Practical Prescribed Time Tracking Control With Bounded Time-Varying Gain Under Non-Vanishing Uncertainties

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***Abstract*—This paper investigates the prescribed-time control (PTC) problem for a class of strict-feedback systems subject to non-vanishing uncertainties. The coexistence of mismatched uncertainties and non-vanishing disturbances makes PTC synthesis nontrivial. In this work, a control method that does not involve infinite time-varying gain is proposed, leading to a practical and global prescribed time tracking control solution for the strict-feedback systems, in spite of both the mismatched and non-vanishing uncertainties. Different from methods based on control switching to avoid the issue of infinite control gain that involves control discontinuity at the switching point, in our method a softening unit is exclusively included to ensure the continuity of the control action. Furthermore, in contrast to most existing prescribed-time control works where the control scheme is only valid on a finite time interval, in this work, the proposed control scheme is valid on the entire time interval. In addition, the prior information on the upper or lower bound of g_i is not in need, enlarging the applicability of the proposed method. Both the theoretical analysis and numerical simulation confirm the effectiveness of the proposed control algorithm.**

***Index Terms*—Adaptive control, prescribed time control (PTC), strict-feedback systems, tracking control.**

I. Introduction

SINCE the original work on prescribed time control (PTC) by Song *et al*. [1] for high order nonlinear systems, PTC has attracted considerable attention from the control community during the past few years [2]–[16]. The salient feature of PTC lies in its ability to achieve closed loop system stability within finite time that is independent of system initial conditions and thus can be pre-specified arbitrarily [1]. As the convergence rate is one of the most important factors for any control system, PTC is of particular interest for time-critical systems, e.g., emergency braking [17], missile interception [18], spaceship docking [19], and so forth. Compared with traditional finite-time control [20]–[28] and fixed time control [29]–[31] methods where the settling time is not at user's disposal, PTC has its superiority, and thus has motivated numerous following up studies and extensions since its introduction, including cooperative prescribed-time control for networked multi-agent systems under local communication condition [2], [3], prescribed-time stabilization for strict feedback-like systems with mismatched uncertainties [4], prescribed-time tracking control for nonlinear multi-input and multi-output (MIMO) systems [5], and the inverse prescribed-time optimality control for stochastic strict-feedback nonlinear systems [6], etc.

The prescribed-time control method is systematically proposed by Song *et al*. [1] upon using infinity time-varying gain that diverges to being unbounded as the time approaches the user-defined terminal time. Then, this methodology is extended to various systems or control problems ([1], [2], [4]–[6]). However, numerical problems during the implementation of the controller may be encountered due to any disturbance/noise that is amplified, rendering the PTC method somewhat unpractical. Recently, there have been some efforts devoted to bounded time-varying gain (BTG) based PTC control methods, such as [32]–[35]. Orlov *et al*. [32], [33] constructed a prescribed-time robust differentiator and observer upon using BTG. In [34], a predefined-time method based on BTG is proposed for arbitrary-order differentiators. However, it is rather difficult to extend the methods [32]–[34] to more general systems with unknown and time-varying control gain. More recently, a novel approach based on BTG is proposed in [35] that naturally links finite time control with prescribed time control, avoiding the numerical implementation problem associated with the current PTC method. Nevertheless, the transition time before the prescribed time still depends on the systems' initial states and parameters, which is the first motivation of this work that integrates BTG into the controller design, allowing it to be implemented without the need for control switching.

Also, it is worth noting that most existing PTC methods based on state or time transformation are invalid beyond the prescribed time interval. Efforts have been made in allowing PTC to be functional beyond the prescribed time. In [7], the

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prescribed-time consensus and containment control problem of multi-agent systems is addressed upon using piecewise continuous time-varying scaling function. This idea has been extended to consensus tracking control of nonlinear multi-agent systems satisfying the time-varying Lipschitz condition in [8]. In addition, [9] addresses the prescribed-time cooperative guidance control problem with input saturation. In the aforementioned works [7]–[9], although the methods are valid on the whole time interval, the switching mechanism is involved. This is the second motivation of the work that establishes a new method allowing the system to continue operating beyond the prescribed time interval.

On the other hand, a precise system model is always hard to obtain under unpredictable internal/external environment, which also brings about unmatched uncertainties in the model. Although prescribed time control methods have been developed for strict-feedback systems with unmatched uncertainties [11], [12], those methods are based on vanishing uncertainties/disturbances. Given the presence of non-vanishing uncertainties, it is hard to ensure that the states of the practical system converge to the point of origin after a prescribed time [11], [12]. In practice, system uncertainties and external disturbances (which are possibly non-vanishing) are inevitable, and it is highly desirable to investigate the control problem for such systems subject to non-vanishing uncertainties with high control precision and fast convergence speed. In fact, there have been a lot of efforts devoted to the control problem for systems subject to the non-vanishing uncertainties [5], [36]–[38]. However, the control results for the systems subject to non-vanishing uncertainties with high control accuracy are scare. Further, it is nontrivial to extend the aforementioned stabilization methods to tracking control for general nonlinear systems with mismatched yet non-vanishing uncertainties. This is mainly because among those infinite time-varying gain based system transformation methods, it is not easy to ensure the boundedness of virtual controllers that involve the differential product of infinite time-varying gain function and non-vanishing terms through the backstepping design process. This is the third motivation in the work, where, instead of using state transformation, we introduce BTG directly into controller design and Lyapunov function based analysis.

The above analysis indicates that although rich results on PTC have been reported during the past few years, at least three major issues have not been adequately addressed: 1) practicality of PTC; 2) accommodation of gains and rejection of mismatched yet non-vanishing uncertainties; and 3) operational capability beyond the settling time. In this paper, we present a method aimed at addressing those issues simultaneously. The main contributions of this paper can be summarized as follow:

1) Different from current PTC which normally relies on time-varying feedback control gain growing limitlessly with time and becoming infinite at equilibrium, the proposed method, with the aid of a time-varying scaling function that grows monotonically with time and maintains boundedness at and beyond the settling time, does not involve infinite control gain anytime during system operation making PTC practical and linking PTC with its practical version analytically;

2) Without using any prior information on the upper or lower bound of system control gain, the proposed control is able to settle the tracking error in the neighborhood of the origin within the prescribed time; and

3) The developed solution is truly global, allowing the system to operate on the entire time interval, yet is able to deal with non-vanishing and mismatched uncertainties, leading to practical and global prescribed time tracking control solution for a larger class of nonlinear systems. In addition, a compensation mechanism based on a computable softening unit is introduced to guarantee the continuity of the control action, which solves the infinite control gain problem without involving discontinuity at the switching point.

Notations: Throughout this paper, t_0 denotes the initial time; $\mathbb{R}$ denotes the set of real numbers; $L_\infty := \{\chi(t)|\chi : \mathbb{R}_+ \to \mathbb{R}, \sup_{t\in\mathbb{R}_+} |\chi(t)| < +\infty\}$. We denote by $\chi^{(i)}$ the ith derivative of χ, and by χ^i the ith power of χ.

II. Problem Formulation and Preliminaries

A. Problem Formulation

In this paper, we consider a class of strict-feedback systems with non-vanishing uncertainties, which is modeled by

$$\begin{cases} \dot{x}_i(t) = g_i(\bar{x}_i,t)x_{i+1}(t) + f_i(\bar{x}_i,t) \\ \qquad i = 1,\ldots,n-1 \\ \dot{x}_n(t) = g_n(\bar{x}_n,t)u(t) + f_n(\bar{x}_n,t) \\ \qquad y = x_1 \end{cases} \tag{1}$$

where $x_i(t) \in \mathbb{R}$ denotes the state with $\bar{x}_i = [x_1,\ldots,x_i]^T$ being the state vector, $u(t) \in \mathbb{R}$ denotes the control input and y is the output, $f_i(\bar{x}_i,t) \in \mathbb{R}$ denotes the lumped uncertainty, which is unknown non-vanishing smooth function, and $g_i \in \mathbb{R}$ is the unknown time-varying control gain. The planar systems (1) are generally employed to describe the dynamics of practical systems such as the "wing-rock" unstable motion of some high-performance aircraft [39], electromechanical system [40], etc.

The objective here is to develop a control strategy for the system (1) such that the output signal $y(t)$ closely synchronizes with $x_r(t)$, i.e., the tracking error $e(t) = y(t) - x_r(t)$ converges to a small residual set containing origin within prescribed time and maintains synchronization thereafter.

To this end, we impose the following assumptions on system model (1) and the desired trajectory $x_r(t)$, respectively.

Assumption 1: The control gain $g_i(t)$ $(i = 1,\ldots,n)$ involved in the systems dynamic model (1) is unknown and time-varying yet bounded away from zero, namely, there exist unknown constants $\underline{g}_i$ and $\overline{g}_i$ such that $0 < \underline{g}_i \le |g_i(t)| \le \overline{g}_i < +\infty$ and the control direction is definite (without loss of generality, we assume that $\mathrm{sgn}(g_i(t)) = 1$).

Remark 1: With Assumption 1, the resultant control scheme becomes more practical and more elegant because it does not require the upper bound or lower bound of the virtual or actual control gain g_i $(i = 1,\ldots,n)$, in contrast to most of existing schemes that normally demand certain prior bound information on the virtual and/or the actual control gains ([1], [16], [37]).

Assumption 2: For the non-vanishing uncertain term $f_i(\bar{x}_i,t)$ $(i=1,\ldots,n)$, there exists an unknown constant $\nu_i>0$ and a known continuously differentiable scalar function $\vartheta_i(\bar{x}_i)$ such that

$$|f_i(\bar{x}_i,t)|\le \nu_i\vartheta_i(\bar{x}_i). \tag{2}$$

In addition, $\vartheta_i(\bar{x}_i)$ is either bounded unconditionally for any $\bar{x}_i$ or bounded only if $\bar{x}_i$ is bounded.

Remark 2: In Assumption 2, $\vartheta_i(\bar{x}_i)$ $(i=1,\ldots,n)$ denotes a computable scalar function carrying "core" information of the system that is independent of system parameters.

Assumption 3: For all $t\in[t_0,+\infty)$, the desired trajectory $x_r(t)\in\mathbb{R}$ and its pth $(p=1,\ldots,n-1)$ order derivatives are known, bounded, and piecewise continuous.

Remark 3: Assumption 3 is a commonly required condition in addressing the tracking control problem (see [5], [13], [37]).

B. The Preliminaries

Inspired by [1], we put forward a time-varying function defined on the whole time interval as follows:

$$\mu(t)=\begin{cases}\left(\dfrac{T^*}{T^*+t_0-t}\right)^q, & t\in[t_0,T^*+t_0-\epsilon)\\[2ex] \left(\dfrac{T^*}{\epsilon}\right)^q, & t\in[T^*+t_0-\epsilon,+\infty)\end{cases} \tag{3}$$

where $q\ge\max\{n,2\}$ is any user-defined real number, T^* is any physically allowable finite time pre-specified by the designer, and ϵ is a small constant satisfying $0<\epsilon<T^*$. It is noted that μ so defined is continuous, monotonically increasing, and bounded for $t\in[t_0,+\infty)$ with $\mu(t_0)=1$, $\mu(t)\to(\frac{T^*}{\epsilon})^q$ as $t\to(T^*+t_0-\epsilon)^-$ and $\mu(t)=(\frac{T^*}{\epsilon})^q$ for all $t\ge T^*+t_0-\epsilon$.

