# A Tutorial on Quantized Feedback Control

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***Abstract*—In this tutorial paper, we explore the field of quantized feedback control, which has gained significant attention due to the growing prevalence of networked control systems. These systems require the transmission of feedback information, such as measurements and control signals, over digital networks, presenting novel challenges in estimation and control design. Our examination encompasses various topics, including the minimal information needed for effective feedback control, the design of quantizers, strategies for quantized control design and estimation, achieving consensus control with quantized data, and the pursuit of high-precision tracking using quantized measurements.**

***Index Terms*— Consensus control, high-precision control, networked control, quantized estimation, quantized feedback control, robust control.**

## I. Introduction

CONTROL systems play a crucial role in diverse domains of modern society, with applications ranging from autonomous vehicles to power grid management. Traditionally, control theory assumes a continuous feedback signal, requiring analog controller processing to generate an appropriate control signal. However, in networked control systems, the feedback signal undergoes quantization, where it is discretized into a limited number of levels before being processed by a digital controller.

In networked control systems, the feedback channel is shared with other network functions and has limited data transmission capacity. This introduces several undesirable distortions to the feedback signals in the control loop, including quantization errors, time delays, transmission errors, and packet dropouts. Overlooking the impact of quantization effects when implementing control designs can lead to degraded performance, noise, oscillations, and even system instability.

To address these challenges, various quantization methods have been developed to mitigate the adverse effects of quantization and enhance the performance of control systems. These methods aim to minimize quantization errors and optimize control signal accuracy in the presence of limited communication resources.

Quantized feedback control has emerged as a vital technique in the realm of networked control systems, primarily driven by the wide availability of digital sensors, communication links and embedded devices. The application of quantized feedback control spans diverse fields such as power electronics, communication systems, robotics, and aerospace. In power electronics, it enables precise regulation of voltage and current in power converters. Communication systems benefit from quantized feedback control by facilitating control over signal amplitude and phase. Robotics employs quantized feedback control to govern the position and velocity of robotic arms, while in aerospace, it ensures effective control over the flight of aircraft and spacecraft.

The objective of this tutorial paper is to explore the field of quantized feedback control, addressing the challenges posed by networked control systems. We delve into various aspects and topics, aiming to provide a comprehensive understanding of the subject matter. Specifically, we examine the minimal information required for effective feedback control, the design of quantizers for efficient signal quantization, strategies for quantized control design and estimation, achieving consensus control using quantized data, and the pursuit of high-precision tracking using quantized measurements.

By delving into these topics, we aim to shed light on the current state of research and highlight key advancements and methodologies that have been developed to tackle the challenges posed by quantization in networked control systems. Ultimately, this survey paper serves as a valuable resource for researchers, engineers, and practitioners interested in the field of quantized feedback control, providing insights, methodologies, and avenues for further exploration and development in this rapidly evolving domain.

The utilization of quantized information in control and estimation has roots in the early stages of control research. The investigation of the quantized linear quadratic Gaussian (LQG) control problem, wherein the feedback information must be quantized by a fixed-rate quantizer, commenced in the early 1960s. Noteworthy contributions to this problem can be found in the works of [1]–[5]. The early works by Kalman [6] and Widrow [7] have significantly contributed to the field of quantized feedback control. They investigated the effects of quantization errors in sampled-data feedback systems, shedding light on the challenges and limitations imposed by the quantization process. Their research provided valuable insights into the trade-offs between quantization levels, signal resolution, and control system performance, enabling the development of techniques to mitigate the detrimental effects of quantization and enhance the performance and robustness of control systems operating under quantization constraints.

The advent of networked control systems, particularly in the

Manuscript received August 9, 2023; revised September 10, 2023; accepted September 13, 2023. This work was partially supported by National Natural Science Foundation of China (62350710214, U23A20325) and Shenzhen Key Laboratory of Control Theory and Intelligent Systems (ZDSYS2022033 0161800001). Recommended by Associate Editor Xiaohua Ge.

Citation: M. Fu, "A tutorial on quantized feedback control," *IEEE/CAA J. Autom. Sinica*, vol. 11, no. 1, pp. 5–17, Jan. 2024.

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Digital Object Identifier 10.1109/JAS.2023.123972

realm of industrial control and automation, has reignited interest in quantized feedback control. Numerous studies and advancements have been made in this area, as evidenced by the works of [8]−[27]. More specifically, [8] started the study of quantized state feedback. The works of [9], [10] initiated the research on finite bandwidth constraint for feedback. These issues were followed up in [11]−[13], [17]. Control under noisy feedback channel was studied in [18]. The problem of minimal feedback information rate for stabilization was solved by [14], [15], which extended to the work of [23]. The work of [16] introduced the use of logarithmic quantizers in static quantized feedback control. This led to the sector bound approach to quantized feedback control [19], [20], [22]. Markov jump systems were studied in [21]. The quantized LQG problem also received significant attention, as demonstrated by the research conducted by [24]−[27].

Similar to the significance of state estimation in classical control theory, estimation based on quantized information plays a pivotal role in quantized feedback control.

Just like how state estimation holds importance in classical control theory, estimation using quantized data assumes a central role in the realm of quantized feedback control. This aspect has been duly recognized in the aforementioned references, as well as in the works of quantized state estimation (e.g., [9], [28]). Additionally, the applications of quantized estimation extend beyond feedback control. They encompass sensor network-based estimation and tracking, as explored in the works of [29], [30], as well as consensus networks examined by [31], [32]. Notably, there are instances where quantized estimation contributes to the broader context of network-based estimation, addressing challenges such as transmission delays, packet dropouts, and related issues, as discussed in the studies by [29], [30], [33].

The quantized average consensus problem involves a network of interconnected agents aiming to collaboratively compute the average of their initial values or states, despite limited communication and quantization constraints [34], [35]. This challenge necessitates the design of quantization strategies that allow agents to exchange information in a distributed manner while mitigating the effects of quantization errors, ultimately achieving consensus on the average value.

The problem of quantized regulation control (i.e., regulation control using quantized measurement) [36]−[38] aims to control a system's behavior using quantized measurements of the system output. This challenge entails developing control strategies that can effectively utilize coarse, discretized measurements to steer the system towards a desired trajectory or reference, despite the inherent limitations introduced by quantization.

In addition to the works on quantized control above, an explosive number of recent developments have been reported in the literature on quantized feedback control. Here we only provide some samples. For networked control systems, [39] gave distributed observer designs; [40] investigated distributed quantized feedback designs for consensus tracking; [41] carried out works on formation tracking; [42] did research on distributed containment control. Adaptive schemes for quantized feedback control were offered in [43]–[45]. Unreliable feedback channels were studied in [46]. Markov jump systems were treated in [47], [48]. Switched control behaviors for quantized feedback systems were analyzed in [49]. Observer-based feedback with quantized inputs and outputs was considered in [50]. The work of [51] studied the use of of spherical polar coordinate quantizers in feedback control. Event-triggered quantized feedback control was discussed in [52]−[54]. Impact of network attacks to average consensus was dealt with in [55]. Many applications of quantized feedback control were reported, including mobile vehicles [57], unmanned marine vehicles [56], induction machine drives [58], and quantized motion control for hysteresis systems [59].

In the rest of this paper, we introduce an array of specialized tools and findings that address diverse challenges in quantized feedback control as discussed above. The topics to be discussed include the utilization of minimal feedback information for stabilization (Section II), the design of feedback controls employing both static and dynamic quantization (Sections III and IV), quantized state estimation (Section V), quantized linear quadratic Gaussian (LQG) control (Section VI), quantized average consensus control (Section VII), and quantized regulation for periodic signals (Section VIII).

In addition to its tutorial purpose, the paper's content also provides a glimpse into the author's individual exploration and perspective within this realm of research, which is why the cited references should not be considered exhaustive in any way.

## II. Minimal Feedback Information for Stabilization

Nair and Evans [14], [15] have made significant contributions to the field of stabilization with data-rate-limited feedback, focusing on determining the tightest achievable bounds for stabilizing systems under such constraints. Their work addresses the challenge of limited communication capacity in feedback systems and aims to establish fundamental limits on the achievable performance. By applying tools from control theory and information theory, Nair and Evans have derived rigorous theoretical bounds that characterize the trade-off between the available data rate for feedback and the achievable stability and performance of the system. Their research provides valuable insights into the fundamental limitations of data-rate-limited feedback and offers guidelines for designing controllers that optimize the trade-off between performance and communication constraints.

Consider the following linear time-invariant system:

$$\begin{cases} x(k+1) = Ax(k)+Bu(k) \\ y(k) = Cx(k) \end{cases} \tag{1}$$

where $x(k)\in\mathbb{R}^n$ is the state, $u(k)\in\mathbb{R}^m$ is the control input, $y(k)\in\mathbb{R}^p$ is the measured output, $A\in\mathbb{R}^{n\times n}, B\in\mathbb{R}^{n\times m}$ and $C\in\mathbb{R}^{p\times n}$ are given. We assume that $(A,B,C)$ is a minimal realization. The transfer function from $u(k)$ to $y(k)$ will be denoted by $G(z)$.

The quantized feedback control problem is depicted in Fig. 1. The feedback channel involves two components: a *quantizer* and a *controller*. Between the quantizer and controller is a

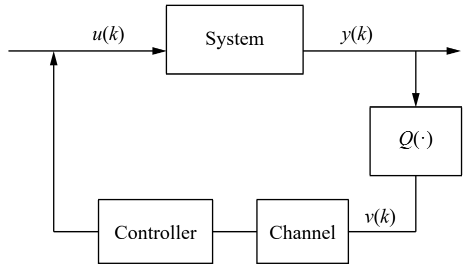


Fig. 1. Quantized feedback control.

*channel* with data rate of $R$, i.e., the quantizer's output takes a value from the set $\mathbb{Z}_\mu = \{1, 2, \ldots, \mu\}$ with $\mu = 2^R$. In the most general setting, the quantizer takes the form of an *encoder*

$$v(k) = \gamma_k(\tilde{y}(k), \tilde{v}(k-1)) \tag{2}$$

and the controller takes the form below:

$$u(k) = \delta_k(\tilde{v}(k-1)) \tag{3}$$

where $\tilde{y}(k) = \{y(i)\}_{i=0}^k$, $\tilde{v}(k)$ is similarly defined, and $\gamma_k(\cdot): \mathbb{R}^{p\times(k+1)} \times \mathbb{Z}_\mu^k \to \mathbb{Z}_\mu$ and $\delta_k(\cdot): \mathbb{Z}_\mu^k \to \mathbb{R}^m$ are (generally time-varying) encoder-controller functions to be designed.

