## Letter

# Fixed-Time Consensus-Based Nash Equilibrium Seeking

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Dear Editor,

This letter examines the fixed-time stability of the Nash equilibrium (NE) in non-cooperative games. We propose a consensus-based NE seeking algorithm for situations where players do not have perfect information and communicate via a topology graph. The proposed algorithm can achieve NE in a fixed time that does not depend on initial conditions and can be adjusted in advance. In this strategy, players use their estimates of other players' actions to update their own actions. We present sufficient conditions that ensure the fixed-time stability of the NE through rigorous Lyapunov stability analysis. Finally, we provide an example to verify the feasibility of the theoretical result.

Game theoretic methods have become prevalent in engineering applications, such as power allocation [1], cooperative control [2], [3], energy consumption control [4], and self-driving [5]. Once a specific problem has been modeled as a game, the question becomes how to find the NE. In a game, each player aims to find a strategy that minimizes its own cost function. The NE of a game is a set of actions for which players can no longer decrease their cost functions by solely changing their own actions. In [1], the zero-sum game of two networks of agents was investigated. The potential game and aggregate game were considered in [2] and [4], respectively. For more generalized non-cooperative games, an extreme seeking based method was developed in [6], but it required global information, which may not be applicable to practical problems. To tackle this issue, researchers have paid attention to studying NE seeking strategies under imperfect information. In [7], a gossip-based algorithm was designed for discrete-time NE seeking. In [8], the authors proposed a continuous-time NE seeking algorithm that incorporates a consensus protocol [9]–[14]. A passivity-based approach was developed in [15] for nonlinear and heterogeneous players. The papers [16]–[18] studied NE seeking under disturbance, control input saturation, and switching topologies, respectively.

The convergence rate is an important index for evaluating system performance. While the aforementioned results all achieved NE with an asymptotic convergence rate, where the fastest rate is exponential, the infinite convergence time usually does not meet the requirement of practical systems. To acquire NE more quickly, Fang *et al*. [19] proposed two finite-time NE seeking algorithms that employ signum and saturation functions. However, the convergence time of the finite-time result is related to the initial conditions, which are not always available in practice. To overcome this disadvantage, the authors of [20] proposed a fixed-time NE seeking algorithm based on extreme seeking. The prescribed-time algorithms were developed based on the motion-planning method in [21] and the time base generator in [22].

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Inspired by the fixed-time leader-following consensus protocol in [13], a new algorithm for fixed-time NE seeking under a communication graph is proposed by integrating leader-following consensus and gradient play in this letter. It is not a trivial extension, and difficulties arise from two aspects. Firstly, in the fixed-time leader-following consensus problem, the leader's input is commonly assumed to be bounded. In the consensus-based NE seeking algorithm, the action update law is considered the leader's input. However, to achieve fixed-time NE seeking, the action update law cannot be bounded. Secondly, the traditional quadratic form of the Lyapunov function is not applicable. The nonlinearity of the gradient play exists in the action update law, making the stability analysis more difficult.

The contributions of this letter are summarized as follows. First, the proposed method extends the asymptotic consensus-based NE seeking strategies [8], [16], [18] to achieve a fixed-time convergence result. A new Lyapunov function is designed to prove fixed-time convergence. Moreover, it is a distinct method from that presented in [20]. The explicit form of the settling time is given, which is independent of the initial conditions and only relies on the design parameters, allowing it to be predetermined prior to system operation. Second, in contrast to the NE seeking strategy studied in [22], which steers actions to a neighborhood of the NE with size dependent on the initial conditions, the proposed algorithm in this letter attains the exact NE.

**Preliminaries:** Suppose a network of $N$ players exchanges information via an undirected graph $\mathcal{G}=\{\mathcal{V},\mathcal{E}\}$. The corresponding Laplacian matrix $\mathcal{L}=[l_{ij}]$ is defined as $l_{ij}<0$ if $(j,i)\in\mathcal{E}$, $l_{ij}=0$ otherwise for $i\neq j$, and $l_{ii}=-\sum_{j=1,j\neq i}^{N} l_{ij}$. A more detailed definition of graph theory can be founded in [11]. The dynamic of each player is

$$\dot{x}_i(t)=u_i(t) \tag{1}$$

where $x_i$ and $u_i$ are the action and control input of player $i$, respectively. Each player has a cost function $J_i(\boldsymbol{x})$ that depends on its own action and the actions of the other players, where $\boldsymbol{x}=[x_1,x_2,\dots,x_N]^T$ is the action profile. Denote $\boldsymbol{x}_{-i}=[x_1,\dots,x_{i-1},x_{i+1},\dots,x_N]^T$, the cost function for player $i$ can be expressed as $J_i(\boldsymbol{x})=J_i(x_i,\boldsymbol{x}_{-i})$. The objective of the players in the game is to minimize their own cost functions by adjusting their actions in response to the actions of other players. However, they do not know the actions of other players and only have immediate access to their neighboring players via a graph. This letter aims to design a NE seeking strategy that steers $\boldsymbol{x}$ towards the NE $\boldsymbol{x}^*$, where no player has the incentive to deviate, within a fixed time.

Assumption 1: The players in the game communicate via an undirected and connected graph $\mathcal{G}$.

Assumption 2: The cost function $J_i(\boldsymbol{x})\in C^2$.

