Analysis and Design of Time-Delay Impulsive Systems Subject to Actuator Saturation

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***Abstract*—This paper investigates the exponential stability and performance analysis of nonlinear time-delay impulsive systems subject to actuator saturation. When continuous dynamics is unstable, under some conditions, it is shown that the system can be stabilized by a class of saturated delayed-impulses regardless of the length of input delays. Conversely, when the system is originally stable, it is shown that under some conditions, the system is robust with respect to sufficient small delayed-impulses. Moreover, the design problem of the controller with the goal of obtaining a maximized estimate of the domain of attraction is formulated via a convex optimization problem. Three examples are provided to demonstrate the validity of the main results.**

***Index Terms*—Delayed impulses, impulsive control, impulsive disturbance, nonlinear systems, saturation.**

I. Introduction

ALL actuators have limited capabilities in real control systems since practical control can only deliver limited magnitudes and rates of signals due to physical constraints. As is known to all, input saturation may cause performance deterioration and even instability [1]. When input saturation is encountered, it makes sense to explore effective strategies to alleviate undesirable effects. Over the past decades many useful methods have been developed in this field. Recently, two types of methods to deal with the saturation function are widely applicable. In the first, polytopic differential inclusion is utilized to describe saturation nonlinearity [2], [3]. The second main approach uses global/regional sector conditions which places saturation nonlinearity into a linear sector [4], [5]. Stability analysis of nonlinear systems with input saturation has been extensively studied over the past years [6]–[9].

Impulsive systems have been extensively investigated as they provide effective mathematical models to deal with plants with discontinuous input [10]−[13]. For example, impulsive phenomenon is ubiquitous in biology [14], mechanics [15] and neural networks [16]. In general, there are two main kinds of impulsive effects: impulsive control and impulsive disturbance. More specifically, the first kind of impulsive effects corresponds to the case where impulses are used to control the continuous dynamics, while the second one concerned with the case where the behaviour of the system is subject to impulsive disturbance. Over the past decades, a large amount of results concerning different impulsive effects can be found in [17]−[19]. In the process of transmission and sampling of the information, time delays are always inevitable [20]–[22]. For instance, in the application of neural networks, time delays in dynamical nodes expresses response time and coupling delays refer to communication delays; in a financing institution, the decision of an investor is often influenced by both current transaction information and past transaction information of other investors, as shown in [23]−[26]. On the other hand, saturated impulse is ubiquitous in practical applications, such as impulsive synchronization of neural networks in which signal transmission is limited due to the inherent physical constraints and instantaneous acceleration of mechanical systems in which performance is constrained by digital implementation. However, the relevant theoretical results related to impulsive saturation has been relatively less developed [27]−[29] on account of the complex coupling effects between impulses dynamics and input saturation. Recently, existence of a solution for impulsive differential equations under saturation was studied in [27]. Reference [28] developed a linear differential inclusion method for exponential stability of nonlinear impulsive system with input saturation. Impulsive control of a time-delay system under input saturation was investigated in [29]. However, both continuous dynamics and discrete dynamics were required to be stable/stabilized in [29]. In addition, input delay was excluded in afore mentioned works. More recently, delayed-impulses control for discrete systems with input saturation was addressed in [30], where two classes of impulses, i.e., stabilizing impulses and destabilizing impulses, were studied, respectively. Nevertheless, the estimation of the stability region was excluded in [30], which is essential to the research of saturated systems. Moreover, some limitations on impulse intervals and delays were imposed, which brings more conservativeness. Therefore, the existing literature on the problems of stability and performance analysis for nonlinear systems with delayed impulses and input saturation were not effectively

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solved.

Motivated by the above discussion, we shall investigate the exponential stability of time-delay impulsive systems involving input saturation. With the help of Razumikhin-type technique, a relatively maximized estimate of the stability region is obtained by means of an optimization algorithm. The novelty and distinctiveness of the proposed results is that we remove the restriction on the length of input delays, i.e., the length of input delays has no implicit relationship with impulse intervals. Moreover we fully considered the relationship between impulsive actions, impulse intervals and stability region.

The remainder of the paper is organized as follows. Section II introduces the model of impulsive systems under delayed impulses and input saturation. Main results including the problems of exponential stability/stabilization and estimation of the domain of attraction are investigated in Section III. In Section IV, three numerical simulations are proposed to demonstrate the applicability of our results. Section V summarizes this paper.

Notations: Let symbols $\mathbb{R},\mathbb{R}^{n\times m},\mathbb{Z}_+,\mathbb{R}_+,I,co\{\cdot\}$ denote accordingly the set of real numbers, all $n\times m$ real matrices, positive integer numbers, non-negative real numbers, the unit matrix of appropriate dimension, and the convex hull of a set, respectively. For a matrix Q, Q^T denote the transpose of the matrix Q, $Q>0$ if Q is a symmetric positive definite matrix, $\lambda_{\max}(Q)$ and $\lambda_{\min}(Q)$ represent the maximum and minimum eigenvalue of symmetric matrix Q, respectively. For $m,n\in\mathbb{R}$, let $PC([m,n],\mathbb{R}^n)$ denote the set of piecewise right continuous functions $\chi:[m,n]\to\mathbb{R}^n$ with the norm defined by $\|\chi\|_\tau=\sup_{m\le s\le n}\|\chi(s)\|$. Let $p\vee q$ be the maximum value of p and q. Given two integers s_1 and s_2 with $s_1<s_2$, let $\mathcal{I}[s_1,s_2]=\{s_1, s_1+1,\ldots,s_2\}$. Given a matrix $P\in\mathbb{R}^{n\times n}>0$ and positive constants r,ρ, define ellipsoid $\mathcal{E}(P,\rho)\doteq\{x\in\mathbb{R}^n: x^TPx\le\rho\}$ and set $\mathcal{M}(P,\rho)=\{\varphi\in PC([-r,0],\mathbb{R}^n):\varphi(\theta)\in\mathcal{E}(P,\rho),\forall\theta\in[-r,0]\}$. Given an matrix $U\in\mathbb{R}^{m\times n}$, let u_i be the ith row of the matrix U and denote $\mathcal{L}(U)\doteq\{x\in\mathbb{R}^n:|u_ix|\le 1, i\in\mathcal{I}[1,m]\}$. Define $\mathcal{D}\in\mathbb{R}^{m\times m}$ as the set of diagonal matrices whose diagonal elements are either 1 or 0. Let each element of $\mathcal{D}$ be D_i and denote $D_i^-=I-D_i$, $i\in\mathcal{I}[1,2^m]$.

II. Preliminaries

Consider the following nonlinear impulsive system with saturated delayed impulses:

$$\dot{x}(t)=Ax(t)+f(t,x(t-\tau(t))),\ t\ne t_k \tag{1a}$$

$$x(t_k)=B\text{sat}(u(t_k^- -\xi_k))+Cx(t_k^-),\ k\in\mathbb{Z}_+ \tag{1b}$$

$$x(t_0+\theta)=\varphi(\theta),\ \theta\in[-r,0] \tag{1c}$$

where $x\in\mathbb{R}^n$ and $u\in\mathbb{R}^m$ are the state and the input of the system, $A\in\mathbb{R}^{n\times n},B\in\mathbb{R}^{n\times m},C\in\mathbb{R}^{n\times n}$ are constant matrices, $\tau(t)$ is a time-varying delay satisfying $0\le\tau(t)\le\varsigma$, and $\{\xi_k\ge 0, k\in\mathbb{Z}_+\}$ are impulse input delays satisfying $\xi=\max_{k\in\mathbb{Z}_+}\{\xi_k\}<\infty$. Define diagonal matrix $L=\text{diag}(l_j)$ for later use. Saturation function sat is defined as $\text{sat}(u)=col\{\text{sat}(u_1)\ \cdots\ \text{sat}(u_m)\}$, where $\text{sat}(u_i)=\text{sign}(u_i)\min\{1,|u_i|\}$, $i\in\mathcal{I}[1,m]$. The function $\varphi\in PC([-r,0],\mathbb{R}^n)$ is the initial state of the system and $r=\max\{\varsigma,\xi\}$. $\{t_k,k\in\mathbb{Z}_+\}$ is a sequence of impulse instants which satisfy $0\le t_0<t_1<\cdots<t_k\to+\infty$ as $k\to+\infty$, and set $\mathcal{F}$ is used to denote such kind of impulse sequences. In particular, we define $\mathcal{F}_{\max}(\beta)$ and $\mathcal{F}_{\min}(\beta)$ as the set of impulse sequences which satisfy $\sup_{k\in\mathbb{Z}_+}\{t_k-t_{k-1}\}\le\beta$ and $\inf_{k\in\mathbb{Z}_+}\{t_k-t_{k-1}\}\ge\beta$, respectively.

