Cyclic-Pursuit-Based Circular Formation Control of Mobile Agents with Limited Communication Ranges and Communication Delays

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***Abstract*—This article addresses the circular formation control problem of a multi-agent system moving on a circle in the presence of limited communication ranges and communication delays. To minimize the number of communication links, a novel distributed controller based on a cyclic pursuit strategy is developed in which each agent needs only its leading neighbour's information. In contrast to existing works, we propose a set of new potential functions to deal with heterogeneous communication ranges and communication delays simultaneously. A new framework based on the admissible upper bound of the formation error is established so that both connectivity maintenance and order preservation can be achieved at the same time. It is shown that the multi-agent system can be driven to the desired circular formation as time goes to infinity under the proposed controller. Finally, the effectiveness of the proposed method is illustrated by some simulation examples.**

***Index Terms*—Circular formation control, communication delays, cyclic pursuit, heterogeneous communication ranges.**

I. Introduction

CIRCULAR formation control has attracted much attention in recent years due to its wide range of applications, such as satellite clustering [1], target enclosing [2], and perimeter defence [3]. In circular formation control, a group of mobile agents is tasked to converge to/move on a common circle. This problem was initially studied for cases where agents are spread evenly on the circle [4]−[8]. Recent studies have focused on a more general case in which agents can achieve any desired formation pattern [3], [9]−[13]. For example, distributed control laws that can preserve spatial order of the agents on the circle have been proposed for multi-agent systems (MASs) with integrator dynamics [3], [9]−[12]. In these works, the desired spacing pattern in [3], [9] was decided by using the heterogeneous maximum velocities of MASs such that the optimal coverage of the circle could be reached. In [10], any desired spacing pattern was achievable with consensus-based controllers. Note that the MASs were assumed to locate initially on the circle in [3], [9], [10], which was then relaxed to any initial positions on a two-dimensional mission space in [11]. More recently, the results of these works [3], [9]−[11] were extended to MASs with double-integrator dynamics [12] and nonholonomic dynamics [13]−[15].

Most of the aforementioned works assume that each agent is able to communicate with its immediate anticlockwise and clockwise neighbouring agents. The assumption of interaction with two neighbouring agents can be relaxed to unidirectional interaction to further reduce the amount of information exchange in MASs. To this end, cyclic-pursuit-based controllers were proposed for integrator-type MASs [4]−[6], [16] and nonholonomic MASs [7], [8]. This design is inspired by animal behaviours in which each member of a group follows the member immediately in front of it [17]. Cyclic-pursuit-based controllers are first developed by analysing cyclic matrices and their properties [4]−[8]. In [16], cyclic pursuit-fuzzy PD controllers are proposed to further improve the system's robustness. However, this result only works in an MAS with two agents. Although this strategy has the advantage of requiring minimum communication links since each agent needs only its leading neighbour's information for its control law, only evenly-spaced formation tasks with order preservation can be achieved in [4]−[8], [16]. To the best of our knowledge, limited attention has been paid to solving the circular formation problem with any desired formation pattern via a cyclic pursuit strategy.

Another typical assumption in most existing works [3]−[13] on circular formation control is that each agent can always exchange information with its neighbouring agent regardless of their Euclidean distance. In practice, however, the communication ranges of agents are rather limited. When the distance between neighbouring agents goes beyond their communication range, the information cannot be exchanged, and a disconnected communication topology may occur. Thus, considerable effort has recently been devoted to achieving the desired formation pattern for MASs subject to limited communication ranges [18]−[21]. One approach is to design gradient-descent control laws based on a potential function [18], [19]

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whose value increases as the distance between neighbouring agents increases. For example, a potential function was designed in [18] where an attractive force was applied to each agent to preserve existing links. In [19], another potential function was constructed based on a local estimation of the network's algebraic connectivity. Differently from potential-function-based controllers, a switching control strategy was proposed in [20] where the controller was designed to use the maximum interaction range if the inter-agents' distances were unavailable. The study also showed that the MAS could achieve the prescribed circular formation pattern even if network connectivity was not always maintained. Unlike above-mentioned works with fixed interaction ranges, authors in [21] considered an MAS with adjustable interaction ranges and proposed a model predictive strategy to realize communication management.

In addition, in many practical systems, communication delays often occur when agents exchange information through networks, and such delays can lead to performance degradation or even instability of the concerned system [22]−[24]. The problem of communication delays was initially investigated for consensus problems of MASs [25], [26] via the Lyapunov-Krasovskii functional. The same problem of communication delays was subsequently considered for other cooperative control problems, such as the circular formation control problem [27]−[29]. It is worth noting, however, that the circular formation problem for MASs with both limited communication ranges and communication delays has not been addressed in the available literature.

Motivated by the above observations, this article investigates the cyclic-pursuit-based circular formation control problem for MASs with limited communication ranges and communication delays. Solving this problem involves three main challenges. The first challenge is to design potential functions for MASs with both heterogeneous communication ranges and communication delays. Existing works [18]−[20] only discuss MASs with identical communication ranges. The potential functions proposed in [18], [19] cannot be directly applied to the formation problem under consideration here. The second challenge is to conduct the convergence analysis of the closed-loop system. Unidirectional communication under cyclic pursuit will lead to a Laplacian matrix that is not diagonalizable. Thus, the convergence analysis provided in [10]−[12], [20] with a diagonalizable Laplacian matrix cannot be adopted for this work. The final challenge is preserving spatial order of the MAS while simultaneously maintaining network connectivity. Most existing studies, such as [10]−[12], [20], preserve spatial order of the agents without considering network connectivity. Some other works [14], [15] do not consider order preservation problem and thus collisions may happen between neighbouring agents. The use of potential functions will introduce nonlinearity into the closed-loop system, with which spatial order of the MAS cannot be preserved in the same way as in [10]−[12], [20].

In this article, a novel cyclic-pursuit-based circular formation control scheme is proposed for MASs with heterogeneous communication ranges and time-varying communication delays. Our method contains the following steps. First, the admissible upper bounds for formation errors are derived so that the distances of neighbouring agents can simultaneously satisfy network connectivity and order preservation constraints. Based on these upper bounds and the formation errors, a set of bounded potential functions are designed. Second, a gradient-descent-based controller is proposed using the designed potential functions, where a sufficient condition for the control gain is found using Lyapunov functionals and matrix analysis. Third, we show that the proposed controller guarantees network connectivity and order preservation simultaneously by using the induction method. Finally, the convergence of the formation errors to zero is proved by the Barbalat's lemma. The main contributions of this work can be summarized as follows:

1) A novel distributed controller based on a cyclic pursuit strategy is designed to solve the circular formation problem with any desired formation pattern. Compared with [4]−[8], [16], where only evenly-spaced formation patterns can be achieved under a cyclic pursuit strategy, the controller in this work can achieve any desired formation pattern. Unlike [3], [9]−[12], [14], [15], where each agent is able to communicate with its immediate anticlockwise and clockwise neighbouring agents, the design in this work only needs unidirectional communication.

2) Novel potential functions are proposed by simultaneously taking into account heterogeneous communication ranges and communication delays. Unlike [3]−[16], [27], [28], where the communication ranges of MASs are assumed to be infinite, our work considers an MAS with limited communication ranges. Unlike [18]−[20] that only consider MASs with identical communication ranges, the potential functions in this work can handle MASs with both heterogeneous communication ranges and communication delays. More specifically, a Lyapunov functional is proposed to analyse the delayed system. A sufficient condition for the control gain is then derived to guarantee that this functional is non-increasing.

3) A new framework based on the admissible upper bound of the formation error is developed to simultaneously guarantee connectivity maintenance and order preservation. Under this framework, the admissible upper bound for the formation error is designed such that the Euclidean distance between neighbouring agents is guaranteed to be both greater than zero and less than their communication range. Then, network connectivity and order preservation can be achieved by simply keeping the absolute value of the formation error within the designed admissible upper bound. Compared with [27], [28], which study the unevenly-spaced circular formation problem with different types of communication delays, the controller proposed in our work can preserve the agent's spatial order and reduce the risk of inter-agent collisions.

