Letter

Relaxed Stability Criteria for Delayed Generalized Neural Networks via a Novel Reciprocally Convex Combination

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Dear Editor,

This letter examines the stability issue of generalized neural networks (GNNs) with time-varying delay based on a novel reciprocally convex combination (RCC). By considering a new matrix polynomial, the proposed novel reciprocally convex method leads to a tight bound for integral inequality combination and encompasses several existing approaches as special cases. The relaxed stability conditions with less conservatism are developed by employing the proposed reciprocally convex combination and the Lyapunov-Krasovskii (L-K) functional. Finally, several numerical examples are conducted to show the superiorities of the stability conditions.

GNNs are composed of an enormous amount of connected units or nodes called artificial neurons, which are connected to simulate neurons in a biological brain. As we know, time-varying delays are an inevitable factor in GNNs because of the limited switching speed and communication bandwidth. Stability is a prerequisite for dynamic system analysis and application, but time delays may lead to unacceptable dynamic responses or instability [1], [2]. Thus, it is fundamental to focus on the stability of the GNNs with time delay.

The convex combination $((1/\varrho)\eta_1^T Z\eta_1 + 1/(1-\varrho)\eta_2^T Z\eta_2)$ is frequently encountered when evaluating the integral term in the L-K functional. RCC plays a key role in dealing with the combination in the stability analysis of delayed GNNs. In the last few years, a number of improvements in RCC have been obtained [3]–[5]. In [3], an RCC was developed by introducing a high-order matrix polynomial. A ϱ^2-dependent RCC was proposed in [4] by introducing ϱ^2 related terms to study the dissipativity issue of delayed GNNs. By employing m-degree matrix-valued polynomial, a generalized RCC was established in [5] to investigate delayed neural networks. From the above study, it is clear that matrix-valued polynomials have been widely employed in RCC. These slack matrix variables are valid for reducing the RCC estimation errors. However, few studies developed the non-affine RCC for the study of delayed GNNs. There is much room for development in the RCC to estimate the combination accurately.

The second rote is constructing the L-K functional that contains more information about the neuron activation function and system state. A suitable L-K functional can build the connection among neuron activation function and system vectors of GNNs. Thus, an L-K functional that contains more information is conclusive for diminishing the conservatism of the conditions.

This letter establishes an RCC by considering a new matrix polynomial to evaluate the convex combination. The proposed RCC includes some inequalities in [5]–[7] as exceptional cases and can be instantly dealt with by linear matrix inequality (LMI). Then, the stability conditions with less conservatism are derived by exploiting the new RCC and the L-K functional with delay-product terms. Several numerical examples are implemented to verify the effectiveness of the stability conditions.

Notation: $\mathbb{S}^n(\mathbb{S}^{n+})$ denotes the set of $n\times n$ symmetric (positive-definite) real matrices. $He\{O\} = O^T + O$. Symmetric terms are expressed by "*". $C_m^n = \frac{m!}{(m-n)!n!}$.

Problem statement: Consider a category of delayed GNNs as follows:

$$\begin{cases}\dot{x}(t) = -Ax(t) + W_0 f(W_2 x(t)) + W_1 f(W_2 x(t-d_t))\\ x(t) = \omega(t), \quad t\in[-\bar{d},0]\end{cases} \tag{1}$$

where $\omega(t)$ and $x(t)\in\mathbb{R}^n$ represent the initial condition and system neuron state, respectively. W_i are the interconnection weight matrices. $f(W_2x(t)) = col\{f_1(W_{21}x(t)),\ldots,f_n(W_{2n}x(t))\}$ stands for the neuron activation function. A is a diagonal matrix consisting of positive numbers. The time-varying delay d_t satisfying

$$0\le d_t\le\bar{d},\ \ \underline{u}\le\dot{d}_t\le\bar{u}. \tag{2}$$

The activation function $f(*)$ satisfies $f_i(0)=0$ and

$$k_{mi}\le\frac{f_i(\tau_1)-f_i(\tau_2)}{\tau_1-\tau_2}\le k_{pi} \tag{3}$$

where $\tau_1\ne\tau_2$, k_{mi} and k_{pi} are given constants. Let $K_p = \mathrm{diag}\{k_{p1},\ldots,k_{pn}\}$ and $K_m = \mathrm{diag}\{k_{m1},\ldots,k_{mn}\}$.

