Letter

Novel Criteria on Finite-Time Stability of Impulsive Stochastic Nonlinear Systems

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Dear Editor,

This letter considers the finite-time stability (FTS) problem of generalized impulsive stochastic nonlinear systems (ISNS). By employing the stochastic Lyapunov and impulsive control approach, some novel criteria on FTS are presented, where both situations of stabilizing and destabilizing impulses are considered. Furthermore, new impulse-dependent estimation strategies of stochastic settling time (SST) are proposed. These estimation strategies establish quantitative relationships between the impulsive effects and the mathematical expectation of SST, which can directly assess the influence of impulses on the system performance. Finally, an example is given to validate the effectiveness of the presented results.

Introduction: Impulsive systems are a special class of hybrid systems involving continuous-time dynamics and discrete-time dynamics. It has been intensively researched and well used in many fields such as electronic circuit systems, aircraft, etc., [1] and [2]. In the actual world, many man-made and physical systems could be modeled as stochastic nonlinear systems due to the inevitable effects of uncertain factors and white noise [3]–[5]. As more generalized dynamical systems, the ISNS has wider theoretical and applied significance and has been studied [6]–[8].

However, the existing results of ISNS mainly focus on the infinite-time asymptotic stability. Due to security reasons or improving productivity, a large number of practical applications require strict limits on response time. That is why, FTS has been extensively investigated for controlled systems [9]. For deterministic systems, Lyapunov criteria of FTS were presented in [10], and then some improved and extended results were given in [11] and [12]. However, these theorems cannot be directly applied to ISNS. A fundamental technical hurdle is that stochastic disturbances bring both the integral term and the Hessian in the stochastic Lyapunov analysis. On the other hand, for stochastic systems, the definition and Lyapunov criteria of FTS were presented in [13], and then some improved and extended results were given in [14]–[16].Due to the existence of impulsive effects which will lead to discontinuities in the system, these existing theorems for stochastic systems also cannot be directly applied to ISNS. Therefore, establishing the FTS criterion of generalized ISNS is a fundamental and yet to be solved problem in this field.

Motivated by the above discussion, this letter aims to establish the FTS criterion of generalized ISNS. The main contributions of this work are that: by employing the stochastic Lyapunov and impulsive control approach, some novel criteria on FTS are proved under both situations of stabilizing and destabilizing impulses. What is more, new impulse-dependent estimation strategies of SST are proposed. These estimation strategies establish quantitative relationships between the impulses and the mathematical expectation of SST. It is shown that stabilizing impulse can improve the convergence rate, and correspondingly decrease the SST. On the contrary, destabilizing impulses may reduce the convergence rate, and increase the SST.

Problem formulation: Consider the following ISNS:

$$\begin{cases} dz(t) = f(t,z)dt + g(t,z)dB(t), & t \notin \mathcal{P} \\ z(t) = h(z(t^-)), & t \in \mathcal{P} \end{cases} \tag{1}$$

where $z(t) \in \mathcal{Z} \subseteq \mathbb{R}^n$ is the system state vector, $B(t) \in \mathbb{R}^m$ represents Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t>t_0}, \mathbb{P})$. The functions $f\colon \mathbb{R}^+_{t_0} \times \mathcal{Z} \to \mathbb{R}^n$ and $g\colon \mathbb{R}^+_{t_0} \times \mathcal{Z} \to \mathbb{R}^{n\times m}$ are continuous with respect to t, z and satisfy $f(\cdot,0) = 0$, $g(\cdot,0) = 0$; The function $h\colon \mathcal{Z} \to \mathcal{Z}$ is continuous with respect to z and satisfy $h(0) = 0$. Impulsive time sequence $\mathcal{P} = \{t_k,\ k \in \mathbb{Z}^+\}$ is strictly increasing on (t_0, ∞).

Definition 1 [13]: For an impulsive time sequence $\mathcal{P}$, the ISNS (1) is said to be FTS in probability, if it is stable and finite-time attractive in probability, that is, if for any $0 < \varepsilon < 1$ and $r > 0$, there exists a $\delta = \delta(\varepsilon, r) > 0$ such that $\mathbb{P}\{|z(t)| < r, \forall t > t_0\} \geq 1-\varepsilon$, whenever $|x_0| < \delta$, and the SST $T_{z_0}(\mathcal{P}) = inf\{t > t_0 : z(t) = 0\}$ satisfies $\mathbb{P}\{T_{z_0}(\mathcal{P}) < \infty\} = 1$ for any initial value z_0.