Notations: $\mathbb{R}^n$ and $\mathbb{R}^{m\times n}$ denote the space of all real n-dimensional vectors and the space of all real $m\times n$ matrices, respectively. For a vector $x\in\mathbb{R}^n$, $\|x\|$ denotes its Euclidean norm. $\mathbf{0}_n$ and $\mathbf{1}_n\in\mathbb{R}^n$ represent vectors with all their elements being 0 and 1, respectively. $I_n\in\mathbb{R}^{n\times n}$ is the identity matrix. For a matrix $M\in\mathbb{R}^{m\times n}$, M^T denotes its transpose, $\mathrm{col}_k(M)$ is its kth column and $\text{col}(M) := \{\text{col}_k(M): 1 \le k \le n\}$. For symmetric matrices $M \in \mathbb{R}^{n\times n}$ and $N \in \mathbb{R}^{n\times n}$, $M \succ N$ and $M \succcurlyeq N$ denote that $M - N$ is positive definite and positive semi-definite, respectively. Let $\mathcal{G}^* = \{\mathcal{V}, \mathcal{E}^*\}$ denote a directed cyclic graph where $\mathcal{V} = \{1,\dots,n\}$ represents the set of agents in the system and $\mathcal{E}^* = \{(i,i+1): i \in \mathcal{V} \setminus \{n\}\} \cup \{(n,1)\}$ denotes the set of edges.

The remainder of this article is organized as follows. The preliminaries and problem formulation are introduced in Section II. The distributed controller and analysis of the closed-loop system are presented in Section III. Some simulation examples are provided in Section IV. Finally, some conclusions are given in Section V.

II. Preliminaries and Problem Statement

Consider a group of n agents moving on a circle of radius R_0. Suppose that each agent is described by

$$\dot{\theta}_i(t) = u_i(t) \tag{1}$$

where θ_i is the angular position of agent i on the circle and u_i is the control input. According to their initial positions in the anticlockwise direction from the horizontal real axis, the agents are labelled from 1 to n as follows:

$$0 \le \theta_1(0) < \cdots < \theta_n(0) < 2\pi. \tag{2}$$

The desired formation pattern is described by a constant vector $\boldsymbol{c} = [c_1,\dots,c_n]^T$, where each element c_i denotes the desired angular distance between the $(i+1)$th and the ith agents. It is noted that the sum of all elements equals 2π. Define

$$d_i(t) := \begin{cases} \theta_{i+1}(t) - \theta_i(t), & i = 1,\dots,n-1 \\ \theta_1(t) + 2\pi - \theta_n(t), & i = n \end{cases} \tag{3}$$

which denotes the angular distance between two neighbouring agents. Similarly to [10], [30], the definition of order preservation is given as follows.

Definition 1 (Order Preservation): For the MAS (1), given the initial condition (2), we say that the spatial order of the MAS is preserved under a control law u_i if the trajectory of the system (1) satisfies $d_i(t) > 0, \forall i \in \mathcal{V}$ over time.

In our setting, the communication ranges between neighbouring agents are supposed to be limited and heterogeneous. In particular, it is assumed that the $(i+1)$th agent with limited communication range r_{i+1} will only communicate with the ith agent if their Euclidean distance is within its communication range, i.e., $2R\sin\frac{d_i}{2} < r_{i+1}$.

The communication range r_{i+1} of the $(i+1)$th agent can be equivalently described in terms of angular distance defined as $\Delta_{i+1} := 2\arcsin\frac{r_{i+1}}{2R}$. It is assumed that the communication range Δ_{i+1} is larger than the desired formation angular distance c_i for all $i \in \{1,\dots,n\}$. The above-mentioned coordinate frame and notation can be found in Fig. 1.

Denote the formation error by $e_i(t)$, where $e_i(t) = d_i(t) - c_i$. Rewrite it in the following compact form:

$$\boldsymbol{e}(t) = -E^T\boldsymbol{\theta}(t) - \boldsymbol{c}' \tag{4}$$

where $\boldsymbol{e}(t) = [e_1(t),\dots,e_n(t)]^T$, $\boldsymbol{\theta}(t) = [\theta_1(t),\dots,\theta_n(t)]^T$, $\boldsymbol{c}' = [c_1,\dots,c_n - 2\pi]^T$, and

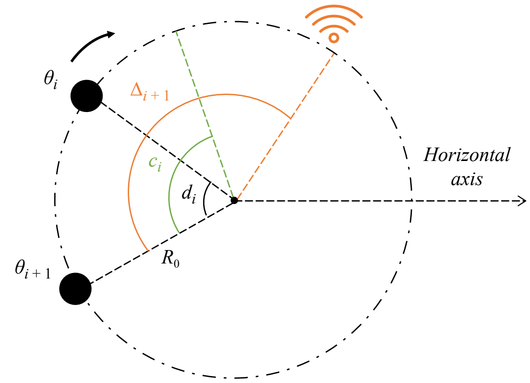


Fig. 1. Coordinate frame and notation.

$$E = \left(\begin{array}{cccccc|c} 1 & 0 & 0 & \cdots & 0 & & -1 \\ -1 & 1 & 0 & \cdots & 0 & & 0 \\ 0 & -1 & 1 & \cdots & 0 & & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 & & 0 \\ 0 & 0 & 0 & \cdots & -1 & & 1 \end{array}\right). \tag{5}$$

It is noted that matrix E is the incidence matrix of the directed cyclic graph $\mathcal{G}^* = \{\mathcal{V}, \mathcal{E}^*\}$ and its rank is $(n-1)$. This incidence matrix can be decomposed into two parts as follows:

$$E = [E_s \quad E_c] \tag{6}$$

where $E_s \in \mathbb{R}^{n\times(n-1)}$ denotes the full-column-rank incidence matrix of a spanning tree, and $E_c \in \mathbb{R}^n$ denotes the incidence matrix of the remaining one edge (the cyclic edge). It can be observed from [31] that matrix E_c can be further expressed by the linear combination of $\text{col}(E_s)$, i.e., $E_c = E_sT$ with $T = -\mathbf{1}_{n-1}$. Define $R := \left[I_{n-1} \quad T\right] \in \mathbb{R}^{(n-1)\times n}$ and one can obtain an alternative expression of the incidence matrix E as follows:

$$E = [E_s \quad E_c] = E_s[I_{n-1} \quad T] = E_sR \tag{7}$$

which is then used for reducing the order of the formation error vector $\boldsymbol{e}(t)$. Let $\boldsymbol{e}_s(t) \in \mathbb{R}^{n-1}$ and $e_c(t) \in \mathbb{R}$ denote the first $(n-1)$ entries and the last entry of the formation error vector $\boldsymbol{e}(t)$, respectively. Since $c_n - 2\pi = -\sum_{i=1}^{n-1} c_i$, it can be verified that

$$\boldsymbol{e}(t) = \begin{pmatrix} \boldsymbol{e}_s(t) \\ e_c(t) \end{pmatrix} = R^T\boldsymbol{e}_s(t). \tag{8}$$

Moreover, matrix R has the following properties:

P1: Every column sum of matrix R equals zero.

P2: R is a full-row-rank matrix.

To study the connectivity maintenance issue, an assumption on initial network connectivity is made as follows.

Assumption 1: The initial communication graph of the MAS (1) is a strongly connected directed cyclic graph $\mathcal{G}^* = \{\mathcal{V}, \mathcal{E}^*\}$.

Note that communication delays between agents are also considered in this work. Assume that the delay $\tau_d(t)$ is time-varying and identical for all agents. To facilitate the convergence analysis of the system in the presence of communication delays, the following standard assumption is introduced [27], [32], [33].

Assumption 2: The time delay $\tau_d(t)$ is known to all the agents and is upper bounded by a known positive constant $\bar{\tau}_d$. The derivative of $\tau_d(t)$ satisfies $\dot{\tau}_d(t) < d_m < 1$. Initially, the MAS stays at rest and has stayed at rest longer than the upper bound of the transmission delay, that is, $\dot{\theta}_i(\xi) = 0$ for $\xi \in [-\bar{\tau}_d, 0], \forall i \in \mathcal{V}$.

Now, the circular formation control problem considered in this work can be formally stated as follows.

