control inputs with multiple saturation functions are constructed. Through a synthesis of an optimization method, a consensus algorithm and time-varying/adaptive gain designs, a fully distributed Nash equilibrium seeking strategy with bounded control inputs is established.

2) As first- and second-order dynamics are special cases of high-order ones, the presented methods provide alternative approaches to the problem considered in [9], while covering more general cases and removing the requirements on any global information. In addition, the presented methods can accommodate the heterogeneity on system orders and require less computation expenditure than that of [10], especially when the order of systems is high.

3) The proposed strategy is fully distributed in the sense that no centralized control gains are involved and no knowledge on any global information is required for players. In particular, compared with the adaptive designs under undirected graphs [11]–[13], the proposed strategy allows directed graphs and only requires one-hop communication, which is preferable for distributed systems.

4) The proposed methods are analytically studied and it is theoretically proven that the Nash equilibrium is globally asymptotically stable under the proposed methods. Several special cases are discussed to provide more insights on the connections with the existing works.

The remaining parts of the paper are given in the following order. Section II presents the formulated problem, and Section III proposes the control laws to achieve the goal of the paper. The context in Section IV focuses on some discussions on the presented results. Numerical verification is given in Section V and conclusions are presented in Section VI.

II. Problem Statement

This paper considers a network of high-order integrator-type players with labels from 1 to N, successively, where $N>1$ is an integer. The state of player i, denoted as $x_i \in \mathbb{R}^{m_i}$, is generated by

$$\begin{cases}\dot{x}_i = A_i x_i + B_i u_i\\ y_i = C_i x_i\end{cases}$$

where $A_i = \begin{bmatrix}\mathbf{0}_{m_i-1} & I_{m_i-1}\\ 0 & \mathbf{0}^T_{m_i-1}\end{bmatrix} \in \mathbb{R}^{m_i\times m_i}$, $B_i = [0\ \cdots\ 0\ 1]^T \in \mathbb{R}^{m_i\times 1}$, $C_i = [1\ 0\ \cdots\ 0\ 0] \in \mathbb{R}^{1\times m_i}$ and $m_i > 1$ is a positive integer. Moreover, $u_i \in \mathbb{R}$ and $y_i \in \mathbb{R}$ are the control input and output/action of player i, respectively. Assume that each player has a local objective function defined as $f_i(\boldsymbol{y}) = f_i(y_i, \boldsymbol{y}_{-i})$, where $\boldsymbol{y} = [y_1, y_2, \ldots, y_N]^T$, $\boldsymbol{y}_{-i} = [y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_N]^T$, and each player aims at minimizing $f_i(y_i, \boldsymbol{y}_{-i})$ through adjusting its own action y_i, i.e.,

$$\min_{y_i}\ f_i(y_i, \boldsymbol{y}_{-i}). \tag{1}$$

Suppose that each player cannot directly access all other players' actions and

$$|u_i| \le U_i \tag{2}$$

where U_i is a positive constant denoting the actuator limitation of player i.

The paper aims to design fully distributed u_i that satisfies (2) to drive the players' actions $\boldsymbol{y}$ to the Nash equilibrium $\boldsymbol{y}^*$, whose definition is given below.

Definition 1 (Nash equilibrium): An action profile $\boldsymbol{y}^* = (y_i^*, \boldsymbol{y}^*_{-i}) \in \mathbb{R}^N$ is a Nash equilibrium if for all $y_i \in \mathbb{R}, i \in \mathcal{V}$,

$$f_i(y_i^*, \boldsymbol{y}^*_{-i}) \le f_i(y_i, \boldsymbol{y}^*_{-i}) \tag{3}$$

where $\mathcal{V}$ is the player set given as $\mathcal{V} = \{1, 2, \ldots, N\}$.

Remark 1: It is worth mentioning that in the paper $x_i \in \mathbb{R}^{m_i}$, where m_i for $i \in \mathcal{V}$ can be different from each other, indicating that the heterogeneity on the order of the players' dynamics is allowed. To avoid any confusion, this paper says that Nash equilibrium seeking strategy is distributed if players update their actions by utilizing local information (including their own information and their neighbors' information) only, with possibility of involving some control gains whose bounds depend on global information. In contrast, the seeking strategy is said to be fully distributed if players update their actions based on their local information and all the control gains can be determined in a decentralized or distributed fashion.

For notational clarity, define $[\chi_i]_{\text{vec}}$ as a column vector whose i-th entry is χ_i. Moreover, let $[\chi_{ij}]_{\text{vec}}$ ($\text{diag}\{\chi_{ij}\}$) for $i, j \in \mathcal{V}$ be a column vector (diagonal matrix) whose entries are $\chi_{11}, \chi_{12}, \ldots, \chi_{1N}, \chi_{21}, \ldots, \chi_{NN}$, respectively. In addition, $[\chi_{ij}]$ is a matrix whose (i,j)-th entry is χ_{ij}.

The remaining sections are based on the assumptions below.

Assumption 1: The players' objective functions $f_i(\boldsymbol{y})$ for $i \in \mathcal{V}$ are twice-continuously differentiable with their gradients $\nabla_i f_i(\boldsymbol{y})$ being globally Lipschitz, i.e., for $\boldsymbol{y}, \boldsymbol{z} \in \mathbb{R}^N$

$$\|\nabla_i f_i(\boldsymbol{y}) - \nabla_i f_i(\boldsymbol{z})\| \le l_i \|\boldsymbol{y} - \boldsymbol{z}\|,\ \forall i \in \mathcal{V} \tag{4}$$

for some positive constant l_i, where $\nabla_i f_i(\boldsymbol{y}) = \frac{\partial f_i(\boldsymbol{y})}{\partial y_i}$ and $\nabla_i f_i(\boldsymbol{z}) = \frac{\partial f_i(\boldsymbol{y})}{\partial y_i}\big|_{\boldsymbol{y}=\boldsymbol{z}}$.

Assumption 2: For $\boldsymbol{y}, \boldsymbol{z} \in \mathbb{R}^N$

$$(\boldsymbol{y} - \boldsymbol{z})^T([\nabla_i f_i(\boldsymbol{y})]_{\text{vec}} - [\nabla_i f_i(\boldsymbol{z})]_{\text{vec}}) \ge \omega \|\boldsymbol{y} - \boldsymbol{z}\|^2 \tag{5}$$

for some positive constant ω.

To design fully distributed control laws, it is assumed that there is a directed communication graph among players, which is described by $\mathcal{G}(\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ and $\mathcal{E} \subseteq \mathcal{V}\times\mathcal{V}$ stand for the node set and edge set, respectively. Let $(i,j) \in \mathcal{E}$ and a_{ji} be an edge from node i to j and its weight, respectively. If $(i,j) \in \mathcal{E}$, $a_{ji} > 0$, otherwise, $a_{ji} = 0$. In this paper,

$a_{ii}=0$. A directed path is defined as a sequence of edges of the form (i_1,i_2), $(i_2,i_3),\ldots$. A directed graph is strongly connected if for every pair of distinct nodes, there is a path. Define $\mathcal{A}=[a_{ij}]$ as the adjacency matrix of $\mathcal{G}$. Then, $\mathcal{L}=\mathcal{D}-\mathcal{A}$, where $\mathcal{D}=\text{diag}\{\sum_{j=1}^N a_{ij}\}$, is the Laplacian matrix of $\mathcal{G}$ [14].

Assumption 3: The directed graph $\mathcal{G}$ is strongly connected.

Remark 2: Assumptions 1−3 are commonly adopted conditions (see, e.g., [5] and many other references therein). Assumption 2 is employed to characterize a unique Nash equilibrium, which is achieved at

$$\nabla_i f_i(\boldsymbol{y})=0,\ \ \forall i\in\mathcal{V} \tag{6}$$

and is globally exponentially stable under the gradient play for games with globally Lipschitz gradients (Assumption 1) [9]. While existing fully distributed Nash equilibrium seeking schemes [11]–[13] are established for undirected communication topologies, Assumption 3 suggests that asymmetric information exchange among the players is sufficient for the developed methods.

III. Main Results

This section develops a fully distributed Nash equilibrium seeking strategy for the considered problem, under which the associated convergence analysis is provided.