Definition 2 [4]: For any $C^{2,1}$ function $V\colon \mathbb{R}^+_{t_0} \times \mathcal{Z} \to \mathbb{R}^+$, the differential operator $\mathcal{L}$ associated with the ISNS (1) is defined by

$$\mathcal{L}V(t,z) = \frac{\partial V(t,z)}{\partial t} + \frac{\partial V(t,z)}{\partial z} f(t,z) + \frac{1}{2} g^T(t,z) \frac{\partial^2 V(t,z)}{\partial z^2} g(t,z).$$

Based on It$\hat{o}$'s formula, it turns out that

$$dV(t,z) = \mathcal{L}V(t,z)dt + \frac{\partial V(t,z)}{\partial z} g(t,z)dB(t).$$

Main results: In this section, we will propose novel criteria of FTS for the generalized ISNS (1), where two different types of impulses will be considered, including stabilizing and destabilizing impulses.

Theorem 1: If there are two $\mathcal{K}_\infty$ class functions ψ_1, ψ_2, a positive-definite continuous function $V\colon \mathbb{R}^+_{t_0} \times \mathcal{Z} \to \mathbb{R}^+$ and some positive constants β, γ , $0 < \varepsilon$, $\theta < 1$, such that for any solution $z(t; z_0)$

$$\psi_1(|z(t)|) \leq V(t,z) \leq \psi_2(|z(t)|) \tag{2}$$

$$V(t, h(z(t))) \leq \theta^{\frac{1}{1-\varepsilon}} V(t^-, z(t^-)) \tag{3}$$

$$\mathcal{L}V(t,z) \leq -\beta V(t,z) - \gamma V^{\varepsilon}(t,z), \quad t \notin \mathcal{P} \tag{4}$$

then, the ISNS (1) is FTS for any impulse sequences $\mathcal{P}$. Furthermore, if the impulse sequence $\mathcal{P} = \{t_i, i = 1, 2, \ldots, p\}$ satisfies

$$t_p \leq t_0 + \frac{\ln\left[\eta^{p-1} \frac{\eta-\theta}{1-\theta}\left(1 + \frac{\beta}{\gamma} V_0^{1-\varepsilon}\right)\right]}{\beta(1-\varepsilon)} \tag{5}$$

for $\theta < \eta < 1$, then the SST $T_{z_0}(\mathcal{P})$ is estimated as

$$\mathbb{E}[T_{z_0}(\mathcal{P})] \leq \mathcal{T}_1 = t_0 + \frac{\ln\left[\eta^{p}\left(1 + \frac{\beta}{\gamma} V_0^{1-\varepsilon}\right)\right]}{\beta(1-\varepsilon)}. \tag{6}$$

Proof: For any $r > 0$ and $\varepsilon \in (0,1)$, define $\sigma_r = inf\{t \geq t_0\colon |z(t; z_0)| > r\}$. From It$\hat{o}$'s formula, it is easily derived that

$$V(t, z(\sigma_r \wedge t)) = V(t_i, z(t_i)) + \int_{t_i}^{\sigma_r \wedge t} \mathcal{L}V(s, z(s))ds + \int_{t_i}^{\sigma_r \wedge t} \frac{\partial V(s, z(s))}{\partial z} g(s, z(s))dB(s), \quad t \in [t_i, t_{i+1}). \tag{7}$$

Taking the expectation of (7), together with the condition (4), yields

$$\mathbb{E}V(t, z(\sigma_r \wedge t)) \leq V(t_i, z(t_i)), \quad t \in [t_i, t_{i+1}). \tag{8}$$

Since $0 < \theta < 1$, it can be seen from (8) and the condition (3) that

$$\mathbb{E}V(t, z(\sigma_r \wedge t)) \leq V(t_0, z_0), \quad \forall t \geq t_0. \tag{9}$$

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Citation: L. F. Hua, H. Zhu, S. M. Zhong, K. B. Shi, and J. D. Cao, "Novel criteria on finite-time stability of impulsive stochastic nonlinear systems," *IEEE/CAA J. Autom. Sinica*, vol. 10, no. 7, pp. 1634–1636, Jul. 2023.