Problem 1: Given a circular formation described by radius R_0 and a desired formation pattern $\boldsymbol{c} = [c_1, \ldots, c_n]^T$, consider an MAS with heterogeneous communication ranges in terms of angular distances $[\Delta_1, \ldots, \Delta_n]$. Design a cyclic-pursuit-based controller such that the following control objectives can be achieved in the presence of time-varying communication delays.

i) The agents eventually reach their desired spacing pattern, that is, $\lim_{t\to\infty} d_i(t) = c_i, \forall i \in \mathcal{V}$.

ii) The angular distance between each pair of agents satisfies $d_i(t) < \Delta_{i+1}$, for all $i \in \mathcal{V}$ and all $t \geq 0$, that is, network connectivity is guaranteed.

iii) The angular distance between each pair of agents satisfies $d_i(t) > 0$, for all $i \in \mathcal{V}$ and all $t \geq 0$, that is, spatial order of the MAS is preserved.

III. Main Results

We begin this section by establishing admissible upper bounds for the formation errors such that connectivity maintenance and order preservation can be achieved simultaneously. We then propose a distributed controller based on an approach of potential functions. Finally, we present an analysis of the derived closed-loop system.

A. Admissible Upper Bounds for Formation Errors

Define an admissible upper bound for the absolute value of formation error $|e_i(t)|$ as follows:

$$\delta_i := \min\{c_i, \Delta_{i+1} - c_i\}. \tag{9}$$

One can claim that $0 < d_i(t)$ and $d_i(t) < \Delta_{i+1}$ are achieved simultaneously by just keeping $|e_i(t)|$ within the corresponding admissible upper bound, that is,

$$|e_i(t)| = |d_i(t) - c_i| < \delta_i. \tag{10}$$

In fact, since $\delta_i \leq \Delta_{i+1} - c_i$ and $-\delta_i \geq -c_i$, it follows that:

$$0 \leq c_i - \delta_i < d_i(t) < c_i + \delta_i \leq \Delta_{i+1} \tag{11}$$

which is exactly what is claimed. Therefore, connectivity maintenance and order preservation of the MAS can be guaranteed simultaneously if (10) is always ensured. For subsequent analysis, define $\delta_M := \max_{i\in\mathcal{V}} \delta_i$ and $\delta_m := \min_{i\in\mathcal{V}} \delta_i$.

Remark 1: Unlike previous works [34], [35], where the collision avoidance is guaranteed by proposing a repulsive function based on the distance of two neighbouring agents, in our work, the same problem is alternatively considered by preserving the initial order of the MAS on a circle. Moreover, some potential functions based on the formation errors will be constructed in the next subsection such that the connectivity problem can also be solved.

Based on δ_i, define a time-varying graph $\mathcal{G}(t) = \{\mathcal{V}, \mathcal{E}(t)\}$ for the MAS, where $\mathcal{V} = \{1, \ldots, n\}$ represents the set of agents in the system and $\mathcal{E}(t) = \{(i, i+1) : |e_i(t)| < \delta_i, i \in \mathcal{V} \setminus \{n\}\} \cup \{(n, 1) : |e_n(t)| < \delta_n\}$ denotes the set of edges at time instant t.

Regarding the initial conditions of the MAS, the following assumption is introduced.

Assumption 3: The MAS is initially positioned on the circle such that $|e_i(0)| < \delta_i - \epsilon, \forall i \in \mathcal{V}$, where $0 < \epsilon < \delta_m$.

Remark 2: It can be observed from Assumption 3 that the inequality in (10) holds at $t = 0$ and thus $\mathcal{G}(0) = \mathcal{G}^*$. Now keeping $|e_i(t)| < \delta_i$, for all $i \in \mathcal{V}$ and all $t \geq 0$ implies that $\mathcal{G}(t)$ is maintained to be the same as the fixed topology $\mathcal{G}^*$. In addition, similarly to [32], the parameter ϵ in Assumption 3 is to introduce the effect of hysteresis and will be used in the proof of Lemma 1.

B. Controller Design

From the previous subsection, it is known that Problem 1 can be solved if the following objectives are achieved.

i) The time-varying graph $\mathcal{G}(t)$ for the MAS is maintained to be $\mathcal{G}^*$, $\forall t \geq 0$.

ii) The formation errors eventually converge to zero, that is, $\lim_{t\to\infty} e_i(t) = 0, \forall i \in \mathcal{V}$.

Inspired by the approach of potential functions in [32], [33], we propose a set of potential functions to keep $|e_i(t)|$ staying inside the admissible upper bound as follows:

$$\psi_i(e_i^2(t)) = \frac{e_i^2(t)}{\delta_i^2 - e_i^2(t) + \frac{\delta_i^2}{q}},\ e_i^2(t) < \delta_i^2 \tag{12}$$

where q is a positive constant parameter to be determined later. It is noted that each potential function is defined on an interval $\left[0, \delta_i^2\right)$ and satisfies the following properties, 1) it is monotonously increasing with respect to $e_i^2(t)$; and 2) it reaches its minimum at $e_i(t) = 0$.

Define

$$p_i(e_i(t)) := \begin{cases} \dfrac{2(\delta_i^2 + \frac{\delta_i^2}{q})}{(\delta_i^2 - e_i^2(t) + \frac{\delta_i^2}{q})^2}, & e_i^2(t) < \delta_i^2 \\ 0, & \text{otherwise} \end{cases} \tag{13}$$

which is non-negative. It can be seen that $\frac{d\psi_i(e_i^2(t))}{de_i(t)} = p_i(e_i(t))e_i(t)$. For subsequent analysis, define matrix $P(\boldsymbol{e}(t)) := \text{diag}(p_1(e_1(t)), \ldots, p_n(e_n(t)))$.

In the presence of time-varying communication delays, the ith agent will receive delayed information $\theta_{i+1}(t - \tau_d(t))$ from the $(i+1)$th agent at time instant t. In this case, the controller $u_i(t)$ uses the delayed formation error, that is, $e_i(t - \tau_d(t)) = d_i(t - \tau_d(t)) - c_i$. We thus propose the following controller:

$$\begin{aligned} u_i(t) &= kp_i(e_i(t - \tau_d(t)))e_i(t - \tau_d(t)) \\ &= k\rho_i(t - \tau_d(t)), i \in \mathcal{V} \end{aligned} \tag{14}$$

where $k > 0$ and

$$\rho_i(t) := p_i(e_i(t))e_i(t). \tag{15}$$

Rewrite (14) into the following compact form:

$$\boldsymbol{u}(t) = k\boldsymbol{\rho}(t - \tau_d(t)) \tag{16}$$

where $\boldsymbol{u}(t) = [u_1(t), \ldots, u_n(t)]^T$ and $\boldsymbol{\rho}(t - \tau_d(t)) = [\rho_1(t - \tau_d(t)),$

$\ldots,\rho_n(t-\tau_d(t))]^T$.

By taking the time derivative of (4) and noting (16), one has

$$\dot{\boldsymbol{e}}(t) = -E^T\boldsymbol{u}(t) = -kE^T\boldsymbol{\rho}(t-\tau_d(t)). \tag{17}$$

It can be calculated from (13) that

$$\dot{p}_i(e_i(t)) = \frac{8(\delta_i^2+\frac{\delta_i^2}{q})e_i(t)}{(\delta_i^2-e_i^2(t)+\frac{\delta_i^2}{q})^3}\dot{e}_i(t).$$

Define

$$\tilde{\delta}_i := \delta_i^2 + \frac{\delta_i^2}{q}. \tag{18}$$

By taking the time derivative of $\rho_i(t)$, one has

$$\begin{aligned}\dot{\rho}_i(t) &= \dot{p}_i(e_i(t))e_i(t) + p_i(e_i(t))\dot{e}_i(t)\\ &= \left[\frac{8\tilde{\delta}_i e_i^2(t)}{(\tilde{\delta}_i - e_i^2(t))^3} + p_i(e_i(t))\right]\dot{e}_i(t)\\ &= \bar{p}_i(t)\dot{e}_i(t)\end{aligned} \tag{19}$$

where

$$\bar{p}_i(t) = \frac{8\tilde{\delta}_i e_i^2(t)}{(\tilde{\delta}_i - e_i^2(t))^3} + \frac{2\tilde{\delta}_i}{(\tilde{\delta}_i - e_i^2(t))^2},\ e_i^2(t) < \delta_i^2. \tag{20}$$

Define matrix $\bar{P}(\boldsymbol{e}(t)) := \mathrm{diag}(\bar{p}_1(t),\ldots,\bar{p}_n(t))$. Since $\bar{p}_i(t) > 0,\ \forall i\in\mathcal{V}$, it can be seen that $\bar{P}(\boldsymbol{e}(t))$ is positive definite. By substituting (17) into the compact form of (19), the closed-loop system can be obtained as follows:

$$\dot{\boldsymbol{\rho}}(t) = -k\bar{P}(\boldsymbol{e}(t))E^T\boldsymbol{\rho}(t-\tau_d(t)). \tag{21}$$

C. Analysis of the Closed-Loop System

Consider a candidate Lyapunov functional as follows:

$$\begin{aligned}V(t) &:= V_1(t)+V_2(t)\\ &:= 2\sum_{i=1}^n \psi_i(e_i^2(t)) + \int_{t-\tau_d(t)}^t (s-t+\bar{\tau}_d)\|\dot{\boldsymbol{\rho}}(s)\|^2 ds\end{aligned} \tag{22}$$

where $V_1(t)$ is defined as the sum of the proposed potential functions, and $V_2(t)$ is added to deal with communication delays.