Assumption 3: For any $z_1,z_2\in\mathbb{R}^N$, there exists a constant, $m>0$ such that $(z_1-z_2)^T(G(z_1)-G(z_2))\geq m\|z_1-z_2\|^2$ where $G(z)=[\frac{\partial J_1(z)}{\partial x_1}, \frac{\partial J_2(z)}{\partial x_2},\dots,\frac{\partial J_N(z)}{\partial x_N}]^T$.

Assumption 4: Define matrix $H(\boldsymbol{x})=[h_{ij}]$ where $h_{ij}=\frac{\partial^2 J_i(\boldsymbol{x})}{\partial x_i\partial x_j}$. Each element $h_{ij}$ is bounded for any $\boldsymbol{x}\in\mathbb{R}^N$.

Remark 1: Assumption 3 guarantees that $H(\boldsymbol{x})+H^T(\boldsymbol{x})\geq 2mI_N$. Assumption 4 ensures that for any $z_1,z_2\in\mathbb{R}^N$, $|\frac{\partial J_i(z_1)}{\partial x_i}-\frac{\partial J_i(z_2)}{\partial x_i}|\leq l_i\|z_1-z_2\|$, where $l_i$ is the global Lipschitz constant.

Remark 2: Assumptions 3 and 4 guarantee the existence and uniqueness of the NE, and $G(\boldsymbol{x})=\boldsymbol{0}$ if and only if $\boldsymbol{x}=\boldsymbol{x}^*$.

Lemma 1 [11]: Consider the system (1), if there exists a Lyapunov function $V(\boldsymbol{x}(t)):\mathbb{R}^N/\{\boldsymbol{x}^*\}\to\mathbb{R}^+, \boldsymbol{x}^*\to 0$ such that $\dot{V}(\boldsymbol{x}(t))\leq -aV^b(\boldsymbol{x}(t))-cV^d(\boldsymbol{x}(t))$, where $a,c>0$, $d>1$, and $0<b<1$, the NE is fixed-time stable. The settling time is $T_{\max}=\frac{1}{a(1-b)}+\frac{1}{c(d-1)}$.

Lemma 2 [19]: The Laplacian matrix $\mathcal{L}$ associated with an undirected and connected graph is semi-positive definite. 0 is a simple eigenvalue of it, and $\mathbf{1}_N$ is the corresponding eigenvector. Let $B=\text{diag}\{a_{11},a_{12},\dots,a_{1N},\dots,a_{N1},\dots,a_{NN}\}$, then $Q=\mathcal{L}\otimes I_N+B$ is positive definite.

Lemma 3 [9]: For $\chi_1,\chi_2,\dots,\chi_N\in\mathbb{R}$, $(\sum_{i=1}^N|\chi_i|)^b\leq\sum_{i=1}^N|\chi_i|^b\leq N^{1-b}(\sum_{i=1}^N|\chi_i|)^b$ holds for $b\in(0,1]$; and $N^{1-d}(\sum_{i=1}^N|\chi_i|)^d\leq\sum_{i=1}^N|\chi_i|^d\leq(\sum_{i=1}^N|\chi_i|)^d$ holds for $d\in(1,+\infty)$.

Lemma 4 [12]: For any $\phi_1,\phi_2 \in \mathbb{R}$ and $\alpha_1,\alpha_2 \in \mathbb{R}^+$, one has $|\phi_1|^{\alpha_1}|\phi_2|^{\alpha_2} \le \frac{\alpha_1}{\alpha_1+\alpha_2}\epsilon|\phi_1|^{\alpha_1+\alpha_2} + \frac{\alpha_2}{\alpha_1+\alpha_2}\epsilon^{-\frac{\alpha_1}{\alpha_2}}|\phi_2|^{\alpha_1+\alpha_2}$, where $\epsilon$ is a positive real number.

**Main results:** The following NE seeking strategy is designed to achieve the NE in the fixed time:

$$\dot{x}_i = -\eta_1 \mathrm{sig}^2\left(\frac{\partial J_i(\boldsymbol{y}_i)}{\partial x_i}\right) - \eta_2 \mathrm{sig}^{\frac{1}{2}}\left(\frac{\partial J_i(\boldsymbol{y}_i)}{\partial x_i}\right) \tag{2}$$

$$\begin{aligned}\dot{y}_{ij} =& -c_1\mathrm{sig}^3(\xi_{ij}(t)) - c_2\mathrm{sign}(\xi_{ij}(t)) - c_3\xi_{ij}(t) - c_4\mathrm{sig}^2(\xi_{ij}(t))\\ &- c_5\mathrm{sig}^{\frac{1}{2}}(\xi_{ij}(t)) - c_6\mathrm{sig}^{\frac{1}{3}}(\xi_{ij}(t)) - c_7\mathrm{sig}^{\frac{5}{3}}(\xi_{ij}(t))\end{aligned} \tag{3}$$