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Digital Object Identifier 10.1109/JAS.2023.123276

It is noteworthy that $|x(\sigma_r \wedge t)| = |x(\sigma_r)| = r$ if $\sigma_r \le t$. Hence, it follows from (2) that:

$$\mathbb{P}(\sigma_r \le t)\psi_1(r) \le \mathbb{E}[I_{\{\sigma_r \le t\}} V(\sigma_r, z(\sigma_r))] \le \mathbb{E}V(\sigma_r, z(\sigma_r)) \le V(t_0, z_0) \le \psi_2(|z_0|). \quad (10)$$

Let $\delta = \psi_2^{-1}(\psi_1(r)\varepsilon)$, it can be deduced that $\mathbb{P}(\sigma_r \le t) \le \varepsilon$ whenever $|z_0| \le \delta$. Then, let $t \to \infty$, it turns out that $\mathbb{P}(\sigma_r \le \infty) \le \varepsilon$, which implies that $\mathbb{P}\{\sup_{t \ge t_0} |z(t)| \le r\} \ge 1 - \varepsilon$. Therefore, the ISNS (1) is stable in probability.

Then, define $\tau_k = inf\{t \ge t_0 : |z(t)| \notin (\frac{1}{k}, k)\}$, in which $k \in \mathbb{N}^+$ and satisfied $\frac{1}{k} < |z_0| < k$. By Definition 2, it follows that:

$$\begin{aligned}\mathcal{L}\,[e^{\beta(1-\varepsilon)t} V^{1-\varepsilon}(t,z)] &= \beta(1-\varepsilon)e^{\beta(1-\varepsilon)t} V^{1-\varepsilon}(t,z) + (1-\varepsilon)e^{\beta(1-\varepsilon)t} V^{-\varepsilon}(t,z) \\ &\quad \times \mathcal{L}V(t,z) - \frac{1}{2}\varepsilon(1-\varepsilon)e^{\beta(1-\varepsilon)t} V^{-\varepsilon-1}(t,z) \\ &\quad \times \Big[\frac{\partial V(t,z)}{\partial z} g(t,z)\Big]^T \Big[\frac{\partial V(t,z)}{\partial z} g(t,z)\Big] \\ &\le \beta(1-\varepsilon)e^{\beta(1-\varepsilon)t} V^{1-\varepsilon}(t,z) \\ &\quad + (1-\varepsilon)e^{\beta(1-\varepsilon)t} V^{-\varepsilon}(t,z)\mathcal{L}V(t,z).\end{aligned}$$

From Itô's formula, one has

$$\begin{aligned} e^{\beta(1-\varepsilon)(\tau_k \wedge t)} V^{1-\varepsilon}(\tau_k \wedge t, z(\tau_k \wedge t)) &= e^{\beta(1-\varepsilon)t_i} V^{1-\varepsilon}(t_i, z(t_i)) \\ &\quad + \int_{t_i}^{\tau_k \wedge t} \beta(1-\varepsilon)e^{\beta(1-\varepsilon)s} V^{1-\varepsilon}(s, z(s))ds \\ &\quad + \int_{t_i}^{\tau_k \wedge t} (1-\varepsilon)e^{\beta(1-\varepsilon)s} V^{-\varepsilon}(s,z(s))\mathcal{L}V(s,z(s))ds \\ &\quad + \int_{t_i}^{\tau_k \wedge t} (1-\varepsilon)e^{\beta(1-\varepsilon)t} V^{-\varepsilon}(t,z)\frac{\partial V(s,z(s))}{\partial z} \\ &\quad \times g(s,z(s))dB(s), \quad t \in [t_i, t_{i+1}). \end{aligned} \quad (11)$$