Before presenting the main results, some technical lemmas are introduced.

Lemma 1: Define

$$T_e := \sum_{i=1}^n \frac{2(\delta_i-\epsilon)^2}{\delta_i^2-(\delta_i-\epsilon)^2+\frac{\delta_i^2}{q}} \tag{23a}$$

$$l_i := \delta_i\sqrt{\frac{1+\frac{1}{q}}{1+\frac{2}{T_e}}},\ \forall i\in\mathcal{V}. \tag{23b}$$

Under Assumptions 1–3, if the constant q is chosen satisfying

$$q > \frac{n(\delta_M-\epsilon)^2}{\delta_M^2-(\delta_M-\epsilon)^2} \tag{24}$$

then one has

$$\delta_i - \epsilon < l_i < \delta_i. \tag{25}$$

Proof: It is noted that

$$\begin{aligned}\frac{d\frac{\delta_i^2}{(\delta_i-\epsilon)^2}}{d\delta_i} &= \frac{2\delta_i(\delta_i-\epsilon)^2-2\delta_i^2(\delta_i-\epsilon)}{(\delta_i-\epsilon)^4}\\ &= \frac{-2\delta_i\epsilon(\delta_i-\epsilon)}{(\delta_i-\epsilon)^4} < 0.\end{aligned}$$

This implies that $\frac{\delta_i^2}{(\delta_i-\epsilon)^2}$ decreases as δ_i increases. It then follows that:

$$\begin{aligned}T_e &= \sum_{i=1}^n \frac{2}{(1+\frac{1}{q})\frac{\delta_i^2}{(\delta_i-\epsilon)^2}-1}\\ &\le \sum_{i=1}^n \frac{2}{(1+\frac{1}{q})\frac{\delta_M^2}{(\delta_M-\epsilon)^2}-1}\\ &< \frac{2n(\delta_M-\epsilon)^2}{\delta_M^2-(\delta_M-\epsilon)^2} < 2q.\end{aligned} \tag{26}$$

The last inequality holds due to (24). Then, it can be obtained that $l_i = \delta_i\sqrt{\frac{1+\frac{1}{q}}{1+\frac{2}{T_e}}} < \delta_i$. Moreover, rewrite T_e in terms of l_i as follows:

$$T_e = \frac{2l_i^2}{\delta_i^2 - l_i^2 + \frac{\delta_i^2}{q}}. \tag{27}$$

It follows from (23a) and (27) that:

$$\begin{aligned}&\frac{2l_i^2}{\delta_i^2-l_i^2+\frac{\delta_i^2}{q}} - \frac{2(\delta_i-\epsilon)^2}{\delta_i^2-(\delta_i-\epsilon)^2+\frac{\delta_i^2}{q}}\\ &= \sum_{j\in\mathcal{V}\setminus\{i\}} \frac{2(\delta_j-\epsilon)^2}{\delta_j^2-(\delta_j-\epsilon)^2+\frac{\delta_j^2}{q}} > 0\end{aligned}$$

which implies that $l_i > \delta_i - \epsilon$. ■

The following lemma gives the condition on the control gain k under which the candidate Lyapunov functional is non-increasing.

Lemma 2: Consider the MAS (1) and the candidate Lyapunov functional (22). Assume that $|e_i(t)| < l_i, \forall i\in\mathcal{V}$, where l_i is defined in (23b). Then, there exists a positive constant k_0 such that $\dot{V}(t)\le 0$ if the control gain k is chosen satisfying $k < k_0$.

Proof: First, we show that the time derivative of the candidate Lyapunov functional $V(t)$ satisfies

$$\dot{V}(t) \le -\boldsymbol{\rho}^T(t-\tau_d(t))ES(t)E^T\boldsymbol{\rho}(t-\tau_d(t)) \tag{28}$$

where

$$S(t) = kI_n - \bar{\tau}_d k^2 I_n - \bar{\tau}_d k^2\bar{P}^2(\boldsymbol{e}(t)). \tag{29}$$

Taking the time derivative of $V(t)$ in (22) gives

$$\dot{V}(t) = \dot{V}_1(t) + \dot{V}_2(t). \tag{30}$$

It then follows from (15) and (17) that:

$$\begin{aligned}\dot{V}_1(t) &= \frac{dV_1(t)}{d\boldsymbol{e}(t)}\dot{\boldsymbol{e}}(t) = -2k\boldsymbol{\rho}^T(t)E^T\boldsymbol{\rho}(t-\tau_d(t))\\ &= -2k(\boldsymbol{\rho}^T(t-\tau_d(t)) + \int_{t-\tau_d(t)}^{t}\dot{\boldsymbol{\rho}}^T(s)ds)E^T\boldsymbol{\rho}(t-\tau_d(t))\\ &\overbrace{\le}^{①} -2k\boldsymbol{\rho}^T(t-\tau_d(t))E^T\boldsymbol{\rho}(t-\tau_d(t))\\ &\quad + \int_{t-\tau_d(t)}^{t}\|\dot{\boldsymbol{\rho}}(s)\|^2ds + \tau_d(t)k^2\|E^T\boldsymbol{\rho}(t-\tau_d(t))\|^2\\ &= -k\boldsymbol{\rho}^T(t-\tau_d(t))(E^T+E)\boldsymbol{\rho}(t-\tau_d(t))\\ &\quad + \int_{t-\tau_d(t)}^{t}\|\dot{\boldsymbol{\rho}}(s)\|^2ds + \tau_d(t)k^2\|E^T\boldsymbol{\rho}(t-\tau_d(t))\|^2\\ &\overbrace{=}^{②} -k\boldsymbol{\rho}^T(t-\tau_d(t))EE^T\boldsymbol{\rho}(t-\tau_d(t))\\ &\quad + \int_{t-\tau_d(t)}^{t}\|\dot{\boldsymbol{\rho}}(s)\|^2ds + \tau_d(t)k^2\|E^T\boldsymbol{\rho}(t-\tau_d(t))\|^2\\ &\le -(k-\bar{\tau}_dk^2)\|E^T\boldsymbol{\rho}(t-\tau_d(t))\|^2 + \int_{t-\tau_d(t)}^{t}\|\dot{\boldsymbol{\rho}}(s)\|^2ds.\end{aligned} \tag{31}$$

Note that the inequalities tagged by ① and ② can be derived from the Cauchy-Schwarz inequality and the special matrix E (5), respectively.

Let $\frac{dF_1(s)}{ds} = (s+\bar{\tau}_d)\|\dot{\boldsymbol{\rho}}(s)\|^2$ and $\frac{dF_2(s)}{ds} = t\|\dot{\boldsymbol{\rho}}(s)\|^2$. Then, it can be obtained that $V_2(t) = \int_{t-\tau_d(t)}^{t} dF_1(s) - \int_{t-\tau_d(t)}^{t} dF_2(s)$. Taking the time derivative of V_2 yields (32) (see the bottom of page 6). The inequality tagged by ③ holds due to Assumption 2.