where $\eta_1,\eta_2,c_i > 0$, $i \in \{1,2,\ldots,7\}$. $y_{ij}$ denotes player $i$'s estimate on player $j$'s action, and $\xi_{ij}(t) = \sum_{m=1}^N a_{im}(y_{ij}(t)-y_{mj}(t)) + a_{ij}(y_{ij}(t)-x_j(t))$. Let $\boldsymbol{y}_i = [y_{i1},y_{i2},\ldots,y_{iN}]^T$ represent player $i$'s estimate on $\boldsymbol{x}$. Define $\boldsymbol{y} = [\boldsymbol{y}_1^T,\boldsymbol{y}_2^T,\ldots,\boldsymbol{y}_N^T]^T$, $\boldsymbol{e}_i = \boldsymbol{y}_i - \boldsymbol{x}$, $\boldsymbol{e} = \boldsymbol{y} - \boldsymbol{1}_N \otimes \boldsymbol{x} = [\boldsymbol{e}_1,\boldsymbol{e}_2,\ldots,\boldsymbol{e}_N]^T$, $G(\boldsymbol{y}) = [\frac{\partial J_1(\boldsymbol{y}_1)}{\partial x_1},\frac{\partial J_2(\boldsymbol{y}_2)}{\partial x_2},\ldots,\frac{\partial J_N(\boldsymbol{y}_N)}{\partial x_N}]^T$, and $\boldsymbol{\xi}(t) = [\xi_{11},\xi_{12},\ldots,\xi_{1N},\xi_{21},\ldots,\xi_{N1},\ldots,\xi_{NN}]^T = Q\boldsymbol{e}$. Then, the seeking strategy (2) and (3) can be rewritten as

$$\dot{\boldsymbol{x}} = -\eta_1\mathrm{sig}^2(G(\boldsymbol{y})) - \eta_2\mathrm{sig}^{\frac{1}{2}}(G(\boldsymbol{y})) \tag{4}$$

$$\begin{aligned}\dot{\boldsymbol{y}} =& -c_1\mathrm{sig}^3(\boldsymbol{\xi}(t)) - c_2\mathrm{sign}(\boldsymbol{\xi}(t)) - c_3\boldsymbol{\xi}(t) - c_4\mathrm{sig}^2(\boldsymbol{\xi}(t))\\ &- c_5\mathrm{sig}^{\frac{1}{2}}(\boldsymbol{\xi}(t)) - c_6\mathrm{sig}^{\frac{1}{3}}(\boldsymbol{\xi}(t)) - c_7\mathrm{sig}^{\frac{5}{3}}(\boldsymbol{\xi}(t)).\end{aligned} \tag{5}$$

Remark 3: The first five terms and the seventh term in (3) or (5) are necessary to compensate for the positive terms in the time derivative of the Lyapunov function. The last two terms are needed to complement the format given in Lemma 1. More details are included in the proof of Theorem 1.

Before presenting Theorem 1, we introduce the following lemma.

Lemma 5: For the real numbers $c$ and $d$, $|\mathrm{sig}^{\frac{1}{2}}(c) - \mathrm{sig}^{\frac{1}{2}}(d)| \le \sqrt{2}|c-d|^{\frac{1}{2}}$ and $|\mathrm{sig}^2(c) - \mathrm{sig}^2(d)| \le |c-d|^2 + 2|d||c-d|$.

Proof: Four cases should be considered.

Case 1: $c \ge 0$, $d \ge 0$, then $|\mathrm{sig}^{\frac{1}{2}}(c) - \mathrm{sig}^{\frac{1}{2}}(d)| = ||c|^{\frac{1}{2}} - |d|^{\frac{1}{2}}| = \frac{||c|-|d||}{||c|^{\frac{1}{2}}+|d|^{\frac{1}{2}}|} \le \frac{|c-d|}{|c-d|^{\frac{1}{2}}} = |c-d|^{\frac{1}{2}}$ and $|\mathrm{sig}^2(c) - \mathrm{sig}^2(d)| = |c^2-d^2| = |c+d||c-d| = |c-d+2d||c-d| \le |c-d|^2 + 2|d||c-d|$.

Case 2: $c < 0$, $d < 0$, then $|\mathrm{sig}^{\frac{1}{2}}(c) - \mathrm{sig}^{\frac{1}{2}}(d)| = |-|c|^{\frac{1}{2}} + |d|^{\frac{1}{2}}| \le |c-d|^{\frac{1}{2}}$ and $|\mathrm{sig}^2(c) - \mathrm{sig}^2(d)| = |-c^2+d^2| \le |c-d|^2 + 2|d||c-d|$ following the same argument as in Case 1.

Case 3: $c \ge 0$, $d < 0$, then $|\mathrm{sig}^{\frac{1}{2}}(c) - \mathrm{sig}^{\frac{1}{2}}(d)| = ||c|^{\frac{1}{2}} + |d|^{\frac{1}{2}}| \le \sqrt{2}(|c|+|d|)^{\frac{1}{2}} = \sqrt{2}|c-d|^{\frac{1}{2}}$ and $|\mathrm{sig}^2(c) - \mathrm{sig}^2(d)| = |c^2+d^2| \le |c-d|^2$ following Lemma 3.

Case 4: $c < 0$, $d \ge 0$, then $|\mathrm{sig}^{\frac{1}{2}}(c) - \mathrm{sig}^{\frac{1}{2}}(d)| = |-|c|^{\frac{1}{2}} - |d|^{\frac{1}{2}}| \le \sqrt{2}|c-d|^{\frac{1}{2}}$ and $|\mathrm{sig}^2(c) - \mathrm{sig}^2(d)| = |-c^2-d^2| \le |c-d|^2$ following the same argument as in Case 3.

These conclude the lemma. ■

Let $\lambda_1(\cdot)$ and $\lambda_N(\cdot)$ denote the smallest and largest eigenvalues of "·", respectively. Define $\bar{l} = \max_i\{l_i\}$ and $\bar{H} = \sup_{\boldsymbol{x}\in\mathbb{R}^N}\|H(\boldsymbol{x})\|$.