Main results: The RCC is a crucial lever in the study of GNNs with time-varying delay. Then, a novel RCC is achieved by introducing a matrix polynomial.

Proposition 1: For a given m and $Z\in\mathbb{S}_+^n$, if there exist $V_j\in\mathbb{R}^{n\times n}(j=0,\ldots,m)$, $W_i\in\mathbb{S}^n(i=1,\ldots,2m)$, such that

$$\mathcal{S}-\mathcal{G}_1^T\begin{bmatrix}G & S\\ * & -G\end{bmatrix}\mathcal{G}_1>0 \tag{4}$$

for $\forall\varrho\in(0,1)$, then the following condition:

$$\begin{bmatrix}\frac{1}{\varrho}Z & O\\ * & \frac{1}{1-\varrho}Z\end{bmatrix}>\sum_{k=0}^{m}\varrho^k\mathcal{U}_k \tag{5}$$

holds for $\forall\varrho\in(0,1)$, where

$$\mathcal{S}=\begin{bmatrix}\mathcal{S}_0+\mathcal{Z} & \frac{1}{2}\mathcal{S}_1 & & & \\ \frac{1}{2}\mathcal{S}_1 & \mathcal{S}_2 & & & \\ & \ddots & \ddots & \ddots & \\ & & & \mathcal{S}_{2[\frac{m}{2}]-2} & \frac{1}{2}\mathcal{S}_{2[\frac{m}{2}]-1}\\ & & & \frac{1}{2}\mathcal{S}_{2[\frac{m}{2}]-1} & \mathcal{S}_{2[\frac{m}{2}]}\end{bmatrix}$$

$$\mathcal{S}_j=-\sum_{i=1}^{m}C_i^j(-1)^j\mathcal{W}_i-\mathcal{V}_j,\ \ \mathcal{Z}=\mathrm{diag}\{Z,Z\}$$

$$\mathcal{U}_j=\begin{cases}\begin{bmatrix}Z+W_1 & V_0\\ * & Z\end{bmatrix}, & j=0\\ \sum_{i=1}^{m-1}C_i^j(-1)^j\mathcal{X}_{1i}+\mathcal{X}_{2j}+C_m^j(-1)^j\mathcal{X}_3, & j=1,\ldots,m-1\\ \begin{bmatrix}-W_{2m-1} & V_m\\ * & (-1)^{m+1}W_{2m}\end{bmatrix}, & j=m\end{cases}$$

$$\mathcal{W}_i=\begin{bmatrix}O & O\\ * & W_{2i}\end{bmatrix},\ \mathcal{V}_j=\begin{bmatrix}W_{2j-1} & V_j\\ * & O\end{bmatrix},\ \mathcal{X}_3=\begin{bmatrix}O & O\\ * & -W_{2m}\end{bmatrix}$$

$$\mathcal{X}_{1i}=\begin{bmatrix}O & O\\ * & W_{2i+2}-W_{2i}\end{bmatrix},\ \mathcal{X}_{2i}=\begin{bmatrix}W_{2i+1}-W_{2i-1} & V_i\\ * & O\end{bmatrix}.$$

Proof: According to (4) and Theorem 1 in [8], the following condition with ϱ holds:

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$$0<\sum_{j=0}^{m}\varrho^j\mathcal{S}_j+\mathcal{Z}. \tag{6}$$

The following condition is yielded by substituting $\mathcal{S}_j$ into (6):

$$\begin{aligned}0&<-\sum_{j=0}^{m}\left(\sum_{i=1}^{m}C_i^j(-1)^j\mathcal{W}_i+\mathcal{V}_j\right)\varrho^j+\mathcal{Z}\\&=-\sum_{i=0}^{m}\left((1-\varrho)^i\mathcal{W}_i+\varrho^i\mathcal{V}_i\right)+\mathcal{Z}.\end{aligned} \tag{7}$$