A. Strategy Design

To deal with players' dynamics, a transformation is firstly conducted by defining $x_i=\Upsilon_i\bar{x}_i$ to convert (1) to

$$\dot{\bar{x}}_i=\bar{A}_i\bar{x}_i+\bar{B}_iu_i \tag{7}$$

in which $\bar{A}_i=\begin{bmatrix}0 & \theta_i^{m_i-1} & \theta_i^{m_i-2} & \cdots & \theta_i\\ 0 & 0 & \theta_i^{m_i-2} & \cdots & \theta_i\\ 0 & 0 & 0 & \cdots & \theta_i\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & 0\end{bmatrix}$ and $\bar{B}_i=[1\ 1\ \cdots\ 1]^T$, $\Upsilon_i=\boldsymbol{R}(A_i,B_i)\boldsymbol{R}(\bar{A}_i,\bar{B}_i)^{-1}$ is a non-singular matrix, $\boldsymbol{R}(A,B)$ denotes the controllability matrix of (A,B), $\theta_i\in(0,\frac{1}{2})$ is a constant and the inverse of a non-singular matrix $\boldsymbol{R}$ is denoted as $\boldsymbol{R}^{-1}$ [15]. Based on the above transformation, the fully distributed bounded control input u_i is designed as

$$\begin{aligned}u_i=&-\sum_{k=1}^{m_i-1}\theta_i^k\phi_i(\bar{x}_{i(m_i-k+1)})\\ &-\theta_i^{m_i}\phi_i(\bar{x}_{i1}+\prod_{k=1}^{m_i-1}\theta_i^k\int_0^t\nabla_i f_i(z_i(\tau))d\tau)\end{aligned} \tag{8}$$

in which $z_i=[z_{i1},z_{i2},\ldots,z_{iN}]^T$ and

$$\begin{cases}\dot{z}_{ij}=-(c_{ij}+\rho_{ij})(\sum_{k=1}^N a_{ik}(z_{ij}-z_{kj})\\ \qquad +a_{ij}(z_{ij}+\int_0^t\nabla_j f_j(z_j(\tau))d\tau))\\ \dot{c}_{ij}=\rho_{ij}.\end{cases} \tag{9}$$

In addition,

$$\rho_{ij}=(\sum_{k=1}^N a_{ik}(z_{ij}-z_{kj})+a_{ij}(z_{ij}+\int_0^t\nabla_j f_j(z_j(\tau))d\tau))^2$$

and $c_{ij}(0)>0$. Moreover, $\phi_i(\cdot)$ is a saturation function defined as $\phi_i(\varsigma)=\text{sign}(\varsigma)\min\{|\varsigma|,\Delta_i\}$, where Δ_i is a positive constant that can be adjusted according to the actuator limitation.

Remark 3: The saturation function utilized in the control design ensures the boundedness of the control inputs. More specifically, given any positive constant U_i, one can choose Δ_i such that

$$\frac{\theta_i}{1-\theta_i}\Delta_i<U_i \tag{10}$$

to ensure that $|u_i|\le U_i$.

Remark 4: The control design (8) and (9) can be viewed as a two-modular structure. The bounded control force (8) serves as a state regulator that drives players' states to a trajectory generated by (9). The auxiliary vector z_i represents player i's local estimate on $\boldsymbol{y}$ and is produced by a leader-following consensus protocol with adaptive gains. The adaptive design, similar to [16], ensures that $\rho_{ij}(t)$ is non-negative and $c_{ij}(t)$ is monotonically increasing as $\dot{c}_{ij}(t)$ is non-negative. Moreover, θ_i can be determined in a decentralized fashion. Therefore, all the control gains are independent of any global information. In addition, the update of the auxiliary variables z_{ij} depends only on local information exchange. Hence, the control input (8) and (9) are fully distributed. Note that as the communication graph is directed in this paper, existing fully distributed Nash equilibrium seeking strategies cannot be applied.

Remark 5: In the considered scenario, each player's objective directly relies on every players' action, i.e., f_i depends on $\boldsymbol{y}$ for every $i\in\mathcal{V}$. Therefore, $z_i\in\mathbb{R}^N$ is produced to estimate $\boldsymbol{y}$. If players' objective functions only explicitly depend on a portion of other players' actions, i.e., f_i is a function of y_j for $j\in\mathcal{V}_i$ where $\mathcal{V}_i\subseteq\mathcal{V}$, interference graphs can be introduced for physical interaction descriptions among players [17]. With further designs on communication graphs, each player only needs to produce estimates on the players' actions that directly affect its objective function [17]. Note that this paper focuses on fully distributed strategy designs for high-order players with actuator limitations and directed graphs. Hence, it is directly considered that each player's objective function explicitly depends on all players' actions to avoid any distraction of readers' attention.

B. Convergence Analysis

In this section, the methods (8) and (9) are analytically investigated. Before proceeding to convergence analysis, the following supportive lemmas are given.

Lemma 1: For each $\theta_i\in(0,\frac{1}{2})$, there exists a constant $T(\theta_i)\ge 0$ such that for all $i\in\mathcal{V}$

$$|\bar{x}_{ik}(t)|\le\Delta_i,\ \ \forall t\ge T,\ \ k\in\{2,\ldots,m_i\}. \tag{11}$$

Proof: See Appendix A for the proof. ■

Remark 6: Lemma 1 demonstrates that there exists a non-negative constant T such that for $t\ge T$, $|\bar{x}_{ik}(t)|$ for all $k=2,\ldots,m_i-1,m_i$ can evolve into the unsaturated region, indicating that the effects of the saturation function on $|\bar{x}_{ik}(t)|$ for $k=2,\ldots,m_i-1,m_i$ vanish within finite time. Based on this conclusion, the stability analysis for the closed-loop system is largely simplified.

Now, we focus on the evolution of $\bar{x}_{i1}(t)$ by considering a reduced system as

$$\dot{\bar{x}}_{i1} = -\theta_i^{m_i}\phi_i(\bar{x}_{i1} + \prod_{k=1}^{m_i-1}\theta_i^k \int_0^t \nabla_i f_i(z_i(\tau))d\tau). \tag{12}$$

Let

$$\tilde{x}_{i1} = \bar{x}_{i1} + \prod_{k=1}^{m_i-1}\theta_i^k \int_0^t \nabla_i f_i(z_i(\tau))d\tau. \tag{13}$$

Then,

$$\dot{\tilde{x}}_{i1} = -\theta_i^{m_i}\phi_i(\tilde{x}_{i1}) + \prod_{k=1}^{m_i-1}\theta_i^k \nabla_i f_i(z_i(t)). \tag{14}$$

Consequently, the subsequent result can be derived.

Lemma 2: Suppose that

$$|\nabla_i f_i(z_i(t))| \leq \nu_1, \ \forall t > 0 \tag{15}$$

and there is a constant $\tilde{T}_1 \geq 0$ such that for all $t \geq \tilde{T}_1$

$$\Delta_i > \frac{2}{\theta_i^{m_i}}|\prod_{k=1}^{m_i-1}\theta_i^k \nabla_i f_i(z_i(t))|. \tag{16}$$

Then, the trajectory $\tilde{x}_{i1}(t)$ generated by (14) stays bounded for all $t \geq 0$ and there exists a $\beta \in \mathcal{KL}$ and a $\gamma \in \mathcal{K}$ such that for $t \geq \tilde{T}$

$$|\tilde{x}_{i1}(t)| \ \leq \ \beta(|\tilde{x}_{i1}(\tilde{T})|, t-\tilde{T}) + \gamma(\sup_{\tilde{T}<\tau<t} |\nabla_i f_i(z_i(\tau))|)$$

for some positive constant $\tilde{T} \geq \tilde{T}_1$.

Proof: See Appendix B for the proof. ■

Lemma 2 demonstrates that with bounded $\nabla_i f_i(z_i(t))$, the trajectory of $\tilde{x}_{i1}(t)$ will always stay bounded. In addition, if $|\nabla_i f_i(z_i(t))|$ is decreasing to be sufficiently small and stays therein thereafter, $|\tilde{x}_{i1}(t)|$ will be upper-bounded by $\beta(|\tilde{x}_{i1}(\tilde{T})|, t-\tilde{T}) + \gamma(\sup_{\tilde{T}<\tau<t}|\nabla_i f_i(z_i(\tau))|)$, indicating that if $|\nabla_i f_i(z_i(t))|$ vanishes to zero as $t \to \infty$

$$\lim_{t\to\infty} |\bar{x}_{i1}(t) + \int_0^t \prod_{k=1}^{m_i-1}\theta_i^k \nabla_i f_i(z_i(\tau))d\tau| = 0. \tag{17}$$

To this end, one needs to further consider the evolution of $\nabla_i f_i(z_i(t))$, which is investigated by considering the following auxiliary system:

$$\begin{cases}\dot{z}_{ij} = -(c_{ij}+\rho_{ij})\xi_{ij} \\ \dot{c}_{ij} = \rho_{ij}.\end{cases} \tag{18}$$

where

$$\begin{cases}\rho_{ij} = \xi_{ij}^2 \\ \xi_{ij} = \sum_{k=1}^{N} a_{ik}(z_{ij} - z_{kj}) + a_{ij}(z_{ij} + \int_0^t \nabla_j f_j(z_j(\tau))d\tau)\end{cases} \tag{19}$$

and $c_{ij}(0) > 0$. Let $\xi = [\xi_{ij}]_{\rm vec}$, $\boldsymbol{z} = [z_{ij}]_{\rm vec}$, $H = \mathcal{L} \otimes I_N + A_0$, $A_0 = {\rm diag}\{a_{ij}\}$, $c = {\rm diag}\{c_{ij}\}$ and $\rho = {\rm diag}\{\rho_{ij}\}$. Then

$$\xi = H(\boldsymbol{z} + \mathbf{1}_N \otimes [\int_0^t \nabla_i f_i(z_i(\tau))d\tau]_{\rm vec}) \tag{20}$$

and

$$\dot{\xi} = H(-(c+\rho)\xi + \mathbf{1}_N \otimes [\nabla_i f_i(z_i(t))]_{\rm vec}). \tag{21}$$

The following result can be obtained.