Theorem 1: Under Assumptions 1−4, the players achieve the NE in a fixed time by utilizing (2) and (3) if there exists suitable $\nu_1,\nu_2,\nu_3,\nu_4,\nu_5,\epsilon_1,\epsilon_2 > 0$ such that $\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,b_3 \ge 0$ and $\alpha_7,b_1 > 0$, where $\alpha_1 = \frac{4}{\lambda_N^2(Q)}(\frac{c_1}{N^2}\lambda_1^4(Q) - \frac{\eta_1\bar{l}^2\bar{H}}{2\nu_1} - \frac{\eta_1\bar{l}\bar{H}}{2\nu_3\nu_4})$, $\alpha_2 = (\frac{2}{\lambda_N(Q)})^{\frac{1}{2}}(c_2\lambda_1(Q) - \frac{1}{2\nu_2}(2\sqrt{N}\bar{l})^{\frac{1}{2}}\eta_2\bar{H})$, $\alpha_3 = \frac{2}{\lambda_N(Q)}(c_3\lambda_1^2(Q) - \frac{1}{2\epsilon_1}\sqrt{N}\lambda_N(Q))$, $\alpha_4 = (\frac{2}{\lambda_N(Q)})^{\frac{3}{2}}(\frac{c_4}{N}\lambda_1^3(Q) - \eta_1\sqrt{N}\lambda_N(Q)\bar{l}^2)$, $\alpha_5 = (\frac{2}{\lambda_N(Q)})^{\frac{3}{4}}(c_5\lambda_1^{\frac{3}{2}}(Q) - (2N^{\frac{3}{2}}\bar{l})^{\frac{1}{2}}\eta_2\lambda_N(Q))$, $\alpha_7 = (\frac{2}{\lambda_N(Q)})^{\frac{4}{3}}(\frac{c_7}{N^{\frac{2}{3}}}\lambda_1^{\frac{8}{3}}(Q) - \frac{3}{2}\epsilon_2^{-\frac{1}{3}}\eta_1\sqrt{N}\lambda_N(Q)\bar{l})$, $b_1 = \frac{\eta_1^2\nu_5 m}{N} - \frac{\epsilon_2}{2}\eta_1\sqrt{N}\lambda_N(Q)\bar{l} - \frac{\nu_4}{2\nu_3}\eta_1\bar{l}\bar{H}$, and $b_3 = (1-\nu_5)m - \frac{\epsilon_1}{2}\sqrt{N}\lambda_N(Q) - \frac{\nu_1}{2}\eta_1\bar{l}^2\bar{H} - \frac{\nu_2}{2}(2\sqrt{N}\bar{l})^{\frac{1}{2}}\eta_2\bar{H} - \nu_3\eta_1\bar{l}\bar{H}$.

Proof: Design the Lyapunov function as $V = V_1 + V_2$ where $V_1 = \sum_{i=1}^N\int_0^{\frac{\partial J_i(\boldsymbol{x})}{\partial x_i}}(\eta_1\mathrm{sig}^2(\tau) + \eta_2\mathrm{sig}^{\frac{1}{2}}(\tau))d\tau$ and $V_2 = \frac{1}{2}\boldsymbol{e}^T Q\boldsymbol{e}$. It can be easily checked that $V$ is positive definite and radially unbounded. Solve for the integral in $V_1$ by applying Lemma 3 results in $V_1 = \frac{\eta_1}{3}\sum_{i=1}^N|\frac{\partial J_i(\boldsymbol{x})}{\partial x_i}|^3 + \frac{2}{3}\eta_2\sum_{i=1}^N|\frac{\partial J_i(\boldsymbol{x})}{\partial x_i}|^{\frac{3}{2}} \le \frac{\eta_1}{3}(\sum_{i=1}^N|\frac{\partial J_i(\boldsymbol{x})}{\partial x_i}|^2)^{\frac{3}{2}} + \frac{2}{3}\eta_2 N^{\frac{1}{4}}(\sum_{i=1}^N|\frac{\partial J_i(\boldsymbol{x})}{\partial x_i}|^2)^{\frac{3}{4}} = \frac{\eta_1}{3}\|G(\boldsymbol{x})\|^3 + \frac{2}{3}\eta_2 N^{\frac{1}{4}}\|G(\boldsymbol{x})\|^{\frac{3}{2}}$. Using Lemma 4, one has $\|G(\boldsymbol{x})\|^2 = (\|G(\boldsymbol{x})\|^{\frac{4}{3}})^1(\|G(\boldsymbol{x})\|^{\frac{1}{3}})^2 \le \frac{1}{3}\|G(\boldsymbol{x})\|^4 + \frac{2}{3}\|G(\boldsymbol{x})\|$. Together with Lemma 3, $N^{-\frac{1}{3}}V_1^{\frac{4}{3}} + V_1^{\frac{2}{3}} \le (\frac{\eta_1}{3})^{\frac{4}{3}}\|G(\boldsymbol{x})\|^4 + [(\frac{2}{3}\eta_2 N^{\frac{1}{4}})^{\frac{4}{3}} + (\frac{\eta_1}{3})^{\frac{2}{3}}]\|G(\boldsymbol{x})\|^2 + (\frac{2}{3}\eta_2 N^{\frac{1}{4}})^{\frac{2}{3}}\|G(\boldsymbol{x})\| \le d_1\|G(\boldsymbol{x})\|^4 + d_2\|G(\boldsymbol{x})\|$, where $d_1 = (\frac{\eta_1}{3})^{\frac{4}{3}} + \frac{1}{2}(\frac{2}{3})^{\frac{7}{3}}\eta_2^{\frac{4}{3}}N^{\frac{1}{3}} + (\frac{1}{3})^{\frac{5}{3}}\eta_1^{\frac{2}{3}}$ and $d_2 = (\frac{2}{3})^{\frac{7}{3}}\eta_2^{\frac{4}{3}}N^{\frac{1}{3}} + 2(\frac{1}{3})^{\frac{5}{3}}\eta_1^{\frac{2}{3}} + (\frac{2}{3})^{\frac{2}{3}}\eta_2^{\frac{2}{3}}N^{\frac{1}{6}}$.