By pre-multiplying and post-multiplying (7) by $\mathrm{diag}\left\{\sqrt{\frac{1-\varrho}{\varrho}}I,\sqrt{\frac{\varrho}{1-\varrho}}I\right\}$, we have

$$\begin{aligned}\mathcal{Z}&>\sum_{i=0}^{m}\left((1-\varrho)^{i-1}\varrho\mathcal{W}_i+\varrho^i\begin{bmatrix}\frac{1-\varrho}{\varrho}W_{2i-1} & V_i\\ * & O\end{bmatrix}\right)\\&=\sum_{i=0}^{m}\left((1-\varrho)^i\begin{bmatrix}O & O\\ * & -W_{2i}\end{bmatrix}+\varrho^i\begin{bmatrix}-W_{2i-1} & V_i\\ * & O\end{bmatrix}\right.\\&\quad\left.+(1-\varrho)^{i-1}\begin{bmatrix}O & O\\ * & W_{2i}\end{bmatrix}+\varrho^{i-1}\begin{bmatrix}W_{2i-1} & O\\ * & O\end{bmatrix}\right)\end{aligned} \tag{8}$$

where $\mathcal{Z}=\begin{bmatrix}\frac{1-\varrho}{\varrho}Z & O\\ * & \frac{\varrho}{1-\varrho}Z\end{bmatrix}$.

By adding $\mathcal{Z}$ to both sides of (8), we obtain

$$\begin{aligned}\begin{bmatrix}\frac{1}{\varrho}Z & O\\ * & \frac{1}{1-\varrho}Z\end{bmatrix}&>\sum_{i=1}^{m-1}((1-\varrho)^i\mathcal{X}_{1i}+\varrho^i\mathcal{X}_{2i})+\mathcal{Z}\\&\quad+\begin{bmatrix}W_1 & V_0\\ * & W_2\end{bmatrix}+(1-\varrho)^m\mathcal{X}_3+\varrho^m\begin{bmatrix}-W_{2m-1} & V_m\\ * & O\end{bmatrix}\\&=\sum_{k=0}^{m-1}\Big(\sum_{i=1}^{m-1}C_i^k(-1)^k\mathcal{X}_{1i}+\mathcal{X}_{2k}+C_m^k(-1)^k\mathcal{X}_3)\varrho^k\\&\quad+\begin{bmatrix}-W_{2m-1} & V_m\\ * & (-1)^{m+1}W_{2m}\end{bmatrix}\varrho^m+\begin{bmatrix}Z & O\\ * & Z+W_2\end{bmatrix}\\&=\sum_{k=0}^{m}\varrho^k\mathcal{U}_k.\end{aligned} \tag{9}$$

■

Remark 1: Proposition 1 is achieved as a generalized reciprocally convex inequality with different orders by considering a new matrix-valued polynomial. Some necessary variable matrices W_i and V_j are integrated into the proposed RCC, which has positive consequences for reducing the conservatism of stability criteria. Furthermore, Proposition 1 includes the RCCs in [5]–[7] as special cases.

1) By considering $V_0=\sum_{j=0}^{m}\bar{M}_j$ and $V_i=\sum_{j=1}^{i}C_i^j(-1)^j\bar{M}_i+\bar{Y}_i$, $(i=1,\dots,m)$, (5) in Proposition 1 is identical to Lemma 4 in [5].

2) By considering $m=2$, (5) decreases to

$$\begin{bmatrix}\frac{1}{\varrho}R & O\\ * & \frac{1}{1-\varrho}R\end{bmatrix}>\begin{bmatrix}M_1 & M_2\\ * & M_3\end{bmatrix} \tag{10}$$

where $M_1=-W_3\varrho^2+(W_3-W_1)\varrho+M_1$, $M_2=V_0+V_1\varrho+V_2\varrho^2$, $M_3=-W_4\varrho^2+(W_2+W_4)\varrho$. Note that the inequality in [6] can be covered by (10).