Lemma 3: Under Assumptions 1−3,

$$\begin{cases}\lim_{t\to\infty} \|-[\int_0^t \nabla_i f_i(z_i(\tau))d\tau]_{\rm vec} - \boldsymbol{y}^*\| = 0 \\ \lim_{t\to\infty} \|\boldsymbol{z}(t) + \mathbf{1}_N \otimes [\int_0^t \nabla_i f_i(z_i(\tau))d\tau]_{\rm vec}\| = 0.\end{cases} \tag{22}$$

Moreover, c_{ij} for all $i, j \in \mathcal{V}$ converge to some finite values.

Proof: See Appendix C for the proof. ■

Based on the above results, the convergence result can be established for the control design (8).

Theorem 1: Under Assumptions 1−3 and the control input (8)

$$\lim_{t\to\infty} \|\boldsymbol{y}(t) - \boldsymbol{y}^*\| = 0. \tag{23}$$

In addition, all other variables stay bounded and converge to finite values.

Proof: See Appendix D for the proof. ■

Theorem 1 illustrates that the Nash equilibrium is globally asymptotically stable though the boundedness of control inputs is considered. Furthermore, all the other variables (i.e., $\boldsymbol{x}_i(t)$, $c_{ij}(t)$ and $z_{ij}(t)$ for all $i, j \in \mathcal{V}$) stay bounded and converge to some finite values.

IV. Discussions on the Presented Results

In this section, some insights on the presented results, in terms of first- and second- order players, undirected graph and no actuator limitation will be provided. This helps to establish a link between the presented results and existing algorithms.

A. First- and Second-Order Integrator-Type Players

For $m_i = 1$, system (1) can be written as

$$\begin{cases}\dot{x}_{i1} = u_i \\ y_i = x_{i1}.\end{cases} \tag{24}$$

Then, one can design u_i as

$$u_i = -\phi_i(x_{i1} + \int_0^t \nabla_i f_i(z_i(\tau))d\tau) \tag{25}$$

where the definitions of other variables follow those in (8) and (9). Following Theorem 1, the subsequent corollary can be obtained.

Corollary 1: Under Assumptions 1−3 and the control input (25),

$$\lim_{t\to\infty} \|\boldsymbol{y}(t) - \boldsymbol{y}^*\| = 0 \tag{26}$$

and all other variables stay bounded and converge to some finite values.

Proof: See Appendix E for the proof. ■

Moreover, for second-order players, the seeking strategy (8) is written as

$$\begin{aligned}u_i = &-\theta_i\phi_i(x_{i2}) \\ &-\theta_i^2\phi_i(\theta_i x_{i1} + x_{i2} + \theta_i \int_0^t \nabla_i f_i(z_i(\tau))d\tau)\end{aligned} \tag{27}$$

with other variables defined in (8).

Compared with bounded controls designed for first- and second-order players [9], the control inputs (25) and (27) provide alternative designs to achieve distributed Nash equilibrium seeking with bounded controls. Moreover, the presented methods have the following elegant features:

1) The presented methods are fully distributed in the sense that they do not contain any global control gains.

2) Note that even with global Lipschitzness of $\nabla_i f_i(\boldsymbol{y})$ for $i \in \mathcal{V}$, only semi-global convergence results can be obtained for second-order players in [9] (see Theorem 5 of [9] and its corresponding analysis). Hence, compared with [9], the proposed method results in global convergence results, which shows one of the advantages of the proposed method.

3) The requirement on the boundedness of $\frac{\partial f_i(\boldsymbol{y})}{\partial y_i \partial y_j}$ for all $i, j \in \mathcal{V}$ is removed from the paper.

B. Undirected Communication Graphs

In [11], adaptive approaches are proposed to achieve fully distributed Nash equilibrium seeking for first-order players under undirected communication graphs. For the case of undirected communication graphs, u_i is designed as

$$\begin{cases} u_i = -\sum_{k=1}^{m_i-1} \theta_i^k \phi_i(\bar{x}_{i(m_i-k+1)}) \\ \qquad -\theta_i^{m_i} \phi_i(\bar{x}_{i1} + \prod_{k=1}^{m_i-1} \theta_i^k \int_0^t \nabla_i f_i(\boldsymbol{z}_i(\tau)) d\tau) \\ \dot{z}_{ij} = -c_{ij}\xi_{ij} \\ \dot{c}_{ij} = \xi_{ij}^2 \end{cases} \tag{28}$$

by following the adaptive design [11].

Correspondingly, the following corollary can be obtained.

Corollary 2: Under Assumptions 1 and 2 and the control input (28),

$$\lim_{t\to\infty} \|\boldsymbol{y}(t) - \boldsymbol{y}^*\| = 0 \tag{29}$$

and all the other variables stay bounded given that the communication graph is undirected and connected.

Proof: See Appendix F for the proof. ■

Corollary 2 indicates that under undirected communication graphs, the adaptive control law [11] can be employed for high-order players. However, the analysis depends on symmetric information exchange among the players and hence, the adaptive designs therein fail to work for directed communication graphs. Therefore, compared with existing fully distributed schemes [11]–[13], this paper has the following advantages:

1) Unlike existing fully distributed schemes that only consider undirected information exchange, the presented design in this paper can accommodate directed graphs.

2) Different from [11] that only considered first-order players, players with multi-integrator type dynamics are addressed, which cover first-order ones as special cases.

3) The control inputs in this paper are restricted within a predefined domain, while actuator limitations were not addressed in existing schemes.

4) Different from [12] and [13] that required two-hop communications, only one-hop communication is needed, which is desirable for distributed systems.

C. Without Actuator Limitation

If the system is without any actuator limitation, the saturation function can be removed from the designed controls, which gives the following control input:

$$u_i = -\sum_{k=1}^{m_i-1} \theta_i^k \bar{x}_{i(m_i-k+1)} - \theta_i^{m_i} \left(\bar{x}_{i1} + \prod_{k=1}^{m_i-1} \theta_i^k \int_0^t \nabla_i f_i(\boldsymbol{z}_i(\tau)) d\tau \right) \tag{30}$$

with other variables defined in (8) and (9).

In this case, the proposed method is still effective and the following corollary can be obtained.

Corollary 3: Under Assumptions 1−3 and (30)

$$\lim_{t\to\infty} \|\boldsymbol{y}(t) - \boldsymbol{y}^*\| = 0. \tag{31}$$

In addition, all other variables stay bounded and converge to finite values.

Proof: The proof can be completed by following Steps 2 and 3 in the proof of Theorem 1. ■

From the discussions above, it is clear that the considered problem covers the problem addressed in [9] as a special case. Moreover, for undirected graphs, the adaptive design [11] can be employed in the control design to find the Nash equilibrium in a fully distributed fashion.

V. A Numerical Example

In this section, a numerical example with 6 players is simulated. In the considered game, each player i's objective function is defined as

$$f_i(\boldsymbol{y}) = y_i^2 + y_i + (y_i - y_{i+1})^2, \ \ i \in \{1, 2, \ldots, 5\}$$
$$f_6(\boldsymbol{y}) = y_6^2 + y_6 + (y_6 - y_1)^2$$

by which the Nash equilibrium is $y_i = -0.5, \forall i \in \{1, 2, \ldots, 6\}$. In the simulation, the communication graph $\mathcal{G}$ is given in Fig. 1, which is directed and strongly connected. In addition, $\theta_i = \frac{1}{4}$, $m_i = 3$, $x_{i1}(0) = i$ and the initial conditions for all other variables in (8) and (9) are set as 1.

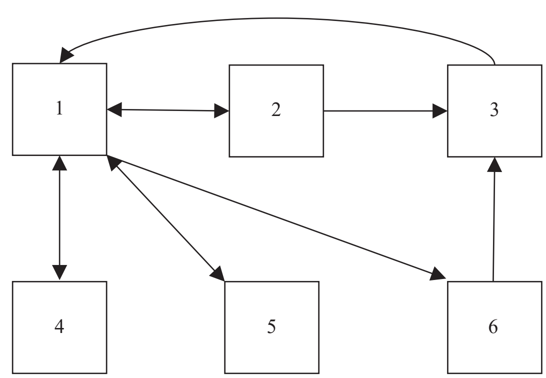

Fig. 1. The strongly connected digraph for players.

In the simulation, it is assumed that $\Delta_i = 1$, which ensures that $|u_i| \leq 0.3281$ for all $i \in \{1, 2, \ldots, 6\}$. With the control input design (8) and (9), the evolution of players' actions and control inputs are shown in Fig. 2, from which it is clear that players' actions are convergent to the Nash equilibrium and control inputs are restricted within the required domain. Moreover, the auxiliary variables $c_{ij}(t)$ and $z_{ij}(t)$ are plotted in Fig. 3, from which it can be seen that they stay bounded and converge to finite values. To this end, the convergence of the developed algorithm has been numerically validated.