Let $P(\boldsymbol{x}) = \eta_1\mathrm{sig}^2(G(\boldsymbol{x})) + \eta_2\mathrm{sig}^{\frac{1}{2}}(G(\boldsymbol{x}))$ and $P(\boldsymbol{y}) = \eta_1\mathrm{sig}^2(G(\boldsymbol{y})) + \eta_2\mathrm{sig}^{\frac{1}{2}}(G(\boldsymbol{y}))$. Following Lemma 3, it is easy to obtain $\|P(\boldsymbol{x})\|^2 = \sum_{i=1}^N(\eta_1\mathrm{sig}^2(\frac{\partial J_i(\boldsymbol{x})}{\partial x_i}) + \eta_2\mathrm{sig}^{\frac{1}{2}}(\frac{\partial J_i(\boldsymbol{x})}{\partial x_i}))^2 \ge \sum_{i=1}^N(\eta_1^2|\frac{\partial J_i(\boldsymbol{x})}{\partial x_i}|^4 + \eta_2^2|\frac{\partial J_i(\boldsymbol{x})}{\partial x_i}|) \ge \eta_1^2 N^{-1}\|G(\boldsymbol{x})\|^4 + \eta_2^2\|G(\boldsymbol{x})\|$.

Differentiate $V_1$ with respect to time yields $\dot{V}_1 = -P(\boldsymbol{x})^T H(\boldsymbol{x}) \times [P(\boldsymbol{x}) + \eta_1(\mathrm{sig}^2(G(\boldsymbol{y})) - \mathrm{sig}^2(G(\boldsymbol{x}))) + \eta_2(\mathrm{sig}^{\frac{1}{2}}(G(\boldsymbol{y})) - \mathrm{sig}^{\frac{1}{2}}(G(\boldsymbol{x})))]$.

Following Lemma 5 and Remark 1, one has $|\mathrm{sig}^{\frac{1}{2}}(\frac{\partial J_i(\boldsymbol{y}_i)}{\partial x_i}) - \mathrm{sig}^{\frac{1}{2}}(\frac{\partial J_i(\boldsymbol{x})}{\partial x_i})| \le (2\bar{l})^{\frac{1}{2}}\|\boldsymbol{e}_i\|^{\frac{1}{2}}$ and $|\mathrm{sig}^2(\frac{\partial J_i(\boldsymbol{y}_i)}{\partial x_i}) - \mathrm{sig}^2(\frac{\partial J_i(\boldsymbol{x})}{\partial x_i})| \le \bar{l}^2\|\boldsymbol{e}_i\|^2 + 2\bar{l} \times |\frac{\partial J_i(\boldsymbol{x})}{\partial x_i}|\|\boldsymbol{e}_i\|$. Further, the stacked vector forms satisfy

$$\|\mathrm{sig}^{\frac{1}{2}}(G(\boldsymbol{y})) - \mathrm{sig}^{\frac{1}{2}}(G(\boldsymbol{x}))\| \le (2\sqrt{N}\bar{l})^{\frac{1}{2}}\|\boldsymbol{e}\|^{\frac{1}{2}} \tag{6}$$

$$\|\mathrm{sig}^2(G(\boldsymbol{y})) - \mathrm{sig}^2(G(\boldsymbol{x}))\| \le \bar{l}^2\|\boldsymbol{e}\|^2 + 2\bar{l}\|G(\boldsymbol{x})\|\|\boldsymbol{e}\|. \tag{7}$$

Substitute them into $\dot{V}_1$,

$$\begin{aligned}\dot{V}_1 \le& -m\|P(\boldsymbol{x})\|^2 + 2\eta_1\bar{l}\bar{H}\|P(\boldsymbol{x})\|\|G(\boldsymbol{x})\|\|\boldsymbol{e}\|\\ &+ \eta_1\bar{l}^2\bar{H}\|P(\boldsymbol{x})\|\|\boldsymbol{e}\|^2 + (2\sqrt{N}\bar{l})^{\frac{1}{2}}\eta_2\bar{H}\|P(\boldsymbol{x})\|\|\boldsymbol{e}\|^{\frac{1}{2}}\\ \le& -\nu_5 m(\eta_1^2 N^{-1}\|G(\boldsymbol{x})\|^4 + \eta_2^2\|G(\boldsymbol{x})\|)\\ &- (1-\nu_5)m\|P(\boldsymbol{x})\|^2 + \frac{1}{2}\eta_1\bar{l}^2\bar{H}(\nu_1\|P(\boldsymbol{x})\|^2 + \frac{1}{\nu_1}\|\boldsymbol{e}\|^4)\\ &+ \frac{\sqrt{2}}{2}(\sqrt{N}\bar{l})^{\frac{1}{2}}\eta_2\bar{H}(\nu_2\|P(\boldsymbol{x})\|^2 + \frac{1}{\nu_2}\|\boldsymbol{e}\|)\\ &+ \nu_3\eta_1\bar{l}\bar{H}\|P(\boldsymbol{x})\|^2 + \frac{1}{2\nu_3}\eta_1\bar{l}\bar{H}(\nu_4\|G(\boldsymbol{x})\|^4 + \frac{1}{\nu_4}\|\boldsymbol{e}\|^4).\end{aligned}$$