Definition 1 ([31], [32]): A function $x\in PC([t_0-r,t_0+\alpha),\mathcal{D})$ with $\alpha>0$ and $\mathcal{D}\subseteq\mathbb{R}^n$ is said to be a solution of (1) if

i) x is continuous at $t\ne t_k$ in $(t_0,t_0+\alpha)$, $k\in\mathbb{Z}_+$;

ii) The derivative of x exists and is continuous at all but at most a finite number of points t in $(t_0,t_0+\alpha)$;

iii) The right-hand derivative of x satisfies the delay difference equation (1a) for all $t\in(t_0,t_0+\alpha)$;

iv) x satisfies the delay differential equation (1b) at each $t_k\in(t_0,t_0+\alpha)$, $k\in\mathbb{Z}_+$;

v) x satisfies the initial condition (1c).

Especially, when $\alpha=+\infty$, $x\in PC([t_0-r,+\infty),\mathcal{D})$ is said to be a global solution of (1) on the interval $[t_0-r,+\infty)$. To show the existence of the solution, we make the following assumptions on f.

H_1) For each fixed $t\in\mathbb{R}_+$, $f(t,\psi)$ is a continuous function of ψ on $PC([-\varsigma,0],\mathbb{R}^n)$;

H_2) For $\sigma\in[0,\alpha)$, if $x\in PC([t_0-\varsigma,t_0+\sigma],\mathbb{R}^n)$ and x is continuous at each $t\ne t_k$ in $(t_0,t_0+\sigma]$, then the composite function v defined by $v(t)=f(t,\psi)$ is an element of the function set $PC([t_0,t_0+\sigma],\mathbb{R}^n)$;

H_3) For $\sigma\in[0,\alpha)$ and each compact set $F\subset\mathbb{R}^n$, there exists $M>0$ such that $|f(t,\psi)|\le M$ for all $(t,\psi)\in[t_0,t_0+\sigma]\times PC([-\varsigma,0],F)$.

H_4) There exists a constant l_j such that $|f_j(t,\psi_j)|\le l_j|\psi_j|$, $j=1,\ldots,n$, for any $\psi=(\psi_1,\ldots,\psi_n)^T\in PC([-\varsigma,0],\mathbb{R}^n)$, where $f=(f_1,\ldots,f_n)^T$. Define diagonal matrix $L=\text{diag}(l_j)$ for later use.

It is shown in [31] that under assumptions $H_1)-H_4$), for any initial condition $\varphi\in PC([-r,0],\mathbb{R}^n)$, system (1) admits a solution $x(t)=x(t,t_0,\varphi)$ that exists on a maximal interval $[t_0-r,t_0+T)$, where $0<T\le\infty$.

Definition 2 ([28]): System (1) is said to be locally uniformly exponentially stable (*LUES*) over the class $\mathcal{F}$, if for any initial value $\varphi\in\mathcal{M}(P,\varrho)$, there exist constants $M_0\ge 1$ and $\iota>0$ such that

$$|x(t,t_0,\varphi)|\le M_0|\varphi|_r e^{-\iota(t-t_0)},\ \forall t\ge t_0,\ \forall\{t_k\}\in\mathcal{F}.$$

Define the domain of the attraction of the origin by $\mathcal{S}=\{\varphi\in PC([-r,0],\mathbb{R}^n):\lim\limits_{t\to\infty}x(t,t_0,\varphi)=0\}$. Obviously, $\mathcal{M}(P,\varrho)$ is included in $\mathcal{S}$.

Lemma 1 ([33]): Given matrices $H,K\in\mathbb{R}^{m\times n}$, for any $x\in\mathbb{R}^n$, if $x\in\mathcal{L}(H)$, then,

$$\text{sat}(Kx)\in co\{D_iKx+D_i^-Hx: i\in\mathcal{I}[1,2^m]\}.$$

Consequently, $\text{sat}(Kx)$ can be expressed as

$$\text{sat}(Kx)=\sum_{i=1}^{2^m}\vartheta_i(D_iKx+D_i^-Hx) \tag{2}$$

where $0\le\vartheta_i\le 1$ and $\sum_{i=1}^{2^m}\vartheta_i=1$. For simplicity, denote $J_i=D_iK+D_i^-H$, $N(\vartheta)=\sum_{i=1}^{2^m}\vartheta_iJ_i$ and $\vartheta=[\vartheta_1\vartheta_2\cdots\vartheta_{2^m}]$ for later use.

In this paper, our interests lie in stability and stabilization analysis, and the estimation of $\mathcal{S}$. Specifically speaking, our objective is to establish exponential stability/stabilization criteria by Lyapunov function method and obtain a maximized estimate of $\mathcal{S}$ of system (1) involving saturated delayed-impulses. For this purpose, let us now employ state feedback $u(t) = Kx(t)$, where $K \in \mathbb{R}^{m\times n}$ is the gain matrix to be designed.

III. Main Results

A. Stability and Stabilization Analysis of Nonlinear Systems Involving Delayed Impulses

In this section, Razumikhin-type method for stability and stabilization of system (1) is developed. Meanwhile, we also present some sufficient conditions for a set to be inside of $\mathcal{S}$.

Theorem 1: Given constant scalar ϱ and matrices $H, K \in \mathbb{R}^{m\times n}, L = \mathrm{diag}(l_j) \in \mathbb{R}^{n\times n}$, if there exist $n\times n$ matrix $P > 0$, $n\times n$ diagonal matrix $M > 0$ and positive constants γ,β, δ,μ_1,μ_2 and $\mu \in (0,1)$ with $\mu_1+\mu_2 < \mu$, such that $\mathcal{E}(P,\varrho/\mu) \subset \mathcal{L}(H)$, $LML < \delta P$,

$$\begin{bmatrix} A^TP+PA+(\frac{\gamma}{\mu}+\frac{\ln\mu}{\beta})P & 0 & P \\ \star & (-\gamma+\delta)P & 0 \\ \star & \star & -M \end{bmatrix} < 0 \quad (3)$$

and

$$\begin{bmatrix} -\mu_1 P & J_i^T B^T PC & J_i^T B^T P \\ \star & C^TPC-\mu_2 P & 0 \\ \star & \star & -P \end{bmatrix} \le 0, i \in \mathcal{I}[1,2^m] \quad (4)$$

then system (1) is LUES over the class $\mathcal{F}_{\max}(\beta)$. Moreover, $\mathcal{M}(P,\varrho)$ is included in $\mathcal{S}$.

Proof: It follows from $\mu_1+\mu_2<\mu$ that there exist positive constants λ,ς,ξ and $h\in(0,1-\mu)$ such that $\mu_1 e^{\lambda\xi}+\mu_2 \le \mu$ and:

$$\Psi_1 = \begin{bmatrix} \Phi_1 & 0 \\ \star & (-\gamma e^{-\lambda\varsigma}+\delta)P \end{bmatrix} < 0 \quad (5)$$

where $\Phi_1 = A^TP+PA+(\lambda+\frac{\gamma}{\mu}+\frac{\ln(\mu+h)}{\beta})P+PM^{-1}P$. On the other hand, it follows from (4) that:

$$\sum_{i=1}^{2^m}\vartheta_i \begin{bmatrix} -\mu_1 P & J_i^T B^T PC & J_i^T B^T P \\ \star & C^TPC-\mu_2 P & 0 \\ \star & \star & -P \end{bmatrix} \le 0 \quad (6)$$

where $\vartheta_i\in[0,1]$ and $\sum_{i=1}^{2^m}\vartheta_i = 1$. By Schur complement, (6) is equivalent to

$$\Theta = \begin{bmatrix} -\mu_1 P + N(\vartheta)^TB^TPBN(\vartheta) & N(\vartheta)^TB^TPC \\ \star & C^TPC-\mu_2 P \end{bmatrix} \le 0.$$