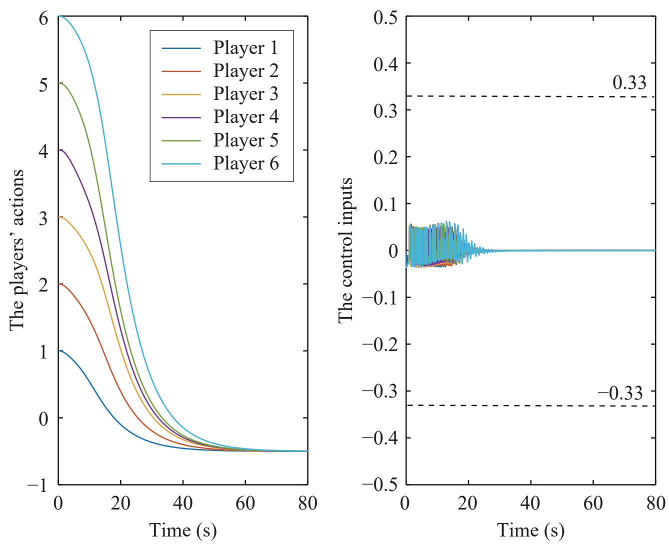

Fig. 2. The players' actions $y_i(t)$ and control signals u_i generated by (8) and (9).

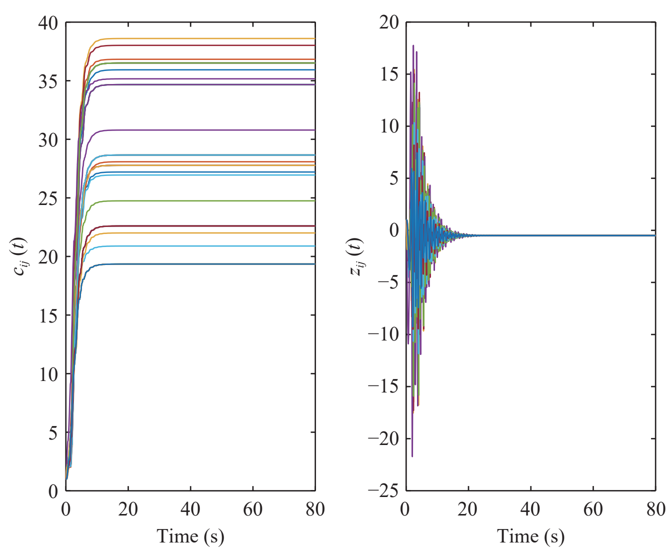

Fig. 3. The evolution of the auxiliary variables $c_{ij}(t)$ and $z_{ij}(t)$ for $i, j \in \{1, 2, \ldots, 6\}$ generated by (8) and (9).

To further illustrate the functionality of the saturation functions in the proposed method, they are removed and correspondingly the method (30) is simulated. With all the settings kept the same as the case with saturation functions, players' actions and control inputs generated by (30) are plotted in Fig. 4. From this figure, it is clear that players' actions are still

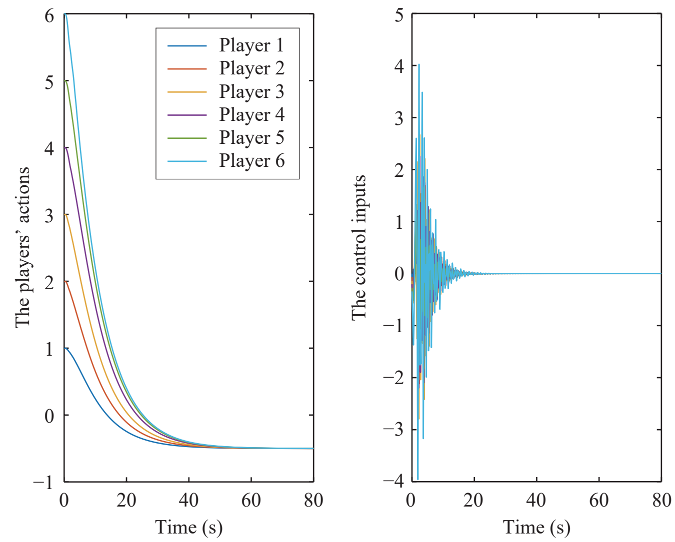

Fig. 4. The players' actions and control inputs generated by (30).

convergent to the Nash equilibrium but controls are sometimes out of $[-0.3281, 0.3281]$. Comparing Fig. 2 with Fig. 4, it can be concluded that the saturation functions are effective to restrict controls within the required domain.

VI. Conclusions

This paper contributes to finding the Nash equilibrium in a fully distributed fashion for high-order players subject to actuator limitations. A linear transformation is firstly applied to the players' dynamics, based on which multiple saturation functions are employed to develop control inputs. With the saturation functions, control inputs can be restricted within the required region. Moreover, control gains are designed to be adaptive, which allow asymmetric information exchange among players and lead to fully distributed schemes. It is proven that, by the designed bounded control inputs, players' actions are convergent to the Nash equilibrium. Distributed Nash equilibrium seeking under various communication constraints (see, e.g., [18]) will be considered in future works.

Appendix A
Proof of Lemma 1

Proof: By (7) and (8), it can be obtained that

$$\begin{aligned}
\dot{\bar{x}}_{im_i} &= -\theta_i\phi_i(\bar{x}_{im_i}) - \sum_{k=2}^{m_i-1}\theta_i^k\phi_i(\bar{x}_{i(m_i-k+1)}) \\
&\quad -\theta_i^{m_i}\phi_i(\bar{x}_{i1} + \prod_{k=1}^{m_i-1}\theta_i^k\int_0^t \nabla_i f_i(z_i(\tau))d\tau) \\
&\leq -\theta_i\phi_i(\bar{x}_{im_i}) + \frac{\theta_i^2\Delta_i}{1-\theta_i}. \qquad (32)
\end{aligned}$$

Define

$$V_{im_i} = \int_0^{\bar{x}_{im_i}} \phi_i(\tau)d\tau. \qquad (33)$$

Then, it can be easily obtained that

$$V_{im_i} = \begin{cases} \dfrac{\Delta_i^2}{2} + (|x_{im_i}| - \Delta_i)\Delta_i, & \text{if } |x_{im_i}| > \Delta_i \\ \dfrac{x_{im_i}^2}{2}, & \text{if } |x_{im_i}| \le \Delta_i. \end{cases}$$

Therefore, V_{im_i} is positive definite and radially unbounded. By Lemma 4.3 in [19], there are $\mathcal{K}_\infty$ functions α_1 and α_2 such that $\alpha_1(|\bar{x}_{im_i}|) \le V_{im_i} \le \alpha_2(|\bar{x}_{im_i}|)$. Taking the time derivative of V_{im_i} gives

$$\begin{aligned} \dot{V}_{im_i} \le& -\theta_i\phi_i^2(\bar{x}_{im_i}) + \frac{\theta_i^2\Delta_i|\phi_i(\bar{x}_{im_i})|}{1-\theta_i} \\ \le& -\sigma_1\theta_i\phi_i^2(\bar{x}_{im_i}) \end{aligned} \tag{34}$$

for $|\phi_i(\bar{x}_{im_i})| \ge \frac{\theta_i\Delta_i}{(1-\sigma_1)(1-\theta_i)}$, where $\sigma_1 \in (0,1)$ is a constant.

Case I ($|\phi_i(\bar{x}_{im_i})| = \Delta_i$): Let θ_i^* be a positive constant such that $\frac{\theta_i^*}{(1-\sigma_1)(1-\theta_i^*)} = 1$. Then, it is clear that $\dot{V}_{im_i} < 0$ is always satisfied for $\theta_i \in (0, \theta_i^*)$, indicating that if $|\bar{x}_{im_i}(0)| > \Delta_i$, $|\bar{x}_{im_i}(t)|$ is bounded and there exists a positive constant $\bar{T}_1$ such that $|\bar{x}_{im_i}(t)| \le \Delta_i$ for $t > \bar{T}_1$.

Case II ($|\phi_i(\bar{x}_{im_i})| = \bar{x}_{im_i}$): In this case, $\dot{V}_{im_i} \le -\sigma_1\theta_i\phi_i^2(\bar{x}_{im_i})$, for all $|\bar{x}_{im_i}| > \frac{\theta_i\Delta_i}{(1-\sigma_1)(1-\theta_i)}$. Note that as V_{im_i} itself is a $\mathcal{K}_\infty$ function, one can choose $\alpha_1(|\bar{x}_{im_i}|) = \alpha_2(|\bar{x}_{im_i}|) = V_{im_i}$ and hence, if $\theta_i < \theta_i^*$, $\frac{\theta_i}{(1-\sigma_1)(1-\theta_i)} < 1$. Then, there exists a class $\mathcal{KL}$ function β_1 and for every $|\bar{x}_{im_i}(0)| < \Delta_i$, there exists a constant $\tilde{T}_1 \ge 0$ such that

$$\begin{cases} |\bar{x}_{im_i}(t)| \le \beta_1(|\bar{x}_{im_i}(0)|, t), & \forall t < \tilde{T}_1 \\ |\bar{x}_{im_i}(t)| < \dfrac{\theta_i\Delta_i}{(1-\sigma_1)(1-\theta_i)}, & \forall t \ge \tilde{T}_1 \end{cases} \tag{35}$$

based on Theorem 4.19 in [19].