Taking the expectation of (11), together with the condition (4), yields

$$\mathbb{E}V^{1-\varepsilon}(\tau_k \wedge t, z(\tau_k \wedge t)) \le e^{\beta(1-\varepsilon)(t_i - \tau_k \wedge t)}\mathbb{E}[V^{1-\varepsilon}(t_i, z(t_i)) + \frac{\gamma}{\beta} - \frac{\gamma}{\beta}e^{\beta(1-\varepsilon)(\tau_k \wedge t - t_i)}], \quad t \in [t_i, t_{i+1}). \quad (12)$$

Since $0 < \theta < 1$, it can be seen from (12) and the condition (3) that

$$\mathbb{E}V^{1-\varepsilon}(\tau_k \wedge t, z(\tau_k \wedge t)) \le e^{\beta(1-\varepsilon)(t_0 - \tau_k \wedge t)}\mathbb{E}[V_0^{1-\varepsilon} + \frac{\gamma}{\beta} - \frac{\gamma}{\beta}e^{\beta(1-\varepsilon)(\tau_k \wedge t - t_0)}], \quad t \ge t_0. \quad (13)$$

It is evident that τ_k is increasing stopping time sequence. Let $k, t \to \infty$, one obtains $\tau_k \wedge t \to T_{z_0}(\mathcal{P})$. Thus, we have from (13) that $\mathbb{E}[T_{z_0}(\mathcal{P})] \le \mathcal{T}_0$. It indicates that $\mathbb{P}\{T_{z_0}(\mathcal{P}) < \infty\} = 1$, that is, the ISNS (1) is finite-time attractiveness in probability. Therefore, it is concluded that the ISNS (1) is FTS for any $\mathcal{P}$.

Next, we turn our attention to proving the SST $T_{z_0}(\mathcal{P})$ can be estimated as (6) if the impulse time sequence $\mathcal{P} = \{t_i, i = 1, 2, \ldots, p\}$ satisfies (5). Due to $\theta < \eta < 1$, it follows from (5) that:

$$t_i \le t_p \le t_0 + \frac{\ln[\eta^{p-1}\frac{\eta-\theta}{1-\theta}(1 + \frac{\beta}{\gamma}V_0^{1-\varepsilon})]}{\beta(1-\varepsilon)} \le t_0 + \frac{\ln[\eta^{i-1}\frac{\eta-\theta}{1-\theta}(1 + \frac{\beta}{\gamma}V_0^{1-\varepsilon})]}{\beta(1-\varepsilon)}.$$

Hence, one obtains

$$\theta\eta^{i-1} + \frac{(1-\theta)e^{\beta(1-\varepsilon)(t_i - t_0)}}{1 + \frac{\beta}{\gamma}V_0^{1-\varepsilon}} \le \eta^i. \quad (14)$$

Based on (12), it is easily seen that for $t \in [t_0, t_1)$

$$\mathbb{E}V^{1-\varepsilon}(\tau_k \wedge t, z(\tau_k \wedge t)) \le e^{\beta(1-\varepsilon)(t_0 - \tau_k \wedge t)}\mathbb{E}[V_0^{1-\varepsilon} + \frac{\gamma}{\beta} - \frac{\gamma}{\beta}e^{\beta(1-\varepsilon)(\tau_k \wedge t - t_0)}] \quad (15)$$

which, together with (3) and (14), obtains that for $t \in [t_1, t_2)$

$$\begin{aligned} &\mathbb{E}V^{1-\varepsilon}(\tau_k \wedge t, z(\tau_k \wedge t)) \\ &\le e^{\beta(1-\varepsilon)(t_1 - \tau_k \wedge t)}\mathbb{E}[V^{1-\varepsilon}(t_1, z(t_1)) + \frac{\gamma}{\beta} - \frac{\gamma}{\beta}e^{\beta(1-\varepsilon)(\tau_k \wedge t - t_1)}] \\ &\le e^{\beta(1-\varepsilon)(t_0 - \tau_k \wedge t)}\mathbb{E}\left\{\left[\theta + \frac{(1-\theta)\frac{\gamma}{\beta}e^{\beta(1-\varepsilon)(t_1 - t_0)}}{V_0^{1-\varepsilon} + \frac{\gamma}{\beta}}\right] \times (V_0^{1-\varepsilon} + \frac{\gamma}{\beta}) - \frac{\gamma}{\beta}e^{\beta(1-\varepsilon)(\tau_k \wedge t - t_0)}\right\} \\ &\le e^{\beta(1-\varepsilon)(t_0 - \tau_k \wedge t)}\mathbb{E}\left[\eta(V_0^{1-\varepsilon} + \frac{\gamma}{\beta}) - \frac{\gamma}{\beta}e^{\beta(1-\varepsilon)(\tau_k \wedge t - t_0)}\right]. \end{aligned} \quad (16)$$