Suppose that $x(t) \doteq x(t,t_0,\varphi)$ is a solution of system (1) through (t_0,φ). Choose Lyapunov function $V(t) = x^T(t)Px(t)$ and we shall prove that for any $\varphi\in\mathcal{M}(P,\varrho)$, when $t\ge t_0-r$,

$$V(t) \le \frac{\varrho}{\mu}e^{-\lambda(t-t_0)}.$$

Denote $\Lambda(t) \doteq (e^{\lambda(t-t_0)}\vee 1)V(t)$. In what follows, we show that for any $\varphi\in\mathcal{M}(P,\varrho)$:

$$\Lambda(t) \le \varrho/\mu,\ t\ge t_0-r. \quad (7)$$

Firstly, it is obvious that $\Lambda(t+\theta) \le \sup_{t_0-r\le s\le t_0} V(s) \le \varrho < \varrho/\mu$ for $t<t_0$ and $\theta\in[-r,0]$, and then, we shall show that

$$\Lambda(t)\le\varrho/\mu,\ t\in[t_0,t_1).$$

Otherwise, there exists $\acute{t} = \inf\{t\in[t_0,t_1)|\Lambda(t)>\varrho/\mu\}$ such that $\Lambda(\acute{t}) = \varrho/\mu$ and $\Lambda(t)\le\varrho/\mu$ for $t\le\acute{t}$. Since $\Lambda(t_0) = V(t_0) \le \varrho<\varrho/\mu$, we know that $\acute{t}>t_0$. Furthermore, there exists $\grave{t} = \sup\{t\in[t_0,\acute{t})|\Lambda(t)\le\varrho\}$ such that $\Lambda(\grave{t}) = \varrho$ and $\Lambda(t)\ge\varrho$ for $t\in[\grave{t},\acute{t}]$. Therefore, when $t\in[\grave{t},\acute{t}]$, $\Lambda(t+\theta)\le\varrho/\mu\le\Lambda(t)/\mu$, $\theta\in[-r,0]$. For $t\in[\grave{t},\acute{t}]$, taking the right-upper Dini derivative of $V(t)$, by the property of $2p^Tq \le p^TMp + Q^TM^{-1}q$, we have

$$\begin{aligned} D^+\Lambda(t) &= e^{\lambda(t-t_0)}(\lambda V(t)+D^+V(t)) \\ &\le e^{\lambda(t-t_0)}(\lambda x^T(t)Px(t)+2x^T(t)P(Ax(t) \\ &\quad + f(x(t-\tau(t))))) \\ &\le e^{\lambda(t-t_0)}(\lambda x^T(t)Px(t)+2x^T(t)PAx(t) \\ &\quad + x^T(t)PM^{-1}Px(t) + f^T(t-\tau(t))Mf(t-\tau(t))) \\ &\le e^{\lambda(t-t_0)}(\lambda x^T(t)Px(t)+2x^T(t)PAx(t) \\ &\quad + x^T(t)PM^{-1}Px(t) \\ &\quad + x^T(t-\tau(t))LMLx(t-\tau(t))). \end{aligned}$$

By the above analysis, according to $\Lambda(t+\theta)\le\Lambda(t)/\mu, LML<\delta P$ and (5), we then get that

$$\begin{aligned} D^+\Lambda(t) &\le e^{\lambda(t-t_0)}x^T(t)(A^TP+PA+\lambda P+PM^{-1}P)x(t) \\ &\quad + \delta e^{\lambda(t-t_0)}x^T(t-\tau(t))Px(t-\tau(t)) \\ &\quad + \gamma(\Lambda(t)/\mu-\Lambda(t-\tau(t))) \\ &\le e^{\lambda(t-t_0)}\eta_1(t)\Psi\eta_1(t)^T \\ &\le e^{\lambda(t-t_0)}\eta_1(t)\Psi_1\eta_1^T(t)+\sigma\Lambda(t) \\ &\le \sigma\Lambda(t) \end{aligned} \quad (8)$$

where $\eta_1(t)\doteq(x^T(t)\ x^T(t-\tau(t))),\sigma\doteq-\frac{\ln(\mu+h)}{\beta}$ and

$$\Psi \doteq \begin{bmatrix} A^TP+PA+(\lambda+\frac{\gamma}{\mu})P+PM^{-1}P & 0 \\ \star & (-\gamma e^{-\lambda\varsigma}+\delta)P \end{bmatrix}.$$

Note that $h\in(0,1-\mu)$, which leads to $\sigma>0$. For impulse time sequence $\{t_k\}\in\mathcal{F}_{\max}(\beta)$, we have

$$\Lambda(\acute{t}) \le e^{\sigma(\acute{t}-\grave{t})}\Lambda(\grave{t}) \le \exp\{-\frac{\ln(\mu+h)}{\beta}\beta\}\varrho < \varrho/\mu \quad (9)$$

which contradicts with the fact that $\Lambda(\acute{t}) = \varrho/\mu$. Hence, it holds that $\Lambda(t)\le\varrho/\mu,\ t\in[t_0,t_1)$. For some $k\in\mathbb{Z}_+$, assume that

$$\Lambda(t)\le\varrho/\mu,\ t\in[t_0-r,t_k). \quad (10)$$

Next, we shall show that

$$\Lambda(t)\le\varrho/\mu,\ t\in[t_k,t_{k+1}). \quad (11)$$

According to $\mathcal{E}(P,\varrho/\mu)\subset\mathcal{L}(H)$, for every $x(t)\in\mathcal{E}(P,\varrho/\mu)$, it deduces from Lemma 1 that $\mathrm{sat}(Kx(t-\xi_k)) = N(\vartheta)x(t-\xi_k)$.

According to $\Theta \le 0$, it is easy to derive that

$$\begin{aligned} V(t_k) &= x^T(t_k)Px(t_k) \\ &= x^T(t_k^- - \xi_k)N(\vartheta)^T B^T PBN(\vartheta)x(t_k^- - \xi_k) \\ &\quad + 2x^T(t_k^- - \xi_k)N(\vartheta)^T B^T PCx(t_k^-) \\ &\quad + x^T(t_k^-)C^T PCx(t_k^-) \\ &= \eta_2(t_k^-)\Theta\eta_2^T(t_k^-) + \mu_1 V(t_k^- - \xi_k) + \mu_2 V(t_k^-) \\ &\le \mu_1 V(t_k^- - \xi_k) + \mu_2 V(t_k^-) \end{aligned} \tag{12}$$

where $\eta_2(t_k^-) \doteq (x^T(t_k^- - \xi_k) \;\; x^T(t_k^-))$. In this case,

$$\begin{aligned} \Lambda(t_k) &= e^{\lambda(t_k - t_0)}V(t_k) \\ &\le e^{\lambda(t_k - t_0)}(\mu_1 V(t_k^- - \xi_k) + \mu_2 V(t_k^-)) \\ &\le \mu_1 e^{\lambda\xi}\Lambda(t_k^- - \xi_k) + \mu_2\Lambda(t_k^-) \\ &\le \frac{\varrho}{\mu}(\mu_1 e^{\lambda\xi} + \mu_2) \le \varrho. \end{aligned}$$

If (11) is not true, then there exists instant $\acute{t} = \inf\{t \in [t_k, t_{k+1})|\Lambda(t) > \varrho/\mu\}$ such that $\Lambda(\acute{t}) = \varrho/\mu$ and $\Lambda(t) \le \varrho/\mu$ for $t \le \acute{t}$. Since $\Lambda(t_k) < \varrho$, we know that $\acute{t} > t_k$. Furthermore, there exists instant $\grave{t} = \sup\{t \in [t_k, \acute{t}) : \Lambda(t) \le \varrho\}$ such that $\Lambda(\grave{t}) = \varrho$ and $\varrho \le \Lambda(t) \le \varrho/\mu$ for $t \in [\grave{t}, \acute{t}]$. After the same steps in the proof of (9), we have $\Lambda(\acute{t}) < \varrho/\mu$, which yields a contradiction. Hence, (11) holds under the assumption (10). Then we conclude that (7) is true. It can be finally deduced that $V(t) \le \frac{\varrho}{\mu}e^{-\lambda(t-t_0)}$, $t \ge t_0 - r$, i.e.,

$$|x(t, t_0, \varphi)| \le \sqrt{\frac{\varrho}{\mu\lambda_{\min}(P)}}e^{-\frac{1}{2}\lambda(t-t_0)}.$$

Based on the description of Definition 1, system (1) is LUES over the class $\mathcal{F}_{\max}(\beta)$. Moreover, the set $\mathcal{M}(P,\varrho)$ is included in $\mathcal{S}$. ■

Remark 1: The idea of the proof in Theorem 1 is based on the Razumikhin technique [34]. In fact, one may find from the proof of Theorem 1 that $D^+\Lambda(t) \le \sigma\Lambda(t)$ ($\sigma > 0$), when $\Lambda(t+\theta) \le \varrho/\mu \le \Lambda(t)/\mu$, $\theta \in [-r, 0]$. Note that $\sigma > 0$ means that when the behavior of the system diverges, we stabilize the system through impulsive control. Recently, exponential stability/stabilization conditions for saturated discrete-time systems were derived in [30]. However, delayed impulses, based on the saturated structure, were not essentially taken into consideration during the process of stability analysis. Moreover, it requires that the size of input delays should be less than the lower bound of the impulse intervals. In addition, we remove the restrictions imposed on the input delays and impulse intervals.