Summarizing the above two cases, one gets that for any initial condition

$$|\bar{x}_{im_i}(t)| \le \Delta_i, \ \ \forall t > T_1 \tag{36}$$

for some $T_1 \ge 0$. Note that for each $\theta_i \in (0, \frac{1}{2})$, $\frac{\theta_i}{(1-\sigma_1)(1-\theta_i)} < 1$ is satisfied and hence, the above conclusion holds for all $\theta \in (0, \frac{1}{2})$. Recalling that

$$\dot{\bar{x}}_{i(m_i-1)} = \theta_i\bar{x}_{im_i} + u_i$$

it can be easily obtained that there is no finite escape time for $x_{i(m_i-1)}(t)$ based on the boundedness of $\bar{x}_{im_i}$ and the control inputs. Therefore, $x_{i(m_i-1)}(t)$ would stay bounded for $t < T_1$. Moreover, for $t \ge T_1$, one has

$$\begin{aligned} \dot{\bar{x}}_{i(m_i-1)} =& \theta_i\bar{x}_{im_i} + u_i \\ =& -\theta_i^2\phi_i(\bar{x}_{i(m_i-1)}) - \sum_{k=3}^{m_i-1}\theta_i^k\phi_i(\bar{x}_{i(m_i-k+1)}) \\ & -\theta_i^{m_i}\phi_i(\bar{x}_{i1} + \prod_{k=1}^{m_i-1}\theta_i^k\int_0^t\nabla_i f_i(z_i(\tau))d\tau). \end{aligned} \tag{37}$$

Define

$$V_{i(m_i-1)} = \int_0^{\bar{x}_{i(m_i-1)}}\phi_i(\tau)d\tau.$$

Then, it can be easily obtained that

$$\dot{V}_{i(m_i-1)} \le -\theta_i^2\phi_i^2(\bar{x}_{i(m_i-1)}) + \frac{\theta_i^3\Delta_i|\phi_i(\bar{x}_{i(m_i-1)})|}{1-\theta_i}.$$

By similar analysis to that for $\bar{x}_{im_i}$, one gets that there exists a positive constant $T_2 \ge T_1$ such that

$$|\bar{x}_{i(m_i-1)}(t)| \le \Delta_i, \ \ \forall t < T_2 \tag{38}$$

given that $\theta_i \in (0, \theta_i^*)$. Repeating the above process, it can be obtained that there exists a constant $T \ge 0$ such that if $t \ge T$

$$|\bar{x}_{ik}(t)| \le \Delta_i \tag{39}$$

for all $k = 2, \ldots, m_i$. ■

Appendix B
Proof of Lemma 2

Proof: As $|\nabla_i f_i(z_i(t))| < \nu_1$ for all $t \ge 0$, one gets from (14) that

$$|\tilde{x}_{i1}(t) - \tilde{x}_{i1}(0)| \le \theta_i^{m_i}\Delta_i t + \prod_{k=1}^{m_i-1}\theta_i^k\nu_1 t \tag{40}$$

by utilizing the comparison lemma [19]. Therefore, for any bounded t, $\tilde{x}_{i1}(t)$ is bounded and the system (14) cannot have finite escape time.

The following analysis is conducted for $t \ge \tilde{T}_1$. Define

$$V_{i1} = \int_{\tilde{T}_1}^{\tilde{x}_{i1}}\phi_i(\tau)d\tau. \tag{41}$$

Then, for $t \ge \tilde{T}_1$

$$\begin{aligned} \dot{V}_{i1} =& \phi_i(\tilde{x}_{i1})(-\theta_i^{m_i}\phi_i(\tilde{x}_{i1}) + \prod_{k=1}^{m_i-1}\theta_i^k\nabla_i f_i(z_i(t))) \\ \le& -\frac{\theta_i^{m_i}\phi_i^2(\tilde{x}_{i1})}{2} \end{aligned} \tag{42}$$

for all $|\phi_i(\tilde{x}_{i1})| \ge \frac{2}{\theta_i^{m_i}}|\prod_{k=1}^{m_i-1}\theta_i^k\nabla_i f_i(z_i(t))|$.

Case I ($|\phi_i(\tilde{x}_{i1})| = \Delta_i$): If this is the case

$$\dot{V}_{i1} \le -\frac{1}{2}\theta_i^{m_i}\phi_i^2(\tilde{x}_{i1}) \tag{43}$$

is always satisfied as for $t \ge \tilde{T}_1$, $\Delta_i \ge \frac{2}{\theta_i^{m_i}}|\prod_{k=1}^{m_i-1}\theta_i^k\nabla_i f_i(z_i(t))|$, indicating that for all $|\tilde{x}_{i1}(\tilde{T}_1)| > \Delta_i$, $|\tilde{x}_{i1}(t)|$ will evolve into the unsaturated region after some finite time.

Case II ($\phi_i(\tilde{x}_{i1}) = \tilde{x}_{i1}$): In this case

$$\dot{V}_{i1} \le -\frac{1}{2}\theta_i^{m_i}\phi_i^2(\tilde{x}_{i1}) \tag{44}$$

for all $|\tilde{x}_{i1}| > \frac{2}{\theta_i^{m_i}}|\prod_{k=1}^{m_i-1}\theta_i^k\nabla_i f_i(z_i(t))|$. Therefore, by Theorem 4.18 in [19], one gets that for $t \ge \tilde{T}_1$

$$\begin{aligned} |\tilde{x}_{i1}(t)| \le& \beta(|\tilde{x}_{i1}(\tilde{T}_1)|, t - \tilde{T}_1) \\ & + \alpha_1^{-1}(\alpha_2(\sup_{\tilde{T}_1<\tau<t}\frac{2|\prod_{k=1}^{m_i-1}\theta_i^k\nabla_i f_i(z_i(\tau))|}{\theta_i^{m_i}})) \\ \le& \beta(|\tilde{x}_{i1}(\tilde{T}_1)|, t - \tilde{T}_1) + \gamma(\sup_{\tilde{T}_1<\tau<t}|\nabla_i f_i(z_i(\tau))|) \end{aligned}$$

where $\gamma(\cdot)$ is a $\mathcal{K}_\infty$ function as $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ are $\mathcal{K}_\infty$ functions (defined in the proof of Lemma 1) for all $|\tilde{x}_{i1}(\tilde{T}_1)| < \Delta_i$. To this end, the conclusions can be easily obtained by combining the above two cases. ■

Appendix C
Proof of Lemma 3

Proof: To show the convergence property of (18), let $V = V_1 + V_2 + V_3$, in which

$$V_1 = \frac{1}{2}\|[-\int_0^t \nabla_i f_i(z_i(\tau))d\tau]_{\text{vec}} - \boldsymbol{y}^*\|^2$$

$$V_2 = \epsilon \sum_{i=1}^N \sum_{j=1}^N p_{ij}(c_{ij} + \frac{\rho_{ij}}{2})\rho_{ij}$$

$$V_3 = \frac{\epsilon \sum_{i=1}^N \sum_{j=1}^N p_{ij}(c_{ij} - c^*)^2}{2}$$

where $P = \text{diag}\{p_{ij}\}$ satisfies $PH + H^T P = Q$, Q is a symmetric positive definite matrix as the communication graph is strongly connected, ϵ and c^* are positive constants to be further quantified. Then

$$\begin{aligned}\dot{V}_2 &= \epsilon \sum_{i=1}^N \sum_{j=1}^N p_{ij}(c_{ij} + \frac{\rho_{ij}}{2})\dot{\rho}_{ij} + \epsilon \sum_{i=1}^N \sum_{j=1}^N p_{ij}(\dot{c}_{ij} + \frac{\dot{\rho}_{ij}}{2})\rho_{ij} \\ &= \epsilon \sum_{i=1}^N \sum_{j=1}^N p_{ij}(c_{ij} + \rho_{ij})\dot{\rho}_{ij} + \epsilon \sum_{i=1}^N \sum_{j=1}^N p_{ij}\dot{c}_{ij}\rho_{ij} \\ &= \epsilon \sum_{i=1}^N \sum_{j=1}^N p_{ij}(c_{ij} + \rho_{ij})\dot{\rho}_{ij} + \epsilon \sum_{i=1}^N \sum_{j=1}^N p_{ij}\rho_{ij}^2. \end{aligned} \tag{45}$$

In addition

$$\begin{aligned}\dot{V}_3 &= \epsilon \sum_{i=1}^N \sum_{j=1}^N p_{ij}(c_{ij} - c^*)\dot{c}_{ij} \\ &= \epsilon \sum_{i=1}^N \sum_{j=1}^N p_{ij}(c_{ij} - c^*)\rho_{ij}. \end{aligned} \tag{46}$$

Combining (45) and (46), one can derive that

$$\begin{aligned}\dot{V}_2 + \dot{V}_3 = \epsilon \sum_{i=1}^N \sum_{j=1}^N p_{ij}(c_{ij} + \rho_{ij})\dot{\rho}_{ij} \\ + \epsilon \sum_{i=1}^N \sum_{j=1}^N p_{ij}(\rho_{ij} + c_{ij} - c^*)\rho_{ij} \end{aligned} \tag{47}$$

in which

$$\begin{aligned}&\epsilon \sum_{i=1}^N \sum_{j=1}^N p_{ij}(c_{ij} + \rho_{ij})\dot{\rho}_{ij} \\ &= 2\epsilon \sum_{i=1}^N \sum_{j=1}^N p_{ij}(c_{ij} + \rho_{ij})\xi_{ij}\dot{\xi}_{ij} \\ &= -\epsilon\xi^T(c+\rho)(PH + H^T P)(c+\rho)\xi \\ &\quad + 2\epsilon\xi^T(c+\rho)PH(\mathbf{1}_N \otimes [\nabla_i f_i(z_i(t))]_{\text{vec}}) \\ &\le -\epsilon\underline{\lambda}\xi^T(c+\rho)(c+\rho)\xi \\ &\quad + 2\epsilon\xi^T(c+\rho)PH(\mathbf{1}_N \otimes [\nabla_i f_i(z_i(t))]_{\text{vec}}) \end{aligned} \tag{48}$$

where $\underline{\lambda}$ is the minimum eigenvalue of Q.