Nair and Evans [14], [15] studies the so-called *$\varrho$-exponential stabilization problem* where the initial state $x(0)$ is assumed to a random variable satisfying $\mathcal{E}[\|x(0)\|^{r+\epsilon}] < \infty$ for some integer $r > 0$ and any $\epsilon > 0$, and the objective is to design the encoder-controller pair for a given $\varrho > 0$ such that the closed-loop system satisfies

$$\varrho^{-kr}\mathcal{E}[\|x_k\|^r] \to 0, \text{ as } k \to \infty. \tag{4}$$

*Theorem 1 [14]:* The system (1) admits a encoder-controller (2) and (3) that $\varrho$-exponentially stabilizes the system in the $r$th absolute moment sense (4) if and only if

$$R > \sum_{|\eta_j|\geq\varrho} \log_2 \frac{|\eta_j|}{\varrho} \tag{5}$$

where $\eta_j$ are the eigenvalues of $A$, counting multiplicities.

*Proof:* The proof of the result is quite technically involved [15]. The basic idea behind the proof is to recursively quantize the initial state. We first consider the case the state dimension $n = 1$. In this case, $A$ is a scalar, and so are $y(k) = x(k)$ and $u(k)$. Without loss of generality, we assume that $\varrho = 1$ and that the only eigenvalue of the system $\eta_1$ is such that $|\eta_1| \geq 1$. At each time instant, the scalar state is allocated a data rate of $R$, which is used to quantize and transmit its state. At the other end of the channel, the quantized state is used in lieu of the true state to design a controller with the aim to drive the state to zero in one time step. Due to the quantization error, the state will be driven only to a small neighbourhood. At the next time instant, this state is magnified by the unstable $A$. This process continues for every time instant. It can be argued that when $R > \log_2|\eta_1|$, the shrinkage of the state by the controller will dominate the magnification of the state by $\eta_1$, thus exponential stabilization is achieved. But when $R < \log_2|\eta_1|$, the shrinkage of the state by the controller becomes insufficient to suppress the growth of the state by $\eta_1$, causing instability.

For the general case of $n > 1$, the idea above can be generalized by decomposing the state into scalar sub-systems, each corresponding to an eigenvalue $\eta_i$ of $A$. By means of a time-sharing protocol, each scalar state component with eigenvalue $\eta_i$ satisfying $|\eta_i| \geq 1$ is allocated an effective data rate slightly greater than $\log_2|\eta_i|$, which is used to quantize and transmit its state component. At the other end of the channel, the controller uses the constructed state estimate to drive the state to zero. Again, the presence of quantization error means that the state can never be driven to zero in general. But as long as the data rate $R$ satisfies (5), the controller can be constructed to dominate the quantization error, leading to exponential stability. Conversely, when (5) fails to hold, the state growth will dominate the controller, causing instability.

*Remark 1:* The work of Nair and Evans [14], [15] holds significant theoretical values because it offers the fundamental limit on the data rate required for feedback stabilization. However, designing quantized feedback controllers according to this data rate is not practical for several reasons. Firstly, most practical systems do not have many very unstable eigenvalues, meaning that the required minimum data rate $R$ by (5) is typically very small. For example, each eigenvalue of $|\eta_i| \approx 1.99$ (which is very unstable for a discrete-time system!) with $\rho = 1$ requires only 1 bit of quantization. This limit would hardly pose any implementation difficulty in practice, making the consideration of minimal data rate not meaningful in practice. Secondly, the associated encoder-controller typically causes huge overshoot, making the resulting closed-loop behavior not practical. Finally, control performance can not be easily accommodated. Typically, much higher data rate is required to achieve satisfactory control performances in practice. Thus, more practical quantized feedback control designs are needed. ■

## III. Static Quantized Feedback Control

Elia and Mitter [16] have made notable contributions to the field of stabilizing linear systems with limited information. Their work focuses on the use of a static quantizer (rather than a dynamic quantizer as in the case of [14], [15]) in quantized feedback control and recognized the significance of logarithmic quantizers (as depicted in Fig. 2). Their work has advanced our understanding of how to achieve stability and control in practical scenarios where information constraints are prevalent, and it has paved the way for further research and developments in this important area.

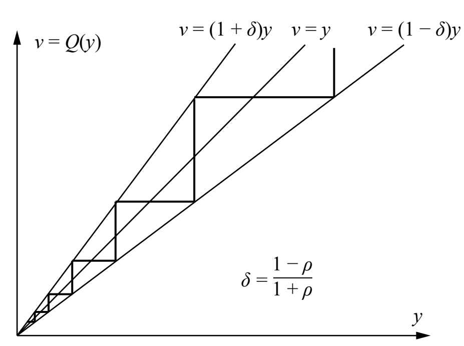


Fig. 2. Logarithmic quantizer.

Returning to the system (1), the *problem of quantized feedback control with static quantization* is to design a static quantizer

$$v(k) = Q(y(k)) \tag{6}$$

and a feedback controller of the form

$$\begin{cases} \hat{x}(k+1) = A_c\hat{x}(k) + B_c v(k), & \hat{x}(0) = 0 \\ u(k) = C_c\hat{x}(k) + D_c v(k) \end{cases} \tag{7}$$

with $\hat{x}(k) \in \mathbb{R}^n$, such that the closed-loop system is stable and that the so-called quantization density [16] is coarsest (i.e., smallest). The quantization density of $Q(\cdot)$ is defined as follows:

$$\eta_Q = \limsup_{\epsilon \to 0} \frac{\#g[\epsilon]}{-\ln \epsilon} \tag{8}$$

where $\#g[\epsilon]$ denotes the number of quantization levels in the interval $[\epsilon,\ 1/\epsilon]$.

The optimal form of a static quantizer turns out to be a *logarithmic quantizer*, which is described by

$$\mathcal{V} = \{\mu_i = \rho^i \mu_0 :\ i = 0, \pm 1, \pm 2, \ldots\} \cup \{0\},\ \mu_0 > 0 \tag{9}$$

where $\rho \in (0,\ 1)$ and

$$Q(y) = \begin{cases} \rho^i \mu_0, & \text{if } \dfrac{1}{1+\delta}\rho^i\mu_0 < y \le \dfrac{1}{1-\delta}\rho^i\mu_0 \\ 0, & \text{if } y = 0 \\ -Q(-y), & \text{if } y < 0 \end{cases} \tag{10}$$

where

$$\delta = \frac{1-\rho}{1+\rho}. \tag{11}$$

A pictorial representation is given in Fig. 2 (shown for $y > 0$ only). The quantization density of a logarithmic quantizer is $\eta_Q = 2/\ln(1/\rho)$, meaning that the smaller the $\rho$, the smaller the $\eta_Q$. Thus, $\rho$ can be interpreted as the *quantization density*.

The description above is for an infinite-level logarithmic quantizer. In practice, it is truncated when the input is too large (by a saturator) or too small (by a dead zone) in magnitude.

Logarithmic quantization outperforms linear quantization in several cases. Firstly, in quantized feedback control, where the goal is to drive the output or state to the origin while quantizing the control or measurement signal [16], [19]. This applies to stabilization, tracking, and disturbance attenuation. Logarithmic quantization reduces the multiplicative quantization error as the input signal decreases, although it increases for larger input signals. Secondly, logarithmic quantization is advantageous in quantized state estimation, where the system's state needs to be estimated using quantized information [28]. While direct quantization of the measured signal may not be suitable due to persistently large measurements, quantizing the estimation error instead benefits from logarithmic quantization. It provides a small quantization error for small estimation errors and allows for a larger quantization error when the estimation error is large. Lastly, logarithmic quantization is beneficial when the signal to be quantized already contains multiplicative noise. Many sensors, such as range sensors used for position measurement, specify accuracies using relative errors. Logarithmic quantization, which also introduces a multiplicative error, simply magnifies the existing noise without altering its structure.

The primary technical finding regarding quantized feedback stabilization using static quantization is presented in [19]. The result builds a fundamental bridge between quantized feedback control and robust control, paving way for a lot of further research on networked control. According to this study, logarithmic quantization provides the optimal quantizer structure for achieving quadratic stabilization of (1). Furthermore, under quadratic stabilization, quantized feedback control is equivalent to robust control with uncertainty bounded by a sector. The coarsest quantization density, corresponding to the smallest $\rho$, can be determined through standard $H_\infty$ optimization, as explained in the following.

*Theorem 2:* The following results hold for the system (1).

1) If the system (1) is quadratically stabilizable[1] via static quantized feedback control (6) and (7), then the quantizer with the coarsest quantization density can take a logarithmic form.

2) For a given quantization density $\rho > 0$, the system (1) is quadratically stabilizable via a static quantized controller (6) and (7) if and only if the following auxiliary system:

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) \\ v(k) = (1+\Delta)Cx(k),\ \ |\Delta| \le \delta \end{cases} \tag{12}$$

is quadratically stabilizable via

$$\begin{cases} x_c(k+1) = A_c x_c(k) + B_c v(k) \\ u(k) = C_c x_c(k) + D_c v(k) \end{cases} \tag{13}$$

where $\delta$, which is the sector bound produced by the quantization error, and $\rho$ are related by (10).