Lemma 1: Let $\psi(t)$ be an unknown bound function and $||\psi||_{[t_0,t]}=\sup_{\tau\in[t_0,t]}|\psi(\tau)|$, and $V(t):[t_0,+\infty)\to\mathbb{R}^+\cup\{0\}$ be a continuously differentiable function, then

1) if

$$\dot{V}(t)\le -k\mu(t)V-\frac{2q}{T^*}\mu(t)^{\frac{1}{q}}V+\frac{\psi(t)}{\mu(t)} \tag{4}$$

for $t\in[t_0,T^*+t_0-\epsilon)$, it holds that

$$V(t)\le\frac{\zeta_1(t)}{\mu(t)^2}V(t_0)+\frac{||\psi||_{[t_0,t]}}{k\mu(t)^2} \tag{5}$$

where k is a finite positive constant, $\zeta_1(t)=\exp\Big(-\frac{kT^*}{q-1}((\frac{T^*}{T^*+t_0-t})^{q-1}-1)\Big)$ is a monotonically decreasing function with $\zeta_1(t_0)=1$ and $\zeta_1(t)\to0$ as $t\to(T^*+t_0-\epsilon)^-$ and $\epsilon\to0$;

2) if

$$\dot{V}(t)\le-\bar{k}\mu V(t)-\frac{2q}{T^*}\mu^{\frac{1}{q}}V(t)+\frac{\psi(t)}{\mu} \tag{6}$$

for $t\in[T^*+t_0-\epsilon,+\infty)$, it holds that

$$V(t)\le\frac{\zeta_2(t)\zeta_1(T^*+t_0-\epsilon)}{\mu^2}V(t_0)+\frac{\zeta_2(t)||\psi||_{[t_0,T^*+t_0-\epsilon)}}{k\mu^2}+\frac{||\psi(t)||_{[T^*+t_0-\epsilon,t]}}{\bar{k}\mu^2} \tag{7}$$

where $\bar{k}$ is a finite positive constant, $\zeta_2(t)=\exp(-\bar{k}(\frac{T^*}{\epsilon})^q(t-T^*-t_0+\epsilon))$ is a monotonically decreasing function with $\zeta_2(T^*+t_0-\epsilon)=1$ and $\zeta_2(t)\to0$ as $t\to+\infty$ and $\epsilon\to0$.

Proof: Solving the differential inequality (4) on $t\in[t_0,T^*+t_0-\epsilon)$ gives that

$$\begin{aligned}V(t)\le{}&\exp\left(\int_{t_0}^t(-k\mu(\tau)-\frac{2q}{T^*}\mu(\tau)^{\frac{1}{q}})d\tau\right)V(t_0)\\&+\int_{t_0}^t\exp\left(\int_\tau^t(-k\mu(s)-\frac{2q}{T^*}\mu(\tau)^{\frac{1}{q}})ds\right)\frac{\psi(\tau)}{\mu(\tau)}d\tau\\\triangleq{}&\phi_1(t)V(t_0)+\phi_2(t)\end{aligned} \tag{8}$$

where $\phi_1(t)=\exp(\int_{t_0}^t(-k\mu(\tau)-\frac{2q}{T^*}\mu(\tau)^{\frac{1}{q}})d\tau)$, and $\phi_2(t)=\int_{t_0}^t\exp(\int_\tau^t(-k\mu(s)-\frac{2q}{T^*}\mu(s)^{\frac{1}{q}})ds)\frac{\psi(\tau)}{\mu(\tau)}d\tau$.

Firstly, by (3), the function $\phi_1(t)$ can be rewritten as

$$\begin{aligned}\phi_1(t)={}&\exp(-k\int_{t_0}^t\mu(\tau)d\tau-\frac{2q}{T^*}\int_{t_0}^t\frac{T^*\,d\tau}{T^*+t_0-\tau})\\={}&\exp(-k\int_{t_0}^t\mu(\tau)d\tau)\exp(-2q\int_{t_0}^t\frac{d\tau}{T^*+t_0-\tau})\\={}&\exp(-\frac{kT^*}{q-1}(\mu(t)^{1-\frac{1}{q}}-1))(\frac{T^*+t_0-t}{T^*})^{2q}\\={}&\mu(t)^{-2}\exp(-\frac{kT^*}{q-1}(\mu(t)^{1-\frac{1}{q}}-1))\end{aligned} \tag{9}$$

which is a monotonically decreasing function.

Next, the function $\phi_2(t)$ is computed as

$$\begin{aligned}\phi_2(t)={}&\int_{t_0}^t\exp(-k\int_\tau^t\mu(s)ds)\\&\times\exp(-2q\int_\tau^t\frac{ds}{T^*+t_0-s})\frac{\psi(\tau)}{\mu(\tau)}d\tau\\={}&\int_{t_0}^t\exp(-k\int_{t_0}^t\mu(s)ds+k\int_{t_0}^\tau\mu(s)ds)\\&\times(\frac{T^*+t_0-t}{T^*+t_0-\tau})^{2q}\frac{\psi(\tau)}{\mu(\tau)}d\tau\\\le{}&\frac{||\psi||_{[t_0,t]}}{\mu(t)^2}\exp(-k\int_{t_0}^t\mu(s)ds)\\&\times\int_{t_0}^t\exp(k\int_{t_0}^\tau\mu(s)ds)\mu(\tau)d\tau\\={}&\frac{||\psi||_{[t_0,t]}}{\mu(t)^2}\exp(-k\int_{t_0}^t\mu(s)ds)\\&\times\int_{t_0}^t\exp(k\int_{t_0}^\tau\mu(s)ds)d(\int_{t_0}^\tau\mu(s)ds)\\={}&\frac{||\psi||_{[t_0,t]}}{\mu(t)^2}\exp(-k\int_{t_0}^t\mu(s)ds)\\&\times\frac{1}{k}\exp(k\int_{t_0}^\tau\mu(s)ds)\Big|_{t_0}^t\\={}&\frac{||\psi||_{[t_0,t]}}{k\mu(t)^2}\left(1-\exp(-\frac{kT^*}{q-1}((\frac{T^*}{T^*+t_0-t})^{q-1}-1))\right)\\\le{}&\frac{||\psi||_{[t_0,t]}}{k\mu(t)^2}\end{aligned} \tag{10}$$

where the relationship that $0\le1-\exp(-\frac{kT^*}{q-1}((\frac{T^*}{T^*+t_0-t})^{q-1}-$

1)) < 1 has been used. By inserting (9) and (10) into (8), we then can derive (5).

Now, we consider the case of $t \in [T^* + t_0 - \epsilon, +\infty)$.

From (5), we get $V(T^* + t_0 - \epsilon) = \lim_{t \to (T^*+t_0-\epsilon)^-} V(t) = 0$ on $t \in [t_0, T^* + t_0 - \epsilon)$ as $\epsilon \to 0$. When $t \in [T^* + t_0 - \epsilon, +\infty)$, it is noted that $\mu = \left(\frac{T^*}{\epsilon}\right)^q$ is constant, and in this case,

$$\dot{V}(t) \le -\bar{k}\mu V - \frac{2q}{T^*}\mu^{\frac{1}{q}} V + \frac{\psi(t)}{\mu}.$$

Obviously,

$$\begin{aligned} V(t) &\le \exp\Big(\big(-\bar{k}\mu - \frac{2q}{T^*}\mu^{\frac{1}{q}}\big)(t - T^* - t_0 + \epsilon)\Big) V(T^* + t_0 - \epsilon) \\ &\quad + \int_{T^*+t_0-\epsilon}^{t} \exp\Big(\big(-\bar{k}\mu - \frac{2q}{T^*}\mu^{\frac{1}{q}}\big)(t-\tau)\Big)\frac{\psi}{\mu} d\tau \\ &\le \exp\Big(\big(-\bar{k}\mu - \frac{2q}{T^*}\mu^{\frac{1}{q}}\big)(t - T^* - t_0 + \epsilon)\Big) \\ &\quad \times V(T^* + t_0 - \epsilon) + \frac{\|\psi(t)\|_{[T^*+t_0-\epsilon, t]}}{\bar{k}\mu^2 + \frac{2q}{T^*}\mu^{\frac{1}{q}+1}} \\ &\quad \times \Big(1 - \exp\Big(\big(-\bar{k}\mu - \frac{2q}{T^*}\mu^{\frac{1}{q}}\big)(t - T^* - t_0 + \epsilon)\Big)\Big) \\ &\le \zeta_2(t) V(T^* + t_0 - \epsilon) + \frac{\|\psi(t)\|_{[T^*+t_0-\epsilon, t]}}{\bar{k}\mu^2}. \end{aligned} \tag{11}$$

By combining (11) with (5), we have

$$\begin{aligned} V(t) &\le \zeta_2(t)\Big(\frac{\zeta_1(T^* + t_0 - \epsilon)}{\mu^2} V(t_0) + \frac{\|\psi\|_{[t_0, t]}}{k\mu^2}\Big) + \frac{\|\psi(t)\|_{[T^*+t_0-\epsilon, t]}}{\bar{k}\mu^2} \\ &= \frac{\zeta_2(t)\zeta_1(T^* + t_0 - \epsilon)}{\mu^2} V(t_0) \\ &\quad + \frac{\zeta_2(t)\|\psi\|_{[t_0, T^*+t_0-\epsilon)}}{k\mu^2} + \frac{\|\psi(t)\|_{[T^*+t_0-\epsilon, t]}}{\bar{k}\mu^2} \end{aligned} \tag{12}$$

which is consistent with (7). By noting that if $\epsilon \to 0$ and $t \to T^* + t_0 - \epsilon$, μ tends to infinity. In such case we can derive that $\lim_{\substack{\epsilon \to 0 \\ t \to (T^*+t_0-\epsilon)^-}} V(t) = \lim_{\substack{\epsilon \to 0 \\ t \to (T^*+t_0-\epsilon)^-}} \left(\frac{\zeta_1(t)}{\mu(t)^2} V(t_0) + \frac{\|\psi\|_{[t_0, t]}}{k\mu(t)^2}\right) = 0$ in (5) and $\lim_{\substack{\epsilon \to 0 \\ t \to (T^*+t_0-\epsilon)^+}} V(t) = \lim_{\substack{\epsilon \to 0 \\ t \to (T^*+t_0-\epsilon)^+}} \left(\frac{\zeta_2(t)\zeta_1(T^*+t_0-\epsilon)}{\mu^2} V(t_0) + \frac{\zeta_2(t)\|\psi\|_{[t_0, T^*+t_0-\epsilon)}}{k\mu^2} + \frac{\|\psi(t)\|_{[T^*+t_0-\epsilon, t]}}{\bar{k}\mu^2}\right) = 0$ in (7). In addition, it further implies (12) that $\lim_{\substack{\epsilon \to 0 \\ t \to +\infty}} V(t) = 0$. ■

Remark 4: Lemma 1 allows us to leverage prescribed time control to its practical (executable) version, making it possible to address the tracking control (not just regulation) problem for strict feedback systems with unknown time varying control gains and mismatched yet non-vanishing uncertainties.