Note that

$$\begin{aligned}&2\epsilon\xi^T(c+\rho)PH(\mathbf{1}_N \otimes [\nabla_i f_i(z_i(t))]_{\text{vec}}) \\ &\le 2\epsilon\|\xi^T(c+\rho)\|\|PH\mathbf{1}_N \otimes \boldsymbol{q}_1\| \\ &\quad + 2\epsilon\|PH\mathbf{1}_N \otimes \boldsymbol{q}_2\|\|\xi^T(c+\rho)\| \\ &\le \frac{\epsilon\underline{\lambda}}{4}\xi^T(c+\rho)(c+\rho)\xi + + \frac{\epsilon^2\underline{\lambda}}{4}\xi^T(c+\rho)(c+\rho)\xi\epsilon_1 \\ &\quad + \frac{4\epsilon N \max\{p_{ij}\}^2\|H\|^2 \max\{l_i\}^2\|H^{-1}\|^2\|\xi\|^2}{\underline{\lambda}} \\ &\quad + \frac{4N^2 \max\{p_{ij}\}^2\|H\|^2 \max\{l_i\}^2\|\boldsymbol{q}_3\|^2}{\underline{\lambda}\epsilon_1} \end{aligned} \tag{49}$$

where

$$\boldsymbol{q}_1 = [\nabla_i f_i(z_i) - \nabla_i f_i([-\int_0^t \nabla_i f_i(z_i(\tau))d\tau]_{\text{vec}})]_{\text{vec}}$$

$$\boldsymbol{q}_2 = [\nabla_i f_i([-\int_0^t \nabla_i f_i(z_i(\tau))d\tau]_{\text{vec}}) - \nabla_i f_i(\boldsymbol{y}^*)]_{\text{vec}}$$

$$\boldsymbol{q}_3 = [-\int_0^t \nabla_i f_i(z_i(\tau))d\tau]_{\text{vec}} - \boldsymbol{y}^*.$$

Moreover

$$\begin{aligned}\epsilon \sum_{i=1}^N \sum_{j=1}^N p_{ij}(\rho_{ij} + c_{ij} - c^*)\rho_{ij} \le \frac{\epsilon\underline{\lambda}}{4}\xi^T(\rho+c)(\rho+c)\xi \\ - (\epsilon\min\{p_{ij}\}c^* - \frac{\max\{p_{ij}^2\}\epsilon}{\underline{\lambda}})\|\xi\|^2. \end{aligned} \tag{50}$$

Summarizing the above inequalities, one can derive that

$$\begin{aligned}\dot{V}_2 + \dot{V}_3 \le -(\epsilon\frac{\underline{\lambda}}{2} - \frac{\epsilon^2\epsilon_1\underline{\lambda}}{4})\xi^T(c+\rho)(c+\rho)\xi \\ + p_1\|[-\int_0^t \nabla_i f_i(z_i(\tau))d\tau]_{\text{vec}} - \boldsymbol{y}^*\|^2 - p_2\|\xi\|^2 \end{aligned} \tag{51}$$

where

$$p_1 = \frac{4N^2 \max\{p_{ij}\}^2\|H\|^2 \max\{l_i\}^2}{\underline{\lambda}\epsilon_1}$$

$$p_2 = \epsilon\min\{p_{ij}\}c^* - \frac{\max\{p_{ij}^2\}\epsilon}{\underline{\lambda}} - \frac{4\epsilon N \max\{p_{ij}\}^2\|H\|^2 \max\{l_i\}^2\|H^{-1}\|^2}{\underline{\lambda}}.$$

Furthermore

$$\begin{aligned}\dot{V}_1 &= -\boldsymbol{r}^T[\nabla_i f_i(z_i)]_{\text{vec}} \\ &= -\boldsymbol{r}^T[\nabla_i f_i([-\int_0^t \nabla_i f_i(z_i(\tau))d\tau]_{\text{vec}})]_{\text{vec}} \\ &\quad - \boldsymbol{r}^T[\nabla_i f_i(z_i) - \nabla_i f_i([-\int_0^t \nabla_i f_i(z_i(\tau))d\tau]_{\text{vec}})]_{\text{vec}} \\ &\le -\omega\|\boldsymbol{r}\|^2 + \max\{l_i\}\|z + \mathbf{1}_N \otimes [\int_0^t \nabla_i f_i(z_i(\tau))d\tau]_{\text{vec}}\|\|\boldsymbol{r}\| \\ &\le -\omega\|\boldsymbol{r}\|^2 + \max\{l_i\}\|H^{-1}\|\|\xi\|\|\boldsymbol{r}\| \\ &\le -(\omega - \frac{\max\{l_i\}\|H^{-1}\|}{2\epsilon_1})\|\boldsymbol{r}\|^2 + \frac{\max\{l_i\}\|H^{-1}\|\epsilon_1}{2}\|\xi\|^2 \end{aligned}$$

where $\boldsymbol{r} = [-\int_0^t \nabla_i f_i(z_i(\tau))d\tau]_{\text{vec}} - \boldsymbol{y}^*$ is defined for notational convenience. Therefore

$$\dot{V} \le -(\frac{\epsilon\underline{\lambda}}{2} - \frac{\epsilon^2\epsilon_1\underline{\lambda}}{4})\xi^T(c+\rho)(c+\rho)\xi$$
$$-(\epsilon\min\{p_{ij}\}c^* - \frac{\max\{p_{ij}^2\}\epsilon}{\underline{\lambda}} - \frac{\max\{l_i\}\|H^{-1}\|\epsilon_1}{2} -$$
$$\frac{4\epsilon N\max\{p_{ij}\}^2\|H\|^2\max\{l_i\}^2\|H^{-1}\|^2}{\underline{\lambda}})\|\xi\|^2 - p_3\|\boldsymbol{r}\|^2$$

where $p_3 = \omega - \frac{4N^2\max\{p_{ij}\}^2\|H\|^2\max\{l_i\}^2}{\underline{\lambda}\epsilon_1} - \frac{\max\{l_i\}\|H^{-1}\|}{2\epsilon_1}$.

Choose ϵ_1 such that

$$\epsilon_1 > \frac{\max\{l_i\}\|H^{-1}\|}{2\omega} + \frac{4N^2\max\{p_{ij}\}^2\|H\|^2\max\{l_i\}^2}{\underline{\lambda}\omega}$$

and $\epsilon < \frac{2}{\epsilon_1}$. In addition

$$c^* > \frac{\max\{p_{ij}^2\}\epsilon}{\underline{\lambda}\epsilon\min\{p_{ij}\}} + \frac{\max\{l_i\}\|H^{-1}\|\epsilon_1}{2\epsilon\min\{p_{ij}\}} + \frac{4\epsilon N\max\{p_{ij}\}^2\|H\|^2\max\{l_i\}^2\|H^{-1}\|^2}{\underline{\lambda}\epsilon\min\{p_{ij}\}}. \tag{52}$$

Then

$$\dot{V} \le 0 \tag{53}$$

and V is bounded so as $[-\int_0^t \nabla_i f_i(z_i(\tau))d\tau]_{\text{vec}}$, ρ_{ij} and c_{ij}. Recalling that $\rho_{ij} = \xi_{ij}^2$, it can be obtained ξ_{ij} is bounded. In addition, for $\dot{V} = 0$, $\|\xi\| = 0$, and $\|-[\int_0^t \nabla_i f_i(z_i(\tau))d\tau]_{\text{vec}} - \boldsymbol{y}^*\| = 0$. By further recalling the definition of c_{ij}, one can obtain that it is monotonically increasing, and hence it converges to some finite value as it is bounded. ■

Appendix D
Proof of Theorem 1

Proof: The proof can be completed by several steps.