By calculation, it is easy to derive that

$$\mathbb{E}V^{1-\varepsilon}(\tau_k \wedge t, z(\tau_k \wedge t)) \le e^{\beta(1-\varepsilon)(t_0 - \tau_k \wedge t)}\mathbb{E}\left[\eta^i(V_0^{1-\varepsilon} + \frac{\gamma}{\beta}) - \frac{\gamma}{\beta}e^{\beta(1-\varepsilon)(\tau_k \wedge t - t_0)}\right], \quad t \in [t_i, t_{i+1}) \quad (17)$$

in which t_{p+1} is defined by $\mathcal{T}_0$. Therefore, it is not hard to verify from (17) that the estimation of the SST can be deduced in the interval $[t_p, t_{p+1}]$, and $\mathbb{E}[T_{z_0}(\mathcal{P})] \le \mathcal{T}_1$. That is, if the impulse time sequence $\mathcal{P} = \{t_i, i = 1, 2, \ldots, p\}$ satisfies (5), then the ISNS (1) is FTS and the corresponding SST is estimated as (6). ■

Note that the impulse studied in Theorem 1 is stabilizing impulse $0 < \theta < 1$. Next, we will propose another FTS result under destabilizing impulse $\theta > 1$.

Theorem 2: If there are two $\mathcal{K}_\infty$ class functions ψ_1, ψ_2, a positive-definite continuous function $V : \mathbb{R}_{t_0}^+ \times \mathcal{Z} \to \mathbb{R}^+$ and some positive constants β, γ, $0 < \varepsilon < 1$, $\theta > 1$, such that for any solution $z(t; z_0)$, the conditions (2)–(4) hold, and the impulse sequences $\mathcal{P}$ satisfies

$$\min\left\{k \in \mathbb{Z}^+ : t_k \ge t_0 + \frac{\ln[\theta^{k-1}(1 + \frac{\beta}{\gamma}V_0^{1-\varepsilon})]}{\beta(1-\varepsilon)}\right\} = P_m < +\infty \quad (18)$$

then the ISNS (1) is FTS, and the SST $T_{z_0}(\mathcal{P})$ is estimated as

$$\mathbb{E}[T_{z_0}(\mathcal{P})] \le \mathcal{T}_2 = t_0 + \frac{\ln[\theta^{P_m - 1}(1 + \frac{\beta}{\gamma}V_0^{1-\varepsilon})]}{\beta(1-\varepsilon)}. \quad (19)$$

Proof: With a similar method as the proof in Theorem 1, it is easily proved that the ISNS (1) is stable in probability. To avoid repeating similar proofs, the detailed proof process is omitted.

Next, we will prove that the ISNS (1) is finite-time attractiveness in probability, and the SST $T_{z_0}(\mathcal{P})$ can be estimated as (19) if $\mathcal{P}$ satisfies (18). By (12), it is founded that for $t \in [t_0, t_1)$

$$\mathbb{E}V^{1-\varepsilon}(\tau_k \wedge t, z(\tau_k \wedge t)) \le e^{\beta(1-\varepsilon)(t_0 - \tau_k \wedge t)}\mathbb{E}\left[V_0^{1-\varepsilon} + \frac{\gamma}{\beta} - \frac{\gamma}{\beta}e^{\beta(1-\varepsilon)(\tau_k \wedge t - t_0)}\right]. \quad (20)$$