3) The largest sector bound $\delta_{\sup}$ for quadratic stabilization (corresponding to the coarsest quantization density $\rho_{\inf}$) is given by

$$\delta_{\sup} = (\inf_{H(z)} \|\bar{G}_c(z)\|_\infty)^{-1} \tag{14}$$

where $\|\cdot\|_\infty$ denotes the $H_\infty$ norm, and

$$\begin{cases} \bar{G}_c(z) = (1 - H(z)G(z))^{-1}H(z)G(z) \\ H(z) = D_c + C_c(zI - A_c)^{-1}B_c. \end{cases}$$

*Proof:* Define the quantization error function by

$$e = v - y = Q(y) - y = \Delta(y)y.$$

Then, any quantizer with quantization density $\rho$ has the property that $|\Delta(y)| \le \delta$ with $\delta$ relating to $\rho$ in (11). The closed-loop system of (1) with (6) and (7) can be expressed as

$$\bar{x}(k+1) = \mathcal{A}(\Delta(y(k))\bar{x}(k) \tag{15}$$

where $\bar{x} = \text{col}\{x, x_c\}$, $\mathcal{A}(\Delta) = \bar{A} + \bar{B}\bar{K}(\bar{C} + \hat{I}\Delta\hat{C})$ and

$$\bar{A} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix},\ \bar{B} = \begin{bmatrix} 0 & B \\ I & 0 \end{bmatrix},\ \bar{C} = \begin{bmatrix} 0 & I \\ C & 0 \end{bmatrix}$$

$$\hat{I} = \begin{bmatrix} 0 \\ I \end{bmatrix},\ \hat{C} = [C\ \ 0],\ \bar{K} = \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix}.$$

[1] A system is said to be quadratically stable if its stability can be asserted by using a quadratic Lyapunov function.

The first step is to show the equivalence of any quantizer with density $\rho$ and a logarithmic quantizer with the same density in the context of quadratic stabilization. More specifically, the quadratic stability of the system (15) requires the existence of some $\bar{P} = \bar{P}' > 0$ such that

$$\bar{x}'[\mathcal{A}'(\Delta(\bar{C}\bar{x}))\bar{P}\mathcal{A}(\Delta(\bar{C}\bar{x})) - \bar{P}]\bar{x} < 0$$

for all $\bar{x} \neq 0$. This can be shown to be equivalent to

$$\mathcal{A}'(\Delta)\bar{P}\mathcal{A}(\Delta) - \bar{P} < 0, \quad \forall|\Delta| \leq \delta. \tag{16}$$

The latter is the same as requiring the system (12) and (13) to be quadratically stable.

The next step is to convert the quadratic stabilization problem of the system (12) to an $H_\infty$ control problem by borrowing the robust control theory. More specifically, we define the "nominal" open-loop transfer function $G(z) = (zI - A)^{-1}B$, the controller transfer function $H(z) = D_c + C_c(zI - A_c)^{-1}B_c$, and the "nominal" closed-loop transfer function $\bar{G}_c(z) = (1 - H(z)G(z))^{-1}H(z)G(z)$. Then, the closed-loop system (12) and (13) is equivalent to the closed-loop system with the open-loop block equal to $\bar{G}_c(z)$ and feedback block equal to $\Delta$. Using the relationship between quadratic stabilization and $H_\infty$ control (see [19]), it can be shown that the closed-loop system (12) and (13) is quadratically stable if and only if $\delta\|\bar{G}_c(z)\|_\infty < 1$.

Finally, the largest $\delta$ can be obtained by minimizing the term $\|\bar{G}_c(z)\|_\infty$. ■

*Remark 2:* Although the discussions above are for a single-input-single-output system, similar results hold for multi-input-multi-output systems, as shown in [19]. Also shown in [19] is that if the observation matrix $C$ can be chosen, the largest sector bound (or coarsest quantization density) for quadratic stabilization is obtained when the nominal open-loop function $G(z)$ is of relative degree one without unstable zeros. In particular, if state feedback is available (i.e., $y(k) = x(k)$), one can always construct a matrix $C$ first to make $G(z)$ satisfy the above condition.

## IV. Dynamic Quantized Feedback Control

A *dynamic quantizer* differs from a static quantizer by utilizing memory to consider past input-output values in order to determine the quantization of current input. This added complexity grants dynamic quantizers greater power and flexibility. The key concept behind dynamic quantization is the use of *dynamic scaling* in conjunction with a static quantizer. By prescaling the input signal to a range more suitable for quantization and dynamically adjusting the scaling parameter online, researchers have explored various strategies. Notable works in this area include [8], [13], [17], [18].

In [8], it is emphasized that, with the use of a quantizer having different levels of sensitivity, a feedback strategy can be devised to drive the closed-loop state arbitrarily close to zero for an extended duration, assuming the system is not excessively unstable. Building upon this idea of a quantizer with sensitivity, [13] extends the concept by demonstrating the existence of a dynamic adjustment in quantizer sensitivity and a corresponding quantized state feedback that achieves asymptotic stabilization of the system. For output feedback scenarios, local or semi-global stabilization results are obtained.

The work of [14], [15] introduced earlier shows that by an appropriate use of dynamic scaling, the minimum *feedback information rate*) (or capacity) can be achieved for dynamic quantized feedback control. But as we cautioned before, this line of work may be impractical due to lack of performance guarantee and poor transient response. It is also pointed out in [60] that capacity results are in general not valid for practical communications channels which are not noise free.

In order to mitigate these concerns, it is necessary to design dynamic scaling with a focus on practical control objectives rather than solely minimizing channel capacity. An investigation into a straightforward dynamic scaling approach is presented in [20]. This method combines a $2N$-level logarithmic quantizer $Q(\cdot)$ with the following scaling technique:

$$v_k = g_k^{-1}Q(g_k y_k) \tag{17}$$

where the scaling gain $g_k$ is adjusted by

$$g_{k+1} = \begin{cases} g_k\gamma_1, & \text{if } |Q(g_k y_k)| = \mu_0 \\ g_k/\gamma_2, & \text{if } |Q(g_k y_k)| = \rho^{N-1}\mu_0 \\ g_k, & \text{otherwise} \end{cases} \tag{18}$$

with some initial $g_0 > 0$, where $\gamma_1, \gamma_2 \in (0, 1)$ are design parameters. The basic idea is to reduce the magnitude of the subsequent input if the current input is excessively large, or conversely, amplify it if the current input is too small in magnitude.

In [20], it is shown that when employing this dynamic scaling method, we can achieve quadratic stabilization using only a finite number of logarithmic quantization levels, provided that the system can be quadratically stabilized via static quantized feedback control.

As explained in the previous section, quadratic stabilization of the system (1) via static quantized feedback control with quantization density $\rho$ implies (16). Since this is a strict inequality, we can choose some $\eta \in (0, 1)$ such that

$$\mathcal{A}'(\Delta)\bar{P}\mathcal{A}(\Delta) \leq (1 - \eta)\bar{P}, \quad \forall|\Delta| \leq \delta.$$

Defining

$$\hat{A} = \begin{bmatrix} A & BC_c \\ 0 & A_c \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} BD_c \\ B_c \end{bmatrix}$$

the scaling parameters for (18) can be chosen according to

$$\sqrt{1-\eta} < \gamma_2 < 1, \quad \gamma_1^2\bar{A}'\bar{P}\bar{A} < (1-\eta)\bar{P}$$

which are always possible by taking $\gamma_1$ sufficiently small and $\gamma_2$ sufficiently close to 1. Then, define $\eta_1$ by

$$1 - \eta_1 = \gamma_2^{-2}(1 + \tau)(1 - \eta)$$

with sufficiently small $\tau$ to ensure $\eta_1 \in (0, 1)$. We have the following main result from [20].

*Theorem 3:* Suppose the system (1) can be quadratically stabilized via a static quantized feedback controller (6) and (7). Let the scaled $2N$-level logarithmic quantizer (9), (10) and (17) be applied with $N > N_0$, where

$$N_0 = 1 + \frac{\ln(\eta_1^{-1}\gamma_2^{-2}(1+\tau^{-1})\hat{B}'\bar{P}\hat{B}\hat{C}\bar{P}^{-1}\hat{C}')}{2\ln(\rho^{-1})}.$$

Then, the state $\bar{x}(k)$ converges to zero asymptotically.

*Proof:* The first step is to show that the scaled state $z_k = g_k$ is bounded by showing that the quadratic Lyapunov function for the scaled state, $V(z_k) = z_k' \bar{P} z_k$, is asymptotically attracted to a bounded set. This is ensured by the careful selection of the parameters $\eta, \gamma_1, \gamma_2, \tau$ and $\eta_1$. The choice of $N_0$ is then to guarantee that the $2N$-level logarithmic quantizer will no longer be saturated when the scaled state gets near this bounded set. The next step is to show that the subsequent scaling gain will keep increasing, i.e., $g_k \to \infty$ as $k \to \infty$. Therefore, the state $\bar{x}_k \to 0$ as $k \to \infty$. ■

Empirical findings indicate that the number of quantization bits per time sample required for most practical control systems is relatively moderate [20]. To illustrate this observation, we examine the system (1) below with two unstable open-loop poles at $1.2 \pm j0.5$:

$$A = \begin{bmatrix} 2.7 & -2.41 & 0.507 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$C = [1 \quad -0.5 \quad 0.04].$$

Fig. 3 shows the state response of the closed-loop system with only a 4-bit logarithmic quantizer.

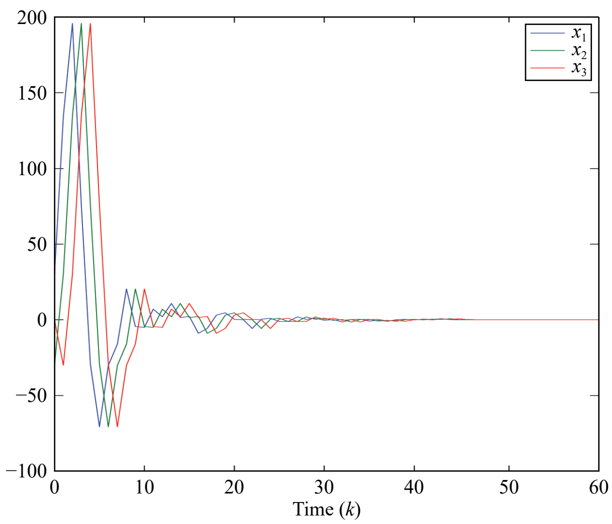


Fig. 3. Closed-loop response with a 4-bit quantizer.

## V. Quantized State Estimation

A dual and equally important quantized feedback problem is *quantized state estimation*, where the state estimation needs to be made using quantized information due to digital transmissions.