Step 1: Analyze the evolution of the system for $t \le T$ and $t > T$, respectively. According to Lemmas 1−3, there is no finite escape time for $\bar{x}_{ik}$, z_{ij} and c_{ij} where $i,j \in \mathcal{V}$ and $k \in \{1,2,\ldots,m_i\}$, indicating that for $t < T$, $\bar{x}_{ik}(t)$, $z_{ij}(t)$ and $c_{ij}(t)$ are all bounded. Moreover, by Lemma 1, it can be obtained that for $t > T$,

$$\begin{cases} \dot{\bar{x}}_{i1} = -\theta_i^{m_i}\phi_i(\bar{x}_{i1} + \prod_{k=1}^{m_i-1}\theta_i^k\int_0^t \nabla_i f_i(z_i(\tau))d\tau) \\ \dot{z}_{ij} = -(c_{ij}+\rho_{ij})\xi_{ij} \\ \dot{c}_{ij} = \rho_{ij} \end{cases} \tag{54}$$

where $\rho_{ij} = \xi_{ij}^2$.

Step 2: Analyze the evolution of $\tilde{x}_{i1}$ for $t \to \infty$. By Lemma 3

$$\lim_{t\to\infty}\|-[\int_0^t \nabla_j f_j(z_j(\tau))d\tau]_{\text{vec}} - \boldsymbol{y}^*\| = 0 \tag{55}$$

and hence, by Barbarlat's Lemma [19], one gets that

$$\lim_{t\to\infty}\nabla_j f_j(z_j(t)) = 0 \tag{56}$$

indicating that there exists a positive constant $T_1 > T$ such that for all $t > T_1$

$$|\tilde{x}_{i1}(t)| \le \beta(|\tilde{x}_{i1}(T_1)|, t-T_1) + \gamma(\sup_{T_1<\tau<t}|\nabla_i f_i(z_i(\tau))|)$$

by Lemma 3. Recalling that $\lim_{t\to\infty}\nabla_j f_j(z_j(t)) = 0$, it is clear that $\lim_{t\to\infty}|\tilde{x}_{i1}(t)| = 0$.

Step 3: Analyze the steady state of $\bar{x}_{ik}$ for $k \in \{2,\ldots,m_i\}$. Recalling the dynamics (7), it can be obtained that for $t > T$

$$\dot{\bar{x}}_{i2} = -\theta_i^{m_i-1}\bar{x}_{i2} - \theta_i^{m_i}\phi_i(\tilde{x}_{i1}). \tag{57}$$

Regard $v_{im_i} = \theta_i^{m_i}\phi_i(\tilde{x}_{i1})$ as a virtual control input. Then, it can be easily obtained that the system (57) is input-to-state stable by defining a Laypunov candidate function as $\bar{V} = \frac{1}{2}\bar{x}_{i2}^2$. As for $t \to \infty$, $|v_{im_i}(t)|$ vanishes to zero, one gets that $\lim_{t\to\infty}|\bar{x}_{i2}(t)| = 0$. Moreover, for $t > T$

$$\dot{\bar{x}}_{i3} = -\theta_i^{m_i-2}\bar{x}_{i3} - \theta_i^{m_i-1}\bar{x}_{i2} - \theta_i^{m_i}\phi_i(\tilde{x}_{i1}). \tag{58}$$

Let $v_{i(m_i-1)} = -\theta_i^{m_i-1}\bar{x}_{i2} - \theta_i^{m_i}\phi_i(\tilde{x}_{i1})$ be the virtual control input, then, it can be easily obtained that (58) is input-to-state stable. Noticing that $\lim_{t\to\infty}|v_{i(m_i-1)}(t)| = 0$, one gets that $\lim_{t\to\infty}|\bar{x}_{i3}(t)| = 0$.

Repeating the above process, one gets that

$$\lim_{t\to\infty}|\bar{x}_{ik}(t)| = 0,\ \ \forall k \in \{2,\ldots,m_i\}.$$

Step 4: Analyze the steady state of $\boldsymbol{y}(t)$. Recalling that $x_i = \Upsilon_i^{-1}\bar{x}_i$, and $y_i = x_{i1}$, one can obtain that

$$y_i = \frac{\bar{x}_{i1}}{\prod_{k=1}^{m_i-1}\theta_i^k} + \sum_{k=2}^{m_i} g_k(\theta_i)\bar{x}_{ik} \tag{59}$$

where $g_k(\theta_i)$ denotes some function of θ_i.

Note that by Lemma 3

$$\lim_{t\to\infty}\|-[\int_0^t \nabla_i f_i(z_i(\tau))d\tau]_{\text{vec}} - \boldsymbol{y}^*\| = 0 \tag{60}$$

and

$$\lim_{t\to\infty}\|\boldsymbol{z}(t) + \mathbf{1}_N \otimes [\int_0^t \nabla_i f_i(z_i(\tau))d\tau]_{\text{vec}}\| = 0 \tag{61}$$

then it is clear that

$$\lim_{t\to\infty}\|\boldsymbol{y}(t) - \boldsymbol{y}^*\| = 0 \tag{62}$$

by further noticing that

$$\lim_{t\to\infty} y_i(t) = \frac{\bar{x}_{i1}(t)}{\prod_{k=1}^{m_i-1}\theta_i^k} \tag{63}$$

and

$$\lim_{t\to\infty}\bar{x}_{i1}(t) = \prod_{k=1}^{m_i-1}\theta_i^k y_i^* \tag{64}$$

for all $i \in \mathcal{V}$. To this end, the conclusions are obtained. ■

Appendix E
Proof of Corollary 1

Proof: In this case

$$\begin{cases} \dot{x}_{i1} = -\phi_i(x_{i1} + \int_0^t \nabla_i f_i(z_i(\tau))d\tau) \\ \dot{z}_{ij} = -(c_{ij}+\rho_{ij})\xi_{ij} \\ \dot{c}_{ij} = \rho_{ij} \end{cases} \tag{65}$$

where $\rho_{ij} = \xi_{ij}^2$.

By Lemma 3

$$\lim_{t\to\infty} \|[-\int_0^t \nabla_i f_i(z_i(\tau))d\tau]_{\text{vec}} - \boldsymbol{y}^*\| = 0 \tag{66}$$

and hence

$$\lim_{t\to\infty} \|\boldsymbol{y}(t) - \boldsymbol{y}^*\| = 0.$$

Following Step 2 in the proof of Theorem 1:

$$\lim_{t\to\infty} |\tilde{x}_{i1}(t)| = 0 \tag{67}$$

in which $\tilde{x}_{i1}(t) = x_{i1} + \int_0^t \nabla_i f_i(z_i(\tau))d\tau$ in this case, from which the conclusions can be easily obtained and thus, the rest of proof is omitted. ■

Appendix F
Proof of Corollary 2

Proof: To prove the result, define an auxiliary system as

$$\begin{cases} \dot{z}_{ij} = -c_{ij}\xi_{ij} \\ \dot{c}_{ij} = \xi_{ij}^2. \end{cases} \tag{68}$$

Define

$$\begin{aligned} V = &\frac{1}{2}\| - [\int_0^t \nabla_j f_j(z_j(\tau))d\tau]_{\text{vec}} - \boldsymbol{y}^*\|^2 \\ &+ (z + \mathbf{1}_N \otimes [\int_0^t \nabla_j f_j(z_j(\tau))d\tau]_{\text{vec}})^T H \\ &\times (z + \mathbf{1}_N \otimes [\int_0^t \nabla_j f_j(z_j(\tau))d\tau]_{\text{vec}}) + \sum_{i=1}^N \sum_{j=1}^N (c_{ij} - c_{ij}^*)^2. \end{aligned}$$

Then, following the proof of Lemma 3 and [11], one gets that:

$$\lim_{t\to\infty} \| - [\int_0^t \nabla_j f_j(z_j(\tau))d\tau]_{\text{vec}} - \boldsymbol{y}^*\| = 0 \tag{69}$$

and

$$\lim_{t\to\infty} \|z + \mathbf{1}_N \otimes [\int_0^t \nabla_j f_j(z_j(\tau))d\tau]_{\text{vec}}\| = 0 \tag{70}$$

for (68). The rest of proof follows those in Theorem 1 and is omitted. ■

References

[1] Y. Wan, J. Qin, F. Li, X. Yu, and Y. Kang, "Game theoretic-based distributed charging strategy for PEVs in a smart charging station," *IEEE Trans. Smart Grid*, vol. 12, no. 1, pp. 538–547, 2021.

[2] N. Xiao, X. Wang, L. Xie, T. Wongpiromsarn, E. Frazzoli, and D. Rus, "Road pricing design based on game theory and multi-agent consensus," *IEEE/CAA J. Autom. Sinica*, vol. 1, no. 1, pp. 31–39, 2014.

[3] J. Koshal, A. Nedic, and U. Shanbhag, "Distributed algorithms for aggregative games on graphs," *Operations Research*, vol. 64, pp. 680–704, 2016.

[4] A. R. Romano and L. Pavel, "Dynamic NE seeking for multi-integrator networked agents with disturbance rejection," *IEEE Trans. Control Network Systems*, vol. 7, no. 1, pp. 129–139, 2020.

[5] M. Ye, D. Li, Q.-L. Han, and L. Ding, "Distributed Nash equilibrium seeking for general networked games with bounded disturbances," *IEEE/CAA J. Autom. Sinica*, vol. 10, no. 2, pp. 376–387, 2023.