By combining (31) and (32), one can obtain that

$$\begin{aligned}\dot{V}(t) &\le -(k-\bar{\tau}_dk^2)\boldsymbol{\rho}^T(t-\tau_d(t))EE^T\boldsymbol{\rho}(t-\tau_d(t))\\ &\quad + \bar{\tau}_dk^2\boldsymbol{\rho}^T(t-\tau_d(t))E\bar{P}^2(\boldsymbol{e}(t))E^T\boldsymbol{\rho}(t-\tau_d(t))\\ &= -\boldsymbol{\rho}^T(t-\tau_d(t))E(kI_n - \bar{\tau}_dk^2I_n\\ &\quad - \bar{\tau}_dk^2\bar{P}^2(\boldsymbol{e}(t)))E^T\boldsymbol{\rho}(t-\tau_d(t))\\ &= -\boldsymbol{\rho}^T(t-\tau_d(t))ES(t)E^T\boldsymbol{\rho}(t-\tau_d(t)).\end{aligned} \tag{33}$$

Thus the inequality (28) is proved.

Next, the condition on the control gain k under which $S(t) \succ \frac{1}{2}kI_n$ will be derived. It follows from (18) and (23b) that $l_i^2 = \frac{T_e\tilde{\delta}_i}{2+T_e}$. Under the assumption that $|e_i(t)| < l_i, \forall i \in \mathcal{V}$, an upper bound of $\bar{p}_i(t), \forall i \in \mathcal{V}$ can be derived from (20) as follows:

$$\begin{aligned}\bar{p}_i(t) &< \frac{8\tilde{\delta}_il_i^2}{(\tilde{\delta}_i - l_i^2)^3} + \frac{2\tilde{\delta}_i}{(\tilde{\delta}_i - l_i^2)^2} = \frac{8\tilde{\delta}_i^2T_e}{(2+T_e)(\frac{2\tilde{\delta}_i}{2+T_e})^3}\\ &\quad + \frac{2\tilde{\delta}_i}{(\frac{2\tilde{\delta}_i}{2+T_e})^2} = \frac{T_e(2+T_e)^2}{\tilde{\delta}_i} + \frac{(2+T_e)^2}{2\tilde{\delta}_i} := B_{p_i}.\end{aligned} \tag{34}$$

It can be observed from (34) that the value of B_{p_i} increases as $\tilde{\delta}_i$ decreases. Define $\lambda_M := T_e(2+T_e)^2/\tilde{\delta}_m + (2+T_e)^2/2\tilde{\delta}_m$. It then follows from (34) that $0 < \bar{p}_i(t) < B_{p_i} \le \lambda_M, \forall i \in \mathcal{V}$. Define

$$k_0 := \frac{1}{2\bar{\tau}_d(1+\lambda_M^2)}. \tag{35}$$

It follows from (29) and (35) that if the control gain k satisfies $k < k_0$, then:

$$S(t) \succ \frac{1}{2}kI_n. \tag{36}$$

Therefore, it follows from (28) that $\dot{V}(t) \le 0$. ■

Remark 3: In Lemme 2, an upper bound for the control gain k is given by (35) using Lyapunov functionals and matrix analysis. It can be seen from (35) that as the upper bound $\bar{\tau}_d$ of the delays increases, a smaller control gain k is needed to guarantee the stability of the closed-loop system.

Now, it is time to present our main results. The following first result shows that the time-varying graph $\mathcal{G}(t)$ is maintained to be $\mathcal{G}^*, \forall t \ge 0$.

Theorem 1: Under Assumptions 1−3, choose q satisfying (24) and $k < k_0$, where k_0 is defined in (35). Then, the distributed controller (14) guarantees that the time-varying graph $\mathcal{G}(t)$ of the MAS (1) is maintained to be $\mathcal{G}^*, \forall t \ge 0$.

Proof: We first show that $|e_i(t)| < l_i$, for all $i \in \mathcal{V}$ and all $t \ge 0$ by contradiction.

It follows from Assumption 2 that $V(0) = V_1(0) = \sum_{i=1}^{n}\frac{2e_i^2(0)}{\delta_i^2 - e_i^2(0) + \frac{\delta_i^2}{q}}$. From Assumption 3, that is, $|e_i(0)| < \delta_i - \epsilon$, $\forall i \in \mathcal{V}$ and Lemma 1, one has the following inequalities at $t = 0$:

$$\begin{aligned}\dot{V}_2(t) &= \lim_{h\to 0}\frac{\left[\int_t^{t+h}dF_1(s) - \int_{t-\tau_d(t)}^{t+h-\tau_d(t+h)}dF_1(s) - \int_t^{t+h}dF_2(s) + \int_{t-\tau_d(t)}^{t+h-\tau_d(t+h)}dF_2(s)\right]}{h} - \lim_{h\to 0}\int_{t+h-\tau_d(t+h)}^{t+h}\frac{h\|\dot{\boldsymbol{\rho}}(s)\|^2}{h}ds\\ &= \lim_{h\to 0}\frac{\left[F_1(s)|_t^{t+h} - F_1(s)|_{t-\tau_d(t)}^{t+h-\tau_d(t+h)} - F_2(s)|_t^{t+h} + F_2(s)|_{t-\tau_d(t)}^{t+h-\tau_d(t+h)}\right]}{h} - \int_{t-\tau_d(t)}^{t}\|\dot{\boldsymbol{\rho}}(s)\|^2ds\\ &= F_1'(t) - F_1'(t-\tau_d(t)) - F_2'(t) + F_2'(t-\tau_d(t)) - \int_{t-\tau_d(t)}^{t}\|\dot{\boldsymbol{\rho}}(s)\|^2ds\\ &= (t+\bar{\tau}_d)\|\dot{\boldsymbol{\rho}}(t)\|^2 - (1-\dot{\tau}_d(t))(t-\tau_d(t)+\bar{\tau}_d)\|\dot{\boldsymbol{\rho}}(t-\tau_d(t))\|^2 - t\|\dot{\boldsymbol{\rho}}(t)\|^2 + t(1-\dot{\tau}_d(t))\|\dot{\boldsymbol{\rho}}(t-\tau_d(t))\|^2 - \int_{t-\tau_d(t)}^{t}\|\dot{\boldsymbol{\rho}}(s)\|^2ds\\ &= \bar{\tau}_d\|\dot{\boldsymbol{\rho}}(t)\|^2 + (1-\dot{\tau}_d(t))(\tau_d(t)-\bar{\tau}_d)\|\dot{\boldsymbol{\rho}}(t-\tau_d(t))\|^2 - \int_{t-\tau_d(t)}^{t}\|\dot{\boldsymbol{\rho}}(s)\|^2ds\\ &\overbrace{\le}^{③} \bar{\tau}_d\|\dot{\boldsymbol{\rho}}(t)\|^2 - \int_{t-\tau_d(t)}^{t}\|\dot{\boldsymbol{\rho}}(s)\|^2ds = \bar{\tau}_dk^2\boldsymbol{\rho}^T(t-\tau_d(t))E\bar{P}^2(\boldsymbol{e}(t))E^T\boldsymbol{\rho}(t-\tau_d(t)) - \int_{t-\tau_d(t)}^{t}\|\dot{\boldsymbol{\rho}}(s)\|^2ds.\end{aligned} \tag{32}$$

$$\begin{cases} V(0) < T_e \\ |e_i(0)| < l_i, \forall i \in \mathcal{V}. \end{cases} \tag{37}$$

Assume that there exists a time instant $t_1 > 0$, such that, for an arbitrary agent $i \in \mathcal{V}$, the absolute value of its formation error reaches l_i for the first time, that is, $|e_i(t)| < l_i$, for $t \in [0, t_1)$ and $|e_i(t_1)| = l_i$. By choosing the control gain k satisfying $k < k_0$ and applying Lemma 2 for $t \in [0, t_1)$, one has

$$V(t) \le V(0) < T_e,\ t \in [0, t_1). \tag{38}$$

At the same time, it can be obtained from the continuity of the formation error $e_i(t)$ that $\lim_{t \to t_1^-} |e_i(t)| = l_i$. Then, one has

$$\lim_{t \to t_1^-} 2\psi_i(e_i^2(t)) = \lim_{t \to t_1^-} \frac{2e_i^2(t)}{\delta_i^2 - e_i^2(t) + \frac{\delta_i^2}{q}} = \frac{2l_i^2}{\delta_i^2 - l_i^2(t) + \frac{\delta_i^2}{q}} = T_e.$$

Consequently, it can be obtained that

$$\begin{aligned} \lim_{t \to t_1^-} V(t) &= \lim_{t \to t_1^-} \left(2\psi_i(e_i^2(t)) + 2 \sum_{j \in \mathcal{V} \setminus \{i\}} \psi_j(e_j^2(t)) + V_2(t) \right) \\ &= T_e + \lim_{t \to t_1^-} \left(2 \sum_{j \in \mathcal{V} \setminus \{i\}} \psi_j(e_j^2(t)) + V_2(t) \right) > T_e \end{aligned}$$

which contradicts the inequality (38). Thus there does not exist a time instant $t_1 > 0$ such that $|e_i(t)|$ can reach l_i at $t = t_1$. Since i is arbitrarily chosen, it can be obtained that

$$|e_i(t)| < l_i \tag{39}$$

for all $i \in \mathcal{V}$ and all $t \ge 0$.