Remark 2: In Theorem 1, note that $V(t_k)$ is related to previous state $V(t_k - \xi_k)$ and $V(t_k^-)$. If we take impulsive sequences with eventually uniformly bounded impulsive frequencies, i.e., $N(t, t_0)/t \ge \frac{1}{\tau} - \bar{r}, \tau > 0, \bar{r} > 0$, then it can be obtained that

$$\begin{aligned} t - t_0 &= (t - t_k) + (t_k - \xi_k - t_{i^{(m)}}) + (t_{i^{(m)}} - \xi_{i^{(m)}} - t_{i^{(m-1)}}) \\ &\quad + \cdots + (t_{i^{(1)}} - \xi_{i^{(1)}} - t_0) \\ &\quad + (\xi_{i^{(m)}} + \xi_{i^{(m-1)}} + \cdots + \xi_{i^{(1)}}) \end{aligned}$$

where $\{t_{i^{(m)}}\} \in \mathcal{F}, i^{(m)} \in \mathbb{Z}_+$ and $k > i^{(m)} > i^{(m-1)} > \cdots > i^{(1)}$. It should be noticed that, once the state of the solution at the moment $t_k - \xi_k$ can be measured, stabilization can be guaranteed by utilizing the feedback of history information. However, in the presence of delayed impulses, we can not figure out which interval $[t_{i-1}, t_i)$ ($i \in \mathbb{Z}_+$) that $t_k - \xi_k$ lies in. Hence, how to estimate the length of interval $(t_k - \xi_k - t_{i^{(m)}})$ and establish the relation between $V(t_k)$ and $V(t_{i^{(m)}})$ are still tricky problems. The above proposed idea in this paper seems infeasible in this case, which requires further research. Especially, if the solution of system (1) at the impulsive moment only relies on the historical state, i.e., $C = 0$, then (1b) can be replaced by

$$x(t_k) = B\text{sat}(u(t_k^- - \xi_k)) \tag{13}$$

and we have following corollary.

Corollary 1: Given constant scalar ϱ and matrices $H, K \in \mathbb{R}^{m\times n}, L = \text{diag}(l_j) \in \mathbb{R}^{n\times n}$, if there exist $n \times n$ matrix $P > 0$, $n \times n$ diagonal matrix $M > 0$ and positive constants γ, β, δ and $\mu < 1$, such that (3), $\mathcal{E}(P, \varrho/\mu) \subset \mathcal{L}(H)$, $LML < \delta P$, and

$$\begin{bmatrix} -\mu P & J_i^T B^T P \\ \star & -P \end{bmatrix} \le 0, i \in \mathcal{I}[1, 2^m] \tag{14}$$

then system (1a) with impulse (13) is LUES over the class $\mathcal{F}_{\max}(\beta)$.

It should be noticed that Theorem 1 mainly considers impulsive control problem. Next we show another result related to impulsive disturbance problem.

Theorem 2: Given constant scalar ϱ and matrices $H, K \in \mathbb{R}^{m\times n}, L = \text{diag}(l_j) \in \mathbb{R}^{n\times n}$, if there exist $n \times n$ matrix $P > 0$, $n \times n$ diagonal matrix $M > 0$ and positive constants γ, β, δ, μ_1, μ_2 and $\mu > 1$ with $\mu > \mu_2/(1-\mu_1) > 0$, such that (4), $\mathcal{E}(P, \mu\varrho) \subset \mathcal{L}(H)$, $LML < \delta P$, and

$$\begin{bmatrix} A^T P + PA + (\mu\gamma + \dfrac{\ln\mu}{\beta})P & 0 & P \\ \star & (-\gamma+\delta)P & 0 \\ \star & \star & -M \end{bmatrix} < 0 \tag{15}$$

then system (1) is LUES over the class $\mathcal{F}_{\min}(\beta)$. Moreover, $\mathcal{M}(P, \varrho)$ is included in $\mathcal{S}$.

Proof: It follows from $\mu > \mu_2/(1-\mu_1) > 0$ and (15) that exist positive scalars λ, ξ, ς and h such that $\mu_1\mu e^{\lambda\xi} + \mu_2 \le \mu$ and:

$$\Psi_2 = \begin{bmatrix} \Phi_2 & 0 \\ \star & (-\gamma e^{-\lambda\varsigma} + \delta)P \end{bmatrix} < 0 \tag{16}$$

where $\Phi_2 \doteq A^T P + PA + (\lambda + \mu\gamma + \frac{\ln(\mu+h)}{\beta})P + PM^{-1}P$. Define $V(t) = V(t, x(t)) = x^T(t)Px(t)$ and construct an auxiliary function $\Lambda(t) \doteq (e^{\lambda(t-t_0)} \vee 1)V(t), t \ge t_0 - r$. For any $\varphi \in \mathcal{M}(P, \varrho)$, we will prove that

$$\Lambda(t) \le \mu\varrho, \; t \ge t_0 - r. \tag{17}$$

To do this, we first show that

$$\Lambda(t) \le \mu\varrho, \; t \in [t_0 - r, t_1). \tag{18}$$

Obviously $\Lambda(t+\theta) \le \sup_{t_0 - r \le s \le t_0} V(s) \le \varrho < \mu\varrho$ for $t \le t_0$ and $\theta \in [-r, 0]$. Next, we claim that for $t \in [t_0, t_1)$, (18) holds. Or else, then there exists instant $\acute{t} = \inf\{t \in [t_0, t_1)|\Lambda(t) > \mu\varrho\}$ such that $\Lambda(\acute{t}) = \mu\varrho$ and $\Lambda(t) < \mu\varrho$ for $t < \acute{t}$. Since $\Lambda(t_0) = V(t_0) \le \varrho < \mu\varrho$, we know that $\acute{t} > t_0$. Furthermore, there exists instant

$\grave{t} = \sup\{t \in [t_0, \acute{t}) | \grave{t} \leq \varrho\}$ such that $\Lambda(\grave{t}) = \varrho$. Therefore, when $t \in [\grave{t}, \acute{t}]$, $\Lambda(t+\theta) \leq \mu\varrho \leq \mu\Lambda(t)$, $\theta \in [-r, 0]$, it then follows from (16) and the fact $LML < \delta P$ that:

$$\begin{aligned} D^+\Lambda(t) &\leq e^{\lambda(t-t_0)}(\lambda V(t) + D^+V(t)) + \gamma(\mu\Lambda(t) - \Lambda(t - r(t))) \\ &\leq e^{\lambda(t-t_0)}\eta_1(t)\Psi_2\eta_1(t)^T + \sigma\Lambda(t) \\ &\leq \sigma\Lambda(t) < 0 \end{aligned} \tag{19}$$

where $\eta_1(t) \doteq (x^T(t)\ x^T(t-\tau(t))), \sigma \doteq -(\ln(\mu+h))/\beta$. According to (19), we have

$$\Lambda(\acute{t}) < \Lambda(\grave{t}) \tag{20}$$

which contradicts the definition of $\grave{t}$ and $\acute{t}$. Hence, (18) is true. Now, suppose that there exists $k \in \mathbb{Z}_+$ such that

$$\Lambda(t) \leq \mu\varrho,\ t \in [t_0 - r, t_k). \tag{21}$$

First, we claim that $\Lambda(t_k^-) \leq \varrho$. If not, we obtain that $\Lambda(t_k^-) > \varrho$. Here, we have two cases to consider.

Case 1: $\Lambda(t) > \varrho$, $t \in [t_{k-1}, t_k)$.

In this case, $\mu\Lambda(t) \geq \mu\varrho \geq \Lambda(t+\theta)$, $\theta \in [-r, 0]$ and $D^+\Lambda(t) < \sigma\Lambda(t)$. Note that $\mu > 1$ leads to $\sigma < 0$. Since the impulse time sequence $\{t_k\} \in \mathcal{F}_{\min}(\beta)$, it can be concluded that $\Lambda(t_k^-) \leq e^{\sigma(t_k - t_{k-1})}\Lambda(t_{k-1}) < \mu\rho/(\mu+h) < \varrho$. This is a contradiction.

Case 2: There is some $t \in [t_{k-1}, t_k)$ such that $\Lambda(t) \leq \varrho$.

In this case, there exists $\acute{t} = \sup\{t \in [t_{k-1}, t_k) : \Lambda(t) \leq \varrho\}$ such that $\Lambda(\acute{t}) = \varrho$ and $\mu\Lambda(t) \geq \mu\varrho \geq \Lambda(t+\theta)$ for $t \in [\acute{t}, t_k)$, $\theta \in [-r, 0]$. In this case, we have $D^+\Lambda(t) < \sigma\Lambda(t) < 0$ for $t \in [\acute{t}, t_k)$, which yields that $\Lambda(t_k^-) < \Lambda(\acute{t}) = \varrho$. This is also a contradiction.