When $t_1 \ge \mathcal{T}_0$, it follows that $P_m = 1$ and the ISNS (1) is finite-time attractiveness in probability. When $t_1 < \mathcal{T}_0$, then $P_m \ge 2$. Since $\theta \ge 1$, it is easy to derive that for $t \in [t_1, t_2)$

$$\begin{aligned} &\mathbb{E}V^{1-\varepsilon}(\tau_k \wedge t, z(\tau_k \wedge t)) \\ &\le e^{\beta(1-\varepsilon)(t_1 - \tau_k \wedge t)}\mathbb{E}\left[V^{1-\varepsilon}(t_1, z(t_1)) + \frac{\gamma}{\beta} - \frac{\gamma}{\beta}e^{\beta(1-\varepsilon)(\tau_k \wedge t - t_1)}\right] \\ &\le e^{\beta(1-\varepsilon)(t_1 - \tau_k \wedge t)}\mathbb{E}\left\{\theta e^{\beta(1-\varepsilon)(t_0 - t_1)}\left[V_0^{1-\varepsilon} + \frac{\gamma}{\beta} - \frac{\gamma}{\beta}e^{\beta(1-\varepsilon)(t_1 - t_0)}\right] + \frac{\gamma}{\beta} - \frac{\gamma}{\beta}e^{\beta(1-\varepsilon)(\tau_k \wedge t - t_1)}\right\} \\ &\le e^{\beta(1-\varepsilon)(t_0 - \tau_k \wedge t)}\mathbb{E}\left[\theta(V_0^{1-\varepsilon} + \frac{\gamma}{\beta}) - \frac{\gamma}{\beta}e^{\beta(1-\varepsilon)(\tau_k \wedge t - t_0)}\right]. \end{aligned} \quad (21)$$

By calculation, it is easy to derive that

$$\mathbb{E}V^{1-\varepsilon}(\tau_k \wedge t, z(\tau_k \wedge t)) \le e^{\beta(1-\varepsilon)(t_0 - \tau_k \wedge t)}\mathbb{E}\left[\theta^{P_m - 1}(V_0^{1-\varepsilon} + \frac{\gamma}{\beta}) - \frac{\gamma}{\beta}e^{\beta(1-\varepsilon)(\tau_k \wedge t - t_0)}\right], \quad t \in [t_{P_m - 1}, t_{P_m}). \quad (22)$$

Therefore, it is not hard to verify from (22) that the ISNS (1) is finite-time attractiveness in probability. That is, if the impulse time sequence $\mathcal{P}$ satisfies (18), then the ISNS (1) is FTS and the corresponding SST is estimated as (19). ■

Numerical example: In this section, an example is given to validate the effectiveness of the presented FTS results. Consider the mass-spring system [6] with linear viscous damping and a hardening spring, which can be modeled by the Duffing's equation as follows:

$$\rho\ddot{x}(t)+\xi\dot{x}(t)+\delta x(t)+\delta\alpha^2x^3(t)=\phi(t) \tag{23}$$

in which $\xi\dot{x}(t)$, $\phi(t)$ are the resistive force and the external force, respectively, and $\delta x(t)+\delta\alpha^2x^3(t)$ represents hardening spring. As we know, for experiments performed in air or other viscous media, there are external forces that keep the mass-spring in motion at some instants t_k. The kinetic energy transfer between the mass-spring and the external forces can be described by impulsive effects. Let $z(t)=[z_1,z_2(t)]^T=[x(t),\dot{x}(t)]^T$ and $\phi(t)=\varphi(t)+\sigma(t)\dot{B}(t)$, where $\dot{B}(t)$ is a 2-dimensional independent white noise, then the system (23) is equivalently rewritten as the ISNS model of the form (1)

$$\begin{aligned}&\begin{bmatrix}dz_1(t)\\dz_2(t)\end{bmatrix}=\begin{bmatrix}z_2(t)\\\mathcal{G}(t)\end{bmatrix}dt+\begin{bmatrix}0\\\sigma(t)/\rho\end{bmatrix}dB(t),\ \ t\notin\mathcal{P}\\&\begin{bmatrix}z_1(t)\\z_2(t)\end{bmatrix}=\begin{bmatrix}\lambda_1z_1(t^-)\\\lambda_2z_2(t^-)+y(t^-)\end{bmatrix},\ \ t\in\mathcal{P}\end{aligned} \tag{24}$$

where $\mathcal{G}(t)=-\frac{1}{\rho}(\xi z_2(t)+\delta z_1(t)+\delta\alpha^2z_1^3(t)-\varphi(t))$, $\mathcal{P}$ is the impulses time sequence, $y(t^-)$ is the change in velocity as a result of the additional force at time t, and $1\le\lambda_1,\ \lambda_2\le 2$.