Lemma 2: Let $\phi(t)$ be an unknown bound function and $\|\phi\|_{[t_0, t]} = \sup_{\tau \in [t_0, t]} |\phi(\tau)|$. If a continuously differentiable function $\chi(t)$ satisfies

$$\dot{\chi}(t) \le -k\mu(t)\chi(t) + \mu(t)\phi(t) \tag{13}$$

with $k > 0$, then

1) For $t \in [t_0, T^* + t_0 - \epsilon)$,

$$\chi(t) \le \exp\left(\frac{kT^*}{q-1}\big(1 - \mu(t)^{1-\frac{1}{q}}\big)\right)\chi(t_0) + \frac{\|\phi\|_{[t_0, t]}}{k}.$$

2) For $t \in [T^* + t_0 - \epsilon, +\infty)$,

$$\begin{aligned} \chi(t) &\le \exp\big(\frac{kT^*}{q-1}\big(1 - \mu(T^* + t_0 - \epsilon)^{1-\frac{1}{q}}\big)\big)\chi(t_0) \\ &\quad + \frac{\|\phi\|_{[t_0, T^*+t_0-\epsilon]}}{k} + \frac{\|\phi\|_{[T^*+t_0-\epsilon, t]}}{k\mu^2}. \end{aligned}$$

Proof: 1) The solution of the differential (13) on $t \in [t_0, T^* + t_0 - \epsilon)$ is derived as

$$\begin{aligned} \chi(t) &\le \exp\big(-k\int_{t_0}^{t} \mu(\tau)d\tau\big)\chi(t_0) \\ &\quad + \int_{t_0}^{t} \exp\big(-k\int_{\tau}^{t} \mu(s)ds\big)\mu(\tau)|\phi(\tau)| d\tau \\ &\le \exp\left(\frac{kT^*}{q-1}\big(1 - \mu(t)^{1-\frac{1}{q}}\big)\right)\chi(t_0) \\ &\quad + \|\phi\|_{[t_0, t]} \exp\big(-k\int_{t_0}^{t} \mu(s)ds\big) \\ &\quad \times \int_{t_0}^{t} \exp\big(k\int_{t_0}^{\tau} \mu(s)ds\big) d\big(\int_{t_0}^{\tau} \mu(s)ds\big) \\ &\le \exp\left(\frac{kT^*}{q-1}\big(1 - \mu(t)^{1-\frac{1}{q}}\big)\right)\chi(t_0) + \frac{\|\phi\|_{[t_0, t]}}{k}. \end{aligned} \tag{14}$$

2) Upon using (13) and (3), we can obtain that

$$\dot{\chi}(t) \le -k\mu\chi(t) + \mu\phi(t), \quad t \in [T^* + t_0 - \epsilon, +\infty)$$

which yields

$$\begin{aligned} \chi(t) &\le \exp\big(\int_{T^*+t_0-\epsilon}^{t} -k\mu d\tau\big)\chi(T^* + t_0 - \epsilon) \\ &\quad + \int_{T^*+t_0-\epsilon}^{t} \exp\big(\int_{\tau}^{t} -k\mu ds\big)\mu|\phi(\tau)| d\tau \\ &\le \exp\big(-k\mu \cdot (t - T^* - t_0 + \epsilon)\big)\chi(T^* + t_0 - \epsilon) \\ &\quad + \mu\|\phi\|_{[T^*+t_0-\epsilon, t]} \int_{T^*+t_0-\epsilon}^{t} \exp\big(\int_{\tau}^{t} -k\mu ds\big) d\tau \\ &\le \exp\big(-k\mu \cdot (t - T^* - t_0 + \epsilon)\big) \\ &\quad \times \big(\exp\big(\frac{kT^*}{q-1}\big(1 - \mu(T^* + t_0 - \epsilon)^{1-\frac{1}{q}}\big)\big)\chi(t_0) \\ &\quad + \frac{\|\phi\|_{[t_0, T^*-t_0+\epsilon]}}{k}\big) + \frac{\|\phi\|_{[T^*+t_0-\epsilon, t]}}{k} \\ &\quad \times \big(1 - \exp\big(-k\mu \cdot (t - T^* - t_0 + \epsilon)\big)\big) \\ &\le \exp\big(\frac{kT^*}{q-1}\big(1 - \big(\frac{T^*}{\epsilon}\big)^{q-1}\big)\big)\chi(t_0) \\ &\quad + \frac{1}{k}\big(\|\phi\|_{[t_0, T^*-t_0+\epsilon]} + \|\phi\|_{[T^*-t_0+\epsilon, t]}\big). \end{aligned} \tag{15}$$

It is noted that $\chi(t)$ is bounded from (15). ■

Lemma 3 [13]: Consider the following one-order differential equation:

$$\dot{\chi}_1(t) = -a\chi_1(t) + b\chi_2(t) \tag{16}$$

where $\chi_2(t)$ is a nonnegative function, $a > 0$, $b > 0$. Then, for any given positive initial state $\chi_1(t_0) \ge 0$, the associated solution $\chi_1(t) \ge 0$ holds for $\forall t \ge t_0$.

Lemma 4 [8]: For any given vectors $\eta_1, \eta_2 \in \mathbb{R}^m$, the following inequality (i.e., Young's Inequality) holds:

$$\|\eta_1\|\|\eta_2\| \leq \frac{\mu\gamma\|\eta_1\|^2}{2} + \frac{\|\eta_2\|^2}{2\mu\gamma}$$

where μ is defined as in (3) and $\gamma > 0$ is a user-design constant.

Remark 5: Lemma 2 is crucial in guaranteeing the boundedness of the updated parameter utilized in the proposed adaptive control scheme. With the aid of Lemma 3, such an estimated parameter is also ensured to be non-negative. In developing Lemma 4, we purposely introduce the design variable μ through Young's inequality, which plays an important role in our later stability analysis.

III. Main Result

A. Controller Design

The error surface z_i $(i = 1,\dots,n)$ is introduced as

$$\begin{aligned} z_1(t) &= y(t) - x_r(t) = x_1(t) - x_r(t) \\ z_i(t) &= x_i(t) - \alpha_{i-1}(t), \quad i = 2,\dots,n \end{aligned} \tag{17}$$

where $\alpha_{i-1}(t) \in \mathbb{R}$ is the virtual control. Let $T = T^* + t_0 - \epsilon$. The main procedure is divided into two stages: $t \in [t_0, T)$ and $t \in [T, +\infty)$.

Stage 1: $t \in [t_0, T)$.

Step 1: From the first equation of (1) and the expression of α_1 in (17), we can obtain that

$$\dot{x}_1 = g_1 x_2 + f_1 = g_1 z_2 + g_1 \alpha_1 + f_1. \tag{18}$$

Then, the time derivative of $\frac{1}{2\underline{g}_1} z_1^2$ along (17) is

$$\begin{aligned} \frac{1}{\underline{g}_1} z_1 \dot{z}_1 &= \frac{1}{\underline{g}_1} z_1(\dot{x}_1 - \dot{x}_r) \\ &= \frac{1}{\underline{g}_1} z_1(g_1 z_2 + g_1\alpha_1 + f_1 - \dot{x}_r). \end{aligned} \tag{19}$$

With the help of Lemma 4, it is not difficult to get that

$$\frac{1}{\underline{g}_1} z_1 g_1 z_2 \leq \frac{1}{\underline{g}_1}|z_1||g_1||z_2| \leq \frac{\mu z_1^2 z_2^2}{2} + \frac{\overline{g}_1^2}{2\underline{g}_1^2\mu} \tag{20}$$

$$\frac{1}{\underline{g}_1} z_1 f_1 \leq \frac{1}{\underline{g}_1}|z_1|\nu_1\vartheta_1 \leq \frac{\gamma_1\mu z_1^2\nu_1^2\vartheta_1^2}{2} + \frac{1}{2\gamma_1\underline{g}_1^2\mu} \tag{21}$$

$$-\frac{1}{\underline{g}_1} z_1 \dot{x}_r \leq \frac{1}{\underline{g}_1}|z_1||\dot{x}_r| \leq \frac{r_1\mu|\dot{x}_r|^2 z_1^2}{2} + \frac{1}{2r_1\underline{g}_1^2\mu} \tag{22}$$

where $\gamma_1 > 0$ and $r_1 > 0$ are constants.

By inserting (20)–(22) into (19), it follows that:

$$\begin{aligned} \frac{1}{\underline{g}_1} z_1 \dot{z}_1 \leq{}& \frac{g_1}{\underline{g}_1} z_1\alpha_1 + \frac{\omega_1\gamma_1\psi_1\mu z_1^2}{2} + \frac{r_1\mu\phi_1 z_1^2}{2} \\ &+ \frac{\overline{g}_1^2}{2\underline{g}_1^2\mu} + \frac{1}{2\gamma_1\underline{g}_1^2\mu} + \frac{1}{2r_1\underline{g}_1^2\mu} + \frac{\mu z_1^2 z_2^2}{2} \end{aligned} \tag{23}$$

where $\omega_1 = \max\{\nu_1^2\}$ is the unknown virtual constant, $\psi_1 = \vartheta_1^2$ and $\phi_1 = |\dot{x}_r|^2$ are the computable bounded functions.

The Lyapunov function is chosen as

$$V_1 = \frac{z_1^2}{2\underline{g}_1} + \frac{\widetilde{\omega}_1^2}{2\mu^2} \tag{24}$$

where $\widetilde{\omega}_1 = \omega_1 - \widehat{\omega}_1$ is the parameter estimation error, $\widehat{\omega}_1$ is the estimation value of parameter ω_1 and μ is defined as in (3). Upon using (23), the time derivative of (24) is

$$\begin{aligned} \dot{V}_1 ={}& \frac{1}{\underline{g}_1} z_1\dot{z}_1 + \frac{1}{\mu^2}\widetilde{\omega}_1\dot{\widetilde{\omega}}_1 - \frac{\dot{\mu}}{\mu^3}\widetilde{\omega}_1^2 \\ \leq{}& \frac{g_1}{\underline{g}_1} z_1\alpha_1 + \frac{\omega_1\gamma_1\mu z_1^2\psi_1}{2} + \frac{r_1\mu\phi_1 z_1^2}{2} + \frac{\overline{g}_1^2}{2\underline{g}_1^2\mu} + \frac{\mu z_1^2 z_2^2}{2} \\ &+ \frac{1}{2r_1\underline{g}_1^2\mu} + \frac{1}{2\gamma_1\underline{g}_1^2\mu} - \frac{1}{\mu^2}\widetilde{\omega}_1\dot{\widehat{\omega}}_1 - 2\frac{q}{T^*}\mu^{\frac{1}{q}}\frac{\widetilde{\omega}_1^2}{2\mu^2}. \end{aligned} \tag{25}$$

The virtual control α_1 is designed as

$$\begin{aligned} &\alpha_1(x_1, x_r, \mu, \widehat{\omega}_1) \\ &\quad = -\left(k_{11} + \frac{\gamma_1\widehat{\omega}_1\psi_1}{2} + \frac{r_1\phi_1}{2}\right)\mu z_1 - \frac{k_{12}q^2}{T^{*2}}\mu^{1+\frac{2}{q}} z_1^3 \end{aligned} \tag{26}$$

with the adaptive law

$$\dot{\widehat{\omega}}_1 = -k_{\omega_1}\mu\widehat{\omega}_1 + \frac{\gamma_1\mu^3}{2}\psi_1 z_1^2, \quad \widehat{\omega}_1(t_0) > 0 \tag{27}$$

where $k_{11} > 0$, $k_{12} \geq \frac{1}{2}$, and k_{ω_1} is designed as $k_{\omega_1} = \overline{k}_{\omega_1} + 2\frac{q}{T^*}(\frac{T^*}{\epsilon})^{1-q}$ with $\overline{k}_{\omega_1} > 0$. From Lemma 3, it is confirmed that $\widehat{\omega}_1 \geq 0$.

Noting the term "$-\frac{1}{\mu^2}\widetilde{\omega}_1\dot{\widehat{\omega}}_1$" in (25) with $\dot{\widehat{\omega}}_1$ in (27), and $\widetilde{\omega}_1\widehat{\omega}_1 = \widetilde{\omega}_1(\omega_1 - \widetilde{\omega}_1) = -\widetilde{\omega}_1^2 + \widetilde{\omega}_1\omega_1$, it is derived that

$$\begin{aligned} -\frac{1}{\mu^2}\widetilde{\omega}_1\dot{\widehat{\omega}}_1 &= -\frac{1}{\mu^2}\widetilde{\omega}_1\left(-k_{\omega_1}\mu\widehat{\omega}_1 + \frac{\gamma_1\mu^3}{2}\psi_1 z_1^2\right) \\ &= -\frac{k_{\omega_1}\mu}{\mu^2}\widetilde{\omega}_1^2 + \frac{k_{\omega_1}\mu}{\mu^2}\widetilde{\omega}_1\omega_1 - \frac{1}{2}\mu\gamma_1\widetilde{\omega}_1\psi_1 z_1^2 \\ &\leq -\frac{k_{\omega_1}\mu}{\mu^2}\widetilde{\omega}_1^2 + \frac{k_{\omega_1}\mu}{\mu^2}\left(\frac{\widetilde{\omega}_1^2}{2} + \frac{\omega_1^2}{2}\right) - \frac{1}{2}\mu\gamma_1\widetilde{\omega}_1\psi_1 z_1^2 \\ &= -\frac{k_{\omega_1}\mu}{2\mu^2}\widetilde{\omega}_1^2 + \frac{k_{\omega_1}\omega_1^2}{2\mu} - \frac{1}{2}\mu\gamma_1\widetilde{\omega}_1\psi_1 z_1^2. \end{aligned} \tag{28}$$