Hence, we conclude that $\Lambda(t_k^-) < \varrho$. In what follows, we shall prove show that for all $t \in [t_k, t_{k+1})$:

$$\Lambda(t) \leq \mu\varrho. \tag{22}$$

According to $\mathcal{E}(P, \mu\varrho) \subset \mathcal{L}(H)$, for every $x(t) \in \mathcal{E}(P, \mu\varrho)$, it deduces from Lemma 1 that $\mathrm{sat}(Kx(t-\xi_k)) = N(\vartheta)x(t-\xi_k)$. When $t = t_k$, it can be derived that (12) holds. Then,

$$\begin{aligned} \Lambda(t_k) &= e^{\lambda(t_k - t_0)}V(t_k) \\ &\leq e^{\lambda(t_k - t_0)}(\mu_1 V(t_k^- - \xi_k) + \mu_2 V(t_k^-)) \\ &\leq \mu_1 e^{\lambda\xi}\Lambda(t_k^- - \xi_k) + \mu_2\Lambda(t_k^-) \\ &\leq (\mu_1\mu e^{\lambda\xi} + \mu_2)\varrho \leq \mu\varrho. \end{aligned}$$

Next, we show that (22) holds for $t \in (t_k, t_{k+1})$. If not, then there exists $\acute{t} = \inf\{t \in [t_k, t_{k+1}) : \Lambda(t) > \mu\varrho\}$ such that $\Lambda(\acute{t}) = \mu\varrho$ and $\Lambda(t) < \mu\varrho$ for $t < \acute{t}$. Furthermore, there exists $\grave{t} = \sup\{t \in [t_k, \acute{t}) : \Lambda(t) \leq \varrho\}$ such that $\Lambda(\grave{t}) = \varrho$. Similar to the discussion of (20), we have $\Lambda(\acute{t}) < \varrho$, which is a contradiction. Hence, (22) is valid under the assumption (21). Then we derive that (17) is true. Based on the description of Definition 1, system (1) is LUES over the class $\mathcal{F}_{\min}(\beta)$. Moreover, the set $\mathcal{M}(P, \varrho)$ is included in $\mathcal{S}$. ■

Remark 3: In general, the research on impulsive effects can be mainly divided into two categories: unstable continuous dynamics with stabilizing impulses (i.e., impulsive control) and stable continuous dynamics with destabilizing impulses (i.e., impulsive disturbance). From the perspective of impulsive control, Theorem 1 investigated the exponential stabilization problem of system (1). Note that a constraint on the upper bound of the impulsive interval length is imposed, i.e., $t_k \in \mathcal{F}_{\max}(\beta)$. It indicates that to guarantee the stabilization of the system, the interval length of contiguous impulse instants cannot be overlong. In addition, in the case of impulsive disturbance, Theorem 2 investigated exponential stability problem of system (1). To maintain the stability property of the system, a constraint on the lower bound of the impulsive intervals is imposed, i.e., $t_k \in \mathcal{F}_{\min}(\beta)$, which reveals that impulse sequences should not happen so frequently to destroy the stability of the system.

Especially, in the absence of input delays, i.e., $\xi_k = 0$, and considering a special case where $C = I$, (1b) can then be replaced by

$$\Delta x(t_k) = B\mathrm{sat}(u(t_k^-)). \tag{23}$$

As a special case, many applications involving impulses can be modelled by (23), such as multi-agent systems [35], network systems [36] and coupled dynamical systems [37]. In what follows, we apply Theorem 2 to investigate the stability property of system (1) with impulse effects (23).

Corollary 2: Given a constant scalar ϱ and matrices $H, K \in \mathbb{R}^{m\times n}, L = \mathrm{diag}(l_j) \in \mathbb{R}^{n\times n}$, if there exist $n\times n$ matrix $P > 0$, $n\times n$ diagonal matrix $M > 0$ and positive constants $\gamma, \beta, \delta, \mu$ with $\mu > 1$, such that (15), $\mathcal{E}(P, \mu\varrho) \subset \mathcal{L}(H)$, $LML < \delta P$, and

$$\begin{bmatrix} -\mu P & (I + BJ_i)^T P \\ \star & -P \end{bmatrix} \leq 0, i \in \mathcal{I}[1, 2^m]$$

then system (1a) with impulse (23) is LUES over the class $\mathcal{F}_{\min}(\beta)$.

B. Controller Design and Estimation of the Domain of Attraction

In this section, we shall introduce an optimization approach to enlarge the estimation of the domain of attraction $\mathcal{S}$ by designing control gain K and choosing appropriate impulse sequences.

As as mentioned in [33], we take a shape reference set $\Xi = co\{\varpi_1, \varpi_2, \ldots, \varpi_p\}$ into consideration, where $p \in \mathbb{Z}_+$ and $\varpi_1, \varpi_2, \ldots, \varpi_p$ are some given points in $\mathbb{R}^n$. Note that the size of the set $\mathcal{M}(P, \varrho)$ is proportional to the ellipsoid $\mathcal{E}(P, \varrho)$. Hence, we shall take into account $\mathcal{E}(P, \varrho)$ to estimate the size of $\mathcal{S}$. In this case, we aim to design matrices K, H, P, M and choose appropriate impulses sequences $\mathcal{F}$ such that $\alpha\Xi \subset \mathcal{E}(P, \varrho)$ with maximized scalar α. To solve this problem, an optimization problem is formulated as follows:

$$\max_{P,K,H,M,\mathcal{F}} \quad \alpha \tag{24}$$

i) $\alpha\Xi \subset \mathcal{E}(P, \varrho)$

ii) $LML < \delta P$

iii) (3) *or* (15)

iv) (4)

v) $\mathcal{E}(P, g(\mu)\varrho) \subset \mathcal{L}(H)$

vi) $\mu > \begin{cases} \mu_1 + \mu_2, & \mu \in (0, 1) \\ \mu_2/(1 - \mu_1), & \mu > 1 \end{cases}$

where $g(\mu) = \mu$ if $\mu > 1$ and $g(\mu) = 1/\mu$ if $\mu \in (0, 1)$.

Note that v) is bilinear since it contains two unknown decision variables P and H, i.e., that is, it is a bilinear matrix inequality (BMI) problem. This fact makes it difficult to solve the optimization problem (24). To solve this problem, linear matrix inequality (LMI) algorithms are developed by performing a classical linearizing change of variables, which corresponds to the introduction of some auxiliary variables defined as follows.

Let $\eta = \varrho/\alpha^2, \Gamma = KW, Z = HW, W = P^{-1}, G = M^{-1}$, and $e_i = (0,\dots,\underset{i}{1},\dots,0)_{1\times m}$, then we rewrite (24) as follows:

$$\min_{Y,Z,W,M,\mathcal{F}} \quad \eta \tag{25}$$

$$\text{i)} \quad \begin{bmatrix} -\eta & \varsigma_i^T \\ \star & -W \end{bmatrix} \le 0,\ i \in \mathcal{I}[1,p]$$

$$\text{ii)} \quad \begin{bmatrix} -\delta W & WL \\ \star & -G \end{bmatrix} \le 0$$

$$\text{iii)} \quad \begin{bmatrix} \mho_1 & 0 & G \\ \star & (-\gamma+\delta)W & 0 \\ \star & \star & -G \end{bmatrix} < 0$$

$$\text{iv)} \quad \begin{bmatrix} -\mu_1 W & \mho_2 & \Gamma^T D_i B^T + Z^T D_i^- B^T \\ \star & (\epsilon^2-\mu_2)W & 0 \\ \star & \star & -W \end{bmatrix} \le 0$$

$$i \in \mathcal{I}[1,2^m]$$

$$\text{v)} \quad \begin{bmatrix} -\frac{1}{g(\mu)\varrho}W & Z^T e_i^T \\ \star & -1 \end{bmatrix} \le 0,\ i \in \mathcal{I}[1,m]$$

$$\text{vi)} \quad \mu > \begin{cases} \mu_1+\mu_2, & \mu \in (0,1) \\ \mu_2/(1-\mu_1), & \mu > 1 \end{cases}$$

where $\mho_1 = WA^T + AW + (g(\mu)\gamma + \frac{\ln\mu}{\beta})W$, $\mho_2 = \epsilon\Gamma^T D_i B^T + \epsilon Z^T D_i^- B^T$. Note that condition iv) in (25) corresponds to the condition of the theorems by taking $C = \epsilon I, \epsilon \in \mathbb{R}$. In the case of $\epsilon = 0$ or 1, we have Corollaries 1 and 2 (moreover, $\xi_k = 0$), respectively. Without loss of generality, let $\varrho = 1$. Denote the optimal value η as η^*, P as P^*, K as K^* and H as H^*. By solving the above problem (25), LUES is guaranteed in the ellipsoid $\sqrt{1/\eta^*}\mathcal{E}(W^{-1},1)$ with all initial conditions with impulsive gain $K^* = \Gamma W^{-1}$ and auxiliary gain $H^* = ZW^{-1}$.