3) By considering $m=1$, (5) decreases to

$$\begin{bmatrix}\frac{1}{\varrho}R & O\\ * & \frac{1}{1-\varrho}R\end{bmatrix}>\begin{bmatrix}R+(1-\varrho)W_1 & \varrho V_0+\varrho V_1\\ * & R+\varrho W_2\end{bmatrix}$$

which are identical to the result in [7]. Before proceeding, some matrices and vectors are defined in [9].

Theorem 1: For given $\bar{d}$, $\underline{\mu}<\bar{\mu}$, $K_p=\mathrm{diag}\{k_1^+,\dots,k_n^+\}$, and $K_m=\mathrm{diag}\{k_1^-,\dots,k_n^-\}$, the system (1) is asymptotically stable if there exist matrices $P_1\in\mathbb{S}_+^{9n}$, $P_2\in\mathbb{S}_+^{9n}$, $Q_1\in\mathbb{S}_+^{6n}$, $Q_2\in\mathbb{S}_+^{6n}$, $R\in\mathbb{S}_+^{n}$, $Z\in\mathbb{S}_+^{2n}$, $G_1\in\mathbb{S}_+^{6n}$, $G_2\in\mathbb{S}_+^{14n}$, $G_3\in\mathbb{S}_+^{14n}$, $X_k\in\mathbb{S}^{3n}$ $(k=1,2,3,4)$, $X_z\in\mathbb{R}^{2n}$, $Y_\iota\in\mathbb{R}^{3n}$ $(\iota=0,1,2)$, diagonal matrices $L_1\in\mathbb{S}_+^n$, $L_2\in\mathbb{S}_+^n$, $H_j\in\mathbb{S}_+^n$, $T_j\in\mathbb{S}_+^n$ $(j=1,2,3)$, and skew-symmetric matrices $N_1\in\mathbb{R}^{6n}$, $N_2\in\mathbb{R}^{14n}$, $N_3\in\mathbb{R}^{14n}$, such that the following inequalities hold:

$$\Upsilon>0,\ \mathcal{E}_U-\mathcal{G}_{[1]}^T\begin{bmatrix}G_1 & N_1\\ * & -G_1\end{bmatrix}\mathcal{G}_{[1]}<0 \tag{11}$$

$$\mathcal{E}_\Theta^{[\bar{u}]}-\mathcal{G}_{[\bar{d}]}^T\begin{bmatrix}G_2 & N_2\\ * & -G_2\end{bmatrix}\mathcal{G}_{[\bar{d}]}<0 \tag{12}$$

$$\mathcal{E}_\Theta^{[\underline{u}]}-\mathcal{G}_{[\bar{d}]}^T\begin{bmatrix}G_3 & N_3\\ * & -G_3\end{bmatrix}\mathcal{G}_{[\bar{d}]}<0. \tag{13}$$

Proof: Consider the following augmented L-K functional:

$$\begin{aligned}V(t)&=\psi_1^T(t)\mathcal{P}_{dt}\psi_1(t)+\int_{t-d_t}^{t}\psi_2^T(t,s)Q_1\psi_2(t,s)ds\\&\quad+\int_{t-\bar{d}}^{t-d_t}\psi_2^T(t,s)Q_2\psi_2(t,s)ds\\&\quad+\bar{d}\int_{t-\bar{d}}^{t}\int_u^t\dot{x}^T(s)R\dot{x}(s)dsdu\\&\quad+\bar{d}\int_{t-\bar{d}}^{t}\int_u^t\psi_3^T(s)Z\psi_3(s)dsdu\\&\quad+2\sum_{i=1}^{n}\int_0^{W_{2i}x(t)}\left(l_{2i}f_i^+(s)+l_{1i}f_i^-(s)\right)ds\end{aligned} \tag{14}$$

where

$$\begin{aligned}&\mathcal{P}_{dt}=d_tP_1+(d-d_t)P_2\\&f_i^+(s)=k_i^+s-f_i(s),f_i^-(s)=f_i(s)-k_i^-s\\&\psi_1(t)=col\{\zeta_1,\zeta_4,d_t\phi_1,d_t\phi_3,d_m\phi_2,d_m\phi_4\}\\&\psi_{21}(t,s)=col\{\textstyle\int_s^t x(u)du,\int_{t-d_t}^s x(u)du\}\\&\psi_{22}(t,s)=col\{\textstyle\int_s^{t-d_t}x(u)du,\int_{t-\bar{d}}^s x(u)du\}\\&\psi_2(t,s)=col\{x(s),\dot{x}(s),\psi_{21}(t,s),\psi_{22}(t,s)\}\\&\psi_3(s)=col\{x(s),f(W_2x(s))\}.\end{aligned}$$