Consider the following linear system:

$$\begin{cases} x(k+1) = Ax(k) + Bw(k), \quad x(0) = x_0 \\ y(k) = Cx(k) + v(k) \end{cases} \tag{19}$$

where $w(k) \in \mathbb{R}^m$ is the process noise, $v(k) \in \mathbb{R}$ is the measurement noise. It is assumed that $x_0 \in \mathbb{R}^n$ is a random variable with mean $\bar{x}_0$ and covariance $\Sigma_0$, and $w$ and $v$ are uncorrelated zero-mean white noises with covariances $\Sigma_w$ and $\Sigma_v$, respectively, and they are uncorrelated with $x_0$.

The quantized estimator is shown in Fig. 4. Instead of quantizing the measured signal directly, we choose to quantize the prediction error of the estimator. The estimator takes the following form [28]:

$$\begin{cases} \hat{x}(k+1) = A\hat{x}(k) + LQ(y(k) - \hat{y}(k)) \\ \hat{y}(k) = C\hat{x}(k) \end{cases} \tag{20}$$

with $\hat{x}(0) = \bar{x}_0$, where $\hat{x}(k) \in \mathbb{R}^n$ is the estimate of $x(k)$, $\hat{y}(k) \in \mathbb{R}$ is the estimate of $y(k)$ based on $\hat{x}(k)$, $Q(\cdot)$ is the quantizer, and $L$ is the estimator gain.

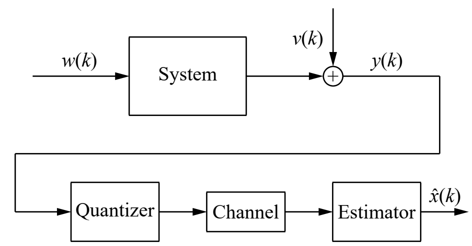


Fig. 4. Quantized state estimation.

Considering the dynamic nature of the estimator, it is evident that the quantized state estimator falls into the category of a dynamic quantizer. It is important to note that the state estimation is solely constructed using the quantized prediction error in the given context. Consequently, under the assumption of an ideal channel, both the transmitter and receiver sides can generate an identical estimate by utilizing the quantized prediction error. Specifically, the construction of $\hat{x}(k)$ on the transmission side does not necessitate the transmission of the estimated state from the receiver side.

Defining the prediction error

$$\varepsilon(k) = y(k) - \hat{y}(k) \tag{21}$$

we consider a static logarithmic quantizer $Q(\cdot)$ in (10) for quantizing the prediction error. Defining the estimation error

$$e(k) = x(k) - \hat{x}(k)$$

the state estimation error dynamics is given by

$$\begin{cases} e(k+1) = Ae(k) + Bw(k) - LQ(\varepsilon(k)) \\ \varepsilon(k) = Ce(k) + v(k). \end{cases} \tag{22}$$

Denoting the state estimation error covariance by

$$E(k) = \mathcal{E}\{e(k)e'(k)\}$$

the *quantized state estimation problem* is to choose the quantization density $\rho$ and the filter gain $L$ such that the trace of $E = \lim_{k\to\infty} E(k)$ is minimized.

Following our earlier analysis of the logarithmic quantizer, it is clear that

$$Q(\varepsilon) - \varepsilon = \Delta(\varepsilon)\varepsilon$$

with $|\Delta(\varepsilon)| \le \delta$. Using the above, we can define an auxiliary uncertain system

$$\begin{aligned} z(k+1) = {} & Az(k) + Bw(k) - L(Cz(k) + v(k)) \\ & + L\Delta_k(Cz(k) + v(k)), \quad |\Delta_k| \le \delta. \end{aligned} \tag{23}$$

The key difference between the term $\Delta_k$ and $\Delta(\varepsilon_k)$ is that the former is arbitrary subject to $|\Delta_k| \le \delta$, whereas the latter is specifically generated by the quantizer. The main result on the quantized state estimator (20) is given below [28].

*Theorem 4:* The estimation error dynamics (22) is quadratically stable if and only if the auxiliary dynamics (23) is quadratically stable. Moreover, if the auxiliary system (12) is quadratically stable, then the covariance matrix $E(k)$ is bounded and asymptotically invariant. Finally, the minimum quantization density $\rho_{\text{inf}}$ for the estimation error dynamics to be quadratically stable is given by

$$\rho_{\text{inf}} = \frac{1-\delta_{\text{sup}}}{1+\delta_{\text{sup}}}$$

with

$$\delta_{\text{sup}} = \frac{1}{\min_L \|C(zI-A+LC)^{-1}L\|_\infty}.$$

*Proof:* The equivalence between the quadratic stability of (22) and that of (23) can be established similarly to Part 2 of Theorem 2. The boundedness and asymptotic invariance of $E(k)$ follow from the quadratic stability of (23). The minimum quantization density result is the dual version of Part 3 of Theorem 2. The details can be found in [28]. ■

To demonstrate quantized state estimation, we consider the system model (19) [28] with

$$A = \begin{bmatrix} 2.4744 & -2.811 & 1.7038 & -.5444 & .0723 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 01 & 0 & 0 & \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$B' = [1 \;\; 0 \;\; 0 \;\; 0 \;\; 0]$$

$$C = [0.245 \;\; 0.236 \;\; 0.384 \;\; 0.146 \;\; 0.035]. \tag{24}$$

$\Sigma_w = 1$ and $\Sigma_v = 1/16$. For each $\delta \in 0,\ 0.3]$, we compare two estimator gains $L$, one taken as the Kalman gain designed by ignoring the quantization error and one being the robust gain computed by treating the quantization error as a multiplicative noise.

In Fig. 5, the simulated values of $\mathrm{Tr}(E)$ are presented. Two key observations can be made: 1) As the quantization becomes coarser (smaller $\rho$ or larger $\delta$), the estimation error increases. 2) The robust gain demonstrates notably better performance compared to the Kalman gain, particularly in scenarios where the quantization becomes coarse (also shown in the figure are the estimates of $\mathrm{Tr}(E)$ which can be ignored for this paper).

When using a finite-level quantizer, truncation introduces additional estimation error. In this scenario, the parameter $\mu_0$ of the quantizer also requires careful design, in addition to $\rho$ [28]. For this example, with approximately $4 \sim 5$ bits of quantization, the quantized estimator exhibits only a slightly larger estimation error variance compared to the case without quantization. Fig. 6 illustrates the relationship between the estimation error and the number of quantization bits $N_b$.

## VI. Quantized LQG Control

Having addressed the quantized estimation problem, our focus now shifts to the *quantized LQG control* problem, which

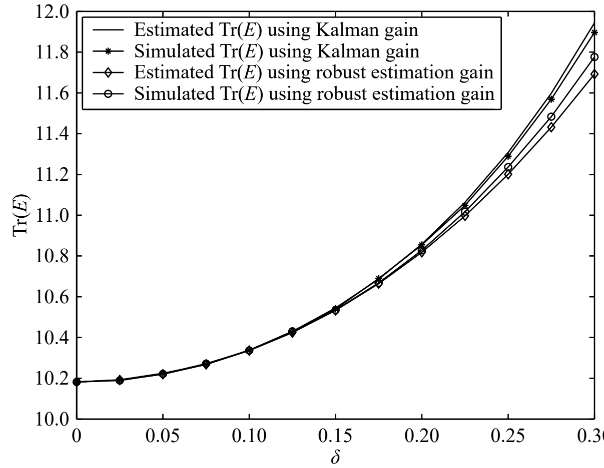


Fig. 5. State estimation error vs. sector bound size.

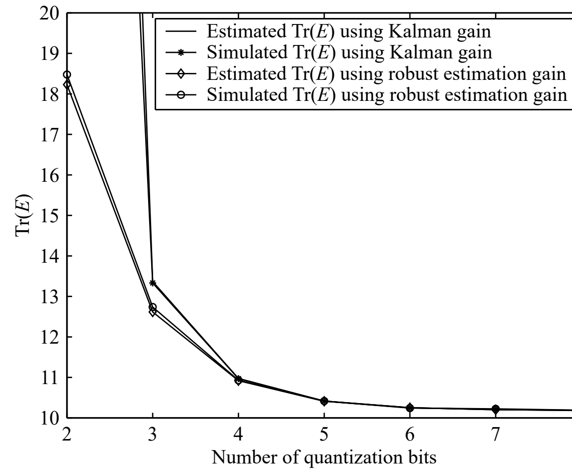


Fig. 6. State estimation error vs. number of quantization bits.

extends from the conventional LQG problem. Here, the feedback channel is a digital link with a predetermined fixed bit rate and the feedback signal must undergo quantization, as depicted in Fig. 7 [26].

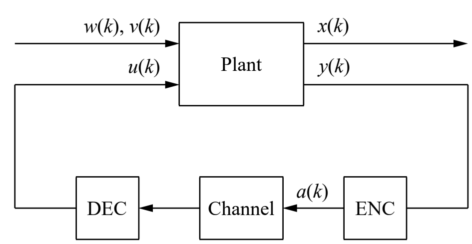


Fig. 7. Quantized LQG control system.

The system we consider is given by

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) + w(k) \\ y(k) = Cx(k) + v(k) \end{cases} \tag{25}$$

where $x(k) \in \mathbb{R}^n$ is the state, $u(k) \in \mathbb{R}^m$ is the control input, $y(k) \in \mathbb{R}^p$ is the measured output, $w(k) \in \mathbb{R}^n$ and $v(k) \in \mathbb{R}^p$ are independent Gaussian random distributions with zero mean and covariances $W_k > 0$ and $V_k > 0$, respectively, and the initial state $x(0)$ is also assumed to be an independent zero-mean Gaussian distribution with covariance $\Sigma_0$.

The encoder (ENC) is a dynamic quantizer of $y_t$, i.e.,

$$a(k) = \alpha_k(\tilde{y}(k)|\tilde{a}(k-1)) \tag{26}$$

where $\alpha_k(\cdot)$ takes values in a finite alphabet set $\mathcal{A}$ with size of $2^R$, and the notation $\tilde{y}(k) = \{y(i)\}_{i=0}^{k}$ is as before. Similarly, the decoder (DEC) is a dynamic controller operating on the received quantized information

$$u(k) = \beta_k(a(k)|\tilde{a}(k-1)). \tag{27}$$

The following linear quadratic cost is to be treated:

$$J = \mathcal{E}\left[x(T)'Q_T x(T) + \sum_{k=0}^{T-1} x(k)'Q_k x(k) + u(k)'S_k u(k)\right] \tag{28}$$

where $\mathcal{E}[\cdot]$ is the expectation, $Q_k = Q_k' \geq 0$ and $S_k = S_k' > 0$ for all $k$.