Remark 4: It can be observed that the optimal value of η^* is not unique since this method is sensitive to the choice of polyhedral set Ξ and parameters $\mu_1,\mu_2,\mu,\beta,\gamma$ and δ. Note that in the condition $LML < \delta P$, a relatively small δ is expected since it describes the performance index of nonlinearity. In addition, an appropriate parameter γ exists in the term $\gamma(\Lambda(t)/\mu - \Lambda(t-\tau(t)))$ is desirable. However, in this paper, we mainly consider the existence of δ and γ. In this case, considering the fixed Ξ and given data (δ,γ,μ), it is expected to find a solution of problem (25) with appropriate parameters μ_1,μ_2 and β Here, take Theorem 1 for example, it is worth noting that the smaller the value of μ_1 and μ_2 which describe the magnitude of impulse control, the greater the optimal value of α, i.e., the larger regions of admissible initial values may be stabilized. However, this idea makes the problem more conservative via permitting less feasible solutions. Moreover, it is unrealistic to take the value of μ_1 and μ_2 as tiny as we desired due to technology and cost constraints. In other words, a larger estimation of $\mathcal{S}$ and lower cost cannot be reachable simultaneously. In this case, a problem arises naturally: how to make a trade-off between μ_1,μ_2,β and η for saving cost and decreasing conservativeness simultaneously during solving problem (25). With this aim, for fixed δ,γ and μ, we propose two schemes:

1) Given β, solve (25) for η, W, Z, Γ, G as well as the maximal μ_1,μ_2;

2) Given μ_1,μ_2, solve (25) for η, W, Z, Γ, G as well as the maximal upper bound of the impulsive interval β.

Remark 5: Recently, [29] studied locally asymptotic stability of time-delay impulsive systems with input saturation. Reference [38] presented some results concerning stabilization of nonlinear time-delay system subject to input saturation via Lyapunov-Krasovskii functional technique. However, the influence of impulsive actions upon the stability region was essentially neglected in their works. Moreover, the implicit connection among impulse action, system structure and the estimate of the stability region, which is crucial to saturation impulsive control, was not specified in their results.

IV. Examples

In this section, numerical simulations are given to show the validity of the proposed results.

Example 1: Consider the following nonlinear time-delay system:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + f(x(t-\tau(t))),\ t \neq t_k \\ \triangle x(t_k) &= B\mathrm{sat}(Kx(t_k^- - \xi_k)) + 0.2x(t_k^-) \end{aligned} \tag{26}$$

where $\tau(t) \in [0,0.1], \xi_k \in [0,0.4], k \in \mathbb{Z}_+$, and

$$A = \begin{pmatrix} 0.5 & 0.4 \\ 0.1 & 0.2 \end{pmatrix},\ B = \begin{pmatrix} 0.5 \\ 0.1 \end{pmatrix},\ f(x) = \begin{pmatrix} 0.1\sin x_1 \\ 0.2\tanh x_2 \end{pmatrix}.$$

It is worth noting that in the absence of impulses, the continuous dynamics of system (26) is diverging (see Fig. 1(a)). In this case, we shall stabilize system (26) with appropriate control gain K and estimate the maximal domain of attraction. Choose reference set $\Xi = co\{\varpi_1, -\varpi_1\}$, where $\varpi_1 = (-0.8, 0.8)^T$, and parameters $\gamma = 1.1, \delta = 0.5, \mu = 0.49, \mu_1 = 0.25$ and $\mu_2 = 0.2$. Then by using the LMI Toolbox in Matlab, some feasible solutions for can be derived from the optimization problem (25). We have the admissible upper bound of $\beta \le 0.2016$, the optimum value $\alpha^* = 6.9881$ and corresponding matrices

$$H^* = (0.2124 \quad 0.1625),\ K^* = (0.2616 \quad 0.1680)$$

$$P^* = \begin{pmatrix} 0.0923 & 0.0515 \\ 0.0515 & 0.2081 \end{pmatrix},\ M = \begin{pmatrix} 2.0615 & 0 \\ 0 & 2.0615 \end{pmatrix}.$$

By Theorem 1, for any bounded impulse input delay ξ, system (26) is LUES within $\mathcal{E}(P^*,1)$ over the class $\mathcal{F}_{\max}(0.2016)$.

In simulations, let impulse interval $\beta = 0.2$. Fig. 1(b) shows the simulation result of system (26) with initial value $\varphi(\theta) = (2.4,0.8)^T, (-2.7,1.6)^T, (-0.7,-1.8)^T, \theta \in [-0.4,0]$, respectively. It depicts that, under saturated impulsive control, the state trajectories starting from the initial state set $\mathcal{E}(P^*,1)$ (the inner ellipsoid) may enter the permissive state set

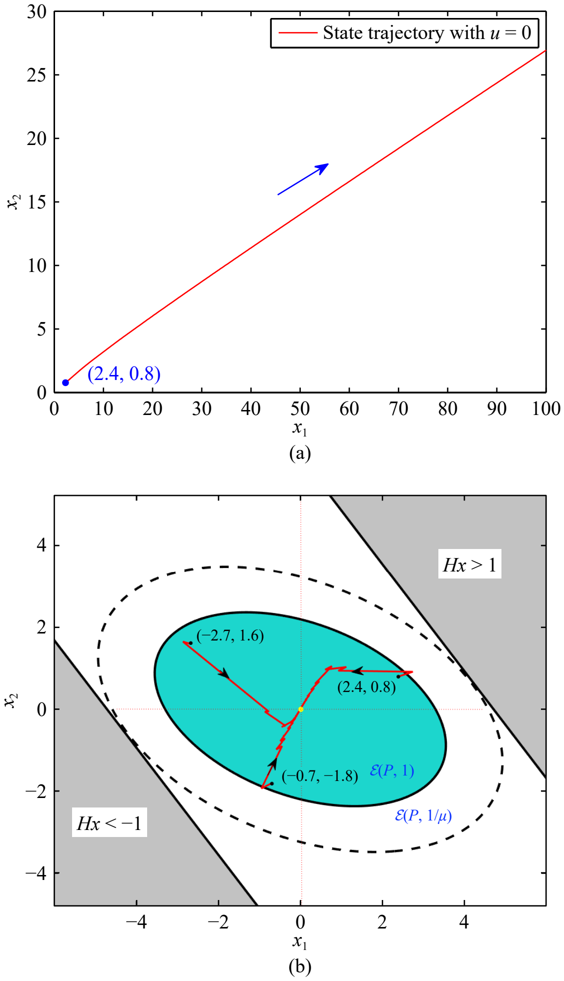

Fig. 1. (a) State responses of system (26) in the absence of stabilizing impulses; (b) State responses of system (26) with saturated stabilizing impulses.

$\mathcal{E}(P^*, 1/u) \subset \mathcal{L}(H)$ (the outer ellipsoid) keeping inside of it and finally converge to the origin.

Under same conditions, consider another case of $\beta = 0.23 > 0.2016$, we shall apply (25) to find the upper bound of μ_1 and μ_2. One can verify that the maximum of μ_1, μ_2, μ is $0.1740, 0.1859$ and 0.3786, respectively. By Theorem 1, for any bounded impulse input delay ξ, system (26) is LUES within $\mathcal{E}(P^{*'}, 1)$ over the class $\mathcal{F}_{\max}(0.23)$ with control gain $K^{*'} = (0.2616 \;\; 0.1680)$, where

$$P^{*'} = \begin{pmatrix} 0.1531 & 0.0262 \\ 0.0262 & 0.1947 \end{pmatrix}.$$

Example 2: Consider another nonlinear system with destabilizing impulses

$$\begin{aligned} &\dot{x}(t) = Ax(t) + 0.1\sin(x(t-\tau)), \; t \neq t_k \\ &\triangle x(t_k) = B\mathrm{sat}(Cx(t_k^- - \xi)), \; k \in \mathbb{Z}_+ \end{aligned} \tag{27}$$

where $\xi, \tau \geq 0$ and

$$A = \begin{pmatrix} -1.8 & 0.3 \\ 0.1 & -1.5 \end{pmatrix}, \; B = \begin{pmatrix} 0.8 \\ 0.8 \end{pmatrix}, \; C = (0.38 \;\; 0.41).$$

Choose parameters $\delta = 0.5$, $\gamma = 1.1$, $\mu = 1.8$, $\mu_1 = 0.32$, $\mu_2 = 1.2$ and reference set $\Xi = co\{\varpi_1, -\varpi_1\}$, where $\varpi_1 = (1.5, -1.5)^T$. In this case, the impulses can be regarded as a perturbation of continuous dynamics, and we shall apply Theorem 2 to choose appropriate impulses sequences such that system (27) is LUES for any bounded impulse input delays ξ. By using LMI Toolbox in Matlab, the obtained allowable maximum length of impulse interval β is 0.9466 for optimization problem (25). By Theorem 2, for any bounded impulse input delay ξ, system (27) is LUES over the class $\mathcal{F}_{\min}(0.9466)$ in the region $\mathcal{E}(P^*, 1)$, where

$$P^* = \begin{pmatrix} 0.0534 & -0.0113 \\ -0.0113 & 0.0603 \end{pmatrix}.$$

In simulations, let $\xi = 1$, $\varsigma = 0.1$ and impulse interval $\beta = 0.9466$, where the simulation of (27) with initial value $\varphi(\theta) = (3, 3.5), \theta \in [0, 1]$ is displayed in Fig. 2. When the continuous dynamics of system (27) is stable (blue curve), it shows that the stability property can be still maintained with certain saturated delayed-impulses disturbance (red curve).