The second inequality above is derived using Lemma 4. Take the time derivative of $V_2$ yields

$$\begin{aligned}\dot{V}_2 =& \boldsymbol{e}^{\mathrm{T}}Q\dot{\boldsymbol{e}} \le \sum_{i=1}^N\sum_{j=1}^N\xi_{ij}\dot{y}_{ij} + \sqrt{N}\lambda_N(Q)\|\boldsymbol{e}\|\|P(\boldsymbol{y})\|\\ \le& -\frac{c_1}{N^2}(\sum_{i=1}^N\sum_{j=1}^N\xi_{ij}^2)^2 - c_2(\sum_{i=1}^N\sum_{j=1}^N\xi_{ij}^2)^{\frac{1}{2}} - c_3\sum_{i=1}^N\sum_{j=1}^N\xi_{ij}^2\\ &- \frac{c_4}{N}(\sum_{i=1}^N\sum_{j=1}^N\xi_{ij}^2)^{\frac{3}{2}} - c_5(\sum_{i=1}^N\sum_{j=1}^N\xi_{ij}^2)^{\frac{3}{4}} - c_6(\sum_{i=1}^N\sum_{j=1}^N\xi_{ij}^2)^{\frac{2}{3}}\\ &- \frac{c_7}{N^{\frac{2}{3}}}(\sum_{i=1}^N\sum_{j=1}^N\xi_{ij}^2)^{\frac{4}{3}} + \frac{1}{2}\sqrt{N}\lambda_N(Q)(\epsilon_1\|P(\boldsymbol{x})\|^2 + \frac{1}{\epsilon_1}\|\boldsymbol{e}\|^2)\\ &+ \eta_1\sqrt{N}\lambda_N(Q)\bar{l}^2\|\boldsymbol{e}\|^3 + (2N^{\frac{3}{2}}\bar{l})^{\frac{1}{2}}\eta_2\lambda_N(Q)\|\boldsymbol{e}\|^{\frac{3}{2}}\\ &+ 2\eta_1\sqrt{N}\lambda_N(Q)\bar{l}\|G(\boldsymbol{x})\|\|\boldsymbol{e}\|^2\\ \le& -\frac{c_1}{N^2}\lambda_1^4(Q)\|\boldsymbol{e}\|^4 - c_2\lambda_1(Q)\|\boldsymbol{e}\| - c_3\lambda_1^2(Q)\|\boldsymbol{e}\|^2\\ &- \frac{c_4}{N}\lambda_1^3(Q)\|\boldsymbol{e}\|^3 - c_5\lambda_1^{\frac{3}{2}}(Q)\|\boldsymbol{e}\|^{\frac{3}{2}} - c_6\lambda_1^{\frac{4}{3}}(Q)\|\boldsymbol{e}\|^{\frac{4}{3}}\\ &- \frac{c_7}{N^{2/3}}\lambda_1^{\frac{8}{3}}(Q)\|\boldsymbol{e}\|^{\frac{8}{3}} + \frac{1}{2}\sqrt{N}\lambda_N(Q)(\epsilon_1\|P(\boldsymbol{x})\|^2 + \frac{1}{\epsilon_1}\|\boldsymbol{e}\|^2)\\ &+ \eta_1\sqrt{N}\lambda_N(Q)\bar{l}^2\|\boldsymbol{e}\|^3 + (2N^{\frac{3}{2}}\bar{l})^{\frac{1}{2}}\eta_2\lambda_N(Q)\|\boldsymbol{e}\|^{\frac{3}{2}}\\ &+ \frac{\epsilon_2}{2}\eta_1\sqrt{N}\lambda_N(Q)\bar{l}\|G(\boldsymbol{x})\|^4 + \frac{3}{2}\epsilon_2^{-\frac{1}{3}}\eta_1\sqrt{N}\lambda_N(Q)\bar{l}\|\boldsymbol{e}\|^{\frac{8}{3}}\end{aligned}$$

by observing that $\|G(\boldsymbol{x})\|\|\boldsymbol{e}\|^2 = \|G(\boldsymbol{x})\|(\|\boldsymbol{e}\|^{\frac{2}{3}})^3 \le \frac{1}{4}\epsilon_2\|G(\boldsymbol{x})\|^4 + \frac{3}{4}\epsilon_2^{-\frac{1}{3}}\|\boldsymbol{e}\|^{\frac{8}{3}}$ following Lemma 4. The second inequality is derived from Lemmas 3 and 4, along with (6) and (7). As $\|\boldsymbol{e}\| \ge \sqrt{\frac{2V_1}{\lambda_N(Q)}}$, one has $\dot{V} \le -\alpha_1 V_2^2 - \alpha_2 V_2^{\frac{1}{2}} - \alpha_3 V_2 - \alpha_4 V_2^{\frac{3}{2}} - \alpha_5 V_2^{\frac{3}{4}} - \alpha_6 V_2^{\frac{2}{3}} - \alpha_7 V_2^{\frac{4}{3}} - b_1\|G(\boldsymbol{x})\|^4 - b_2\|G(\boldsymbol{x})\| - b_3\|P(\boldsymbol{x})\|^2 \le -\beta_1(N^{-\frac{1}{3}}V_1^{\frac{4}{3}} + V_1^{\frac{2}{3}}) - \alpha_6 V_2^{\frac{2}{3}} - \alpha_7 V_2^{\frac{4}{3}} \le -\min\{\alpha_6,\beta_1\} \times V^{\frac{2}{3}} - 2^{-\frac{1}{3}}\min\{\alpha_7, \frac{\beta_1}{N^{\frac{1}{3}}}\}V^{\frac{4}{3}}$, where $\alpha_6 = (\frac{2}{\lambda_N(Q)})^{\frac{2}{3}}c_6\lambda_1^{\frac{4}{3}}(Q)$, $b_2 = \eta_2^2\nu_5 m$, and $\beta_1 = \min\{\frac{b_1}{d_1}, \frac{b_2}{d_2}\}$. By Lemma 1, the upper bound of the settling time is $T_{\max} = \frac{3}{\min\{\alpha_6,\beta_1\}} + \frac{3\times 2^{1/3}}{\min\{\alpha_7,\frac{\beta_1}{N^{1/3}}\}}$. ∎