The problem of *quantized LQG control* is to jointly design the quantizer (encoder) and controller (decoder) to minimize the cost $J$, under the $R$ bit rate constraint.

The investigation of the quantized LQG problem dates back to the 1960s, driven by the demand for digital control. Over the years, numerous efforts have been made to extend the well-established separation principle of the conventional LQG problem [1]−[5], [12], [24], [25]. In 1967, Larson [4] initially proposed that the separation principle could be generalized to quantized LQG control, but subsequent research proved this claim to be incorrect. Fischer [5] revisited the quantized LQG control problem in 1982 and introduced time-varying quantization. He asserted that the separation of control, estimation, and quantization was achievable. Specifically, Fischer claimed that the optimal control $u(k)$ could be expressed as a quantized version of the optimal control $u_o(k)$ for the conventional LQG control problem (without quantization). This quantized control was obtained by minimizing a weighted quantization distortion $\mathcal{E}[(u(k)-u_o(k))'\Omega_k(u(k)-u_o(k))]$, with $\Omega_k$ being a weighting matrix dependent on the cost function. However, this assertion was also again false. Subsequent attempts were made under various restrictive assumptions [12], [24], [25].

The recent research in [26] gives a clear picture of the quantized LQG control problem. It turns out that a weak form of separation principle holds which allows the separate designs of state estimation, state feedback and quantization. However, under the conventional LQG problem, the quantization criterion depends on the control cost function, and optimal quantization can not be done by separately minimizing the quantization errors at different time instants. That is, future evolution of the system dynamics must be taken into account in designing the optimal quantizer. The weak separation principle given below is due to [5].

*Theorem 5:* Consider the quantized LQG control problem for the system (25) with the cost function (28) and $R$-bit fixed-rate quantization. Let

$$u_o(k) = K_k\hat{x}(k) \tag{29}$$

be the optimal control for the conventional LQG control problem (i.e., without quantization), where $\hat{x}(k)$ is the optimal state estimate of $x(k)$ (i.e., the Kalman estimate) and $K_k$ is the optimal state feedback gain matrix given by

$$\begin{cases} K_k = -(S_k + B'P_{k+1}B)^{-1}B'P_{k+1}A \\ P_k = Q_k + A'P_{k+1}A - K_k'(S_k + B'P_{k+1}B)K_k \end{cases} \tag{30}$$

with $k = T-1, T-2, \ldots, 0$ and $P_T = Q_T$. Then, optimal quantized LQG control $u(k)$ is achieved by choosing the encoder (sequence) $\{\alpha(k)\}_{k=0}^{T-1}$ to minimize the following distortion function:

$$D = \sum_{k=0}^{T-1} \mathcal{E}[(u_o(k)-u(k))'\Omega_t(u_o(k)-u(k))] \tag{31}$$

where $\Omega_k = S_k + B'P_{k+1}B$. The corresponding minimum cost function is given by

$$\min J = J_{\text{LQG}} + \min D \tag{32}$$

where $J_{\text{LQG}}$ is the minimum cost of the conventional LQG problem.

*Remark 3:* Despite the successful separation of control design, state estimation and quantization, Fischer [66] incorrectly concluded that each $u_o(k)$ can be separately quantized. It turns out that the encoder sequence $\alpha_0, \alpha_1, \ldots, \alpha_{T-1}$ needs to be jointly designed to minimize the distortion function $D$ in (31), resulting in the lack of a full separation principle. An example showing this lack of separation can be found in [26].

The weak separation principle above implies that the quantized LQG problem is essentially transformed into a *quantized state estimation* problem. In this problem, the system's output signal must undergo quantization using a fixed-rate quantizer, and the quantized information is then utilized to construct an estimate of a linear function of the system's state, which in our case corresponds to the desired control signal. The goal is to minimize a specified distortion function (31).

The fundamental difficulty caused by the lack of full separation principle is that this quantized state estimation problem of minimizing $D$ in (31) cannot be solved causally. That is, at each time instant $k < T-1$, the optimal quantized control $u(k)$ can not be solved without knowing what the future unquantized state estimates $\hat{x}(k')$ are for $k < k' \leq T-1$. Due to this difficulty, we can only design quantizers to approximate $\min D$. That is, the true optimal quantized LQG control can not be achieved in practice.

However, it is worth noting that this quantized state estimation problem can be seen as a *generalized vector quantization* [61] where the whole sequence of $\{u_o(k)\}_{k=0}^{T-1}$ needs to be *jointly* quantized. In [27], it is demonstrated that, when high resolution quantization is assumed (i.e., a large $R$), and a mild rank condition is met, optimal quantization can be achieved using a memoryless quantizer. In this context, memoryless quantizers process each input sample $u_o(k)$ independently.

The aforementioned result, combined with the weak separation principle mentioned earlier, establishes that a *full* separation principle holds for quantized LQG control under the conditions of high resolution quantization and the mild rank condition.

## VII. QUANTIZED AVERAGE CONSENSUS

Distributed consensus represents a major research focus in networked systems with wide-ranging applications encompassing statistical learning, sensor networks, distributed optimization, and computer science [62], [63]. This topic has garnered significant attention in various domains, including distributed estimation, control of multi-agent systems, and more. For example, [64] introduced a distributed algorithm for estimating the relative inter-agent states in multi-agent systems, while [65] devised a distributed procedure for achieving consensus in the frequency domain. Moreover, distributed consensus plays a crucial role in broader applications, including distributed sensing and fusion [66], as well as distributed optimization [67]. A fast convergent algorithm for average consensus is provided in [68].

The basic average consensus problem is a distributed problem which can be easily formulated as follows. Let $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$ be a given connected, undirected graph of $n$ nodes, where $\mathcal{V}$ is the set of nodes, $\mathcal{E}$ is the set of edges, and $\mathcal{A} = [a_{ij}]$ is the adjacency matrix of $\mathcal{G}$. In particular, $a_{ij} = a_{ji}$ with $a_{ij} > 0$ if $(i,\ j) \in \mathcal{E}$, or $a_{ij} = 0$ otherwise. Given the initial values $x(0) = \mathrm{col}\{x_1(0), x_2(0), \ldots, x_n(0)\}$ of the nodes, design a distributed iterative algorithm such that every node can eventually work out the average value of all the nodes. More precisely, average consensus is to be achieved using the following distributed control policy:

$$x_i(k+1) = x_i(k) + hu_i(k) \tag{33}$$

where $x_i(k)$ is the state of the $i$-th node at the $k$-th iteration, $h$ is the step size, and the control $u_i(k)$ depends only on the information passed from node $i$'s neighboring set $\mathcal{N}_i = \{j \in \mathcal{V} : (i,\ j) \in \mathcal{E}\}$. The goal is to design the control policy such that $x_i(k) \to \bar{x}(0)$ as $k \to \infty$, where $\bar{x}(0) = (1/n)\sum_{i=1}^{n} x_i(0)$ is the average initial value.

The standard (and most popular) average consensus algorithm adopts the following policy [63], [62]:

$$u_i(k) = \sum_{j \in \mathcal{N}_i} a_{ij}(x_i(k-1) - x_j(k-1)), \quad i \in \mathcal{V}. \tag{34}$$

It is well known [62], [63] that this algorithm guarantees the asymptotic convergence of $x_i(k)$ to $\bar{x}(0)$. The convergence rate is determined by the second eigenvalue $\lambda_2(\mathcal{L})$ of the Laplacian matrix $\mathcal{L}$ (known as the algebraic connectivity of $\mathcal{G}$), which is defined to be $\mathcal{L} = \mathcal{D} - \mathcal{A}$, where $\mathcal{D} = \mathrm{diag}(\mathrm{deg}_1, \ldots, \mathrm{deg}_n)$ with $\mathrm{deg}_i = \sum_{j=1}^{n} a_{ij}$.

A serious drawback of the algorithm above (and many improved algorithms in the literature) is that the information exchange between nodes must be precise. When the exchanged information is quantized, a naive application of the standard average consensus algorithm above will cause biases in the consensus outcomes, or the steady-state results may even be oscillatory. This is because of the fundamental difficulty that the quantization errors introduced by the neighbouring nodes are not identical and thus can not be cancelled. Therefore, the quantization process must be taken into account in the design of the consensus algorithm so that the impact of information loss caused by the quantizers can be adequately mitigated or even eliminated. Many attempts have been made in the literature to accomplish this goal; see [34] for a survey of relevant literature.

In the sequel, we introduce a quantized average consensus algorithm in [34] which deploys a finite-level dynamic quantizer for encoding the state of each node. The surprising result of [34] is that **asymptotic average consensus can be achieved with any number of quantization bits, even a single bit of quantization**. The encoder of the $j$-th node is given by

$$\begin{cases} \xi_j(0) = 0 \\ \xi_j(k) = g(k-1)\Delta_j(k) + \xi_j(k-1) \\ \Delta_j(k) = Q\left(\dfrac{1}{g(k-1)}(x_j(k) - \xi_j(k-1)\right) \end{cases} \tag{35}$$

where $\xi_j(k)$ is the internal state of the encoder for node $j$, $\Delta_j(k)$ is the output of the encoder to be sent to all the neighbors of node $j$, $g(k) > 0$ is a (common) dynamic scaling parameter to be designed, and $Q(\cdot)$ is a finite-level quantizer (with $2K+1$ levels) given by

$$Q(y) = \begin{cases} 0, & -\dfrac{1}{2} < y < \dfrac{1}{2} \\ i, & \dfrac{2i-1}{2} \le y < \dfrac{2i+1}{2}, \ i = 1, 2, \ldots, K-1 \\ K, & y \ge \dfrac{2K-1}{2} \\ -Q(-y), & y \le -\dfrac{1}{2}. \end{cases} \tag{36}$$

The quantizer is implemented such that when $\Delta_i(k) = 0$, the $i$-th node does not send any information. Thus, the above quantizer takes up $\log_2(2K)$ bits. In particular, the quantizer below:

$$Q(y) = \begin{cases} 0, & -\dfrac{1}{2} < y < \dfrac{1}{2} \\ 1, & y \ge \dfrac{1}{2} \\ -1, & y \le -\dfrac{1}{2} \end{cases} \tag{37}$$

is a one-bit quantizer.