Let $\xi=1.5\rho$, $\delta=2\rho$, $\alpha=\frac{1}{\sqrt{2}}$, $\lambda_1=1.5$, $\lambda_2=1$, and $\varphi(t)=-0.1\rho\times\mathrm{sgn}(2z_1(t)+4z_2(t))$, $\sigma(t)=\frac{1}{\sqrt{2}}\rho z_1^2(t)$, $y(t)=0.2z_2(t)$, and $\mathcal{P}=\{1,2,\ldots,10\}$. Choose the Lyapunov function $V(t,z(t))=z^T(t)Qz(t)+z_1^4(t)$, where $Q=\begin{bmatrix}3&1\\1&2\end{bmatrix}$, then it is not hard to derive that $\mathcal{L}V(t,z(t))\le-V(t,z(t))-0.1\sqrt{V(t,z(t))}$ for $t\notin\mathcal{P}$, and $V(t,z(t))\le 2.25^2V(t^-,z(t^-))$ for $t\in\mathcal{P}$. Hence, based on Theorem 2, it follows that the mass-spring system with destabilizing impulses (24) is FTS. For the initial value $z(0)=[1,-1]^T$, its state trajectories are presented in Fig. 1.

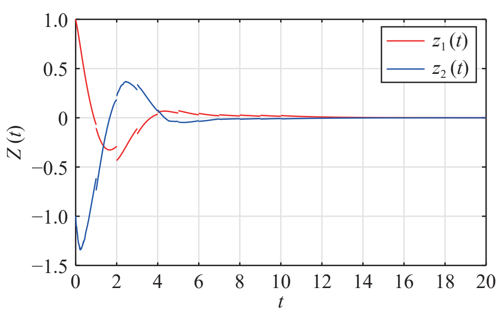

Fig. 1. The state trajectories of the mass-spring system with destabilizing impulses.

Furthermore, the $T_{x_0}(\mathcal{P})$ can be estimated as $\mathbb{E}[T_{z_0}(\mathcal{P})]\le 1.6219(P+3.7544)$, if the impulsive points t_k, $k=1,2,\ldots,P$ satisfies $t_k<2\ln(2.25^{k-1}\times 21)$, and t_{k+1} does not satisfy. When $P=0$ (without impulse) and $P=10$ with $\mathcal{P}=\{1,2,\ldots,10\}$, the state trajectories of the mass-spring system with destabilizing impulses (24) and the SST are presented in Fig. 2, in which the initial value $z(0)=[1,-1]^T$. It can be seen from this that the destabilizing impulses may reduce the convergence rate of the mass-spring system (24), and correspondingly increase the SST.

Conclusion: In this letter, novel criteria on FTS for ISNS have been presented by employing the stochastic Lyapunov and impulsive control approach. Furthermore, new impulse-dependent estimation strategies of SST were proposed. These estimation strategies established quantitative relationships between the impulsive effects and the mathematical expectation of SST, which can directly assess the influence of impulses on the system performance. It is shown that stabilizing impulse can improve the convergence rate, and correspondingly decrease the SST. On the contrary, destabilizing impulses reduce the convergence rate, and correspondingly increase SST.

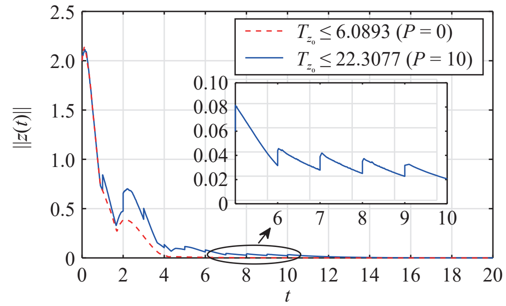

Fig. 2. The state trajectories of the mass-spring system with and without destabilizing impulses.

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