Remark 4: The convergence rate of estimates is faster than that of actions in this letter, so accurate estimated values are used to update actions after some time. In [21], actions and estimates converge almost simultaneously.

**Simulation results:** Consider a network of three players, whose communication topology is shown in Fig. 1. They are playing an energy consumption game of heating ventilation and air conditioning system (HVAC) [4]. Each player has a cost function $J_i(\boldsymbol{x}) = r_i(x_i - \tilde{x}_i)^2 + (k_1\sum_{j=1}^{N} x_j + k_0)x_i$ for $i \in \{1,2,3\}$, where $r_i, k_1, k_0$, and $\tilde{x}_i$ are the thermal coefficient, the elasticity of the electricity rate, the basic electricity rate, and the minimum energy that player $i$ needs to maintain the desired temperature, respectively. The cost function is an aggregate of the load curtailment cost (first term) and the payment for energy consumption (second term). Let $k_1 = 0.001$ USD/kWh$^2$, $k_0 = 0.2$ USD/kWh, $\tilde{x}_1 = 15$ kWh, $\tilde{x}_2 = 20$ kWh, $\tilde{x}_3 = 30$ kWh, and $r_i = 0.1$. The unique NE is $\boldsymbol{x}^* = [13.63, 18.60, 28.55]^T$ kWh. Meanwhile, $m = 0.2010$, $\bar{l} = 0.2020$, and $\bar{H} = 0.2040$.

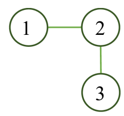


Fig. 1. Communication topology between the three players.

Assume the initial action profile is $\boldsymbol{x}(0) = [6, 25, 16]^T$ kWh and the initial estimates are $\boldsymbol{y}_1(0) = [3, 20, 20]^T$ kWh, $\boldsymbol{y}_2(0) = [10, 22, 13]^T$ kWh, and $\boldsymbol{y}_3(0) = [9, 28, 18]^T$ kWh. The gains are $c_1 = 3700$, $c_2 = 1000$, $c_3 = 4600$, $c_4 = 400$, $c_5 = 100$, $c_6 = 30$, $c_7 = 2000$, $\eta_1 = 2$, and $\eta_2 = 1.5$. This set of parameters satisfies the conditions given in Theorem 1 when $\nu_1 = 1$, $\nu_2 = 0.0005$, $\nu_3 = 1$, $\nu_4 = 0.02$, $\nu_5 = 0.387$, $\epsilon_1 = 0.01$, and $\epsilon_1 = 0.003$.

Fig. 2 displays the evolution of the players' actions under the proposed fixed-time NE seeking algorithm, which obtains the NE in 7 s. Fig. 3 illustrates the players' estimates of the actions, with the estimated values rapidly converging to the actual actions.

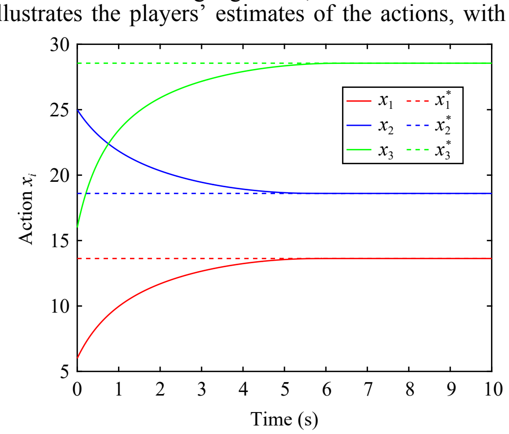


Fig. 2. Plot of the actions of players.

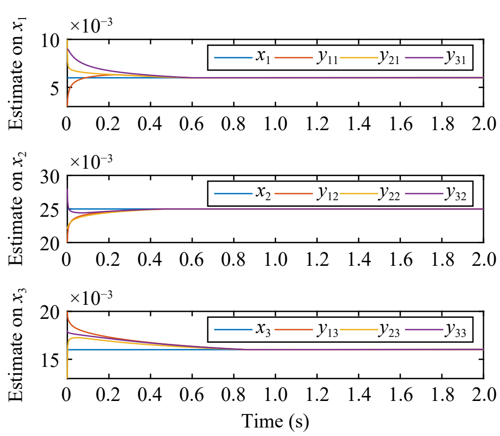


Fig. 3. Plot of the players' estimates on actions.

**Conclusion:** This letter investigates the fixed-time stability of the NE in networked games and provides an upper bound for the settling time. Future work can extend the proposed NE seeking algorithm to achieve prescribed-time convergence, realize fully distributed control, incorporate an event-triggered mechanism, or tackle practical situations such as switching topologies, external disturbances, and players with higher-order dynamics. It can also be adapted to solve different types of games, such as aggregative games and multi-cluster games.