It further follows from Lemma 1 that $|e_i(t)| < l_i < \delta_i$, for all $i \in \mathcal{V}$ and all $t \ge 0$. Then, one can conclude that the time-varying graph $\mathcal{G}(t)$ is maintained to be $\mathcal{G}^*$ for all $t \ge 0$. ■

Next, we present our second main result, that is, the MAS ultimately converges to the desired formation pattern with connectivity maintenance and order preservation.

Theorem 2: Under Assumptions 1−3, choose q satisfying (24) and $k < k_0$, where k_0 is defined in (35). Then, the MAS (1) under the proposed controller (14) will converge to the desired formation pattern with connectivity maintenance and order preservation.

Proof: It has been shown in (39) that $|e_i(t)| < l_i$, for all $i \in \mathcal{V}$ and all $t \ge 0$. Then, Lemma 2 can be applied for all $t \ge 0$ and one can obtain that $\dot{V}(t) \le 0$ if $k < k_0$. Since $V(t) \ge 0$ and $\dot{V}(t) \le 0$, the Lyapunov functional $V(t)$ has a finite limit as time goes to infinity. Thus we will use the Barbalat's lemma [36] to conduct the convergence analysis of the MAS (1). To this end, we need to show that $\ddot{V}$ is bounded. First, given that $V(t) \ge 0$ and $\dot{V}(t) \le 0$, we obtain that $\forall i \in \mathcal{V}$, $e_i(t)$, $\bar{p}_i(t)$, and $\dot{\rho}_i(t)$ are all bounded, which implies that $\dot{e}_i(t)$ and $\ddot{e}_i(t)$ are also bounded. Meanwhile, taking the time derivative of $\bar{p}_i(t)$ in (20) gives

$$\dot{\bar{p}}_i(t) = \left[\frac{48\tilde{\delta}_i e_i^2(t)}{(\tilde{\delta}_i - e_i^2(t))^4} + \frac{24\tilde{\delta}_i}{(\tilde{\delta}_i - e_i^2(t))^3} \right] e_i(t)\dot{e}_i(t).$$

It can be seen that $\dot{\bar{p}}_i(t)$ is also bounded. Since $\ddot{\rho}_i(t) = \dot{\bar{p}}_i(t)\dot{e}_i(t) + \bar{p}_i(t)\ddot{e}_i(t)$, one can conclude that $\ddot{\rho}_i(t)$ is also bounded. It then can be verified that $\ddot{V}(t)$ is bounded. Now, recalling the Barbalat's lemma yields $\lim_{t \to \infty} \dot{V}(t) = 0$.

It follows from (28) that $\lim_{t \to \infty} \dot{V}(t) \le -\lim_{t \to \infty} \boldsymbol{\rho}^T(t - \tau_d(t)) ES(t) E^T \boldsymbol{\rho}(t - \tau_d(t)) \le 0$. Thus, $\lim_{t \to \infty} \boldsymbol{\rho}^T(t - \tau_d(t)) \times ES(t) E^T \boldsymbol{\rho}(t - \tau_d(t)) = 0$. Meanwhile, given (36), it can be obtained that $\lim_{t \to \infty} \boldsymbol{\rho}^T(t - \tau_d(t)) EE^T \boldsymbol{\rho}(t - \tau_d(t)) = 0$, which implies that

$$\lim_{t \to \infty} E^T \boldsymbol{\rho}(t - \tau_d(t)) = \mathbf{0}_n. \tag{40}$$

Since every column sum of matrix E^T equals zero, it follows from (40) that $\lim_{t \to \infty} \boldsymbol{\rho}(t) = \lim_{t \to \infty} \boldsymbol{\rho}(t - \tau_d(t)) = m\mathbf{1}_n$, where $m \in \mathbb{R}$ is a constant. Based on (8) and (15), it can be derived that

$$\lim_{t \to \infty} P(\boldsymbol{e}(t)) R^T \boldsymbol{e}_s(t) = m\mathbf{1}_n. \tag{41}$$

Pre-multiplying (41) by matrix R results in

$$\lim_{t \to \infty} RP(\boldsymbol{e}(t)) R^T \boldsymbol{e}_s(t) = mR\mathbf{1}_n = \mathbf{0}_{n-1}. \tag{42}$$

The last equality in (42) holds because of property P1. Meanwhile, it follows from (13) and property P2 that $RP(\boldsymbol{e}(t))R^T$ is invertible. Therefore, one can conclude that $\lim_{t \to \infty} \boldsymbol{e}_s(t) = \mathbf{0}_{n-1}$. Note also that $\lim_{t \to \infty} \boldsymbol{e}(t) = \lim_{t \to \infty} R^T \boldsymbol{e}_s(t) = \mathbf{0}_n$. Then, $\lim_{t \to \infty} d_i(t) = c_i$ holds for all $i \in \mathcal{V}$.

From Theorem 1, that is, the time-varying graph $\mathcal{G}(t)$ is maintained to be $\mathcal{G}^*, \forall t \ge 0$, one can conclude that

$$|e_i(t)| < \delta_i$$

for all $i \in \mathcal{V}$ and all $t \ge 0$. Then, connectivity maintenance and order preservation are guaranteed simultaneously. ■

Remark 4: The original formation error vector $\boldsymbol{e}(t)$ is alternatively expressed by a reduced-order error vector $\boldsymbol{e}_s(t)$ via matrix R in the proof of Theorem 2 (see (41)). In particular, it is shown that $\boldsymbol{e}_s(t)$ converges to zero by leveraging properties P1 and P2 of matrix R. Then, it follows from (8) that the original formation error vector $\boldsymbol{e}(t)$ converges to zero too.

IV. Simulation

In this section, simulations with a benchmark example are conducted to illustrate that the proposed cyclic-pursuit-based controller can solve the circular formation problem with network connectivity and order preservation in the presence of communication delays for a group of agents with heterogeneous communication ranges.

A. Circular Formation Task with Connectivity Maintenance and Order Preservation without Communication Delays

In this subsection, we compare our controller with some earlier results [5], [20] by a benchmark example to show that it can solve the unevenly-spaced formation problem with guaranteed connectivity maintenance and order preservation. Table I illustrates forms of these three controllers, where $k_1 > 0, k_2 > 0$, and $k_3 > 0$ are the control gains. In this benchmark example, we will show that the cyclic-pursuit-based controller in [5] cannot solve the unevenly-spaced formation problems with order preservation. In the meantime, we will show that under the switching controller in [20], the formation errors cannot converge to zero due to the heterogeneity of

TABLE I
CONTROLLERS IN THE BENCHMARK EXAMPLE

Method	Controller u_i
[5]	$k_1 e_i(t)$
[20]	$k_2(c_{i-1}\min\{d_i(t),\Delta_i\} - c_i\min\{d_{i-1}(t),\Delta_i\})$
This paper	$k_3 p_i(e_i(t))e_i(t)$

communication ranges among agents. By using the framework proposed in this work, the MAS can converge to the prescribed unevenly-spaced formation pattern with guaranteed network connectivity and order preservation. The simulations of this benchmark example are conducted under the same parameters and initial conditions, which are given in Table II.