By inserting (26) into (25), such term as $-\frac{k_{12}q^2}{T^{*2}}\frac{g_1}{\underline{g}_1}\mu^{1+\frac{2}{q}}z_1^4$ would occur. Adding and subtracting $\frac{k_{12}}{\underline{g}_1^2\mu}$ simultaneously, and using the Young's inequality in the opposite direction, namely, $-(\chi_1^2 + \chi_2^2) \leq -2\chi_1\chi_2$, it holds that

$$\begin{aligned} -\frac{k_{12}q^2}{T^{*2}}\frac{g_1}{\underline{g}_1}\mu^{1+\frac{2}{q}}z_1^4 &\leq -\frac{k_{12}q^2}{T^{*2}}\mu^{1+\frac{2}{q}}z_1^4 \\ &= -k_{12}\left(\frac{q^2}{T^{*2}}\mu^{1+\frac{2}{q}}z_1^4 + \frac{1}{\underline{g}_1^2\mu}\right) + \frac{k_{12}}{\underline{g}_1^2\mu} \\ &\leq -2k_{12}\left(\frac{q}{T^*}\sqrt{\mu}\mu^{\frac{1}{q}}z_1^2\right)\times\left(\frac{1}{\underline{g}_1\sqrt{\mu}}\right) + \frac{k_{12}}{\underline{g}_1^2\mu} \\ &= -\frac{4k_{12}}{2\underline{g}_1}\frac{q}{T^*}\mu^{\frac{1}{q}}z_1^2 + \frac{k_{12}}{\underline{g}_1^2\mu}. \end{aligned} \tag{29}$$

By combining (28) and (29), (25) can be rewritten as

$$\begin{aligned}\dot{V}_1(t) \le & -k_{11}\mu z_1^2 - 4k_{12}\frac{q}{T^*}\mu^{\frac{1}{q}}\frac{z_1^2}{2\underline{g}_1}\\ &+\frac{\mu z_1^2 z_2^2}{2} - k_{\omega_1}\mu\frac{\widetilde{\omega}_1^2}{2\mu^2} - 2\frac{q}{T^*}\mu^{\frac{1}{q}}\frac{\widetilde{\omega}_1^2}{2\mu^2} + \frac{\Phi_1}{\mu}\end{aligned} \tag{30}$$

where $\Phi_1 = \frac{\overline{g}_1^2}{2\underline{g}_1^2} + \frac{1}{2\gamma_1\underline{g}_1^2} + \frac{1}{2r_1\underline{g}_1^2} + \frac{k_{\omega_1}\omega_1^2}{2} + \frac{k_{12}}{\underline{g}_1^2}$ is an unknown finite positive constant.

Step 2: The time derivative of $z_2 = x_2 - \alpha_1$ is

$$\begin{aligned}\dot{z}_2 &= g_2 x_2 + f_2 - \dot{\alpha}_1\\ &= f_2 + g_2\alpha_2 + g_2 z_3 - \dot{\alpha}_1.\end{aligned} \tag{31}$$

The second Lyapunov function is chosen as

$$V_2 = V_1 + \frac{1}{2\underline{g}_2}z_2^2 + \frac{\widetilde{\omega}_2^2}{2\mu^2} \tag{32}$$

where $\widetilde{\omega}_2 = \omega_2 - \widehat{\omega}_2$ is the parameter estimation error. The derivative of V_2 along (31) is then taken

$$\begin{aligned}\dot{V}_2 = &\dot{V}_1 + \frac{1}{\underline{g}_2}(z_2 f_2 + g_2 z_2 z_3 + g_2 z_2\alpha_2 - z_2\dot{\alpha}_1)\\ &-\frac{1}{\mu^2}\widetilde{\omega}_2\dot{\widehat{\omega}}_2 - 2\mu^{\frac{1}{q}}\frac{\widetilde{\omega}_2^2}{2\mu^2}.\end{aligned} \tag{33}$$

Similar to the analysis procedures (20)−(22), we have

$$\begin{aligned}&\frac{1}{\underline{g}_2}(z_2 f_2 + g_2 z_2 z_3 - z_2\dot{\alpha}_1)\\ &\le \frac{\mu z_2^2 z_3^2}{2} + \frac{\overline{g}_2^2}{2\mu\underline{g}_2^2} + \frac{z_2 f_2}{\underline{g}_2} - \frac{z_2}{\underline{g}_2}\Big(\frac{\partial\alpha_1}{\partial x_1}(g_1 x_2 + f_1)\\ &\quad+\sum_{j=0}^{1}\frac{\partial\alpha_1}{\partial x_r^{(j)}}x_r^{(j+1)} + \frac{\partial\alpha_1}{\partial\mu}\Big(\frac{q}{T^*}\mu^{1+\frac{1}{q}}\Big) + \frac{\partial\alpha_1}{\partial\widehat{\omega}_1}\dot{\widehat{\omega}}_1\Big)\\ &\le \frac{\mu z_2^2 z_3^2}{2} + \frac{\overline{g}_2^2}{2\mu\underline{g}_2^2} + \frac{\gamma_2\omega_2\mu z_2^2\psi_2}{2}\\ &\quad+\frac{r_2\mu\phi_2 z_2^2}{2} + \frac{2}{2\gamma_2\underline{g}_2^2\mu} + \frac{\overline{g}_1^2+4}{2\mu r_2\underline{g}_2^2}\end{aligned} \tag{34}$$

with $\omega_2 = \max\{\nu_1^2,\nu_2^2\}$, $\psi_2 = (\frac{\partial\alpha_1}{\partial x_1}\vartheta_1)^2 + \vartheta_2^2$ and $\phi_2 = (\frac{\partial\alpha_1}{\partial x_1}x_2)^2 + (\frac{q}{T^*}\mu^{1+\frac{1}{q}}\frac{\partial\alpha_1}{\partial\mu})^2 + (\frac{\partial\alpha_1}{\partial\widehat{\omega}_1}\dot{\widehat{\omega}}_1)^2 + \sum_{j=0}^{1}(\frac{\partial\alpha_1}{\partial x_r^{(j)}}x_r^{(j+1)})^2$ being calculable variables, where Lemma 4 and the derivative of α_1 defined as in (26) has been used.

The virtual controller α_2 is designed as

$$\begin{aligned}\alpha_2 = &-(k_{21} + \frac{\gamma_2\widehat{\omega}_2\psi_2}{2} + \frac{r_2\phi_2}{2})\mu z_2\\ &-\frac{k_{22}q^2}{T^{*2}}\mu^{1+\frac{2}{q}}z_2^3 - \frac{\mu z_1^2 z_2}{2}\end{aligned} \tag{35}$$

with the adaptive law

$$\dot{\widehat{\omega}}_2 = -k_{\omega_2}\mu\widehat{\omega}_2 + \frac{1}{2}\gamma_2\mu^3\psi_2 z_2^2,\ \ \widehat{\omega}_2(t_0) > 0 \tag{36}$$

where $\widehat{\omega}_2$ is the estimation value of parameter ω_2, $k_{21} > 0$, $\gamma_2 > 0$, $r_2 > 0$, $k_{22} \ge \frac{1}{2}$, $k_{\omega_2} = \overline{k}_{\omega_2} + 2\frac{q}{T^*}(\frac{T^*}{\epsilon})^{1-\frac{1}{q}}$, $\overline{k}_{\omega_2} > 0$.

By inserting (35) in (33), we get that

$$\begin{aligned}-\frac{g_2}{\underline{g}_2}z_2\alpha_2 \le &-\frac{g_2}{\underline{g}_2}\Big(k_{21}\mu z_2^2 + \frac{k_{22}q^2}{T^{*2}}\mu^{1+\frac{2}{q}}z_2^4\\ &+\frac{\gamma_2\mu\widehat{\omega}_2\psi_2 z_2^2}{2} + \frac{\mu z_1^2 z_2^2}{2} + \frac{r_2\mu\phi_2}{2}z_2^2\Big)\\ \le &-k_{21}\mu z_2^2 - \frac{4k_{22}}{2\underline{g}_2}\frac{q}{T^*}\mu^{\frac{1}{q}}z_2^2 - \frac{r_2\phi_2}{2}\mu z_2^2\\ &-\frac{\gamma_2\mu\widehat{\omega}_2\psi_2 z_2^2}{2} - \frac{\mu z_1^2 z_2^2}{2} + \frac{k_{22}}{\underline{g}_2^2\mu}.\end{aligned} \tag{37}$$

Substituting (34)−(37) into (33), we obtain

$$\begin{aligned}\dot{V}_2 \le &-\sum_{j=1}^{2}(2k_{j1}\underline{g}_j)\mu\frac{z_j^2}{2\underline{g}_j} - \sum_{j=1}^{2}\frac{4k_{j2}}{2\underline{g}_j}\frac{q}{T^*}\mu^{\frac{1}{q}}z_2^2 + \frac{\mu z_2^2 z_3^2}{2}\\ &-\sum_{j=1}^{2}k_{\omega_j}\mu\frac{\widetilde{\omega}_j^2}{2\mu^2} - 2\frac{q}{T^*}\mu^{\frac{1}{q}}\sum_{j=1}^{2}\frac{\widetilde{\omega}_j^2}{2\mu^2} + \frac{\Phi_2}{\mu}\end{aligned} \tag{38}$$

where $\Phi_2 = \sum_{j=1}^{2}\Big(\frac{\overline{g}_j^2}{2\underline{g}_j^2} + \frac{k_{\omega_j}\omega_j^2}{2} + \frac{k_{j2}}{\underline{g}_j^2} + \frac{j}{2\gamma_j\underline{g}_j^2}\Big) + \Big(\frac{1}{2r_1\underline{g}_1^2} + \frac{\overline{g}_1^2+4}{2r_2\underline{g}_2^2}\Big)$.

Step i ($i = 3,\ldots,n$)*:* The ith Lyapunov function is chosen as

$$V_i = V_{i-1} + \frac{1}{2\underline{g}_i}z_i^2 + \frac{\widetilde{\omega}_i}{2\mu^2} \tag{39}$$

where $\widetilde{\omega}_i = \omega_i - \widehat{\omega}_i$ is the parameter estimation error between the ith unknown factor $\omega_i = \max_{1\le j\le i}\{\nu_j^2\}$ and its estimation $\widehat{\omega}_i$. The virtual controller α_i ($i = 3,\ldots,n-1$) and actual controller α_n ($\alpha_n = u$) can be obtained recursively as follows:

$$\begin{aligned}\alpha_i = &-k_{i1}\mu z_i - \frac{k_{i2}q^2}{T^{*2}}\mu^{1+\frac{2}{q}}z_i^3 - \frac{\gamma_i\widehat{\omega}_i\psi_i}{2}\mu z_i\\ &-\frac{\mu z_{i-1}^2 z_i}{2} - \frac{r_i\phi_i}{2}\mu z_i,\quad i = 3,\ldots,n\end{aligned} \tag{40}$$

with the adaptive law

$$\dot{\widehat{\omega}}_i = -k_{\omega_i}\mu\widehat{\omega}_i + \frac{1}{2}\gamma_i\mu^3\psi_i z_i^2 \tag{41}$$

where $\psi_i = \sum_{j=1}^{i-1}(\frac{\partial\alpha_{i-1}}{\partial x_j}\vartheta_j)^2 + \vartheta_i^2$, $\phi_i = \sum_{j=1}^{i-1}\Big((\frac{\partial\alpha_{i-1}}{\partial x_j}\times x_{j+1})^2 + (\frac{\partial\alpha_{i-1}}{\partial\widehat{\omega}_j}\dot{\widehat{\omega}}_j)^2\Big) + \sum_{j=0}^{i-1}(\frac{\partial\alpha_{i-1}}{\partial x_r^{(j)}}x_r^{(j+1)})^2 + (\frac{q}{T^*}\mu^{1+\frac{1}{q}}\frac{\partial\alpha_{i-1}}{\partial\mu})^2$, $k_{i1} > \iota_i > 0$, $k_{i2} \ge \frac{1}{2}$, $\gamma_i > 0$, $r_i > 0$, $k_{\omega_i} = \overline{k}_{\omega_i} + 2\frac{q}{T^*}(\frac{T^*}{\epsilon})^{1-q}$, $\overline{k}_{\omega_i} > 0$, $\widehat{\omega}_i(t_0) > 0$.