[6] C. Peng, J. Wu, and E. Tian, "Stochastic event-triggered H_∞ control for networked systems under denial of service attacks," *IEEE Trans. Systems, Man, and Cybernetics: Systems*, vol. 52, no. 7, pp. 4200–4210, 2022.

[7] X. Dong, J. Xi, G. Lu, and Y. Zhong, "Formation control for high-order linear time-invariant multiagent systems with time delays," *IEEE Trans. Control of Network Systems*, vol. 1, no. 3, pp. 232–240, 2014.

[8] G. Lin, H. Li, H. Ma, D. Yao, and R. Lu, "Human-in-the-loop consensus control for nonlinear multi-agent systems with actuator faults," *IEEE/CAA J. Autom. Sinica*, vol. 9, no. 1, pp. 111–122, 2022.

[9] M. Ye, "Distributed Nash equilibrium seeking for games in systems with bounded control inputs," *IEEE Trans. Autom. Control*, vol. 66, no. 8, pp. 3833–3839, 2021.

[10] X. Ai and L. Wang, "Distributed adaptive Nash equilibrium seeking and disturbance rejection for noncooperative games of high-order nonlinear systems with input saturation and input delay," *Int. J. Robust and Nonlinear Control*, vol. 31, pp. 2827–2846, 2021.

[11] M. Ye and G. Hu, "Adaptive approaches for fully distributed Nash equilibrium seeking in networked games," *Automatica*, vol. 129, no. 3, p. 109661, 2021.

[12] C. De Persis and S. Grammatico, "Distributed averaging integral Nash equilibrium seeking on networks," *Automatica*, vol. 110, p. 108548, 2019.

[13] M. Bianchi and S. Grammatico, "Continuous-time fully distributed generalized Nash equilibrium seeking for multi-integrator agents," *Automatica*, vol. 129, p. 109660, 2021.

[14] F. Lewis, H. Zhang, K. Hengster-Movric, and A. Das, *Cooperative Control of Multi-Agent Systems: Optimal and Adaptive Design Approaches*, Springer-Verlag London, 2014.

[15] H. J. Sussmann, E. D. Sontag, and Y. Yang, "A general result on stabilization of linear systems using bounded controls," *IEEE Trans. Autom. Control*, vol. 39, no. 12, pp. 2411–2425, 1994.

[16] X. Li, Z. Sun, Y. Tang, and H. R. Karimi, "Adaptive event-triggered consensus of multi-agent systems on directed graphs," *IEEE Trans. Autom. Control*, vol. 66, no. 4, pp. 1670–1685, 2021.

[17] F. Salehisadaghiani and L. Pavel, "Distributed Nash equilibrium seeking in networked graphical games," *Automatica*, vol. 87, pp. 17–24, 2018.

[18] X.-M. Zhang, Q.-L. Han, X. Ge, D. Ding, L. Ding, D. Yue, and C. Peng, "Networked control systems: A survey of trends and techniques," *IEEE/CAA J. Autom. Sinica*, vol. 7, no. 1, pp. 1–17, 2020.

[19] H. K. Khalil, *Nonlinear Systems*, Upper Saddle River, NJ: Prentice Hall, 2002.

Maojiao Ye (Senior Member, IEEE) received the B.Eng. degree in automation from University of Electronic Science and Technology of China in 2012 and the Ph.D. degree from Nanyang Technological University, Singapore, in 2016. She is currently a Professor with the School of Automaton, Nanjing University of Science and Technology. Prior to her current position, she was a Research Fellow at the School of Electrical and Electronic Engineering, Nanyang Technological University, Singapore, from 2016−2017. Her research interests include game theory, distributed optimization, and their applications.

Professor Ye was a recipient of Guan Zhao-Zhi Award in the 36th Chinese Control Conference 2017 (first author), the Best Paper Award in the 15th IEEE International Conference on Control and Automation 2019 (sole author), and the Best Student Paper Award in the 5th Chinese Conference on Swarm Intelligence and Cooperative Control 2021. She is an Early Career Advisory Board Member of Control Engineering Practice and the Secretary of IEEE IES Technical Committee on Network-Based Control Systems and Applications.

Qing-Long Han (Fellow, IEEE) received the B.Sc. degree in mathematics from Shandong Normal University in 1983, and the M.Sc. and the Ph.D. degrees in control engineering from East China University of Science and Technology, in 1992 and 1997, respectively.

Professor Han is Pro Vice-Chancellor (Research Quality) and a Distinguished Professor at Swinburne University of Technology, Australia. He held various academic and management positions at Griffith University and Central Queensland University, Australia. His research interests include networked control systems, multi-agent systems, time-delay sys-

tems, smart grids, unmanned surface vehicles, and neural networks.

Professor Han was awarded the 2021 Norbert Wiener Award (the Highest Award in systems science and engineering, and cybernetics) and the 2021 M. A. Sargent Medal (the Highest Award of the Electrical College Board of Engineers Australia). He was the recipient of the 2021 IEEE/CAA Journal of Automatica Sinica Norbert Wiener Review Award, the 2020 IEEE Systems, Man, and Cybernetics (SMC) Society Andrew P. Sage Best Transactions Paper Award, the 2020 IEEE Transactions on Industrial Informatics Outstanding Paper Award, and the 2019 IEEE SMC Society Andrew P. Sage Best Transactions Paper Award.

Professor Han is a Member of the Academia Europaea (the Academy of Europe) and a Fellow of the Institution of Engineers Australia. He is a Highly Cited Researcher in both Engineering and Computer Science (Clarivate Analytics). He has served as an AdCom Member of IEEE Industrial Electronics Society (IES), a Member of IEEE IES Fellows Committee, a Member of IEEE IES Publications Committee, and Chair of IEEE IES Technical Committee on Networked Control Systems. He is Co-Editor-in-Chief of *IEEE Transactions on Industrial Informatics*, Deputy Editor-in-Chief of *IEEE/CAA Journal of Automatica Sinica*, Co-Editor of *Australian Journal of Electrical and Electronic Engineering*, an Associate Editor for 12 international journals, including the *IEEE Transactions on Cybernetics*, *IEEE Industrial Electronics Magazine*, *Control Engineering Practice*, and *Information Sciences*, and a Guest Editor for 14 Special Issues.

Lei Ding (Senior Member, IEEE) received the B.Eng. degree in automation, and the M.Eng. and the Ph.D. degrees in control theory and control engineering from Dalian Maritime University, in 2007, 2009, and 2014, respectively. From 2010 to 2012, he was a Visiting Ph.D. Student sponsored by China Scholarship Council at the Centre for Intelligent and Networked Systems, Central Queensland University, Australia. From 2015 to 2016, he was a Visiting Research Fellow with Western Sydney University, Australia. From 2016 to 2017, he was a Postdoctoral Associate with Wayne State University, USA. From 2017 to 2019, he worked as a Research Fellow with the Swinburne University of Technology, Australia. He is currently a Full Professor at the Nanjing University of Posts and Telecommunications. His research interests include cooperative control of networked systems and their applications to smart grids.

Professor Ding was the recipient of the 2021 IEEE/CAA Journal of Automatica Sinica Norbert Wiener Review Award, the 2020 IEEE Transactions on Industrial Informatics Outstanding Paper Award, and the 2019 IEEE Systems, Man, and Cybernetics (SMC) Society Andrew P. Sage Best Transactions Paper Award (IEEE Transactions on Cybernetics). He received the 2018 IEEE Transactions on Cybernetics Outstanding Reviewer Award.

Professor Ding is an Early Career Advisory Board Member of *IEEE/ CAA Journal of Automatica Sinica* and *Control Engineering Practice*. He has served as a Guest Editor of *Control Engineering Practice*, *IEEE Transactions on Systems, Man*, *and Cybernetics: Systems* and *Information Sciences*; the Chair of IEEE IES Technical Committee on Network-Based Control Systems and Applications; and the Technical Track Co-Chair for several international conferences.

Shengyuan Xu received the B.Sc. degree from Hangzhou Normal University in 1990, the M.Sc. degree from Qufu Normal University in 1996, and the Ph.D. degree from Nanjing University of Science and Technology in 1999. From 1999 to 2000, he was a Research Associate in the Department of Mechanical Engineering at the University of Hong Kong, China. From December 2000 to November 2001, and December 2001 to September 2002, he was a Postdoctoral Researcher in CESAME at the University Catholique De Louvain, Belgium, and the Department of Electrical and Computer Engineering at the University of Alberta, Canada, respectively. In November 2002, he joined the School of Automation at Nanjing University of Science and Technology, as a Professor.

Professor Xu was a recipient of the National Excellent Doctoral Dissertation Award in the year 2002 from the Ministry of Education of China. He obtained a grant from the National Science Foundation for Distinguished Young Scholars of China in 2006. He was awarded a Cheung Kong Professorship in the year 2008 from the Ministry of Education of China. He is a Member of the Editorial Boards of the *Transactions of the Institute of Measurement and Control*, and the *Journal of the Franklin Institute*. His current research interests include robust filtering and control, singular systems, time-delay systems, neural networks, multidimensional systems, and nonlinear systems.