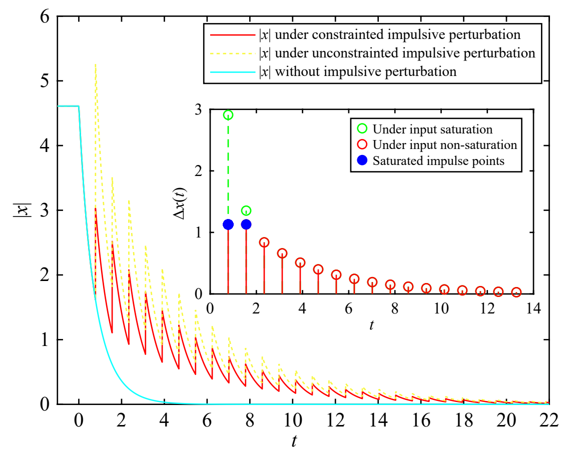

Fig. 2. State trajectory of system (27) under bounded delayed impulsive perturbation.

Example 3: Consider a two-neuron network as follows:

$$\begin{cases} \dot{x}(t) = Ax(t) + M_0 f(x(t)) + M_1 f(x(t-\tau)) \\ x(s) = \varphi(s), \quad s \in [-\tau, 0] \end{cases} \tag{28}$$

where $\tau = 1$ and

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \; M_0 = \begin{bmatrix} 2 & -0.1 \\ -5 & 3 \end{bmatrix}, \; M_1 = \begin{bmatrix} -1.5 & -0.1 \\ -0.2 & -2.5 \end{bmatrix}.$$

It was shown in [39] that system (28) admits chaotic behavior with $f(x(t)) = (\tanh(x_1(t)), \tanh(x_2(t)))^T$ and initial condition $\varphi = (0.4, 0.6)^T$, see Fig. 3(a). Reference [40] pointed out that the chaotic time-delay neural network (28) realizes synchronization under certain stabilizing impulses. In the case that the states are subjected to uncertain input delays, a novel impulsive control strategy was established to guarantee the synchronization of system (28).

Consider the slave system

$$\dot{v}(t) = Av(t) + M_0 f(v(t)) + M_1 f(v(t-1)) \tag{29a}$$

$$v(t_k) = Cu(t_k^- - \xi) + Dv(t_k^-) + (I - D)x(t_k) \tag{29b}$$

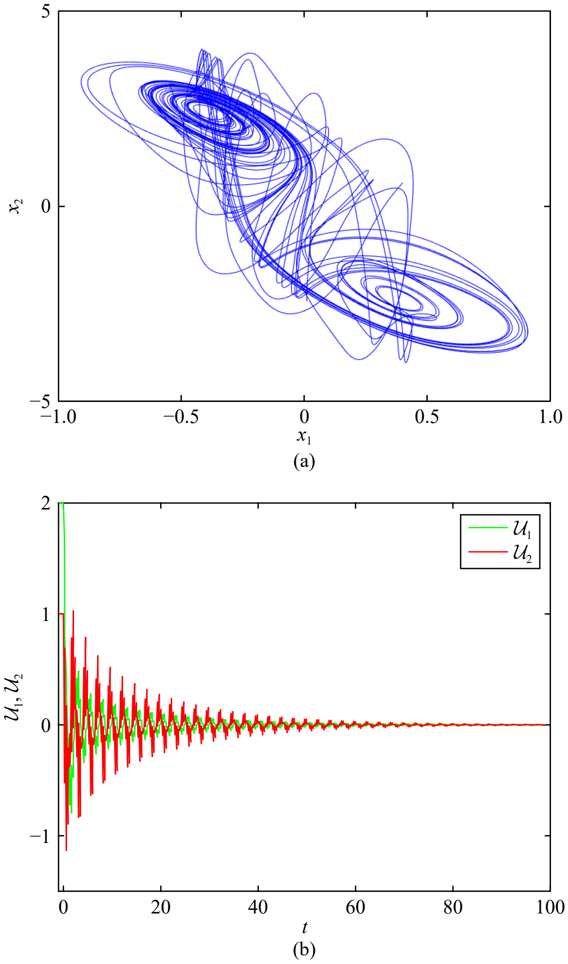

Fig. 3. (a) Chaotic phenomenon of system (28) with initial condition $\varphi = (0.4, 0.6)^T$; (b) State trajectory of error system (31) with initial condition $\varphi = (2,1)^T$ under saturated impulsive control.

$$v(s) = \varphi(s), \quad s \in [-1, 0] \tag{29c}$$

where $k \in \mathbb{Z}_+, \xi = 0.2$ and C, D are two known parameter matrices given by

$$C = \begin{bmatrix} 0.6 \\ 0.75 \end{bmatrix}, \quad D = \begin{bmatrix} 0.13 & 0 \\ 0 & 0.11 \end{bmatrix}.$$

Define synchronization error as $\mathcal{U}(\cdot) := v(\cdot) - x(\cdot)$. Then we have the following error system:

$$\begin{cases} \dot{\mathcal{U}}(t) = A\mathcal{U}(t) + M_0 F(\mathcal{U}(t)) + M_1 F(\mathcal{U}(t-1)) \\ \mathcal{U}(t_k) = Cu(t_k^- - 0.2) + D\mathcal{U}(t_k^-) \\ \mathcal{U}(s) = \varphi(s), \quad s \in [-1, 0] \end{cases} \tag{30}$$

where $k \in \mathbb{Z}_+, u(\cdot) = K\mathcal{U}(\cdot)$. In fact, in view of impulsive saturation, system (30) can be modified as

$$\begin{cases} \dot{\mathcal{U}}(t) = A\mathcal{U}(t) + M_0 F(\mathcal{U}(t)) + M_1 F(\mathcal{U}(t-1)) \\ \mathcal{U}(t_k) = C\text{sat}(u(t_k^- - 0.2)) + D\mathcal{U}(t_k^-),\ k \in \mathbb{Z}_+ \\ \mathcal{U}(s) = \varphi(s), \quad s \in [-1, 0]. \end{cases} \tag{31}$$

Choose parameters $\delta = 0.1, \gamma = 1, \mu = 0.8, \mu_1 = 0.3, \mu_2 = 0.4$ and reference set $\Xi = co\{\varpi_1, -\varpi_1\}$, where $\varpi_1 = (0.2,\ -0.2)^T$. By solving the optimization problem (25), we have the feasible solution $\eta^* = 0.6944$, $\beta \le \beta_{\max} = 0.364$, and gain matrix $K = [0.6489\ 0.5990]$. By Theorem 1 system (28) achieves exponential synchronization under saturated impulsive control over the class $\mathcal{F}_{\max}(0.364)$. In simulations, take the impulse time sequence $\{t_k\} \in \mathcal{F}_{\max}(0.364)$ as follows: $t_k = 0.36k - 0.12$, $k \in \mathbb{Z}_+$. State trajectory of error system (31) with initial condition $\varphi = (2,1)^T$ is depicted in Fig. 3(b).

V. Conclusion

In this paper, LUES of nonlinear systems with saturated delayed impulses have been considered. Our results show that under actuator saturation, the time-delay systems processing destabilizing continuous dynamics become stable by choosing appropriate impulsive time sequences. On the other hand, a nonlinear system subject to input saturation has robust stability with respect to sufficiently small impulsive disturbance. Then *LMI*-based methods have been established for enlarging the estimation of the stability region as well as for control design in a convex optimization problem. Finally, the proposed control method was validated by simulation results. In the future, we will try to extend the obtained results to impulsive sequences satisfying average-type dwell time conditions or those with eventually uniformly bounded impulsive frequencies.

References

[1] G. Stein, “Respect the unstable,” *IEEE Control Systems*, vol. 23, no. 4, pp. 12–25, 2003.

[2] T. Hu and Z. Lin, *Control Systems With Actuator Saturation: Analysis and Design*. Springer Science & Business Media, 2001.

[3] Y.-Y. Cao and Z. Lin, “Stability analysis of discrete-time systems with actuator saturation by a saturation-dependent Lyapunov function,” *Automatica*, vol. 39, no. 7, pp. 1235–1241, 2003.