By inserting the virtual and actual control inputs (40) as well as the adaptive law (41) into the derivative of the Lyapunov function, we then arrive at

$$\begin{aligned}\dot{V}_m \le &-\sum_{j=1}^{m}(2k_{j1}\underline{g}_j)\mu\frac{z_j^2}{2\underline{g}_j} - \sum_{j=1}^{m}4k_{j2}\frac{q}{T^*}\mu^{\frac{1}{q}}\frac{z_j^2}{2\underline{g}_j} + \frac{\mu z_m^2 z_{m+1}^2}{2}\\ &-\sum_{j=1}^{m}k_{\omega_j}\mu\frac{\widetilde{\omega}_j^2}{2\mu^2} - 2\frac{q}{T^*}\mu^{\frac{1}{q}}\sum_{j=1}^{m}\frac{\widetilde{\omega}_j^2}{2\mu^2} + \frac{\Phi_m}{\mu}\end{aligned}$$

for $m = 3,\ldots,n-1$, where $\Phi_m = \sum_{j=1}^{m}\Big(\frac{\overline{g}_j^2}{2\underline{g}_j^2} + \frac{k_{\omega_j}\omega_j^2}{2} + \frac{k_{j2}}{\underline{g}_j^2} + \frac{j}{2\gamma_j\underline{g}_j^2}\Big) + \Big(\frac{1}{2r_1\underline{g}_1^2} + \sum_{j=2}^{m}\frac{\sum_{k=1}^{m-1}\overline{g}_k^2 + 2\cdot j}{2r_j\underline{g}_j^2}\Big)$, and

$$\dot{V}_n \leq -\sum_{j=1}^{n}(2k_{j1}\underline{g}_j)\mu\frac{z_j^2}{2\underline{g}_j} - \sum_{j=1}^{n}\frac{4k_{j2}}{2\underline{g}_j}\frac{q}{T^*}\mu^{\frac{1}{q}}z_j^2 - \sum_{j=1}^{n}k_{\omega_j}\mu\frac{\widetilde{\omega}_j^2}{2\mu^2} - 2\frac{q}{T^*}\mu^{\frac{1}{q}}\sum_{j=1}^{n}\frac{\widetilde{\omega}_j^2}{2\mu^2} + \frac{\Phi_n}{\mu} \tag{42}$$

where $\Phi_n = \sum_{j=1}^{n-1}\big(\frac{\overline{g}_j^2}{2\underline{g}_j^2}\big) + \sum_{j=1}^{n}\big(\frac{k_{\omega_j}\omega_j^2}{2} + \frac{k_{j2}}{\underline{g}_j^2} + \frac{j}{2\gamma_j\underline{g}_j^2}\big) + \big(\frac{1}{2r_1\underline{g}_1^2} + \sum_{j=2}^{n}\frac{\sum_{k=1}^{n-1}\overline{g}_k^2 + 2\cdot j}{2r_j\underline{g}_j^2}\big)$.

Stage 2: $t \in [T, +\infty)$

Following the similar design procedure as in Stage 1, here we only give the expression form of virtual/actual controllers and adaptive laws.

The virtual controller α_1 is consistent with (26) as in the first step. It is noted that in the conventional design of the backstepping method, repeated derivatives of the virtual controller are involved. In such cases, the function μ defined in (3) is a constant. The derivative of μ is zero so that $\phi_i (i = 2, \ldots, n)$ is not uniform, which affects the design of virtual control strategies. Now, we introduce the following calculable control action softening unit:

$$\delta_i(T) = \Big(\frac{\gamma_i(\widehat{\omega}_i\overline{\psi}_i - \widehat{\omega}_i\psi_i)}{2}\mu z_i + \frac{r_i(\overline{\phi}_i - \phi_i)}{2}\mu z_i\Big)\Big|_{t=T} \tag{43}$$

to remedy the discontinuity of the control input at $t = T$. To this end, the virtual/actual controller laws are designed as

$$\alpha_i = -k_{i1}\mu z_i - \frac{k_{i2}q^2}{T^{*2}}\mu^{1+\frac{2}{q}}z_i^3 - \frac{\gamma_i\widehat{\omega}_i\overline{\psi}_i}{2}\mu z_i - \frac{\mu z_{i-1}^2 z_i}{2} - \frac{r_i\overline{\phi}_i}{2}\mu z_i + \delta_i(T), \quad i = 2, \ldots, n \tag{44}$$

where $\overline{\psi}_i = \sum_{j=1}^{i-1}(\frac{\partial\alpha_{i-1}}{\partial x_j}\vartheta_j)^2 + \vartheta_i^2$, $\overline{\phi}_i = \sum_{j=1}^{i-1}\big((\frac{\partial\alpha_{i-1}}{\partial x_j} \times x_{j+1})^2 + (\frac{\partial\alpha_{i-1}}{\partial\widehat{\omega}_j}\dot{\widehat{\omega}}_j)^2\big) + \sum_{j=0}^{i-1}\big(\frac{\partial\alpha_{i-1}}{\partial x_r^{(j)}}x_r^{(j+1)}\big)^2$, $k_{i1} > \iota_i > 0$, $k_{i2} \geq \frac{1}{2}$, $\gamma_i > 0$, $r_i > 0$, and the actual control input u is equal to α_n, i.e., $u = \alpha_n$.

Correspondingly, the parameter update laws are designed as

$$\dot{\widehat{\omega}}_i = -k_{\omega_i}\mu\widehat{\omega}_i + \frac{1}{2}\gamma_i\mu^3\overline{\psi}_i z_i^2, \quad i = 1, \ldots, n \tag{45}$$

where $k_{\omega_i} = \overline{k}_{\omega_i} + 2\frac{q}{T^*}(\frac{T^*}{\epsilon})^{1-q}$, $\overline{k}_{\omega_i} > 0$, $\widehat{\omega}_i > 0$, and $\widehat{\omega}_i$ is the estimation of $\omega_i = \max_{1\leq j\leq i}\{\nu_j^2\}$.

B. Theoretical Analysis

The main result is given in the following theorem.

Theorem 1: Consider system (1) under Assumptions 1−3. If the control strategies α_i, u and the adaptive laws $\widehat{\omega}_i$ in Section III-A are applied, then the following objectives are achieved.

1) The tracking error is forced to be bounded within the prescribed time T^* ($T^* > T^* - \epsilon$). That is, practical prescribed-time tracking is achieved. More significantly, for all $t \in [t_0, T^* + t_0 - \epsilon)$,

$$|z_i| \leq \frac{1}{\mu}B_{1i} \tag{46}$$

where $B_{1i} = \sqrt{\zeta_1(t)\sum_{j=1}^{n}(\frac{g_i}{\underline{g}_j}z_j^2(t_0) + \underline{g}_i\widetilde{\omega}_j^2(t_0))} + \sqrt{\frac{2\underline{g}_i\Phi}{k}}$ with $\{\zeta_1(t) = \exp\big(-\frac{kT^*}{q-1}(\mu(t)^{1-\frac{1}{q}} - 1)\big)\}$.

2) The tracking error is maintained with near-zero accuracy after the prescribed time T^*. Particularly, for $t \in [T^* + t_0 - \epsilon, +\infty)$,

$$|z_i| \leq \frac{1}{\mu}B_{2i} \tag{47}$$

where $B_{2i} = \sqrt{\zeta_2(t)\zeta_1(T^* + t_0 - \epsilon)\sum_{j=1}^{n}\big(\frac{g_i}{\underline{g}_j}z_j^2(t_0) + \underline{g}_i\widetilde{\omega}_j^2(t_0)\big)} + \sqrt{2\underline{g}_i\zeta_2(t)\Phi/k} + \sqrt{2\underline{g}_i\overline{\Phi}/\overline{k}}$ with $\zeta_2(t) = \exp\big(-\overline{k}(\frac{T^*}{\epsilon})^q(t - T^* - t_0 + \epsilon)\big)$, from which we see that the tracking is achieved at $T^* + t_0 - \epsilon$, which is less than $T^* + t_0$ with near-zero accuracy and is maintained after that time.

3) All the internal signals, including the virtual control α_i, the control input u and z_i are continuous and remain uniformly bounded for all $t \in [t_0, +\infty)$.

Proof:

Stage 1: $t \in [t_0, T)$

We first prove that the output signal $y(t)$ can track the desired reference trajectory $x_r(t)$ within the prescribed time T^*. The Lyapunov function is chosen as

$$V = \sum_{i=1}^{n}\frac{z_i^2}{2\underline{g}_i} + \sum_{i=1}^{n}\frac{\widetilde{\omega}_i^2}{2\mu^2} \tag{48}$$

whose time derivative along (30), (33), and (42) is

$$\begin{aligned}\dot{V} &\leq -\sum_{i=1}^{n}2k_{i1}\underline{g}_i\mu\frac{z_i^2}{2\underline{g}_i} - \sum_{i=1}^{n}4k_{i2}\frac{q}{T^*}\mu^{\frac{1}{q}}\frac{z_i^2}{2\underline{g}_i} - \sum_{i=1}^{n}k_{\omega_i}\mu\frac{\widetilde{\omega}_i^2}{2\mu^2} - 2\frac{q}{T^*}\mu^{\frac{1}{q}}\sum_{i=1}^{n}\frac{\widetilde{\omega}_i^2}{2\mu^2} + \frac{\Phi}{\mu}\\ &\leq -k\mu V - 2\frac{q}{T^*}\mu^{\frac{1}{q}}V + \frac{\Phi}{\mu}\end{aligned} \tag{49}$$

where $k = \min_{1\leq i\leq n}\{2k_{i1}\underline{g}_i, k_{\omega_i}\}$, $k_{i1} > 0$, $k_{i2} \geq \frac{1}{2}$, $k_{\omega_i} = \overline{k}_{\omega_i} + 2\frac{q}{T^*}(\frac{T^*}{\epsilon})^{1-q}$, $\overline{k}_{\omega_i} > 0$ and $\Phi = \sum_{j=1}^{n-1}(\frac{\overline{g}_j^2}{2\underline{g}_j^2}) + \sum_{j=1}^{n}(\frac{k_{\omega_j}\omega_j^2}{2} + \frac{k_{j2}}{\underline{g}_j^2} + \frac{j}{2\gamma_j\underline{g}_j^2}) + (\frac{1}{2r_1\underline{g}_1^2} + \sum_{j=2}^{n}\frac{\sum_{k=1}^{n-1}\overline{g}_k^2 + 2\cdot j}{2r_j\underline{g}_j^2})$.

According to Lemma 1, we derive from (49) that

$$\begin{aligned}V(t) &\leq \frac{\zeta_1}{\mu(t)^2}V(t_0) + \frac{\Phi}{k\mu(t)^2}\\ &= \frac{\zeta_1}{\mu(t)^2}\Big(\sum_{j=1}^{n}\frac{z_j^2(t_0)}{2\underline{g}_j} + \sum_{j=1}^{n}\frac{\omega_j^2(t_0)}{2}\Big) + \frac{\Phi}{k\mu(t)^2}.\end{aligned} \tag{50}$$

In light of (48), we can obtain that

$$|z_i| \leq \sqrt{2\underline{g}_i V(t)}, \quad (i = 1, \ldots, n) \tag{51}$$

and

$$|\widetilde{\omega}_i| \leq \sqrt{2\mu^2 V(t)}, \quad (i = 1, \ldots, n). \tag{52}$$

By (50)−(52), it is not difficult to derive (46) and $|\widetilde{\omega}_i| \leq \sqrt{\zeta_1(t)\sum_{j=1}^{n}(z_j^2(t_0)/\underline{g}_j + \omega_j^2(t_0))} + \sqrt{2\Phi/k}$, both of which guarantees the boundedness of z_i and $\widetilde{\omega}_i$.

Under Assumptions 2 and 3, we have $\psi_1 = \vartheta_1^2 \in L_\infty$, $\phi_1 =$

$|\dot{x}_r|^2 \in L_\infty$. By recalling that the function μ in (3) is bounded on the whole time interval $[t_0, +\infty)$, it can be seen that the term "μz_1" in (46) is bounded on $[t_0, +\infty)$. Using Lemma 2, we can conclude that $\widehat{\omega}_1 \in L_\infty$ is ensured to be bounded.