For each communication channel $(j, i) \in \mathcal{E}$, the $i$-th node receives $\Delta_i(k)$ and then uses the following decoder to estimate $x_j(k)$:

$$\begin{cases} \hat{x}_{ji}(0) = 0 \\ \hat{x}_{ji}(k) = g(t-1)\Delta_j(k) + \hat{x}_{ji}(t-1). \end{cases} \tag{38}$$

The distributed control policy takes the form

$$u_i(k) = \sum_{j \in \mathcal{N}_i} a_{ij}(\hat{x}_{ji}(k) - \xi_i(k)), \quad i \in \mathcal{V}. \tag{39}$$

We have the following result for the above distributed quantized average consensus control [34].

*Theorem 6:* Suppose $\mathcal{G}$ is connected, $\max_i |x_i(0)| \le C_x$ for some constant $C_x > 0$ and $K \ge 1$. We can take $h \in (0, 2/\lambda_N(\mathcal{L}))$ and

$$\rho_h = \max_{2 \le i \le N} |1 - h\lambda_i(\lambda L)| < 1$$

where $\lambda_i(\mathcal{L})$ is the $i$-th eigenvalue of $\mathcal{L}$ in ascending order.

Choose any $\gamma \in (\rho_h, 1)$ and design the dynamic scaling parameter

$$g(k) = g_0 \gamma^k \tag{40}$$

with a sufficiently large $g_0$ (see [34] for details). Then,

$$\lim_{k\to\infty} x_i(k) = \bar{x}(0), \ \ \forall i \in \mathcal{V} \tag{41}$$

and the convergence rate is bounded by $\gamma$, i.e., $|x_i(k) - \bar{x}(0)| \le \gamma^k C, \forall i \in \mathcal{V}$, for some constant $C > 0$.

*Proof:* We first focus on the encoder side (for each node $j$) and explain how the internal state $\xi_j(k)$ in (35) evolves. If the quantizer $Q(\cdot)$ would not introduce any quantization error (i.e., $Q(\cdot)$ is an identity map), we would see that $\xi_j(k) = x_j(k)$ by substituting the third line into the second line of (35). But quantization error is always present. When the input to the quantizer is bounded by $K$ in magnitude, we see from (36) that the quantization error is bounded by $1/2$. Due to the choice of $g(k)$ in (40) and $\gamma \in (0, 1)$, we see from the second line of (35) that the effect of the quantization error will diminish asymptotically and we will have $\xi_j(k) \to x_j(k)$ asymptotically. The careful choice of $h, \rho_h, \gamma$ and the large initial $g_0$ in the theorem is to keep the input to the quantizer bounded by $K$ in magnitude; see [34] for detailed analysis.

For the decoder side (for every node $i \in \mathcal{N}_j$), it is easy to check from (38) that $x_{ji}(k)$ simply duplicates $\xi_j(k)$. Thus, control signal $u_i(k)$ in (39) asymptotically converges to that in (34) to achieve average consensus asymptotically. ■

*Remark 4:* We assert that the quantized average consensus algorithm above is *scalable* to large networks because the dynamic quantizer (35) at each node operates only on local information and the distributed control policy (38) and (39) uses only neighborhood's information. The only difference for a large network is that more iterations are typically required to achieve the same level of convergence accuracy due to the fact that $\lambda_N(\mathcal{L})$ tends to increase as the network size $N$ increases, but this is expected and natural.

## VIII. Quantized Regulation of Periodic Signals

All of the quantized feedback control problems discussed in the preceding sections share a common characteristic: the quantizer is designed to facilitate efficient communication across a digital connection. In this section, we delve into a very different quantized feedback control scenario. Here, quantization is an intrinsic component of the measurement system (i.e., it is an internal device not to be altered), and the objective of control design is centered around reducing or eliminating the influence of quantization errors.

An illustration of this type of quantized feedback system can be seen in a scanning apparatus driven by an electric motor equipped with an optical encoder for position measurement. The encoder employs light and sensors to convert mechanical movement into a digital signal, utilizing a patterned disk to interrupt the light beam, resulting in varying light intensity. This intensity variation is subsequently translated into a quantized motor position.

The specific control task we consider here is to make a given system with quantized measurements to track a specified periodic signal [36]. The feedback loop is depicted in

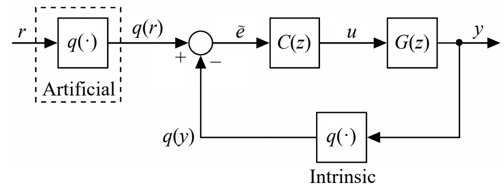


Fig. 8. Quantized feedback control for tracking of periodic signals.

Fig. 8. In this setting, the given system is a linear discrete-time model with transfer function $G(z)$, the control $u(k)$ is to be designed, its output $y(k)$ is measured via a quantizer $q(\cdot)$, i.e., only the quantized signal $q(y(k))$ is available for feedback. To differentiate this type of (given) quantizer from the previous type (to be designed), we use the symbol $q(\cdot)$ here. The quantizer is assumed to be a uniform one with quantization interval $\delta > 0$:

$$q(y) = j\delta, \ \ \text{for } (j-0.5)\delta \le y < (j+0.5)\delta \tag{42}$$

for all integers $j$.

The given reference $r(k)$ is a periodic signal modeled using a finite-term Fourier series

$$r(k) = \delta_0 a_0 + \sum_{i=1}^{m} a_i \cos(\omega_i k + \theta_i) \tag{43}$$

where $m \ge 1$ is the number of sinusoidal terms, $a_i > 0$ are the magnitudes, $\omega_i$ are the angular frequencies, and $\theta_i$ are the initial phases. The term $\delta_0 = 0$ or 1 to allow $r$ with or without a constant term or not.

A mild technical assumption is imposed on the reference signal and quantizer.

*Assumption 1:* The values of $m$ and all $\omega_i$ are known. The quantization interval $\delta$ is such that parameters $a_i$ and $\theta_i$ in the reference $r$ in (43) are uniquely identified from the quantized signal $q(r)$.

Since $r$ contains only $2m$ or $2m+1$ parameters, the assumption above should hold easily for a relatively small $\delta$.

The quantized feedback control problem here is to design a feedback control law $u$, as a function of $q(y)$ and $r$, such that the output $y(k) \to r(k)$ as $k \to \infty$, if possible.

This seems an impossible mission, given that only a quantized version of $y$ is measurable. However, in the work of [36], it is shown that if the $C(z)$ is appropriately designed, the mission can indeed be achieved. The result uses the concept of discrete-time positive realness as defined below:

*Definition 1:* A real rational matrix function $F(z)$ is said to be discrete-time positive-real (DTPR) if $F(z)$ is analytic in $|z| > 1$ and $F(z) + F^T(z^\star) \ge 0$ for all $|z| > 1$.

*Theorem 7 [36]:* Suppose Assumption 1 holds and the controller $C(z)$ is designed such that the open-loop transfer function $H(z) = C(z)G(z)$ satisfies the following conditions:

1) $H(z)$ is DTPR.

2) If $\delta_0 = 1$, then $H(z)$ contains a simple pole at $z = 1$.

3) For $i = 1, \ldots, m$, $H(z)$ contains a simple pole pair at $z = e^{\pm j\omega_i}$.

4) All other poles of $H(z)$ are in $|z| < 1$.

Then, the closed-loop system is stable and the tracking error $e(k) = r(k) - y(k) \to 0$ as $k \to \infty$.

*Proof:* The basic idea is to exploit the in-phase property of both $H(z)$ and the quantizer. That is, the forward-path signals $\tilde{e}$ and $y$ are in-phase (i.e., $\mathrm{Re}[\tilde{e}(z)y(z)] \geq 0$) due to the DTPR assumption, and the feedback-path signals $y$ and $q(y)$ are also in-phase ($y(k)q(y(k)) \geq 0$) due to (42). The proof involves two steps.

The first step exploits the in-phase property of $H(z)$. Take $(A,B,C,D)$ to be a minimal realization of $H(z)$ and take $x(k)$ to be its state at time $k$. Then, use the well-known positive-real lemma [69] to obtain a positive-real matrix $P$ and real matrices $L$ and $W$ such that

$$\begin{aligned} P - A'PA &= L'L \\ C' - A'PB &= L'W \\ D' + D - B'PB &= W'W. \end{aligned}$$

Then, use the Lyapunov function $V(x) = x'Px$ to show that the following decaying property holds for all $k \geq 0$:

$$V(k+1) - V(k) \leq -e(k)\tilde{e}(k).$$

The second step exploits the in-phase property of the quantizer to show that $e(k)\tilde{e}(k) \geq 0$. Together with the decay property above, this leads to $\tilde{e}(k) \to 0$ asymptotically. Finally, use Assumption 1 to show that the above leads to $e(k) \to 0$ asymptotically. ∎

To make the controller design practical, the class of $H(z)$ which satisfies the conditions 1)–4) in Theorem 7 has been worked out in [36] to take the following form:

$$H(z) = k_0\delta_0 \frac{z}{z-1} + \sum_{i=1}^{m} k_i \frac{z(z-\cos\omega_i)}{z^2 - 2z\cos\omega_i + 1} + \sum_{i=m+1}^{m+v} k_i \frac{z}{z-p_i} + \sum_{i=m+v+1}^{m+v+w} k_i \frac{z(z-g_i)}{(z-p_{i1})(z-p_{i2})} \quad (44)$$

where

- $k_i > 0$ for all $i = 0,1,\ldots,m+v+w$.
- $p_i \geq 0; |p_i| < 1$ for all $i = m+1,\ldots,m+v$.
- Either i) or ii) below holds for all $i = m+v+1,\ldots,m+v+w$:

i) $p_{i1}, p_{i2} = \rho_i e^{\pm j\phi_i}$ with $0 < \rho_i < 1, g_i = \rho_i \cos\phi_i$;

ii) $-1 \leq p_{i2} \leq p_{i1} < 1$ with $p_{i1} \geq 0$ and $p_{i2} \leq g_i \leq 1$.