By differentiating $V(t)$ along with the trajectory of (1), we yield

$$\begin{aligned}\dot{V}&=\zeta_t^T(\Pi_1^T\mathcal{P}_{dt}\Pi_2+\Pi_1^T\hat{\mathcal{P}}_{dt}\Pi_1+\Pi_3^TQ_1\Pi_3-d_t\Pi_4^TQ_1\Pi_4\\&\quad+d_t\Pi_4^TQ_2\Pi_4-\Pi_5^TQ_2\Pi_5+\Pi_6Q_1\Pi_7+\Pi_8^TQ_2\Pi_7\\&\quad+\bar{d}^2(\ell_s^TR\ell_s+\Pi_9^TZ\Pi_9)+He\{((\ell_8-K_mW_2\ell_1)L_1\\&\quad+(K_pW_2\ell_1-\ell_{11})L_2)W_2\ell_s\})\zeta_t+\mathcal{W}_1(t)+\mathcal{W}_2(t)\end{aligned} \tag{15}$$

where $\mathcal{W}_1(t)=-\bar{d}\int_{t-\bar{d}}^{t}\dot{x}^T(s)R\dot{x}(s)ds,\mathcal{W}_2(t)=-\bar{d}\int_{t-\bar{d}}^{t}\psi_3^T(s)Z\psi_3(s)ds$.

For the integral terms $\mathcal{W}_1(t)$ and $\mathcal{W}_2(t)$, the following conditions hold by employing auxiliary function-based inequality and Proposition 1:

$$\begin{aligned}\mathcal{W}_1(t)&\le\zeta_t^T\left(-\frac{\bar{d}}{d_t}\Pi_{10}^T\mathcal{R}\Pi_{10}-\frac{\bar{d}}{\bar{d}-d_t}\Pi_{11}^T\mathcal{R}\Pi_{11}\right)\zeta_t\\&\le-\zeta_t^T\Pi_{14}^T\mathcal{U}\Pi_{14}\zeta_t\end{aligned} \tag{16}$$

$$\begin{aligned}\mathcal{W}_2(t)&\le\zeta_t^T\left(-\frac{\bar{d}}{d_t}\Pi_{12}^TZ\Pi_{12}-\frac{\bar{d}}{\bar{d}-d_t}\Pi_{13}^TZ\Pi_{13}\right)\zeta_t\\&\le-\zeta_t^T\Pi_{15}^T\Upsilon\Pi_{15}\zeta_t\end{aligned} \tag{17}$$

where $\mathcal{U}=d_t^2\mathcal{R}_2+d_t\mathcal{R}_1+\mathcal{R}_0$.

According to the constraints of the activation function (3), some common positive definite items are given as follows:

$$0\le\kappa_1(t,H_1)+\kappa_1(t-d_t,H_2)+\kappa_1(t-\bar{d},H_3) \tag{18}$$

$$\begin{aligned}0&\le\kappa_2(t,t-d_t,T_1)+\kappa_2(t-d_t,t-\bar{d},T_2)\\&\quad+\kappa_2(t,t-\bar{d},T_3).\end{aligned} \tag{19}$$

By substituting (16), (17) and (19) into (15), one obtains

$$\begin{aligned}\dot{V}(t)&\le\zeta_t^T\mho_{[d_t,\dot{d}_t]}\zeta_t\\&=\zeta_t^T(d_t^2\Theta_{2[\dot{d}_t]}+d_t\Theta_{1[\dot{d}_t]}+\Theta_{0[\dot{d}_t]})\zeta_t\end{aligned} \tag{20}$$

where $\mho_{[d_t,\dot{d}_t]}$ and $\Theta_{j[\dot{d}_t]}$ are defined in Theorem 1. $\Theta_{j[\dot{d}_t]}$ are the coefficients of the matrix-valued polynomial $\mho_{[d_t,\dot{d}_t]}$.