By recalling that $0 < \mu^{\frac{2}{q}-2} \leq 1$ $(q \geq \max\{n, 2\})$, (46), and $0 < \zeta_1(t) \leq 1$, we have

$$
\begin{aligned}
&|\alpha_1(x_1, x_r, \mu, \widehat{\omega}_1)| \\
&\quad \leq (k_{11} + \frac{\gamma_1 \widehat{\omega}_1 \psi_1}{2} + \frac{r_1 \phi_1}{2})\mu|z_1| + \frac{k_{12} q^2}{T^{*2}} \mu^{\frac{2}{q}-2} (\mu|z_1|)^3 \\
&\quad \leq (k_{11} + \frac{\gamma_1 \widehat{\omega}_1 \psi_1}{2} + \frac{r_1 \phi_1}{2}) B_{11} + \frac{k_{12} q^2}{T^{*2}} B_{11}^3 \in L_\infty.
\end{aligned}
$$

It is noted that ϕ_i and ψ_i $(i = 2, \ldots, n)$ associated with μ in Stage 1 of Section III-A may be large yet bounded and computable. In the controller design, tunable parameters γ_i and r_i are added to allow the computation to develop on schedule. According to Lemma 2, $\widehat{\omega}_i$ is bounded. Note that $0 < \mu^{-2} \leq 1$, we then have

$$
\begin{aligned}
|\alpha_i| \leq\ & (k_{i1} + \frac{\gamma_i \widehat{\omega}_i \psi_i}{2} + \frac{r_i \phi_i}{2} \\
& + \frac{1}{2} z_{i-1}^2)(\mu|z_i|) + \frac{k_{i2} q^2}{T^{*2}} \mu^{\frac{2}{q}-2} (\mu z_i)^3 \\
\leq\ & (k_{i1} + \frac{\gamma_i \widehat{\omega}_i \psi_i}{2} + \frac{r_i \phi_i}{2} + \frac{1}{2} B_{1\,i-1}^2) B_{1i} + \frac{k_{i2} q^2}{T^{*2}} B_{1i}^3 \in L_\infty
\end{aligned}
$$

from which we see that $\alpha_i \in L_\infty$, $(i = 2, \ldots, n)$ after which the control input u is bounded.

Stage 2: $t \in [T, +\infty)$

We prove that the output error remains bounded beyond the finite time T.

The control strategies are proved to be continuous at $t = T$. In light of the definition of the control action softening unit $\delta_i(T)$ in (43), we see that

$$
\begin{aligned}
\alpha_i(T) =\ & -k_{i1}\mu(T) z_i(T) - \frac{k_{i2} q^2}{T^{*2}} \mu(T)^{1+\frac{2}{q}} z_i(T)^3 \\
& - \frac{\gamma_i \widehat{\omega}_i(T) \overline{\psi}_i(T)}{2} \mu(T) z_i(T) \\
& - \frac{\mu(T) z_{i-1}^2(T) z_i(T)}{2} - \frac{r_i \overline{\phi}_i(T)}{2} \mu(T) z_i(T) \\
& + \frac{\gamma_i \widehat{\omega}_i(T)(\overline{\psi}_i(T) - \psi_i(T))}{2} \mu(T) z_i(T) \\
& + \frac{r_i}{2}(\overline{\phi}_i(T) - \phi_i(T)) \mu(T) z_i(T) \\
=\ & -k_{i1}\mu(T) z_i(T) - \frac{k_{i2} q^2}{T^{*2}} \mu(T)^{1+\frac{2}{q}} z_i(T)^3 \\
& - \frac{\gamma_i \widehat{\omega}_i(T) \psi_i(T)}{2} \mu(T) z_i(T) - \frac{\mu(T) z_{i-1}^2(T) z_i(T)}{2} \\
& - \frac{r_i}{2} \phi_i(T) \mu(T) z_i(T) \\
=\ & \lim_{t \to T^-} \alpha_i(t).
\end{aligned} \tag{53}
$$

At the same time, $\alpha_i (i = 1, \ldots, n-1)$ and u can also be ensured to be continuous at the breakpoint $t = T$. It is easily concluded that the control action softening unit $\delta_i(T)$ is a finite constant. Adding such a term to the virtual control α_i to ensure the continuity of the control action does not impact the subsequent time derivative of the virtual control in step $i+1$ because this term is a constant value.

In the following, we analyze the system convergence on $[T, +\infty)$.

The Lyapunov function for the overall system is chosen as in (48). The controller strategies are given as in Stage 2 of Section III-A. By taking similar analysis progresses as in Stage 1 of Section III-A, here we only need to handle the term $\frac{g_j}{\underline{g}_j} z_j \delta_j(T)$. By Young's inequality, we have $\frac{g_j}{\underline{g}_j} z_j \delta_j(T) \leq \frac{\overline{g}_j}{\underline{g}_j} |z_j||\delta_j(T)| \leq \frac{\iota_j \mu z_j^2}{2} + \frac{\overline{g}_j^2 \delta_j^2}{2\iota_j \mu \underline{g}_j^2}$, where $\iota_j (j = 2, \ldots, n)$ is an any given positive constant. The derivative of V is computed as

$$
\begin{aligned}
\dot{V} \leq\ & -\sum_{j=1}^{n} 2(k_{j1} - \iota_j) \underline{g}_j \mu \frac{z_j^2}{2\underline{g}_j} - \sum_{j=1}^{n} 4k_{j2} \frac{q}{T^*} \mu^{\frac{1}{q}} \frac{z_j^2}{2\underline{g}_j} \\
& - \sum_{i=1}^{n} \overline{k}_{\omega_j} \mu \frac{\widetilde{\omega}_j^2}{2\mu^2} - 2\frac{q}{T^*} \mu^{\frac{1}{q}} \sum_{j=1}^{n} \frac{\widetilde{\omega}_j^2}{2\mu^2} + \frac{\overline{\Phi}}{\mu} \\
\leq\ & -\overline{k}\mu V - 2\frac{q}{T^*} \mu^{\frac{1}{q}} V + \frac{\overline{\Phi}}{\mu}
\end{aligned} \tag{54}
$$

where $\overline{k} = \min\{2(k_{j1} - \iota_j)\underline{g}_j, \overline{k}_{\omega_j}\}$, $k_{j1} > \iota_j > 0$, $k_{j2} \geq \frac{1}{2}$ $(j = 1, \ldots, n)$, $\overline{\Phi} = \sum_{j=1}^{n-1} \left(\frac{\overline{g}_j^2}{2\underline{g}_j^2}\right) + \sum_{j=1}^{n} \left(\frac{k_{\omega_j} \varpi_j^2}{2} + \frac{k_{j2}}{\underline{g}_j^2} + \frac{j}{2\gamma_j \underline{g}_j^2}\right) + \sum_{j=2}^{n} \left(\frac{\sum_{k=1}^{n-1} \overline{g}_k^2 + 2 \cdot j - 1}{2r_j \underline{g}_j^2} + \frac{\overline{g}_j^2 \delta_j^2}{2\iota_j \underline{g}_j^2}\right) + \frac{1}{2r_1 \underline{g}_1^2}$.

On the basis of Lemma 1, we have

$$
V(t) \leq \frac{\zeta_2(t)\zeta_1(T)}{\mu^2} V(t_0) + \frac{\zeta_2(t)\Phi}{k\mu^2} + \frac{\overline{\Phi}}{\overline{k}\mu^2}
$$

from which we derive (47) and $|\widetilde{\omega}_i| \leq \sqrt{\zeta_2(t)(\zeta_1(T^* + t_0 - \epsilon) \sum_{i=1}^{n} (z_i^2(t_0)/\underline{g}_i + \widetilde{\omega}_i^2(t_0)))} + \sqrt{2\zeta_2(t)\Phi/k} + \sqrt{2\overline{\Phi}/\overline{k}}$. Therefore, $z_i \in L_\infty$ and $\widehat{\omega}_i \in L_\infty$ for any initial conditions. Especially, $z_i \to 0$ as $t \to +\infty$ and $\epsilon \to 0$. From (47), it is straightforward that "$\mu|z_i| \leq B_{2i}$" is bounded, which provides strong assurance for the boundedness analysis of controllers. Note that ψ_1 and ϕ_1 are bounded, it follows from (26) and Assumption 3 that $\alpha_1 \in L_\infty$. The boundedness of α_i $(i = 2, \ldots, n-1)$ and u are also able to be established by following a similar procedure to the above analysis. ■

Remark 6: In both virtual controller α_i $(i = 1, ..., n-1)$ and the actual controller u in Section III-A, only the finite time-varying gain $\mu(t)$ and finite constant gain are involved. Thus, the proposed control scheme does not involve infinite control gain. However, the control gain g_i $(i = 1, \ldots, n)$ involved in the system dynamic model (1) is supposed to be bounded in Assumption 1 to ensure the controllability of the system, which is commonly needed in any controllable system.

Remark 7: The virtual and actual controllers as given in Section III-A $(\alpha_i, i = 1, \ldots, n)$ are continuous everywhere including "$T = T^* + t_0 - \epsilon$" because it depends on the calculable control action softening unit $(\delta_i(T),\ i = 1, \ldots, n)$. The idea of adding such a softening unit originates from [41] and is verified in (53). This is based on the fact that the derivative of time-varying function (3) is piecewise continuous. With this compensation, the potential discontinuity of the control at the time instance $t = T^* + t_0 - \epsilon$ is avoided gracefully.

Remark 8: Three salient features of the proposed method are worth mentioning: 1) The control scheme is valid on the whole time interval, making it different from existing PTC results (such as [1], [4]–[6], [15]) where the control gains grow unbounded as time approaches the terminal time and, consequently, the control schemes are only valid on $[t_0, T^*)$ rather than on $[t_0, +\infty)$. Although the work by [11] addresses the PTC problem on the whole time interval, it only considers regulation rather than tracking, and furthermore, the nonlinear function "f" is assumed to be known and vanishing; 2) The control scheme does not need either the upper or lower bound information of the virtual or actual control gain g_i ($i = 1, \ldots, n$), while most existing PTC works ([1], [16], [37]) depend on availability of the bound of gains; and 3) different from the existing PTC works ([11], [13], [15]) where the control scheme is discontinuous at the prescribed time T^*, the proposed scheme remains continuous and bounded throughout the whole time interval.

Remark 9: It is noted that although most PTC methods are robust against model uncertainties, [11] is the first adaptive PTC for state regulation of strict feedback systems with known (unit) control gains and vanishing uncertainties, wherein control is switched off and the corresponding adaptive law stops updating at and beyond the terminal time. Whereas the proposed method is a practical robust and adaptive solution capable of dealing with output tracking for systems with unknown time varying gains and mismatched yet nonvanishing uncertainties. It is interesting to note that no priori information on virtual and actual control gains is required in building the control scheme and the feedback gain remains bounded anywhere during system operation without the need to switch off the control action.

IV. Numerical Simulation

To verify the effectiveness of the proposed control method, two simulation examples are given.

A. Example 1 (Application Example)

To compare the performance of the proposed control strategy with that developed by [37], we employ the same model as that in [37] for the numerical simulation

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = g(\cdot)u + f(\cdot) \\ y = x_1 \end{cases} \tag{55}$$

which represents the "wing-rock" unstable motion of some high-performance aircraft [39]. As in [37], for simulation purpose, we still consider $f(\cdot) = 1 + \cos(t)x_1 + 2\sin(2t)x_2 + 2|x_1|x_2 + 3|x_2|x_2 + x_1^3$ and $g(\cdot) = 2 + 0.4\sin(t)$. The desired trajectory is $x_r(t) = \sin(2t)$. Obviously, $g(t)$, $f(t)$, and $x_r(t)$ satisfy Assumptions 1–3, respectively. It is straightforward that $|f(\cdot)| \le \nu\vartheta(x_1, x_2)$ with $\vartheta(x_1, x_2) = 1 + |x_1| + |x_2| + |x_1 x_2| + x_2^2 + |x_1|^3$, which satisfies Assumption 2.