[4] S. Tarbouriech, G. Garcia, J. M. G. da Silva Jr, and I. Queinnec, *Stability and Stabilization of Linear Systems With Saturating Actuators*. Springer Science & Business Media, 2011.

[5] L. Zaccarian and A. R. Teel, *Modern Anti-Windup Synthesis: Control Augmentation for Actuator Saturation*, vol. 36. Princeton University Press, 2011.

[6] X. You, C. Hua, and X. Guan, “Event-triggered leader-following consensus for nonlinear multiagent systems subject to actuator saturation using dynamic output feedback method,” *IEEE Trans. Automatic Control*, vol. 63, no. 12, pp. 4391–4396, 2018.

[7] H. Min, S. Xu, and Z. Zhang, “Adaptive finite-time stabilization of stochastic nonlinear systems subject to full-state constraints and input saturation,” *IEEE Trans. Automatic Control*, vol. 66, no. 3, pp. 1306–1313, 2020.

[8] S. Ling, H. Wang, and P. X. Liu, “Adaptive fuzzy dynamic surface control of flexible-joint robot systems with input saturation,” *IEEE/CAA J. Autom. Sinica*, vol. 6, no. 1, pp. 97–107, 2019.

[9] B. Wang, W. Chen, and B. Zhang, “Semi-global robust tracking consensus for multi-agent uncertain systems with input saturation via metamorphic low-gain feedback,” *Automatica*, vol. 103, pp. 363–373, 2019.

[10] V. Lakshmikantham, P. S. Simeonov, *et al.*, *Theory of Impulsive Differential Equations*, vol. 6. World scientific, 1989.

[11] W. He, Z. Mo, Q.-L. Han, and F. Qian, “Secure impulsive synchronization in lipschitz-type multi-agent systems subject to deception attacks,” *IEEE/CAA J. Autom. Sinica*, vol. 7, no. 5, pp. 1326–1334, 2020.

[12] Y. Wang, X. Li, and S. Song, “Input-to-state stabilization of nonlinear impulsive delayed systems: An observer-based control approach,” *IEEE/CAA J. Autom. Sinica*, vol. 9, no. 7, pp. 1273–1283, 2022.

[13] B. Liu, M. Yang, T. Liu, and D. J. Hill, “Stabilization to exponential input-to-state stability via aperiodic intermittent control,” *IEEE Trans.*

Automatic Control, vol. 66, no. 6, pp. 2913–2919, 2020.

[14] T. Stamov and I. Stamova, "Design of impulsive controllers and impulsive control strategy for the mittag-leffler stability behavior of fractional gene regulatory networks," *Neurocomputing*, vol. 424, pp. 54–62, 2021.

[15] P. G. Howlett, P. J. Pudney, and X. Vu, "Local energy minimization in optimal train control," *Automatica*, vol. 45, no. 11, pp. 2692–2698, 2009.

[16] T. Wei, X. Li, and V. Stojanovic, "Input-to-state stability of impulsive reaction-diffusion neural networks with infinite distributed delays," *Nonlinear Dynamics*, vol. 103, pp. 1733–1755, 2021.

[17] J. Lu, B. Jiang, and W. X. Zheng, "Potential impacts of delay on stability of impulsive control systems," *IEEE Trans. Automatic Control*, vol. 67, no. 10, pp. 5179–5190, 2022.

[18] W. He, F. Qian, Q.-L. Han, and G. Chen, "Almost sure stability of nonlinear systems under random and impulsive sequential attacks," *IEEE Trans. Automatic Control*, vol. PP, no. 99, pp. 1–1, 2020.

[19] L. Zou, Z. Wang, H. Geng, and X. Liu, "Set-membership filtering subject to impulsive measurement outliers: A recursive algorithm," *IEEE/CAA J. Autom. Sinica*, vol. 8, no. 2, pp. 377–388, 2021.

[20] J. Xu, Z. Zhang, and T. Caraballo, "Non-autonomous nonlocal partial differential equations with delay and memory," *Journal of Differential Equations*, vol. 270, no. 1, 2021.

[21] J. Xu, Z. Zhang, and T. Caraballo, "Mild solutions to time fractional stochastic 2D-stokes equations with bounded and unbounded delay," *Journal of Dynamics and Differential Equations*, 2019.

[22] Z. Xu, X. Li, and V. Stojanovic, "Exponential stability of nonlinear state-dependent delayed impulsive systems with applications," *Nonlinear analysis. Hybrid systems: An Int. Multidisciplinary Journal*, vol. 42, 2021.

[23] W.-H. Chen and W. X. Zheng, "Exponential stability of nonlinear time-delay systems with delayed impulse effects," *Automatica*, vol. 47, no. 5, pp. 1075–1083, 2011.

[24] W.-H. Chen, Z. Ruan, and W. X. Zheng, "Stability and l_2-gain analysis for linear time-delay systems with delayed impulses: An augmentation-based switching impulse approach," *IEEE Trans. Automatic Control*, vol. 64, no. 10, pp. 4209–4216, 2019.

[25] S. Luo, F. Deng, and W.-H. Chen, "Stability and stabilization of linear impulsive systems with large impulse-delays: A stabilizing delay perspective," *Automatica*, vol. 127, p. 109533, 2021.

[26] Z.-W. Liu, G. Wen, X. Yu, Z.-H. Guan, and T. Huang, "Delayed impulsive control for consensus of multiagent systems with switching communication graphs," *IEEE Trans. Cybernetics*, vol. 50, no. 7, pp. 3045–3055, 2019.

[27] L. Li, C. Li, and W. Zhang, "Existence of solution, pulse phenomena and stability criteria for state-dependent impulsive differential equations with saturation," *Communications in Nonlinear Science and Numerical Simulation*, vol. 77, pp. 312–323, 2019.

[28] X. Li and C. Zhu, "Saturated impulsive control of nonlinear systems with applications," *Automatica*, p. 110375, 2022.

[29] H. Li, C. Li, D. Ouyang, and S. K. Nguang, "Impulsive stabilization of nonlinear time-delay system with input saturation via delay-dependent polytopic approach," *IEEE Trans. Systems, Man, and Cybernetics: Systems*, vol. 51, no. 11, pp. 7087–7098, 2020.

[30] L. Li, C. Li, and W. Zhang, "Delayed-impulsive control for difference systems with actuator saturation and its synchronisation application," *IET Control Theory & Applications*, vol. 13, no. 8, pp. 1129–1136, 2019.

[31] X. Liu and K. Zhang, *Impulsive Systems on Hybrid Time Domains*. Springer, 2019.

[32] Q. Wang and X. Liu, "Impulsive stabilization of delay differential systems via the Lyapunov-Razumikhin method," *Applied Mathematics Letters*, vol. 20, no. 8, pp. 839–845, 2007.

[33] T. Hu, Z. Lin, and B. M. Chen, "An analysis and design method for linear systems subject to actuator saturation and disturbance," *Automatica*, vol. 38, no. 2, pp. 351–359, 2002.

[34] Y.-Y. Cao, Z. Lin, and T. Hu, "Stability analysis of linear time-delay systems subject to input saturation," *IEEE Trans. Circuits and Systems I: Fundamental Theory and Applications*, vol. 49, no. 2, pp. 233–240, 2002.

[35] X. Liu, J.-W. Xiao, D. Chen, and Y.-W. Wang, "Dynamic consensus of nonlinear time-delay multi-agent systems with input saturation: an impulsive control algorithm," *Nonlinear Dynamics*, vol. 97, no. 2, pp. 1699–1710, 2019.

[36] K. Chen, C. Li, and L. Li, "Locally exponential stability of discrete-time complex networks with impulsive input saturation," *Int. Journal of Control, Automation and Systems*, vol. 17, no. 4, pp. 948–956, 2019.

[37] H. Li, C. Li, W. Zhang, and Z. Cao, "Exponential stabilization for nonlinear coupled dynamical systems via impulsive and sampled-data control with input constraints," *Int. Journal of Robust and Nonlinear Control*, vol. 29, no. 17, pp. 6126–6144, 2019.

[38] Y. Wei and G.-P. Liu, "Composite control for switched impulsive time-delay systems subject to actuator saturation and multiple disturbances," *Nonlinear Analysis: Hybrid Systems*, vol. 35, p. 100825, 2020.

[39] H. Lu, "Chaotic attractors in delayed neural networks," *Physics Letters A*, vol. 298, no. 2–3, pp. 109–116, 2002.

[40] Z.-H. Guan, Z.-W. Liu, G. Feng, and Y.-W. Wang, "Synchronization of complex dynamical networks with time-varying delays via impulsive distributed control," *IEEE Trans. Circuits and Systems I: Regular Papers*, vol. 57, no. 8, pp. 2182–2195, 2010.

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