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## References

[1] B. Gharesifard and J. Cortés, "Distributed convergence to Nash equilibria in two-network zero-sum games," *Automatica*, vol. 49, no. 6, pp. 1683–1692, 2013.

[2] J. R. Marden, G. Arslan, and J. S. Shamma, "Cooperative control and potential games," *IEEE Trans. Syst., Man, Cyber., B*, vol. 39, no. 6, pp. 1393–1407, 2009.

[3] B. Ning, Q.-L. Han, Z. Zuo, J. Jin, and J. Zheng, "Collective behaviors of mobile robots beyond the nearest neighbor rules with switching topology," *IEEE Trans. Cybern.*, vol. 48, no. 5, pp. 1577–1590, 2018.

[4] M. Ye and G. Hu, "Game design and analysis for price-based demand response: An aggregate game approach," *IEEE Trans. Cyber.*, vol. 47, no. 3, pp. 720–730, 2017.

[5] X. Na and D. Cole, "Theoretical and experimental investigation of driver noncooperative-game steering control behavior," *IEEE/CAA J. Autom. Sinica*, vol. 8, no. 1, pp. 189–205, 2021.

[6] P. Frihauf, M. Krstic, and T. Basar, "Nash equilibrium seeking in noncooperative games," *IEEE Trans. Autom. Control*, vol. 57, no. 5, pp. 1192–1207, 2012.

[7] F. Salehisadaghiani and L. Pavel, "Distributed Nash equilibrium seeking: A gossip-based algorithm," *Automatica*, vol. 72, pp. 209–216, 2016.

[8] M. Ye and G. Hu, "Distributed Nash equilibrium seeking by a consensus based approach," *IEEE Trans. Autom. Control*, vol. 62, no. 9, pp. 4811–4818, 2017.

[9] Z. Zuo and L. Tie, "Distributed robust finite-time nonlinear consensus protocols for multi-agent systems," *Int. J. Syst. Sci.*, vol. 47, no. 6, pp. 1366–1375, 2016.

[10] L. Ding, Q.-L. Han, X. Ge, and X -M. Zhang, "An overview of recent advances in event-triggered consensus of multiagent systems," *IEEE Trans. Cyber.*, vol. 48, no. 4, pp. 1110–1123, 2018.

[11] J. Liu, Y. Zhang, Y. Yu, and C. Sun, "Fixed-time event-triggered consensus for nonlinear multiagent systems without continuous communications," *IEEE Trans. Syst., Man, Cyber., Syst.*, vol. 49, no. 11, pp. 2221–2229, 2019.

[12] J. Ni and P. Shi, "Adaptive neural network fixed-time leader-follower consensus for multiagent systems with constraints and disturbances," *IEEE Trans. Cyber.*, vol. 51, no. 4, pp. 1835–1848, 2021.

[13] J. Liu, Y. Wu, M. Sun, and C. Sun, "Fixed-time cooperative tracking for delayed disturbed multi-agent systems under dynamic event-triggered control," *IEEE/CAA J. Autom. Sinica*, vol. 9, no. 5, pp. 930–933, 2022.

[14] B. Ning, Q.-L. Han, Z. Zuo, L. Ding, Q. Lu, and X. Ge, "Fixed-time and prescribed-time consensus control of multiagent systems and its applications: A survey of recent trends and methodologies," *IEEE Trans. Ind. Inf.*, vol. 19, no. 2, pp. 1121–1135, 2023.

[15] D. Gadjov and L. Pavel, "A passivity-based approach to Nash equilibrium seeking over networks," *IEEE Trans. Autom. Control*, vol. 64, no. 3, pp. 1077–1092, 2019.

[16] M. Ye, D. Li, Q.-L. Han, and L. Ding, "Distributed Nash equilibrium seeking for general networked games with bounded disturbances," *IEEE/CAA J. Autom. Sinica*, vol. 10, no. 2, pp. 376–387, 2023.

[17] M. Ye, Q.-L. Han, L. Ding, and S. Xu, "Fully distributed Nash equilibrium seeking for high-order players with actuator limitations," *IEEE/CAA J. Autom. Sinica*, vol. 10, no. 6, pp. 1434–1444, 2023.

[18] M. Ye and G. Hu, "Distributed Nash equilibrium seeking in multiagent games under switching communication topologies," *IEEE Trans. Cyber.*, vol. 48, no. 11, pp. 3208–3217, 2018.

[19] X. Fang, J. Lü, and G. Wen, "Distributed finite-time Nash equilibrium seeking for non-cooperative games," *CSIAM Trans. Appl. Math.*, vol. 2, no. 1, pp. 162–174, 2021.

[20] J. I. Poveda, M. Krstić, and T. Başar, "Fixed-time Nash equilibrium seeking in time-varying networks," *IEEE Trans. Autom. Control*, vol. 68, no. 4, pp. 1954–1969, 2023.

[21] Y. Zhao, Q. Tao, C. Xian, Z. Li, and Z. Duan, "Prescribed-time distributed Nash equilibrium seeking for noncooperation games," *Automatica*, vol. 151, p. 110933, 2023.

[22] J. Liu and P. Yi, "Predefined-time distributed Nash equilibrium seeking for noncooperative games with event-triggered communication," *IEEE Trans. Circuits and Syst. II, Exp. Briefs*, vol. 70, no. 9, pp. 3434–3438, 2023.