Denoting each term above as $\mathcal{N}_i(z)/\mathcal{D}_i(z)$, $i = 0,1,\ldots,m+v+w$, we note that the terms corresponding to $i = 0,1,\ldots,m$ are necessary for the asymptotic tracking of the individual terms in $r$, whereas the extra terms corresponding to $i = m+1,\ldots,m+v+w$ are used for loop shaping to enhance the transient tracking response. The gains $k_i$ are also used to tune to transient response.

To illustrate the practicality of this quantized feedback design, we consider the control of a linear motor as described in [36]. The transfer function from $u$ (in volts) to $y$ (in μm) is given by

$$G(z) = \frac{b(z+1)}{(z-1)(z-e^{-aT})} \quad (45)$$

with $T = 1$ ms, $a = 9.4$ and $b = 1.7 \times 10^7$. The reference signal to be tracked is

$$r(k) = a_0 + a_1 \cos(\omega_1 k) \quad (46)$$

with $a_0 = 0$ for the first 20 s and then switched to 31 μm afterwards, $a_1 = 7.5$ μm and $\omega_1 = 2\pi$ rad/s. The quantization level $\delta = 10$ μm is used, which is very large compared to the reference signal.

Following the design in Theorem 7, the forward-path transfer function is taken to be:

$$\begin{aligned} H(z) &= C(z)G(z) \\ &= \frac{k_0 z}{z-1} + \frac{k_1 z(z-\cos\omega_1)}{z^2 - 2z\cos\omega_1 + 1} + \frac{k_2 z}{z - e^{-aT}}. \end{aligned}$$

The parameters are set to $k_0 = k_1 = k_2 = 10^{-3}$.

The controller $C(z)$ obtained from $H(z)$ above turns out to be non-causal. To get around this difficulty, an approximate version is used by removing a zero at the origin

$$\tilde{C}(z) = 10^{-4} \frac{3.55z^3 - 10.61z^2 + 10.59z - 3.52}{(z^2 - 2z\cos\omega_1 + 1)(z+1)}.$$

This approximation has very marginal effect on the frequency response of the system.

Fig. 9 shows the system response (where $t = kT$). We see that despite the quantized sensor, the output converges to the desired reference after only a few cycles of transient response. The steady state tracking error is almost eligible compared to the quantization level of $\delta = 10$ μm.

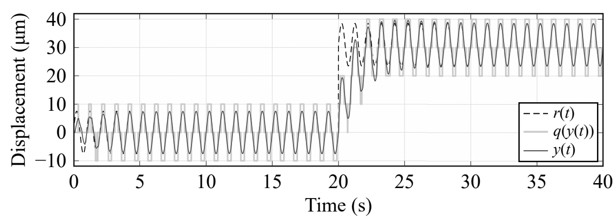


Fig. 9. Example of periodic signal tracking control.

The study above is for discrete-time systems. Similar quantized feedback control results for continuous-time systems can be found in [37], [38].

## IX. Conclusion

While quantization is extensively studied in signal processing and digital communications, we emphasize the caution needed when directly applying techniques from these domains to control issues. This caution arises due to the fundamental presence of feedback in control systems, which yields two significant implications: the re-entry of quantized signals into the system through feedback, and the lack of *a priori* knowledge regarding the boundedness of the input signal to the quantizer. Both of these implications significantly enhance the complexity of quantizer analysis and design.

These distinctive characteristics of quantization within feedback control systems necessitate specialized technical tools. In this tutorial paper, we have introduced such tools to address various quantized feedback control challenges, encompassing topics such as minimal feedback information for stabilization, feedback control design employing static and dynamic quantization, quantized state estimation, quantized LQG control, quantized average consensus, and quantized regulation of periodic signals. Our focus in this paper has been purely on discrete-time systems, with the reason that quanitzed information is typically transmitted in discrete time. For continuous-time systems, sampled-data models can be created, which convert the system into discrete-time, before applying quantized feedback control/estimation methods.

Future research endeavors in this field should undertake interdisciplinary approaches by amalgamating insights from control theory, information theory, communication networks, sensor networks, and quantization theory. This would not only address quantization problems but also extend to resolving other challenges induced by network-based control and estimation issues. Several critical aspects need to be considered. First, there is a need for the development of more advanced quantization schemes that minimize the information loss during the quantization process, allowing for more efficient and accurate control and estimation. This involves exploring novel techniques in quantization, including adaptive quantization and distributed quantization, to adapt to changing system dynamics and enable distributed control in large-scale systems. Second, focus should be made on the integration of machine learning and artificial intelligence techniques to enhance the performance of quantized feedback control systems. These approaches can help in learning optimal quantization parameters and decision thresholds, leading to improved control in complex and dynamic environments. Finally, research efforts can also delve into the development of practical implementation strategies and hardware solutions for quantized feedback control in real-world applications, ensuring that these theoretical advancements translate into tangible benefits across various industries.

It is important to acknowledge that this tutorial paper presents the viewpoints of the author exclusively and may not cover all available results. Quantized feedback control constitutes a relatively recent research domain, rife with open and formidable inquiries.

## REFERENCES


[1] J. B. Lewis and J. T. Tou, "Optimum sampled-data systems with quantized control signals," *Trans. AIEE*, vol. 82, no. 2, pp. 195–201, Jul. 1965.

[2] L. Meier, "Combined optimal control and estimation," in *Proc. Allerton Conf. Circuit and System Theory*, pp. 109–120, 1965.

[3] J. T. Tou, *Optimum Design of Digital Control Systems*, New York: Academic, 1963.

[4] R. E. Larson, "Optimum quantization in dynamic systems," *IEEE Trans. Automatic Control*, vol. AC-12, no. 2, pp. 162–168, 1967.

[5] T. R. Fischer, "Optimal quantized control," *IEEE Trans. Autom. Control*, vol. AC-27, no. 4, pp. 996–998, 1982.

[6] R. E. Kalman, "Nonlinear aspects of sampled-data control systems," in *Proc. Symp. Nonlinear Circuit Theory,* vol. VII. Brooklyn, NY: Polytechnic Press, 1956.

[7] B. Widrow, "Statistical analysis of amplitude-quantized sampled-data systems," *Trans. AIEE*, vol. 79, no. 2, pp. 555–567, January. 1961.

[8] D. F. Delchamps, "Stabilizing a linear system with qunatized state feedback," *IEEE Trans. Automatic Control*, vol. 35, no. 8, pp. 916–924, 1990.

[9] W. S. Wong and R. W. Brockett, "Systems with finite communication bandwidth constraints I: State estimation problems," *IEEE Trans. Automatic Control*, vol. 42, no. 9, pp. 1294–1299, 1997.

[10] W. S. Wong and R. W. Brockett, "Systems with finite communication bandwidth constraints II: Stabilization with limited information feedback," *IEEE Trans. Automatic Control*, vol. 44, no. 5, pp. 1049–1053, 1999.

[11] J. Baillieul, "Feedback designs in information-based control," in *Stochastic Theory and Control Workshop*, Kansis, Springer, pp. 35–57, 2001.

[12] V. Borkar and S. Mitter, "LQG control with communication constraints," in *Communications, Computation, Control and Signal Processing: A Tribute to Thomas Kailath.* Norwell, MA: Kluwer, 1997.

[13] R. W. Brockett and D. Liberzon, "Quantized feedback stabilization of linear systems," *IEEE Trans. Automatic Control*, vol. 45, no. 7, pp. 1279–1289, 2000.

[14] G. N. Nair and R. J. Evans, "Stabilization with data-rate-limited feedback: Tightest attainable bounds," *Systems and Control Letters*, vol. 41, pp. 49–56, 2000.

[15] G. N. Nair and R. J. Evans, "Exponential stabilisability of finite-dimensional linear systems with limited data rates," *Automatica*, vol. 39, pp. 585–593, 2003.

[16] N. Elia and K. Mitter, "Stabilization of linear systems with limited information," *IEEE Trans. Automatic Control*, vol. 46, no. 9, pp. 1384–1400, 2001.

[17] S. Tatikonda and S. Mitter, "Control under communication constraints," *IEEE Trans. Automatic Control*, vol. 49, no. 7, pp. 1056–1068, 2004.

[18] S. Tatikonda and S. Mitter, "Control over noisy channels," *IEEE Trans. Automatic Control*, vol. 49, no. 7, pp. 1196–1201, 2004.

[19] M. Fu and L. Xie, "The sector bound approach to quantized feedback control," *IEEE Trans. Automatic Control*, vol. 50, no. 11, pp. 1698–1711, Nov. 2005.

[20] M. Fu and L. Xie, "Finite-level quantization feedback control for linear systems," *IEEE Trans. Automatic Control*, vol. 54, no. 5, pp. 1165–1170, May 2009.

[21] N. Xiao, L. Xie, and M. Fu, "Stabilization of Markov jump linear systems using quantized state feedback," *Automatica*, vol. 46, no. 10, pp. 1696–1702, 2010.

[22] D. Coutinho, M. Fu, and C. E. de Souza, "Input and output quantized feedback linear systems," *IEEE Trans. Automatic Control*, vol. 55, no. 3, pp. 761–766, 2010.

[23] D. Liberzon, "On stabilization of linear systems with limited information," *IEEE Trans. Automatic Control*, vol. 48, pp. 304–307, 2003.

[24] S. Taikonda, A. Sahai, and S. Mitter, "Stochastic linear control over a communication channel," *IEEE Trans. Automatic Control*, vol. 49, no. 9, pp. 1549–1561, 2004.

[25] A. S. Matveev and A. V. Savkin, "The problem of LQG optimal control via a limited capacity communication channel," *Systems and Control Letters*, vol. 53, pp. 51–64, 2004.

[26] M. Fu, "Lack of separation principle for quantized linear quadratic Gaussian control," *IEEE Trans. Automatic Control*, vol. 57, no. 9, pp. 2385–2390, 2012.