The control scheme is directly taken from Section III-A, where $k_{11} = 1$, $k_{12} = 0.5$, $k_{21} = 0.5$, $k_{22} = 0.5$, $\bar{k}_{\omega_2} = 0.1$, $q = 4$, $\gamma_2 = 0.2$, $r_1 = 0.00001$, $r_2 = 0.0001$, $t_0 = 0$, $T^* = 1.2$ s, $\epsilon = 0.3$, $t_{\text{end}} = 1.5$ s. Then $T^* + t_0 - \epsilon = 0.9$ s and $k_{\omega_2} = \bar{k}_{\omega_2} + 2\frac{q}{T^*}(\frac{T^*}{\epsilon})^{1-q} = 0.2042$. The initial conditions of the system are given by three different conditions $[x_1(0), x_2(0)] = [-0.6, -0.3]$, $[0.3, 0.1]$, $[0.6, 0.4]$, and the initial value of the adaptive law is taken as $\widehat{\omega}(0) = 0.1$.

The simulation results are shown in Fig. 1. From Fig. 1(a), we can see that the system states can track the desired trajectory within the finite time point 0.9s which is less than the prescribed time $T^* = 1.2$ s under different initial condition. In addition, precise tracking can be maintained after the prescribed-time and the system can remain operating beyond T^*, distinguishing itself from the method in [1]. From Fig. 2, it is observed that the control signal is continuous throughout the time interval. From Fig. 3, it is shown that the parameter estimate $\widehat{\omega}$ is also bounded for any initial state.

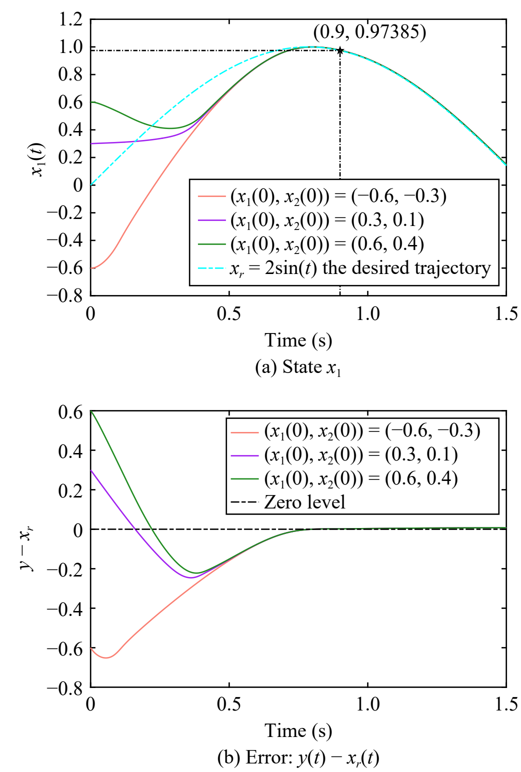


Fig. 1. The performances of state x under different initial conditions.

The simulation results compared with that in [37] (all parameters are consistent) are shown in Figs. 4 and 5. The performance of $x_1(t)$ is shown in Fig. 4, from which we can see that the output x_1 can track the desired trajectory x_r within the prescribed time $T^* = 0.8$ s. The error between state x_1 and x_r is shown in Fig. 5(a) and the error between x_2 and $\dot{x}_r$ is shown in Fig. 5(b). This is the same as that in [37], and further, the precise tracking can be maintained after the prescribed time T^*. From the simulation results in Fig. 5, we can

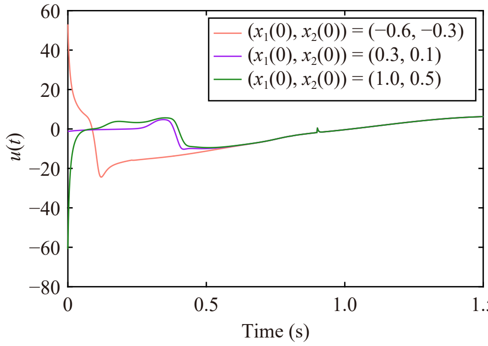


Fig. 2. The practical control input signal of u under different initial conditions.

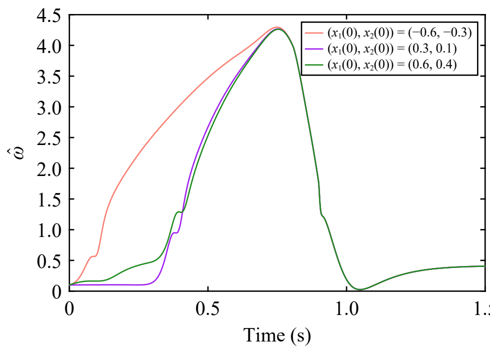


Fig. 3. The performances of $\hat{\omega}$ with different initial conditions.

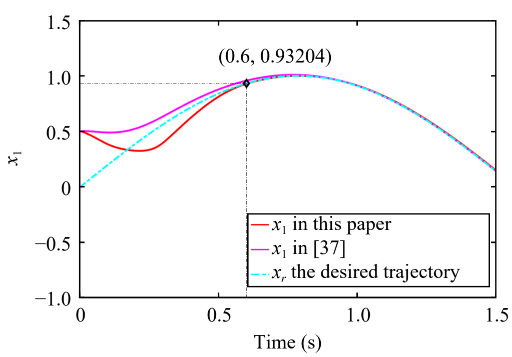


Fig. 4. Compared with [37], the performances of x_1, $([x_1(t_0),\ x_2(t_0)] = [0.5, 0.3]$, $\epsilon = 0.2$, $T^* = 0.8$ s, $t_{\rm end} = 1.5$ s).

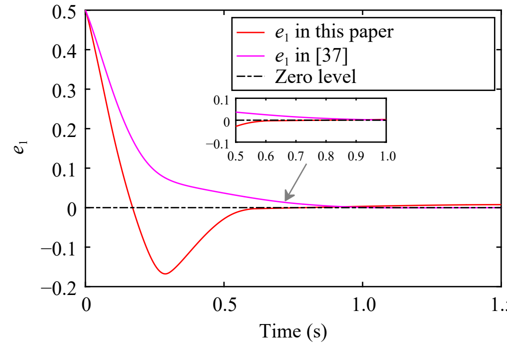
see that an excellent tracking control performance is guaranteed under the proposed control scheme, where the tracking error is on the verge of zero at $t = 0.6$ s before prescribed time T^*, distinguishing itself from that in [37].

B. Example 2 (Numerical Example)

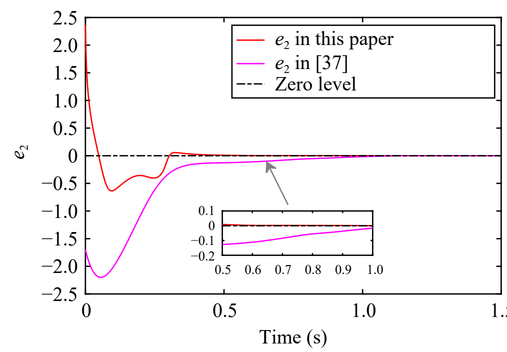
We conduct the simulation on the following strict-feedback nonlinear systems from [42] under Assumptions 1 and 2:

$$\begin{cases} \dot{x}_1 = g_1(x_1)x_2 + f_1(x_1) \\ \dot{x}_2 = g_2(\bar{x}_2)u + f_2(\bar{x}_2) \\ y = x_1 \end{cases} \tag{56}$$

in which $g_1(x_1) = 2 + 0.5\sin(x_1)$, $g_2(\bar{x}_2) = 3 + 0.2\cos(x_1x_2)$, $f_1(\cdot) = x_1^2 + 0.1\cos(0.5x_1)$, $f_2(\cdot) = 0.1x_1x_2 + x_1\exp(-|x_2|) + 0.05\sin(x_1x_2)$. Although these items ($g_1$, g_2, f_1, and f_2) are



(a) Error $e_1 = x_1 - x_r$



(b) Error $e_2 = x_2 - \dot{x}_r$

Fig. 5. The tracking error e under different control schemes.

given explicitly, they are not used in the controller design. Obviously, $\vartheta_1 = x_1^2 + 1$ and $\vartheta_2 = |x_1x_2| + |x_1| + 1$. The desired signal is chosen as $x_r = 2\sin(t)$, which meets Assumption 3. Meanwhile, the design parameters are selected as: $t_0 = 0$, $\epsilon = 0.8$, $T^* = 2.8$ s, $t_{\rm end} = 5$ s, $T^* + t_0 - \epsilon = 2$ s, $q = 3$, $k_{11} = 2$, $k_{12} = 0.5$, $\gamma_1 = 0.5$, $r_1 = 0.0001$, $\bar{k}_{\omega_1} = 0.5$, $k_{\omega_1} = 0.6749$, $k_{21} = 0.5$, $k_{22} = 0.5$, $\gamma_2 = 0.2$, $r_2 = 0.001$, $\bar{k}_{\omega_2} = 0.25$, $k_{\omega_2} = 0.4292$, $[x_1(0), x_2(0)] = [-1, -0.5]$, $[0, 0], [1, 0.5]$. The simulation results are depicted in Figs. 6−9. The state x_1 under different initial conditions is given in Fig. 6 and the tracking error is shown in Fig. 7, from which we see that the system x_1 is capable of tracking the desired trajectory x_r within the finite time 2 s and precise tracking can be maintained after the prescribed time 2.8 s. Obviously, the prescribed time T^* is independent of the initial state and other design parameters of systems. The control signal is shown in Fig. 8, from which we see that the control input signal u is continuous under the proposed control method. The evolution of the adaptive law is presented in Fig. 9, from which we see that the adaptive parameters are bounded.

V. Conclusion

In this paper, a practical prescribed time tracking control method is proposed for a class of strict-feedback systems with mismatched yet non-vanishing uncertainties. The method proposed here, by means of BTG, avoids the numerical problem during the implementation of the controller, which makes PTC practical and bridges PTC and its executable version analytically. It is shown that, without the need for any prior control gain information of system, the tracking error between the

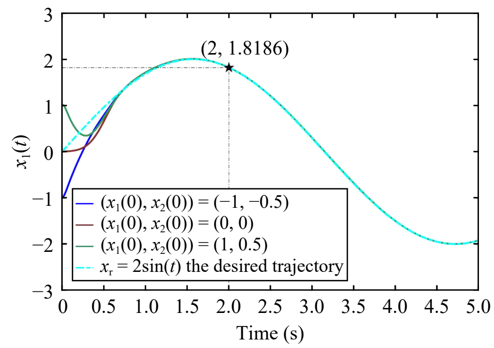


Fig. 6. The performance of x_1 under different initial conditions in Example 2.

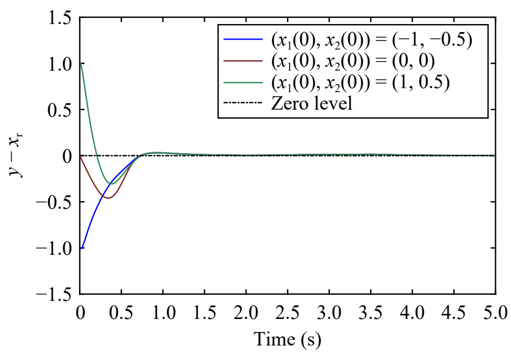


Fig. 7. The tracking error $y - x_r$ under different initial conditions in Example 2.