Consider the circular formation problem for five agents with single integrator dynamics, and the desired formation pattern is described by a constant vector $\boldsymbol{c} = [120°, 100°, 80°, 10°, 50°]^T$. The agents are constrained to move on a circle of radius $R_0 = 10$ m. The heterogeneous communication ranges among agents are given as $r_1 = 20\sin(\frac{11\pi}{36})$ m, $r_2 = 20\sin(\frac{13\pi}{36})$ m, $r_3 = r_4 = 20\sin(\frac{5\pi}{12})$ m, and $r_5 = 20\sin(\frac{\pi}{12})$ m. In terms of angular distance, the heterogeneous communication ranges are given as $\Delta_1 = 110°$, $\Delta_2 = 130°$, $\Delta_3 = \Delta_4 = 150°$, and $\Delta_5 = 30°$. The initial positions of the agents in this benchmark example are chosen as $\boldsymbol{\theta} = [10°, 132°, 204°, 325°, 330°]^T$. The control gains in [5] and [20] are given as $k_1 = 0.05$, $k_2 = 0.0002$, respectively. It is noted that the control gain k_2 in [20] needs to satisfy that $k_2 < \frac{1}{2\max\{\boldsymbol{c}\}}$ to preserve the agents' initial order. Furthermore, the admissible upper bounds in our work can be designed as $\delta_1 = 10°$, $\delta_2 = 50°$, $\delta_3 = 70°$, $\delta_4 = 10°$, and $\delta_5 = 50°$. Moreover, the parameter in Assumption 3 is selected as $\epsilon = 4$. It can be verified that the initial positions of the agents satisfy the initial conditions stated in Assumption 3. The parameters in the proposed controller (14) are chosen as $k = 0.05$ and $q = 41$.

The time responses of the angular positions for all agents under controllers proposed in [5], [20], and our work are shown in Figs. 2(a)–2(c), respectively. In these figures, the value of θ_i along the vertical axis represents the angular position of agent i on the circular mission space, as defined in (1). It can be seen from the zoomed window in Fig. 2(a) that the agents represented in red and green arrive at the same angular positions on the circle. Therefore, the cyclic-pursuit-based controller in [5] cannot always maintain the agents' spatial order for an unevenly-spaced formation task. On the contrary, it can be observed from Figs. 2(b) and 2(c) that the agents' initial order on the circle is always preserved while performing the formation task. The evolutions of the formation errors for all agents under controllers proposed in [5], [20] and our work are shown in Figs. 2(d)–2(f), respectively. It can be observed from Figs. 2(d) and 2(f) that the formation errors converge to zero as time goes to infinity, that is, $\lim_{t\to\infty} d_i(t) - c_i = 0$. However, it can be seen from Fig. 2(e) that the formation errors cannot converge to the desired formation pattern under the switching control law proposed in [20] for agents with heterogeneous communication ranges.

Furthermore, from the analysis in Section III-A, it is known that if the time-varying graph $\mathcal{G}(t)$ for the MAS is maintained to be $\mathcal{G}^*$, $\forall t \geq 0$, then network connectivity and spatial order of the MAS are always maintained. To this end, we need to check whether the absolute values of each formation error $|e_i(t)|$ are smaller than its corresponding admissible upper bound δ_i along the whole evolution. In Fig. 3, the solid curves represent the absolute values of formation errors, that are, $|e_1(t)|, \ldots, |e_5(t)|$. The dotted lines represent the admissible upper bounds of each agent, that are, $\delta_1, \ldots, \delta_5$, obtained by (9). Since each solid curve is always lower than the dotted line with the same colour, Fig. 3 illustrates that $|e_i(t)|$ stays inside the corresponding admissible upper bound δ_i for all $i \in \mathcal{V}$ and all $t \geq 0$, which implies that network connectivity and spatial order of the MAS are always maintained. Therefore, it can be concluded from this benchmark example that the controller proposed in this work can achieve the circular formation task with guaranteed network connectivity and order preservation.

B. Circular Formation Task with Connectivity Maintenance and Order Preservation in the Presence of Communication Delays

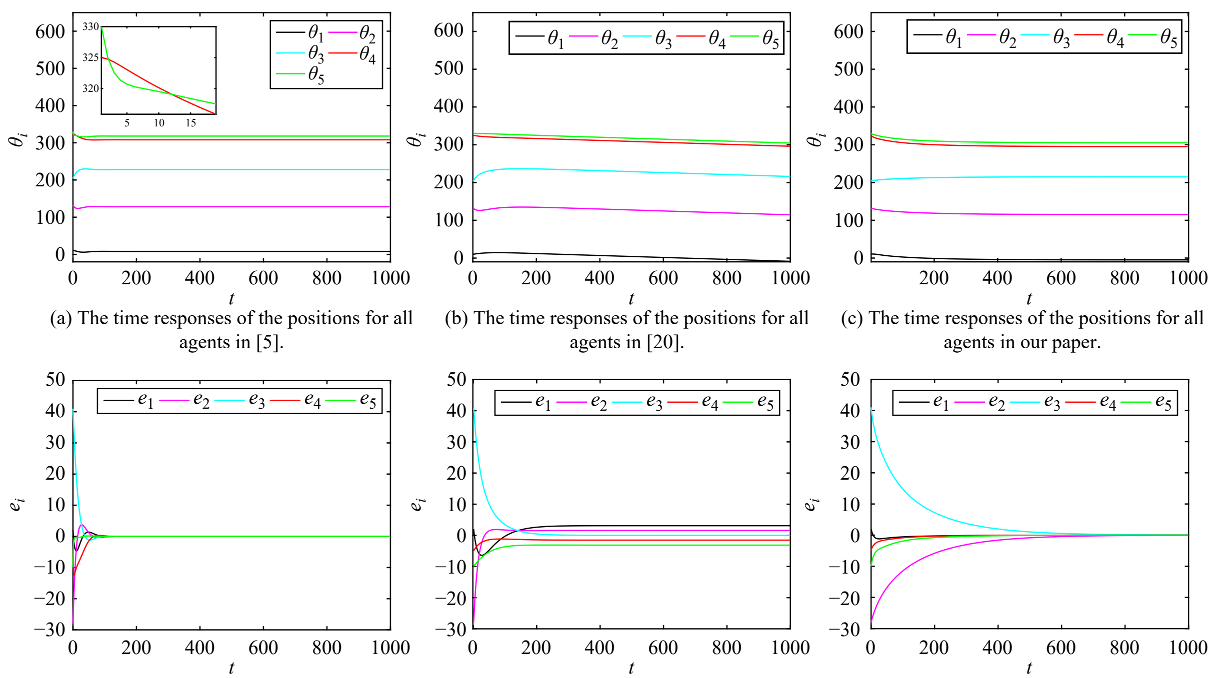
In this subsection, the effectiveness of the proposed controller in the presence of communication delays is verified and the effects of these delays on the circular formation task are also illustrated.

The simulation parameters are the same as in Section IV-A, Table II. To demonstrate the influence of communication delays, we consider and compare the following three different cases: 1) Without communication delays; 2) With delays described by $\tau_{d_1}(t) = 0.1\sin(t) + 0.2$; and 3) With delays described by $\tau_{d_2}(t) = 0.5\sin(t) + 0.6$. The upper bounds for delays $\tau_{d_1}(t)$ and $\tau_{d_2}(t)$ are $\bar{\tau}_{d_1} = 0.3$ and $\bar{\tau}_{d_2} = 1.1$, respectively. It can be verified that the derivatives of these time-varying communication delays satisfy the conditions in Assumption 2. For these three cases, simulations are run with control gains k_3 chosen to be 0.05, 0.02, and 0.01, respectively.