[27] M. Fu, "Linear quadratic Gaussian control with quantized feedback," in *Proc. American Control Conf.*, St. Louis, July 2009.

[28] M. Fu and C. E. de Souza, "State estimation for linear discrete-time systems using quantized measurements," *Automatica*, vol. 45, no. 12, pp. 2937–2945, 2009.

[29] M. Epstein, L. Shi, A. Tiwari, and R. Murry, "Probabilistic performance of state estimation across a lossy network," *Automatica*, vol. 44, no. 12, pp. 3046–3053, 2008.

[30] A. Tiwari, M. Jun, D. Jeffcoat, and R. Murray, "Analysis of dynamic sensor coverage problem using Kalman filters for estimation," in *Proc. 16th IFAC World Congr.*, Prague, Czech Republic, 2005.

[31] R. Carli, F. Fagnani, P. Frasca, and S. Zampieri, "Efficient quantized techniques for consensus algorithms," in *Proc. NeCST Workshop*, Nancy, France, 2001.

[32] R. Carli, F. Fagnani, P. Frasca, and S. Zampieri, "A probabilistic analysis of the average consensus algorithm with quantized communication," in *Proc. 17th IFAC World Congr.*, Seoul, Korea, 2008.

[33] N. Xiao, L. Xie, and M. Fu, "Kalman filtering over unreliable communication networks with bounded Markovian packet dropouts," *Int. J. Robust and Nonlinear Control*, vol. 19, no. 16, pp. 1770–1786, 2009.

[34] M. Fu, “Distributed consensus with limited communication data rate,” *IEEE Trans. Automatic Control*, vol. 56, no. 2, pp. 279–292, 2011.

[35] W. Chen, Z. Wang, D. Ding, and H. Dong, “Consensusability of discrete-time multi-agent systems under binary encoding with bit errors,” *Automatica*, vol. 133, p. 109867, 2021.

[36] A. Salton, J. V. Flores, J. Zheng, and M. Fu, “Asymptotic tracking of discrete-time systems subject to uniform output quantization,” *IEEE Control Systems Letters*, vol. 7, no. 2, pp. 829–834, 2023.

[37] A. T. Salton, M. Fu, J. V. Flores, and J. Zheng, “High precision over long range: A macro-micro approach to quantized positioning systems,” *IEEE Trans. Control Systems Technology*, vol. 29, no. 6, pp. 2406–2415, 2021.

[38] A. T. Salton, J. Zheng, J. V. Flores, and M. Fu, “High-precision tracking of periodic signals: A macro-micro approach with quantized feedback,” *IEEE Trans. Industrial Electronics*, vol. 69, no. 8, pp. 8325–8334, 2022.

[39] F. C. Rego *et al*., “A distributed Luenberger observer for linear state feedback systems with quantized and rate-limited communications,” *IEEE Trans. Automatic Control*, vol. 66, no. 9, pp. 3922–3937, 2021.

[40] Y. H. Choi and S. J. Yoo, “Distributed quantized feedback design strategy for adaptive consensus tracking of uncertain strict-feedback nonlinear multiagent systems with state quantizers,” *IEEE Trans. Cybernetics*, vol. 52, no. 7, pp. 7069–7083, 2022.

[41] Z. Zhang, J. Hu, and H. Huang, “Formation tracking for nonlinear uncertain multi-agent systems via adaptive output feedback quantized control,” *IEEE Access*, vol. 7, 2019.

[42] L. Zhang, C. Hua, H. Yu, and X. Guan, “Distributed adaptive fuzzy containment control of stochastic pure-feedback nonlinear multiagent systems with local quantized controller and tracking constraint,” *IEEE Trans. Systems*, *Man*, *and Cybernetics: Systems*, vol. 49, no. 4, pp. 787–796, 2019.

[43] Z. Zhang, C. Wen, L. Xing, and Y. Song, “Adaptive output feedback control of nonlinear systems with mismatched uncertainties under input/output quantization,” *IEEE Trans. Automatic Control*, vol. 67, no. 9, pp. 4801–4808, 2022.

[44] X. Xia, T. Zhang, Y. Fang, and G. Kang, “Adaptive quantized control of output feedback nonlinear systems with input unmodeled dynamics based on backstepping and small-gain method,” *IEEE Trans. Systems*, *Man*, *and Cybernetics: Systems*, vol. 51, no. 9, pp. 5686–5697, 2021.

[45] X. Yu and Y. Lin, “Adaptive backstepping quantized control for a class of nonlinear systems,” *IEEE Trans. Automatic Control*, vol. 62, no. 2, pp. 981–985, 2017.

[46] S. Dey, A. Chiuso, and L. Schenato, “Feedback control over lossy SNR-limited channels: Linear encoder-decoder-controller design,” *IEEE Trans. Automatic Control*, vol. 62, no. 6, pp. 3054–3061, 2017.

[47] M. Zhang, P. Shi, L. Ma, J. Cai, and H. Su, “Quantized feedback control of fuzzy Markov jump systems,” *IEEE Trans. Cybernetics*, vol. 49, no. 9, pp. 3375–3384, 2019.

[48] M. Xu, Z. Xu, L. Ma, and H. Que, “Quantized $H_\infty$ feedback control of semi-Markov jump systems with limited mode information,” *IEEE Access*, vol. 8, 2020.

[49] A. V. Papadopoulos, F. Terraneo, A. Leva, and M. Prandini, “Switched control for quantized feedback systems: Invariance and limit cycle analysis,” *IEEE Trans. Automatic Control*, vol. 63, no. 11, pp. 3775–3786, 2018.

[50] M. Wakaiki, T. Zanma, and K.-Z. Liu, “Observer-based stabilization of systems with quantized inputs and outputs,” *IEEE Trans. Automatic Control*, vol. 64, no. 7, pp. 2929–2936, 2019.

[51] J. Wang, “Quantized feedback control based on spherical polar coordinate quantizer,” *IEEE Trans. Automatic Control*, vol. 66, no. 12, pp. 6077–6084, 2021.

[52] W. Liu, Q. Ma, S. Xu, and Z. Zhang, “State quantized output feedback control for nonlinear systems via event-triggered sampling,” *IEEE Trans. Automatic Control*, vol. 67, no. 12, pp. 6810–6817, 2022.

[53] J. Lian and C. Li, “Event-triggered sliding mode control of uncertain switched systems via hybrid quantized feedback,” *IEEE Trans. Automatic Control*, vol. 66, no. 6, pp. 2809–2816, 2021.

[54] M. Abdelrahim, V. S. Dolk, and W. P. M. H. Heemels, “Event-triggered quantized control for input-to-state stabilization of linear systems with distributed output sensors,” *IEEE Trans. Automatic Control*, vol. 64, no. 12, pp. 4952–4967, 2019.

[55] M. Ran, S. Feng, J. Li, and L. Xie, “Quantized consensus under data-rate constraints and DoS attacks: A zooming-in and holding approach,” *IEEE Trans. Automatic Control*, vol. 68, no. 9, pp. 5397–5412, 2023.

[56] L.-Y. Hao, Y. Yu, T.-S. Li, and H. Li, “Quantized output-feedback control for unmanned marine vehicles with thruster faults via sliding-mode technique,” *IEEE Trans. Cybernetics*, vol. 52, no. 9, pp. 9363–9376, 2022.

[57] J. Liu and N. Elia, “Quantized control with applications to mobile vehicles,” in *Proc. IEEE Conf. Decision and Control*, 2002.

[58] S. M. M. Uddin and G. Mirzaeva, “A high performance feedback quantized predictive control of induction machine drives,” in *Proc. IEEE Energy Conversion Congr. and Exposition*, 2020.

[59] C. Zhang, Y. Yu, and M. Zhou, “Finite-time adaptive quantized motion control for hysteretic systems with application to piezoelectric-driven micropositioning stage,” *IEEE/ASME Trans. Mechatronics*, 2023. DOI: 10.1109/TMECH.2023.3250481

[60] A. Sahai, “The necessity and sufficiency of anytime capacity for control over a noisy communication link,” in *Proc. IEEE Conf. Decision and Control*, 2004.

[61] A. Rao, D. Miller, K. Rose, and A. Gersho, “Generalized vector quantization: Jointly optimal quantization and estimation,” in *Proc. IEEE Int. Symp. Information Theory*, 1995.

[62] L. Xiao and S. Boyd, “Fast linear iterations for distributed averaging,” *Systems and Control Letters*, vol. 53, pp. 65–78, 2004.

[63] R. Olfati-Saber and R. M. Murray, “Consensus problem in networks of agents with switching topology and time-delays,” *IEEE Trans. Automatic Control*, vol. 49, no. 9, pp. 1520–1533, 2004.

[64] Y. Xu, Z. Wu, Y. Pan, C. K. Ahn, and H. Yan, “Consensus of linear multiagent systems with input-based triggering condition,” *IEEE Trans. Systems*, *Man and Cybernetics: Systems*, vol. 49, no. 11, pp. 2308–2317, 2018.

[65] Y. Chen and Y. Shi, “Distributed consensus of linear multiagent systems: Laplacian spectra-based method,” *IEEE Trans. Systems*, *Man and Cybernetics: Systems*, vol. 50, no. 2, pp. 700–706, 2020.

[66] J. Zhou, Y. Zhu, Z. You, and E. Song, “An efficient algorithm for optimal linear estimation fusion in distributed multisensor systems,” *IEEE Trans. Systems*, *Man and Cybernetics-Part A: Systems and Humans*, vol. 36, no. 5, pp. 1000–1009, 2006.

[67] S. Yang, Q. Liu, and J. Wang, “Distributed optimization based on a multiagent system in the presence of communication delays,” *IEEE Trans. Systems*, *Man and Cybernetics: Systems*, vol. 47, no. 5, pp. 717–728, 2016.

[68] K. Xie, Q. Cai, Z. Zhang, and M. Fu, “Distributed algorithms for average consensus of input data with fast convergence,” *IEEE Trans. Systems*, *Man and Cybernetics: Systems*, vol. 51, no. 5, pp. 2653–2664, 2021.

[69] L. Hitz and B. D. O. Anderson, “Discrete positive-real functions and their application to system stability,” *Proc. IEE*, vol. 116, no. 1, pp. 153–155, 1969.

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