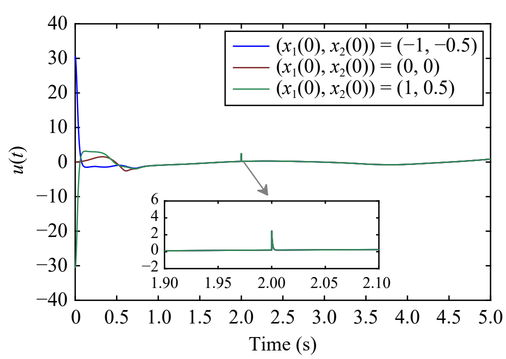
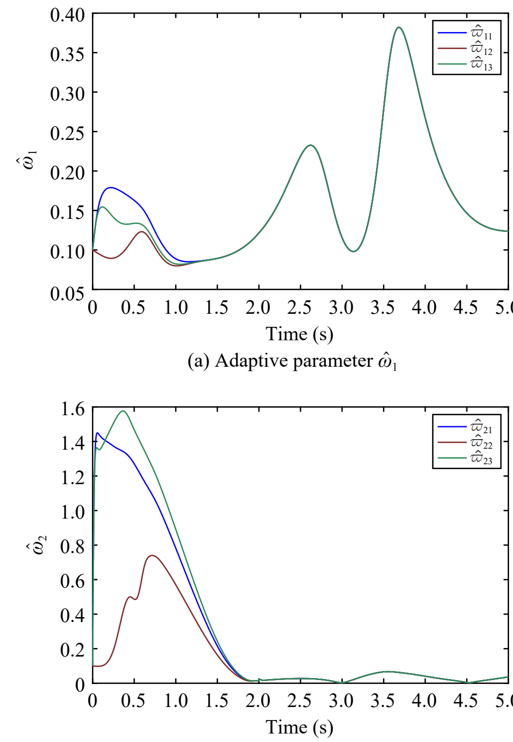


Fig. 8. The performances of u under different initial conditions in Example 2.

output of systems and desired trajectory settles in the neighborhood of origin within the pre-assigned time regardless of the initial condition and other design parameters and particularly, and the neighborhood of origin can be pre-given arbitrarily by simply adjusting the control parameters if the bounds of g are known. The developed solution is truly practical and global, allowing the systems to operate throughout the whole time interval and the states to be started at any initial state. The extension of the proposed PTC method to cooperative control of multi-agent systems under directed topology or to the consensus tracking control of heterogeneous multi-agent systems represents two interesting topics for future research.



(a) Adaptive parameter $\hat{\omega}_1$

(b) Adaptive parameter $\hat{\omega}_2$

Fig. 9. The performances of adaptive parameter $\hat{\omega}$ under different initial conditions.

REFERENCES

[1] Y. D. Song, Y. J. Wang, J. Holloway, and M. Krstic, "Time-varying feedback for regulation of normal-form nonlinear systems in prescribed finite time," *Automatica*, vol. 83, pp. 243–251, Jul. 2017.

[2] Y. J. Wang and Y. D. Song, "Leader-following control of high-order multi-agent systems under directed graphs: Pre-specified finite time approach," *Automatica*, vol. 87, pp. 113–120, Jan. 2018.

[3] B. D. Ning, Q. L. Han, Z. Y. Zuo, L. Ding, Q. Lu, and X. H. Ge, "Fixedtime and prescribed-time consensus control of multiagent systems and its applications: A survey of recent trends and methodologies," *IEEE Trans. Industrial Informatics*, vol. 19, no. 2, pp. 1121–1135, Feb. 2023.

[4] P. Krishnamurthy, F. Khorrami, and M. Krstic, "A dynamic high-gain design for prescribed-time regulation of nonlinear systems," *Automatica*, vol. 115, p. 108860, 2020.

[5] H. F. Ye and Y. D. Song, "Prescribed-time tracking control of MIMO nonlinear systems under non-vanishing uncertainties," *IEEE Trans. Automatic Control*, vol. 68, no. 6, pp. 3664–3671, Jun. 2023.

[6] W. Q. Li and M. Krstic, "Stochastic nonlinear prescribed-time stabilization and inverse optimality," *IEEE Trans. Autom. Control*, vol. 67, no. 3, pp. 1179–1193, Mar. 2022.

[7] Y. J. Wang, Y. D. Song, D. J. Hill, and M. Kristic, "Prescribed-time consensus and containment control of networked multiagent systems," *IEEE Trans. Cyber.*, vol. 49, no. 4, pp. 1138–1147, Apr. 2019.

[8] Y. H. Ren, W. N. Zhou, Z. W. Li, L. Liu, and Y. Q. Sun, "Prescribedtime consensus tracking of multiagent systems with nonlinear dynamics satisfying time-varying lipschitz growth rates," *IEEE Trans. Cyber.*, vol. 53, no. 4, pp. 2097–2109, Apr. 2023.

[9] Z. K. Wang, Y. W. Fang, W. X. Fu, W. H. Ma, and M. G. Wang, "Prescribed-time cooperative guidance law against manoeuvring target with input saturation," *Int. J. Control*, vol. 96, no. 5, pp. 1177–1189, 2023.

[10] Y. H. Ren, W. N. Zhou, Z. W. Li, L. Liu, and Y. Q. Sun, "Prescribedtime cluster lag consensus control for second-order nonlinear leaderfollowing multiagent systems," *ISA Trans.*, vol. 109, pp. 49–60, Mar. 2021.

[11] C. C. Hua, P. Ning, and K. Li, "Adaptive prescribed-time control for a class of uncertain nonlinear systems," *IEEE Trans. Autom. Control*, vol. 67, no. 11, pp. 6159–6166, Nov. 2022.

[12] G. W. Zuo and Y. J. Wang, "Adaptive prescribed finite time control for strict-feedback systems," *IEEE Trans. Autom. Control*, vol. 68, no. 9, pp. 5729–5736, Sept. 2023.

[13] X. Yuan, B. Chen, and C. Lin, "Prescribed finite-time adaptive neural tracking control for nonlinear state-constained systems: Barrier function approach," *IEEE Trans. Neural Networks and Learning Syst.*, vol. 33, no. 12, pp. 7513–7522, Dec. 2022.

[14] Y. Lei, Y. W. Wang, X. K. Liu, and W. Yang, "Prescribed-time stabilization of singularly perturbed systems," *IEEE/CAA J. Autom. Sinica*, vol. 10, no. 2, pp. 569–571, Jan. 2023.

[15] P. Krishnamurthy, F. Khorrami, and M. Krstic, "Robust adaptive prescribed-time stabilization via output feedback for uncertain nonlinear strict-feedback-like systems," *European J. Control*, vol. 55, pp. 14–23, Sept. 2020.

[16] H. F. Ye and Y. D. Song, "Backtepping design embedded with timevarying command filters," *IEEE Trans. Circuits and Systems-II: Express Briefs*, vol. 69, no. 6, pp. 2832–2836, Jun. 2022.

[17] M. Segata and R. Lo Cigno, "Emergency braking: A study of network and application performance," in *Proc. 8th ACM Inter. Workshop on Vehicular Inter-Networking*, 2011, pp. 1–10.

[18] I. S. Jeon, J. I. Lee, and M. J. Tahk, "Impact-time-control guidance law for anti-ship missiles," *IEEE Trans. Control Syst. Technology*, vol. 14, no. 2, pp. 260–266, Feb. 2006.

[19] L. Tian, J. J. Gao, and J. Wang, "Reliability design for rendezvous and docking information stream of Shenzhou spaceship," *Spacecraft Engineering*, vol. 20, no. 6, pp. 50–54, 2011.

[20] V. T. Haimo, "Finite time controller," *SIAM J. Control and Optimization*, vol. 24, no. 4, pp. 760–770, 1986.

[21] Bhat S. P. and D. S. Bernstein, "Finite-time stability of continuous autonomous systems," *SIAM J. Control and Optimization*, vol. 38, no. 3, pp. 751–766, Feb. 2000.

[22] Y. J. Wang, Y. D. Song, and W. Ren, "Distributed adaptive finite-time approach for formation-containment control of networked nonlinear systems under directed topology.," *IEEE Trans. Neural Networks & Learning Syst.*, vol. 29, no. 7, pp. 3164–3175, Jul. 2017.

[23] S. P. Bhat and D. S. Bernstein, "Continuous finite-time stabilization of the translation and rotional double integrator," *IEEE Trans. Autom. Control*, vol. 43, no. 5, pp. 678–682, May 1998.

[24] J. M. Coron and L. Praly, "Adding an integrator for the stabilization problem," *Systems & Control Letters*, vol. 17, no. 2, pp. 89–104, Mar. 1991.

[25] L. Wei and C. Qian, "Adding a power integrator: A tool for global stabilization of high-order lower-triangular systems," *Systems & Control Letters*, vol. 39, no. 5, pp. 339–351, Apr. 2000.

[26] S. P. Bhat and D. S. Bernstein, "Finite-time stability of homogeneous systems," in *Proc. American Control Conf.*, 1997, vol. 43, pp. 2513–2514.

[27] S. Liu, Z. Geng, and J. Sun, "Finite-time attitude control: A finite-time passivity approach," *IEEE/CAA J. Autom. Sinica*, vol. 2, no. 1, pp. 102–108, Jan. 2015.

[28] J. P. Yu, P. Shi, J. P. Liu, and C. Lin, "Neuroadaptive finite-time control for nonlinear MIMO systems with input constraint," *IEEE Trans. Cyber.*, vol. 52, no. 7, pp. 6676–6683, Jul. 2022.

[29] A. Polyakov, "Nonlinear feedback design for fixed-time stabilization of linear control systems," *IEEE Trans. Autom. Control*, vol. 57, no. 8, pp. 2106–2110, Nov. 2012.

[30] B. D. Ning, Z. Y. Zuo, J. Jin, and J. C. Zheng, "Distributed fixed-time coordinated tracking for nonlinear multi-agent systems under directed graphs," *Asian J. Control*, vol. 20, no. 2, pp. 646–658, Mar. 2018.

[31] Y. Liu, H. Y. Li, Z. Y. Zuo, X. D. Li, and R. Q. Lu, "An overview of finite/fixed-time control and its application in engineering systems," *IEEE/CAA J. Autom. Sinica*, vol. 9, no. 12, pp. 2106–2120, Dec. 2022.

[32] Y. Orlov, R. I. V. Kairuz, and L. T. Aguilar, "Prescribed-time robust differentiator design using finite varying gains," *IEEE Control Syst. Letters*, vol. 6, pp. 620–625, Vol. 2022.

[33] R. I. V. Kairuz, Y. Orlov, and L. T. Aguilar, "Robust observer design with prescribed settling-time bound and finite varying gains," *European J. Control*, vol. 68, p. 100667, 2022.

[34] R. Aldana-López, R. Seeber, D. Gómez-Gutiéerrez, M. T. Angulo, and M. Defoort, "A redesign methodology generating predefined-time differentiators with bounded time-varying gains," *Int. J. Robust and Nonlinear Control*, 2022. DOI: 10.1002/rnc.6315.

[35] Y. D. Song and J. Su, "A unified Lyapunov characterization for finite time control and prescribed time control," *Int. J. Robust and Nonlinear Control*, vol. 33, pp. 2930–2949, 2023.

[36] G. W. Zuo and Y. J. Wang, "Asymptotic tracking control for stateconstrained strict-feedback systems with non-vanishing unce-rtainties," *Int. J. Robust and Nonlinear Control*, vol. 32, no. 10, pp. 6017–6034, Mar. 2022.

[37] Y. J. Wang and Y. D. Song, "A general approach to precise tracking of nonlinear systems subject to non-vanishing uncertainties," *Automatica*, vol. 106, pp. 306–314, Aug. 2019.

[38] S. M. Gu, C. J. Qian, and Z. Ni, "Finite-time integral control for a class of nonlinear planar systems with non-vanishing uncertainties," *Automatica*, vol. 136, p. 110016, 2022.

[39] M. M. Monahemi and M. Krstic, "Control of wing rock motion using adaptive feedback linearization," *J. Guidance*, *Control and Dynamics*, vol. 19, no. 4, pp. 905–912, Jul. 1996.

[40] Y. M. Li, S. C. Tong, Y. J. Liu, and T. S. Li, "Adaptive fuzzy robust output feedback control of nonlinear systems with unknown dead zones based on a small-gain approach," *IEEE Trans. Fuzzy Systems*, vol. 22, no. 1, pp. 164–176, Feb. 2014.

[41] Y. D. Song, L. He, and Y. J. Wang, "Globally exponentially stable tracking control of self-restructuring nonlinear systems," *IEEE Trans. Cyber.*, vol. 51, no. 9, pp. 4755–4765, Sept. 2021.

[42] K. Zhao, Y. D. Song, C. L. P. Chen, and L. Chen, "Control of nonlinear systems under dynamic constraints: A unified barrier function-based approach," *Automatica*, vol. 119, p. 109102, 2020.

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