The control inputs for MASs without delays, with delays $\tau_{d_1}(t)$ and $\tau_{d_2}(t)$ are shown in Figs. 4(a), 4(d), and 4(g), respectively. The zoomed window in Fig. 4(a) shows that the control inputs evolve smoothly when there are no delays. However, for the cases with delays shown in Figs. 4(d) and 4(g), the control inputs oscillate until the MASs converge to the desired formation pattern. This is because the controllers therein use outdated error information caused by the sinusoid communication delays. The time responses of the formation errors for MASs without delays, with delays $\tau_{d_1}(t)$ and $\tau_{d_2}(t)$ are shown in Figs. 4(b), 4(e), and 4(h), respectively. It can be observed from Figs. 4(e) and 4(h) that the formation errors converge to zero despite the existence of communication delays. However, it is noted that as the delay upper bound becomes larger, the time responses of the formation errors become slower. This is due to different choices of the control gain k_3 for these three cases. Recall the analysis in Remark 3, a smaller control gain is needed to handle delays with a larger upper bound so that the stability of the closed-loop system can be guaranteed. Finally, it can be observed from Figs. 4(f) and 4(i) that $|e_i(t)|$ stays inside the corresponding admissible upper bound δ_i for all $i \in \mathcal{V}$ and all $t \geq 0$, which implies that net-

TABLE II
SIMULATION PARAMETERS

Parameter	Values
$\boldsymbol{c}$	$[120°, 100°, 80°, 10°, 50°]^T$
R_0	10 m
r_i	$r_1 = 20\sin(\frac{11\pi}{36})$ m, $r_2 = 20\sin(\frac{13\pi}{36})$ m, $r_3 = r_4 = 20\sin(\frac{5\pi}{12})$ m, and $r_5 = 20\sin(\frac{\pi}{12})$ m
Δ_i	$\Delta_1 = 110°$, $\Delta_2 = 130°$, $\Delta_3 = \Delta_4 = 150°$, and $\Delta_5 = 30°$
δ_i	$\delta_1 = 10°, \delta_2 = 50°, \delta_3 = 70°, \delta_4 = 10°$, and $\delta_5 = 50°$
$\boldsymbol{\theta}(0)$	$[10°, 132°, 204°, 325°, 330°]^T$
Others	$\epsilon = 4$, $q = 41$



(a) The time responses of the positions for all agents in [5].

(b) The time responses of the positions for all agents in [20].

(c) The time responses of the positions for all agents in our paper.

(d) The evolutions of the formation errors for all agents in [5].

(e) The evolutions of the formation errors for all agents in [20].

(f) The evolutions of the formation errors for all agents in our paper.

Fig. 2. A benchmark example.

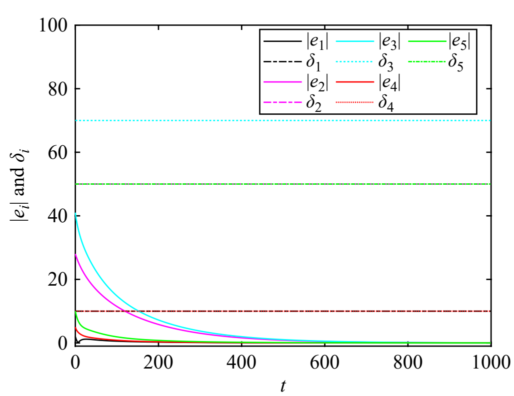
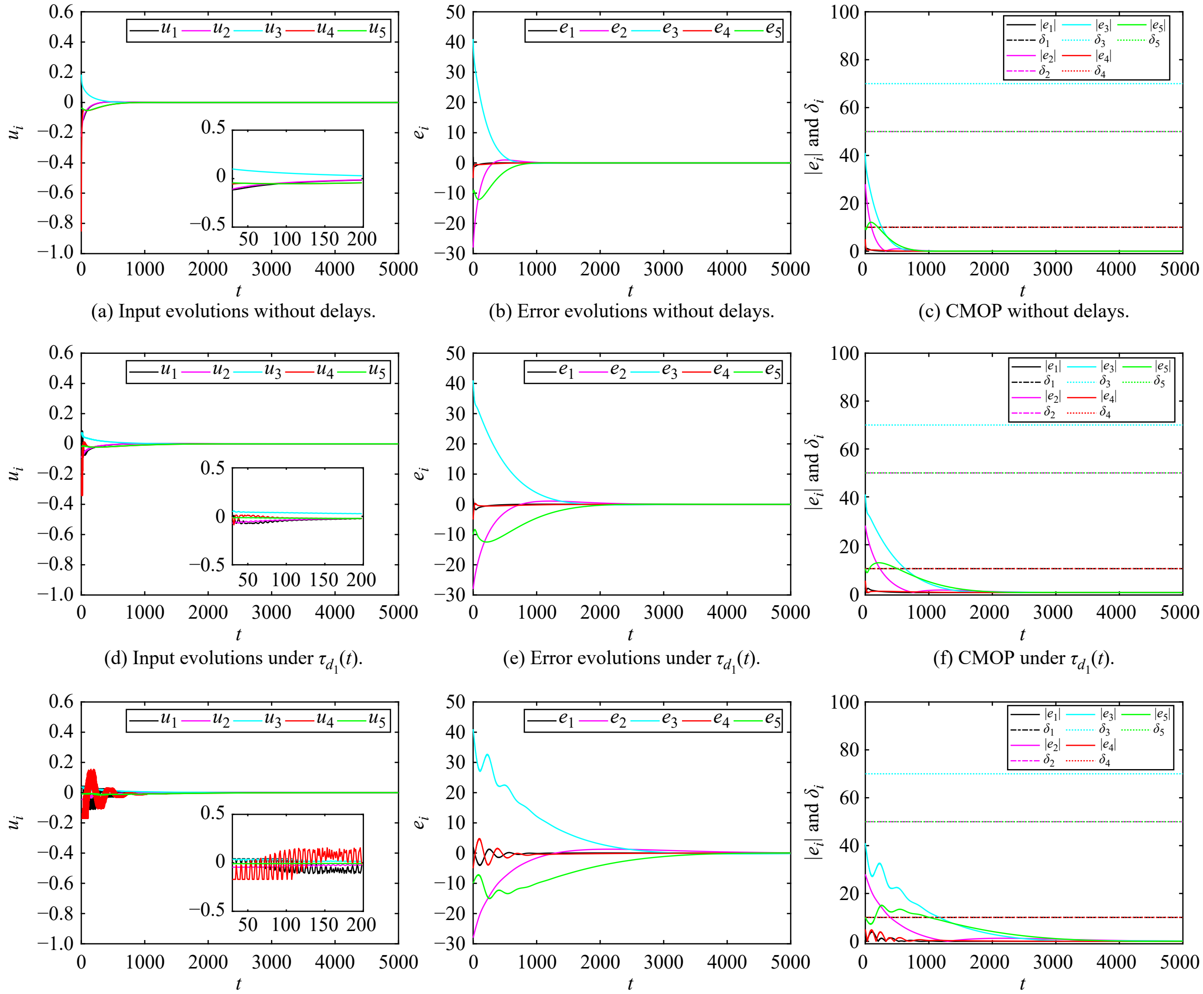


Fig. 3. Connectivity maintenance and order preservation (CMOP).

work connectivity and spatial order of the MAS are always maintained in the presence of communication delays.

V. CONCLUSION

In this article, the circular formation problem for a group of agents subject to limited communication ranges and time-varying communication delays was investigated. To make the distance between neighbouring agents always satisfy network connectivity constraint and order preservation constraint, a set of admissible upper bounds and new potential functions have been designed so that the formation errors stay inside these upper bounds during the whole evolution. Based on these potential functions, a novel gradient-descent-based controller has been proposed in which each agent needs only the information from its leading agent. It has been shown that the proposed controller can render the MAS to converge to the desired formation pattern while maintaining network connectivity and spatial order. In this work, the controller is proposed under the assumption that the communication delays in the MAS are identical. When there exist nonuniform delays, the system matrix will be more complicated. In this case, the properties of the incidence matrix in our design do not hold any longer. It would be interesting to consider a circular formation problem with network connectivity and order preservation in the presence of nonuniform communication delays in the future.



(a) Input evolutions without delays. (b) Error evolutions without delays. (c) CMOP without delays.

(d) Input evolutions under $\tau_{d_1}(t)$. (e) Error evolutions under $\tau_{d_1}(t)$. (f) CMOP under $\tau_{d_1}(t)$.

(g) Input evolutions under $\tau_{d_2}(t)$. (h) Error evolutions under $\tau_{d_2}(t)$. (i) CMOP under $\tau_{d_2}(t)$.

Fig. 4. Simulation results for MASs without delays, with delays $\tau_{d_1}(t)$ and $\tau_{d_2}(t)$.

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