Note that $\mho_{[d_t,\dot{d}_t]}$ is affine on $\dot{d}_t$, $\mho_{[d_t,\dot{d}_t]}<0$ holds for $\forall \dot{d}_t\in[\underline{u},\bar{u}]$ if and only if $\mho_{[d_t,\underline{u}]}<0$ and $\mho_{[d_t,\bar{u}]}<0$. Then LMIs (12) and (13) are derived by utilizing Lemma 1 in [8] to ensure $\mho_{[d_t,\underline{u}]}<0$ and $\mho_{[d_t,\bar{u}]}<0$, respectively. Hence, if the convex optimization conditions (11)–(13) are feasible, $\dot{V}(t)\le -\varsigma x^T(t)x(t)$ holds for a sufficiently small $\varsigma>0$. ■

Remark 2: Some new integral terms of activation function and double integral term $\bar{d}\int_{t-\bar{d}}^{t}\int_{u}^{t}\psi_3^T(s)Z\psi_3(s)dsdu$ are consolidated into the proposed L-K functional to include more information about time delay and activation function. Thus, more cross information about neuron activation function is contained in the L-K functional.

Numerical example: This section provides two extensive delayed GNNs to indicate the superiorities of the derived stability criteria.

Example 1: Consider the GNNs (1) with

$$A=\begin{bmatrix}1.5 & 0\\ 0 & 0.7\end{bmatrix},\; W_0=\begin{bmatrix}0.0503 & 0.0454\\ 0.0987 & 0.2075\end{bmatrix}$$
$$W_1=\begin{bmatrix}0.2381 & 0.9320\\ 0.0388 & 0.5062\end{bmatrix},\; K_p=\begin{bmatrix}0.3 & 0\\ 1 & 0.8\end{bmatrix}$$
$$W_2=I,\; K_m=O.$$

For different $\bar{u}$ and $\underline{u}$, the maximum allowable bounds $\bar{d}$ are evaluated via Theorem 1. The numerical results are summarized in Table 1 with those of diverse approaches. The maximum allowable bound is one of the crucial indicators for evaluating stability criteria. According to Table 1, Theorem 1 improves maximum allowable bounds by 18% more than the leading methods. It can be concluded that Theorem 1 produces less conservative stability criteria for delayed DNNs.

Table 1. Maximum Allowable Delay for Various $\bar{u}$ Values of Example 1

$\bar{u}=-\underline{u}$	0.45	0.5	0.55	NDVs
[10]	16.363	13.060	–	$152.5n^2+23.5n$
[5] ($m=3$)	17.342	14.446	12.670	$191n^2+40n$
Theorem 1	23.290	17.524	14.448	$596.5n^2+64.5n$

Example 2: Consider the GNNs (1) with

$$A=\mathrm{diag}\{7.3458, 6.9987, 5.5949\}$$
$$K_p=\mathrm{diag}\{0.3680, 0.1795, 0.2876\}$$
$$W_2=\begin{bmatrix}13.6014 & -2.9616 & -0.6936\\ 7.4736 & 21.6810 & 3.2100\\ 0.7290 & -2.6334 & -20.1300\end{bmatrix}$$
$$W_1=I,\; W_0=O,\; K_m=O.$$

For this system with time-varying delay, the maximum allowable bounds $\bar{d}$ are derived for $\bar{u}=-\underline{u}=\{0.1, 0.5, 0.9\}$ by solving the optimization problem. Table 2 shows that the bounds $\bar{d}$ based on Theorem 1 are much larger than those based on other present conditions. Then it can be concluded that the technique of Proposition 1 plays a key role in reducing conservatism. Furthermore, the number of decision variables (NDVs) of different methods are recorded in Table 2. How diminishing the NDVs of the proposed methods is an essential research direction.

Table 2. Maximum Allowable Delay for Various $\bar{u}$ Values of Example 2

$\bar{u}=-\underline{u}$	0.1	0.5	0.9	NDVs
[11]	1.1454	0.5806	–	$115n^2+22n$
[5] ($m=3$)	1.1641	0.6396	0.5361	$191n^2+40n$
[12] ($N=2$)	1.1660	0.6480	–	$193n^2+38n$
Theorem 1	1.2054	0.7250	0.5864	$596.5n^2+64.5n$

Conclusion: This letter establishes a new stability criteria for GNNs with time-varying delay. An improved RCC is proposed for delayed GNNs by considering a new matrix-valued polynomial. The proposed RCC encompasses some existing results as exceptional cases. Then, sufficient stability conditions are achieved by exploiting the RCC and L-K functional. Finally, several numerical examples are conducted to reveal the superiorities of the stability scheme.

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References

[1] Y. Chen and G. Chen, "Stability analysis of systems with time-varying delay via a novel Lyapunov functional," *IEEE/CAA J. Autom Sinica*, vol. 6, no. 4, pp. 1068–1073, 2019.

[2] X. Zhang, Q.-L. Han, X. Ge, and B. Zhang, "Delay-variation-dependent criteria on extended dissipativity for discrete-time neural networks with time-varying delay," *IEEE Trans. Neural Networks and Learning Systems*, pp. 1–10, 2021. DOI: 10.1109/TNNLS.2021.3105591

[3] J. Chen, J. H. Park, and S. Xu, "Improvement on reciprocally convex combination lemma and quadratic function negative-definiteness lemma," *J. Franklin Institute*, vol. 359, no. 2, pp. 1347–1360, 2022.

[4] G. Tan and Z. Wang, "α^2-dependent reciprocally convex inequality for stability and dissipativity analysis of neural networks with time-varying delay," *Neurocomputing*, vol. 463, pp. 292–297, 2021.

[5] H. Lin, H. Zeng, X. Zhang, and W. Wang, "Stability analysis for delayed neural networks via a generalized reciprocally convex inequality," *IEEE Trans. Neural Networks and Learning Systems*, pp. 1–9, 2022. DOI: 10.1109/TNNLS.2022.3144032

[6] H. B. Zeng, H. C. Lin, Y. He, K. L. Teo, and W. Wang, "Hierarchical stability conditions for time-varying delay systems via an extended reciprocally convex quadratic inequality," *J. Franklin Institute*, vol. 357, no. 14, pp. 9930–9941, 2020.

[7] A. Seuret and F. Gouaisbaut, "Delay-dependent reciprocally convex combination lemma," 2016. [Online]. Available: http://hal.archives-ouvertes.fr/hal-01257670/.

[8] X. Zhang, Q.-L. Han, and X. Ge, "Novel stability criteria for linear time-delay systems using Lyapunov-Krasovskii functionals with a cubic polynomial on time-varying delay," *IEEE/CAA J. Autom. Sinica*, vol. 8, no. 1, pp. 77–85, 2020.

[9] Y. Wang, "Appendix: Introduction to the main symbols," 2022. [Online]. Available: http://dx.doi.org/10.13140/RG.2.2.14675.66083/1.

[10] F. Long, C. Zhang, Y. He, Q. Wang, and M. Wu, "Stability analysis for delayed neural networks via a novel negative-definiteness determination method," *IEEE Trans. Cybernetics*, vol. 52, no. 6, pp. 5356–5366, 2022.

[11] X. Zhang, Q.-L. Han, and J. Wang, "Admissible delay upper bounds for global asymptotic stability of neural networks with time-varying delays," *IEEE Trans. Neural Networks and Learning Systems*, vol. 29, no. 11, pp. 5319–5329, 2018.

[12] J. Chen, X. Zhang, J. H. Park, and S. Xu, "Improved stability criteria for delayed neural networks using a quadratic function negative-definiteness approach," *IEEE Trans. Neural Networks and Learning Systems*, vol. 33, no. 3, pp. 1348–1354, 2022.